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On Continuous Capacities

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Abstract. Let $(X, \mathcal{W})$ be a balayage space, γ a Choquet capacity on X, $\beta(E)$ the essential base of $E \subset X$ and, for a compact set $K \subset X$, $\alpha(K) = \gamma(\beta(K))$. Then some properties of the set function α are investigated. In particular, it is shown when α is the Choquet capacity. Further, some relation α to the so-called continuous capacity deduced from a kernel on X is given. At last, some open problems from the book [1] by G. Anger are solved.

Keywords: *Capacities, continuous capacities, semipolar sets, essential bases*

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0. Introduction

The negligible (or small) sets play an important role in potential theory. In this note, we will investigate semipolar sets and some set functions describing them.

All our consideration will be done in a balayage space $(X, \mathcal{W})$ (for its definition, basic properties and notions used below we recommend the monograph by J. Bliedtner and W. Hansen [4]). We introduce the notion of α-capacity and study some of its properties. Especially, we give the relation to Borel semipolar sets (see Definition 2.1 and Corollary 2.3). In Theorem 2.1, we solve the problem when an α-capacity is a capacity in Choquet's sense. In Section 3 we deal with a continuous capacity introduced by G. Anger in [1] and we discuss its relation to α-capacities. At last, in Section 4 we solve some open problems from the book by G. Anger [1].

1. Essential base

In this part we recall the notion of essential base and we give some basic properties needed in the following.

Definition 1.1. Let $E \subset X$ and $z \in X$. Then E is said to be ***semipolar*** at z if there exists a fine neighbourhood V of z such that the set $E \cap V$ is semipolar. The set $\beta(E)$ of all points $z \in X$ such that E is not semipolar at z is called the ***essential base*** of E.

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The following lemma is an easy consequence of Definition 1.1.

Lemma 1.1. *Let A, $B \subset X$. Then*

(i) *if $A \subset B$, then $\beta(A) \subset \beta(B)$*

(ii) $\beta(A \cup B) = \beta(A) \cup \beta(B)$

(iii) $\beta(A \cap B) \subset \beta(A) \cap \beta(B)$.

Remark 1.1. The notion of essential base was introduced (into potential theory) by J. Bliedtner and W. Hansen in [3].

Proposition 1.1 (see [4: p. 296]). *Let $E \subset X$. The essential base $\beta(E)$ is the smallest finely closed set $F \subset X$ such that $E \setminus F$ is semipolar.*

Lemma 1.2. *Let E be an arbitrary subset of X. Then*

(i) *if E is finely closed, then $\beta(E) \subset E$*

(ii) *if E is finely open, then $E \subset \beta(E)$*

(iii) *if E is finely open and $A \subset X$, then $\beta(A) \cap E \subset \beta(A \cap E)$.*

In particular, for a compact set $L \subset X$, $\operatorname{int} L \subset \beta(L) \subset L$.

The proof is an easy consequence of Proposition 1.1.

Proposition 1.2. *Let B be a Borel subset of X. Then there exists a sequence $(K_n)_{n=1}^{\infty}$ of compact subsets of B such that*

$$\beta(B) = \beta\left(\bigcup_{n=1}^{\infty} K_n\right) = \overline{\bigcup_{n=1}^{\infty} \beta(K_n)}^{f}$$

where, for a set $E \subset X$, the symbol $\overline{E}^{f}$ denotes the fine closure of E.

Proof. From [4: p. 301] there follows the existence of a sequence $(K_n)_{n=1}^{\infty}$ of compact subsets of B such that $\beta(\cup_{n=1}^{\infty} K_n) = \beta(B)$. The second equality of the assertion follows from [4: p. 297] ∎

Proposition 1.3. *Let $(X, \mathcal{W})$ be a balayage space, $1 \in \mathcal{W}$ and B a Borel subset of X. Then*

$$R_1^{\beta(B)} = \sup\left\{p \in \mathcal{P} : \ p \leq 1 \text{ on } X \text{ and } C(p) \subset B\right\} \tag{1}$$

where the set of functions on the right-hand side is upward directed. In particular,

$$\widehat{R}_1^{\beta(B)} = R_1^{\beta(B)} \qquad \text{and} \qquad C(\widehat{R}_1^{\beta(B)}) \subset \overline{\beta(B)}$$

where, for $u \in \mathcal{W}$, $C(u)$ denotes the superharmonic carrier of u.

Proof. The first part of the assertion follows from [8: p. 502]. Since the function on the right-hand side of (1) is lower semicontinuous it follows that $\widehat{R}_1^{\beta(B)} = R_1^{\beta(B)}$. By [4: p. 252], $C(R_1^{\beta(B)}) \subset \overline{\beta(B)}$, i.e. $C(\widehat{R}_1^{\beta(B)}) \subset \overline{\beta(B)}$ ∎

2. α-capacity

In this section, we introduce the notion of α-capacity and derive some its properties.

Definition 2.1. Let $\mathcal{K}$ denote the set of all compact subsets of X. A set function $\gamma : \mathcal{K} \to [0, \infty]$ is said to be a *Choquet capacity* on X if it satisfies the following conditions:

(i) *Monotonicity*: $\gamma(K) \leq \gamma(L)$ whenever $K, L \in \mathcal{K}$ with $K \subset L$.

(ii) *Strong subadditivity*: $\gamma(K \cap L) + \gamma(K \cup L) \leq \gamma(K) + \gamma(L)$ whenever $K, L \in \mathcal{K}$.

(iii) *Right-continuity*: $\lim_{n\to\infty} \gamma(K_n) = \gamma(K)$ whenever $(K_n)_{n=1}^{\infty}$ is a decreasing sequence of compact sets with intersection K, i.e. if $K_n \downarrow K$.

For an arbitrary set E, we define the *inner capacity* γ_* by

$$\gamma_*(E) = \sup\left\{\gamma(K) : \ K \subset E, \ K \text{compact}\right\}$$

and the *outer capacity* γ^* by

$$\gamma^*(E) = \inf\left\{\gamma_*(U) : \ E \subset U, \ U \text{open}\right\}.$$

We say that $E \subset X$ is *capacitable* if $\gamma_*(E) = \gamma^*(E)$ (for details see, e.g., [4, 5, 10]). In what follows, γ denotes a Choquet capacity.

Lemma 2.1 (see [5: p. 72]). *Every Borel subset of X is capacitable.*

Lemma 2.2 (see [5: p. 70]). *Let $(M_n)_{n=1}^{\infty}$ be an increasing sequence of subsets of X with $\bigcup_{n=1}^{\infty} M_n = M$. Then $\lim_{n\to\infty} \gamma^*(M_n) = \gamma^*(M)$.*

Remark 2.1. It follows from Definition 2.1 that $\gamma(K) = \gamma_*(K)$, whenever K is a compact subset of X. We can extend the set function γ which is defined for compact sets only to capacitable sets $E \subset X$ by defining $\gamma(E) = \gamma_*(E)$. In particular, we write $\gamma(E)$ instead of $\gamma_*(E)$ and $\gamma^*(E)$, whenever E is capacitable.

Definition 2.2. Let $K \subset X$ be a compact set. The *α-capacity* of K is defined as

$$\alpha(K) = \gamma(\beta(K)).$$

If $E \subset X$ is an arbitrary set, then

$$\alpha_*(E) = \sup\left\{\alpha(K) : \ K \subset E, \ K \text{ compact}\right\}$$

is called the *inner α-capacity* of E.

Remark 2.2. It follows from [4: pp. 272 and 297] that $\beta(K)$ is a Borel set. Consequently, the set function α is well defined.

Remark 2.3. It follows from Definition 2.2 that $\alpha_*(K) = \alpha(K)$, whenever $K \subset X$ is a compact set. Further, for an arbitrary subset $A \subset X$, the inequality $\alpha_*(A) \leq \gamma_*(A)$ holds (indeed, if $K \subset A$ is a compact set, then it follows from Lemma 1.2 that $\beta(K) \subset K$ an the rest follows easily from the definition of α_* and the monotonicity of γ_*). The set function α_* is clearly increasing.

Lemma 2.3. *Let K, K_1, K_2 be compact subsets of X. Then for an α-capacity the following assertions are true:*

(i) $0 \leq \alpha(K) \leq \gamma(K)$.

(ii) *Monotonicity: if $K_1 \subset K_2$, then $\alpha(K_1) \leq \alpha(K_2)$.*

(iii) *Strong subadditivity: $\alpha(K_1 \cup K_2) + \alpha(K_1 \cap K_2) \leq \alpha(K_1) + \alpha(K_2)$.*

Proof. The assertion (i) follows from Lemma 1.2 and the monotonicity of γ. The monotonicity of the set operator β and of γ gives assertion (ii). From Lemma 1.1/(ii) and (iii) and the strong subadditivity of γ the assertion (iii) follows ∎

Remark 2.4. In Lemma 2.3, we did not prove the right continuity of an α-capacity on compact sets. The following example shows that this is not true in general:

Consider the potential theory for the heat operator in $\mathbb{R} \times \mathbb{R}$. Let $K = [0,1] \times \{0\}$ and $K_j \subset \mathbb{R} \times \mathbb{R}$ $(j \in \mathbb{N})$ be compact sets such that $K_{j+1} \subset \operatorname{int} K_j$ and $K = \bigcap_{j=1}^{\infty} K_j$. Let further α-^{h}cap denote the α-capacity deduced from the heat capacity ^{h}cap. Then obviously α-${}^h\mathrm{cap}(K) = 0$, since the set K is semipolar. By Lemma 1.2, $K \subset \operatorname{int} K_j \subset \beta(K_j)$ $(j \in \mathbb{N})$. Consequently, ${}^h\mathrm{cap}(K) \leq {}^h\mathrm{cap}(\beta(K_j)) := \alpha\text{-}{}^h\mathrm{cap}(K_j)$ $(j \in \mathbb{N})$. But the heat capacity of K is equal to the Lebesgue measure of K, i.e. ${}^h\mathrm{cap}(K) = \lambda^1(K) = 1$ (see, e.g., [12]), and hence the α-capacity α-^{h}cap is not right continuous on compact sets.

It is natural to ask when the α-capacity is a Choquet capacity and when both notions are identical, i.e., when the equation $\gamma = \alpha$ holds. Theorem 2.1 give us an answer to this question.

Theorem 2.1. *Let $(X, \mathcal{W})$ be a balayage space and γ a Choquet capacity on X. Assume that the condition*

(P) *A compact set $K \subset X$ is polar if and only if $\gamma(K) = 0$*

holds. Then the following conditions are equivalent:

(i) *α is a Choquet capacity on X*

(ii) $\alpha = \gamma$

(iii) *$(X, \mathcal{W})$ satisfies the axiom of polarity, i.e. the semipolar sets in X are polar.*

Proof. Let condition (iii) be satisfied and K be a compact subset of X. By Proposition 1.1, the set $K \setminus \beta(K)$ is semipolar. Using condition (P) we obtain

$$\alpha(K) \leq \gamma(K) \leq \gamma(K \setminus \beta(K)) + \gamma(\beta(K)) = \alpha(K),$$

i.e. $\alpha = \gamma$ (from the validity of condition (P) for compact sets that for Borel subsets of X follows). The implication $(ii) \Rightarrow (i)$ is obvious. Assume that α is a Choquet capacity on X and that in $(X, \mathcal{W})$ the axiom of polarity does not hold, i.e. there exists a non-polar semipolar set $S \subset X$. According to [4: p. 285] there exists a Borel semipolar set S' such that $S' \supset S$. By [4: p. 284] there exists a non-polar compact set $K \subset S'$. Let $(K_n)_{n=1}^{\infty}$ be a sequence of compact sets in X such that

$$K_{n+1} \subset \operatorname{int} K_n \quad (n \in \mathbb{N}) \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} K_n = K.$$

According to Lemma 1.2, $K \subset \operatorname{int} K_n \subset \beta(K_n)$ for all $n \in \mathbb{N}$. Consequently, $\alpha(K_n) \geq \gamma(K) > 0 \quad (n \in \mathbb{N})$ since the set K is non-polar. From the assumption that α is a Choquet capacity on X it follows that $\alpha(K) > 0$. This is a contradiction since the set K is semipolar ∎

Remark 2.5. Let $(X, \mathcal{W})$ be a balayage space and γ a Choquet capacity on X satisfying condition (P) from Theorem 2.1. For a compact set $K \subset X$, let $K \setminus \beta(K)$ be polar. Then the α-capacity α is right continuous on K, i.e. $\lim_{n\to\infty} \alpha(K_n) = \alpha(K)$ whenever $(K_n)_{n=1}^{\infty}$ is a sequence of compact subsets of X such that $K_n \downarrow K$. Indeed, from assumption (P) of Theorem 2.1 it follows that $\gamma(K) = \gamma(\beta(K))$. According to Lemma 1.2, $\beta(K_n) \subset K_n \quad (n \in \mathbb{N})$. Consequently,

$$\alpha(K) \leq \alpha(K_n) = \gamma(\beta(K_n)) \leq \gamma(K_n) \qquad (n \in \mathbb{N}).$$

Since γ is a Choquet capacity, the relations

$$\alpha(K) \leq \lim_{n\to\infty} \alpha(K_n) \leq \gamma(K) = \gamma(\beta(K)) = \alpha(K)$$

hold. Now for a non-polar semipolar set K, the α-capacity α is not right continuous on K.

Theorem 2.2. *Let γ be a Choquet capacity on X satisfying the condition*

(R) *If A is a relatively compact Borel subset of X, then $\gamma(A) = \gamma(\overline{A}^f)$.*

Further, let B be a Borel subset of X. Then $\alpha_*(B) = \gamma(\beta(B))$.

Proof. First let $\alpha_*(B) = \infty$. Choose arbitrary $0 < s \in \mathbb{R}$. Then there exists a compact set $K \subset B$ such that $s < \alpha(K)$. Consequently, $s < \gamma(\beta(B))$ and $\alpha_*(B) = \gamma(\beta(B))$ since s is arbitrary.

Let now $\alpha_*(B) < \infty$ and B be a relatively compact set. Further, let $(K_n)_{n=1}^{\infty}$ $(K_n \subset B)$ as in Proposition 1.2. By Definition 2.2, there exist compact sets $L_n \subset B$ such that $\alpha_*(B) \leq \alpha(L_n) + \frac{1}{n}$ for every $n \in \mathbb{N}$. It follows from Proposition 1.2 and the monotonicity of the operator β that

$$\overline{\bigcup_{n=1}^{\infty} \beta(L_n \cup K_n)}^f = \beta(B). \tag{2}$$

We can assume that $K_n \subset K_{n+1}$ and $L_n \subset L_{n+1}$ for every $n \in \mathbb{N}$. Further,

$$\alpha_*(B) \leq \alpha(L_n \cup K_n) + \frac{1}{n} \qquad \text{and} \qquad \alpha(L_n \cup K_n) \leq \alpha_*(B).$$

This together with Definition 2.2 yields

$$\alpha_*(B) \leq \gamma(\beta(K_n \cup L_n)) + \frac{1}{n} \leq \alpha_*(B) + \frac{1}{n}$$

for every $n \in \mathbb{N}$. By Lemma 2.2, relation (2) and assumption (R), we obtain

$$\alpha_*(B) = \gamma\left(\bigcup_{n=1}^{\infty} \beta(K_n \cup L_n)\right) = \gamma\left(\overline{\bigcup_{n=1}^{\infty} \beta(K_n \cup L_n)}^{f}\right) = \gamma(\beta(B)),$$

i.e. the desired equality.

Let $\alpha_*(B) < \infty$ and B be an arbitrary Borel set. Further, let $(U_n)_{n=1}^{\infty}$ be a sequence of open relatively compact subsets of X such that $U_n \uparrow X$. As proved above,

$$\gamma(\beta(B \cap U_n)) = \alpha_*(B \cap U_n) \leq \alpha_*(B) \qquad (n \in \mathbb{N}).$$

By the monotonicity of γ and Lemma 1.2/(iii) it follows that $\gamma(\beta(B) \cap U_n) \leq \alpha_*(B)$ for all $n \in \mathbb{N}$. Since $\beta(B) \cap U_n \uparrow \beta(B)$, we get according to Lemma 2.2 $\gamma(\beta(B)) \leq \alpha_*(B)$. The converse inequality follows easily from Definition 2.2 ∎

Corollary 2.1. *Let γ be a Choquet capacity on X satisfying condition* (R) *from* Theorem 2.2. *Further, let B be a Borel subset of X and $S \subset X$ a semipolar set. Then* $\alpha_*(B) = \alpha_*(B \setminus S)$.

Proof. First let S be a Borel semipolar set. Obviously, $\beta(B) = \beta(B \setminus S)$. Consequently

$$\alpha_*(B) = \gamma(\beta(B)) = \gamma(\beta(B \setminus S)) = \alpha_*(B \setminus S).$$

If S is an arbitrary semipolar subset of X, then there exists a Borel semipolar set S' such that $S \subset S'$ (see [4: p. 285]). It follows from the monotonicity of α_* that

$$\alpha_*(B \setminus S') \leq \alpha_*(B \setminus S) \leq \alpha_*(B). \tag{3}$$

As proved above, $\alpha_*(B) = \alpha_*(B \setminus S')$. This together with (3) yields the desired equality ∎

Corollary 2.2. *Let γ be a Choquet capacity on X satisfying condition* (R) *from* Theorem 2.2 *and B a Borel subset of X. Then there exists a Borel semipolar set S such that $\alpha_*(B) = \gamma(B \setminus S)$.*

Proof. Let $S = B \setminus \beta(B)$. It follows from [4: pp. 297, 272 and 271] that S is a Borel semipolar set. Further, $B \setminus S \subset \beta(B)$. From the monotonicity of γ and Theorem 2.2 we obtain $\gamma(B \setminus S) \leq \alpha_*(B)$. According to Corollary 2.1 and Remark 2.3 $\alpha_*(B) \leq \gamma(B \setminus S)$ ∎

Corollary 2.3. *Let γ be a Choquet capacity on X satisfying the condition*

(C) *If, for a compact set $K \subset X$, $\gamma(K) = 0$, then K is polar.*

Further, let B be a Borel subset of X. Then B is semipolar if and only if $\alpha_*(B) = 0$.

Proof. Let B be a Borel set and $\alpha_*(B) = 0$. For a compact set $K \subset B$, $\alpha_*(K) = \gamma(\beta(K)) = 0$. Since $\beta(K)$ is a Borel set, it follows from the assumption and [4: p. 248] that the set $\beta(K)$ is polar. But $K = (K \setminus \beta(K)) \cup \beta(K)$. According to Proposition 1.1 the set $K \setminus \beta(K)$ is semipolar. Consequently, every compact set $K \subset B$ is semipolar. By [4: p. 301] it follows that B is semipolar. The rest of the assertion is an easy consequence of the definition of α_* ∎

Corollary 2.4. *Let γ be a Choquet capacity on X satisfying condition* (R) *from Theorem* 2.2 *and let B_1 and B_2 be Borel subsets of X. Then*

$$\alpha_*(B_1 \cup B_2) + \alpha_*(B_1 \cap B_2) \le \alpha_*(B_1) + \alpha_*(B_1).$$

Proof. The assertion follows from Lemma 1.1, Theorem 2.2 and the strong subadditivity and monotonicity of γ ∎

Remark 2.6. As the following example shows, the assumption in Corollary 2.3 that B is a Borel set can not be omitted.

Consider the potential theory for the heat operator in $\mathbb{R} \times \mathbb{R}$. Let T be a set of Berstein type (see, e.g., [11: p. 24] for its existence), $S_1 = \mathbb{R} \times T$ and $S_2 = \mathbb{R} \times CT$ (CA denotes the complement of a set A), and let K be an arbitrary compact subset of S_1. Then $K \subset \mathbb{R} \times L$ for a suitable countable set $L \subset T$. Consequently, $\mathbb{R} \times L$ is semipolar. Let α-${}^h\text{cap}_*$ denote the inner α-capacity deduced from the heat capacity ${}^h\text{cap}$ (the condition from Corollary 2.3 is of course fulfilled). From the monotonicity and the definition of α-${}^h\text{cap}_*$ it follows that α-${}^h\text{cap}(K) = 0$. Consequently, α-${}^h\text{cap}_*(S_1) = 0$. Similarly, α-${}^h\text{cap}_*(S_2) = 0$. Since $S_1 \cup S_2 = \mathbb{R} \times \mathbb{R}$, at least one of the sets S_i $(i = 1, 2)$ is not semipolar. Consequently, there exists a non-semipolar set $A \subset \mathbb{R} \times \mathbb{R}$ such that α-${}^h\text{cap}_*(A) = 0$. By Corollary 2.3, this set cannot be a Borel set.

The sets S_1 and S_2 are an example of sets for which the assertion of Corollary 2.4 does not hold.

Remark 2.7. Let γ be a Choquet capacity on X satisfying condition (R) of Theorem 2.2. For a compact set $K \subset X$, put

$$\tilde{\alpha}(K) = \inf \Big\{ \gamma_*(K \setminus S) : \ S \subset X, \ S \text{ semipolar} \Big\}.$$

Then $\alpha(K) = \tilde{\alpha}(K)$. Indeed, since $S_K = K \setminus \beta(K)$ is semipolar (see Proposition 1.1), we have $\tilde{\alpha}(K) \le \gamma_*(K \setminus S_K) \le \gamma(\beta(K)) = \alpha(K)$. Let $S \subset X$ be an arbitrary semipolar set. According to Corollary 2.1 and Remark 2.3 $\alpha(K) = \alpha_*(K) = \alpha_*(K \setminus S) \le \gamma_*(K \setminus S)$. Taking infimum with respect to $S \subset X$, S semipolar, we get $\alpha(K) \le \tilde{\alpha}(K)$. The proof above shows that the infimum in the definition of $\tilde{\alpha}$ is actually attained.

3. Continuous capacities

In [6], we have investigated the so called **K**-capacity. We recall the basic definitions (cf. [6]).

In the following let X be a locally compact Haussdorff space with a countable base and $\mathcal{M}^+$ the set of all non-negative Radon measures on X. For a set $E \subset X$, let us denote by $\mathcal{M}^+(E)$ the collection of all non-negative Radon measures on X with *compact* support in E (the support of a measure is denoted by supp). A lower semicontinuous function $\mathbf{K}$: $X \times X \to [0, \infty]$ is called a *kernel* on X. The **K**-*potential* of a measure $\mu \in \mathcal{M}^+$ is defined as

$$\mathbf{K}_\mu(x) = \int_X \mathbf{K}(x, y)\, \mu(dy) \qquad (x \in X).$$

For a compact set $L \subset X$, the $\mathbf{K}$-*capacity* (corresponding to the kernel $\mathbf{K}$) is defined by

$$\operatorname{cap}(L) = \sup\Big\{\mu(X) : \ \mu \in \mathcal{M}^+(L),\ \mathbf{K}_\mu \le 1 \ \text{ on } X\Big\}.$$

The adjoint kernel $\widetilde{\mathbf{K}}$ of a kernel $\mathbf{K}$ is defined by $\widetilde{\mathbf{K}}(x,y) = \mathbf{K}(y,x) \quad (x, y \in X)$. Corresponding notions will be noted by a tilde.

By an easy modification of this definition we obtain the notion of a continuous capacity.

Definition 3.1. Let $\mathbf{K}$ be a kernel on X. A set function $\sigma : \mathcal{K} \to [0,\infty]$ defined by

$$\sigma(L) = \sup\Big\{\mu(X) : \ \mu \in \mathcal{M}^+(L),\ \mathbf{K}_\mu \le 1 \text{ and continuous } \mathbf{K} - \text{potential on } X\Big\}$$

is called *continuous* $\mathbf{K}$-*capacity* (corresponding to the kernel $\mathbf{K}$) on X. For $E \subset X$, we define an *inner continuous* $\mathbf{K}$-*capacity* by

$$\sigma_*(E) = \sup\Big\{\sigma(K) : \ K \subset E,\ K \text{ compact } \Big\}.$$

Remark 3.1. The continuous capacity was first introduced into potential theory by G. Anger (see [1: p. 49]). As we will see in Remark 3.3, the continuous capacity is not a Choquet capacity in general. The notation "continuous capacity" is deduced from the requirement of continuity of $\mathbf{K}$-potentials in Definition 3.1.

The following lemma is an easy consequence of Definition 3.1.

Lemma 3.1. *Let c and σ denote the* $\mathbf{K}$-*capacity and the continuous* $\mathbf{K}$-*capacity on X, respectively, and let $K, L \in \mathcal{K}$. Then:*

(i) $0 \le \sigma(L) \le c(L) \le \infty$

(ii) $K \subset L$ *implies* $\sigma(K) \le \sigma(L)$

(iii) $\sigma(K \cup L) \le \sigma(K) + \sigma(L)$.

The question of the relation between a continuous $\mathbf{K}$-capacity and an α-capacity deduced from a $\mathbf{K}$-capacity is natural. Because α-capacity is defined by using the structure of a balayage space and a continuous capacity does not, the kernel $\mathbf{K}$ on X must be in some relation with a balayage space.

From now, we will consider a balayage space $(X, \mathcal{W})$ and a kernel $\mathbf{K}$ on X for which there exists a balayage space $(X, \widetilde{\mathcal{W}})$ with the following properties:

- $1 \in \mathcal{W} \cap \widetilde{\mathcal{W}}$.
- For every $p \in \mathcal{P}(X)$ there exists exactly one measure $\mu \in \mathcal{M}^+$ such that $\mathbf{K}_\mu = p$ and $\operatorname{supp}\mu = C(p)$.
- If $\mu \in \mathcal{M}^+$ and $\overline{\{\mathbf{K}_\mu < \infty\}} = X$, then $\mathbf{K}_\mu \in \mathcal{P}(X)$.
- For every $\tilde{p} \in \widetilde{\mathcal{P}}(X)$ there exists exactly one measure $\mu \in \mathcal{M}^+$ such that $\widetilde{\mathbf{K}}_\mu = \tilde{p}$ and $\operatorname{supp}\mu = C(\tilde{p})$ ($\widetilde{\mathbf{K}}$ is the adjoint kernel of the kernel $\mathbf{K}$).

- If $\mu \in \mathcal{M}^+$ and $\overline{\{\widetilde{\mathbf{K}}_\mu < \infty\}} = X$, then $\widetilde{\mathbf{K}}_\mu \in \widetilde{\mathcal{P}}(X)$.

Here and in the following $\mathcal{P}(X)$ and $\widetilde{\mathcal{P}}(X)$ stand for the set of all potentials (with respect to $\mathcal{W}$ and $\widetilde{\mathcal{W}}$, respectively) on X, $C(u)$ and $C(\widetilde{u})$ denote the carrier of $u \in \mathcal{W}$ and $\widetilde{u} \in \widetilde{\mathcal{W}}$, respectively.

The following theorem deals with the question formulated above.

Theorem 3.1. *Let γ be a* $\mathbf{K}$*-capacity on X, σ a continuous* $\mathbf{K}$*-capacity on X(both corresponding to the kernel* $\mathbf{K}$*) and α the α-capacity deduced from the capacity γ. Then, for every compact set $L \subset X$, $\alpha(L) = \sigma(L)$.*

Proof. Let $L \in \mathcal{K}$. According to Proposition 1.3, $\widehat{R}_1^{\beta(L)} = R_1^{\beta(L)}$ and $C(\widehat{R}_1^{\beta(L)}) \subset \overline{\beta(L)}$. By Lemma 1.2, $\beta(L) \subset L$. Consequently, $\overline{\beta(L)} \subset L$. From here and Remark 2.2 it follows that $\beta(L)$ is a relatively compact Borel set. Let $\mu \in \mathcal{M}^+(\overline{\beta(L)})$ be a measure (which existence follows from [6: p. 97]) with the properties

$$\widehat{R}_1^{\beta(L)} = \mathbf{K}_\mu \qquad \text{and} \qquad \gamma(\beta(L)) = \mu(X) = \alpha(L).$$

Further, let $U \subset X$ be an open, relatively compact set, $L \subset U$. The existence of a measure $\nu \in \mathcal{M}^+(U)$ such that

$$\widetilde{\mathbf{K}}_\nu = 1 \quad \text{on a neighbourhood of } L \tag{4}$$

follows from [6: p. 91]. Using the Fubini theorem, we get

$$\alpha(L) = \int_X \widetilde{\mathbf{K}}_\nu \, d\mu = \int_X \mathbf{K}_\mu \, d\nu = \int_X \widehat{R}_1^{\beta(L)} \, d\nu.$$

By Proposition 1.3 and [5: p. 7] we obtain

$$\begin{aligned}\alpha(L) &= \int_X \sup\Big\{\mathbf{K}_{\mu'} : \ \mu' \in \mathcal{M}^+(L),\ \mathbf{K}_{\mu'} \le 1 \text{ and continuous on } X\Big\}\, d\nu \\ &= \sup\bigg\{\int_X \mathbf{K}_{\mu'}\, d\nu : \ \mu' \in \mathcal{M}^+(L),\ \mathbf{K}_{\mu'} \le 1 \text{ and continuous on } X\bigg\}.\end{aligned}$$

Using the Fubini theorem and equality (4), we get

$$\alpha(L) = \sup\Big\{\mu'(L) : \ \mu' \in \mathcal{M}^+(L),\ \mathbf{K}_{\mu'} \le 1 \text{ and continuous on } X\Big\}$$

and hence $\alpha(L) = \sigma(L)$ what we wanted to prove ∎

Remark 3.2. Let the assumptions of Theorem 3.1 be fulfilled. Then condition (R) of Theorem 2.2 is satisfied. Indeed, let B be a Borel relatively compact subset of X. Then, by [4: p. 273], $\widehat{R}_1^B = \widehat{R}_1^{\overline{B}^f}$. It follows from this and [6: p. 97] that $\gamma(B) = \gamma(\overline{B}^f)$.

Remark 3.3. From Theorem 3.1 and Theorem 2.1 it follows that a continuous capacity is a Choquet capacity if and only if the corresponding balayage space satisfies the axiom of polarity. It is known that the balayage space generated by the heat operator does not satisfy this axiom. Consequently, the continuous heat capacity is not a Choquet capacity.

Corollary 3.1. *Let the assumptions of Theorem 3.1 be fulfilled. If $K, L \in \mathcal{K}$, then*

$$\sigma(K \cup L) + \sigma(K \cap L) \le \sigma(K) + \sigma(L).$$

Proof. The assertion follows from Theorem 3.1 and Lemma 2.3/(iii) ∎

Corollary 3.2. *For all $L \in \mathcal{K}$, let $C(\widehat{R}_1^L) \subset L$ and $C(^{\sim}\widehat{R}_1^L) \subset L$. For all $x \in X$, let the set $\{x\}$ be $\mathcal{W}$- and $\widetilde{\mathcal{W}}$-totally thin, and let σ and $\tilde{\sigma}$ denote the continuous* **K***-capacity and the continuous* $\widetilde{\mathbf{K}}$*-capacity on X, respectively. Then, for all $L \in \mathcal{K}$, $\sigma(L) = \tilde{\sigma}(L)$.*

Proof. Let $\mathcal{P}$, $\widetilde{\mathcal{P}}$, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ denote the system of all $\mathcal{W}$-polar, $\widetilde{\mathcal{W}}$-polar, $\mathcal{W}$-semipolar and $\widetilde{\mathcal{W}}$-semipolar subsets in X, respectively. Further, denote by c and $\tilde{c}$ the **K**-capacity and $\widetilde{\mathbf{K}}$-capacity on X, respectively. By [6: p. 97], $\mathcal{P} = \widetilde{\mathcal{P}}$. According to [9: p. 510] it follows that $\mathcal{S} = \widetilde{\mathcal{S}}$. Now, by [6: p. 92] and Remark 2.7 it follows that, for $L \in \mathcal{K}$,

$$\sigma(L) = \inf\{c_*(L \setminus S) : S \in \mathcal{S}\} \qquad \text{and} \qquad \tilde{\sigma}(L) = \inf\{\tilde{c}_*(L \setminus S) : S \in \tilde{\mathcal{S}}\}.$$

By [6: p. 92] $c = \tilde{c}$. From this and above the desired equality follows ∎

Corollary 3.3. *Let the assumptions of Corollary 3.2 be fulfilled. Further let S be a Borel subset of X. Then the following conditions are equivalent:*

(i) *S is $\mathcal{W}$-semipolar*

(ii) $\sigma_*(S) = 0$

(iii) *S is $\widetilde{\mathcal{W}}$-semipolar*

(iv) $\tilde{\sigma}_*(S) = 0$.

Proof. The equivalence (ii) ⇔ (iv) follows from Corollary 3.2, the equivalence (i) ⇔ (iii) from [9: p. 510], and the equivalence (i) ⇔ (ii) from Corollary 2.3 ∎

4. Some open problems

In this final section we will give partial answers to some unsolved problems from the book [1: pp. 94 and 95]. The numbers associated to these problems are those of [1].

Let c and σ denote the **K**-capacity and the continuous **K**-capacity (corresponding to the kernel **K** on X), respectively, and let the assumptions of Theorem 3.1 be fulfilled.

Problem 19. *Which kernels on X satisfy the C-maximum principle?*

Recall that a kernel **K** on X is defined to satisfy the *C-maximum principle*, if the implication

$$\left.\begin{array}{l}\mu \in \mathcal{M}^+(X), \mathbf{K}_\mu \text{ a continuous } \mathbf{K}\text{ - poten-}\\ \text{tial on } X,\ \mathbf{K}_\mu \le M \text{ on supp}\mu \quad (M \in \mathbb{R})\end{array}\right\} \quad \Longrightarrow \quad \mathbf{K}_\mu \le M \text{ on } X$$

is true. In the considered situation all kernels satisfies the C-maximum principle (see [4: p. 116]).

Problem 21. *For which compact sets $L \subset X$ a continuous* **K***-capacity σ is right continuous on L?*

Let $\mathcal{K}_1$ be the system of all compact subsets of X such that the implication

$$L \in \mathcal{K}_1, L_n \in \mathcal{K} \ \ (n \in \mathbb{N}) \text{ and } L_n \downarrow L \quad \Longrightarrow \quad \sigma(L_n) \to \sigma(L) \text{ for } n \to \infty$$

is true. From Remark 2.5 we get the inclusions

$$\Big\{L \in \mathcal{K} : \ L \setminus \beta(L) \text{ polar}\Big\} \subset \mathcal{K}_1 \subset \mathrm{C}\Big\{L \in \mathcal{K} : \ L \text{ non-polar semipolar}\Big\}.$$

These inclusions solve partially our problem.

Problem 22. *For which kernel* $\mathbf{K}$ *the continuous* $\mathbf{K}$*-capacity* σ *is right continuous?*

A full answer to this question is given by Theorem 2.1. The continuous $\mathbf{K}$-capacities σ are right continuous only for kernels $\mathbf{K}$ on X for which the corresponding balayage space satisfies the axiom of polarity.

Problem 23. *Does there exist a kernel* $\mathbf{K}$ *on* $\mathbb{R}$ *such that the continuous* $\mathbf{K}$*-capacity* σ *is right continuous?*

The Riesz kernels N_α $(0 < \alpha < 1)$ on $\mathbb{R}$ have by Theorem 2.1 the desired property.

Problem 29. *Which relation between a continuous* $\mathbf{K}$*-capacity and a continuous* $\widetilde{\mathbf{K}}$*-capacity holds?*

An answer to this problem is given by Corollary 3.2.

Problem 30. *Which sets* $B \subset X$ *are* σ*-capacitable?*

Recall that a set $B \subset X$ is *σ-capacitable* if

$$\sigma_*(B) = \inf\Big\{\sigma_*(U) : \ U \supset B,\, U \text{ open}\Big\} =: \sigma^*(B).$$

If $P \subset X$ is polar, then by [6: p. 97] $c^*(P) = 0$. Further,

$$0 \le \sigma_*(P) \le \sigma^*(P) \le c^*(P) \le 0.$$

Hence, polar sets are σ-capacitable. Set

$$\Sigma = \Big\{L \cup P : \ L \in \mathcal{K},\, P \subset X,\, P \text{ and } L \setminus \beta(L) \text{ polar}\Big\}$$

and let $A = L \cup P \in \Sigma$. Further, let

$$L_n \in \mathcal{K} \ \ (n \in \mathbb{N}) \quad \text{with} \quad L_n \downarrow L \quad \text{and} \quad L \subset \operatorname{int} L_n \ \ (n \in \mathbb{N}).$$

Then

$$\sigma^*(L \cup P) \le \sigma^*(L) + \sigma^*(P) \le \sigma_*(\operatorname{int} L_n) \le \sigma_*(L_n) = c(\beta(L_n)) \le c(L_n).$$

Since c is a Choquet capacity, $\sigma^*(L \cup P) \le c(L)$. Further,

$$\sigma_*(L \cup P) \le \sigma^*(L \cup P) \le c(L) = c(\beta(L)) = \sigma(L) = \sigma_*(L) \le \sigma_*(L \cup P).$$

The sets $A \in \Sigma$ are hence σ-capacitable. Further, non-polar semipolar sets are not σ-capacitable (see Remark 2.5).

References

[1] Anger, G.: *Funktionalanalytische Betrachtungen bei Differentialgleichungen unter Verwendung von Methoden der Potentialtheorie.* Vol. I. Berlin: Akademie-Verlag 1967.

[2] Bauer, H.: *Harmonische Räume und ihre Potentialtheorie.* Lect. Notes Math. 22 (1966), 1 - 175.

[3] Bliedtner, J. and W. Hansen: *Simplicial cones in potential theory.* Inv. Math. 29 (1975), 83 - 110.

[4] Bliedtner, J. and W. Hansen: *Potential Theory. An Analytic and Probabilistic Approach to Balayage.* Berlin: Springer-Verlag 1986.

[5] Brelot, M.: *Éléments de la théorie classique du potentiel.* 2^e^ ed. Paris: Centre Docum. Univ. 1961.

[6] Brzezina, M.: *Kernels and Choquet capacities.* Aequ. Math. 45 (1993), 89 - 99.

[7] Constantinescu, C. and A. Cornea: *Potential Theory on Harmonic Spaces.* Berlin: Springer-Verlag 1972.

[8] Hansen, W.: Semi-polar sets are almost negligible. J. reine angew. Math. 314 (1980), 217 - 220.

[9] Hansen, W.: *Semi-polar sets and quasi-balayage.* Math. Ann. 257 (1981), 495 - 517.

[10] Helms, L. L.: *Introduction to Potential Theory.* New York: Wiley Intersci. 1969.

[11] Oxtoby, C. J.: *Measure and Category.* Berlin: Springer-Verlag 1971.

[12] Watson, N. A.: *Thermal capacity.* Proc. London Math. Soc. (3) 37 (1978), 342 - 362.

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