



Geometry — *Hyperbolicity for Deligne-Mumford analytic stacks and Brody's theorem*, by SIMONE BORGHESI and GIUSEPPE TOMASSINI, communicated on 11 May 2012.

Dedicated to Franco Bassani

ABSTRACT. — We give the overview on a program leading to the proof of the Brody Theorem for Deligne-Mumford analytic stacks, starting from the definitions of Kobayashi and Brody hyperbolicity for these objects. Complete proofs will appear in [2].

KEY WORDS: Homotopical algebra, stacks, Kobayashi hyperbolicity.

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1. INTRODUCTION

In complex geometry the classic theorem of Brody [3] states that a compact complex space is *Kobayashi hyperbolic* [9] if the only holomorphic maps from \mathbb{C} to it are constant. For the basic results in hyperbolicity, implications and conjectures in complex geometry we refer to [9], [4]. The purpose of this paper is to provide notions of hyperbolicity for *analytic stacks* and to outline the proof of Brody's Theorem for compact Deligne-Mumford analytic stack (see Section 7). These objects generalize complex spaces and their existence can be motivated by the need of using algebraic and geometric techniques on objects which lack a scheme or complex space structure, such as moduli spaces or by keeping track of a higher level of information attached to certain objects, like quotients by actions of Lie groups. At the present state of knowledge, we necessarily have to begin by setting the notions of Kobayashi and Brody hyperbolicity. We emphasize that such notions ought at least to generalize the known ones for complex spaces and be categorical equivalence invariant (thus presentation invariant). The classical Brody hyperbolicity condition for a complex space Y , summarized in the bijectivity of $p^* : \text{Hom}_{\text{holo}}(X, Y) \rightarrow \text{Hom}_{\text{holo}}(\mathbb{C} \times X, Y)$ for all complex spaces X and $p : \mathbb{C} \times X \rightarrow X$ being the projection, can be extended to \mathcal{S} -groupoids \mathcal{G} in two ways by requiring the bijectivity of $p^* : \text{Hom}_{\text{Grp}/\mathcal{S}}(\mathcal{X}, \mathcal{G}) \rightarrow \text{Hom}_{\text{Grp}/\mathcal{S}}(\mathbb{C} \times \mathcal{X}, \mathcal{G})$ for all complex spaces $\mathcal{X} = X$ or for all groupoids \mathcal{X} . In analogy with the classical definition we choose the latter; this explains why both the zeroth and first holotopy presheaf (see Section 3) are involved in the Brody hyperbolicity condition, thus the same had to hold for the Kobayashi hyperbolicity. Already in the

paper [1] we have introduced the concept of Brody hyperbolicity for simplicial presheaves of sets over the site of complex spaces with the strong topology. The work in [6] enables us to use this definition in the context of \mathcal{S} -groupoids and this led us to the Definition 4.1. With some more work, we rephrased it in terms of *holotopy* presheaves of the \mathcal{S} -groupoid: Definition 4.3. In this form, the \mathcal{S} -groupoid equivalence invariance of Brody hyperbolicity is evident as well as the fact that it extends the same classical property for complex spaces. We point out that contradicting Brody hyperbolicity of an analytic stack provides a way of showing the existence of \mathbb{C} parametrized families of objects in the moduli problem associated with the stack. Several definitions of Kobayashi hyperbolicity have been considered for presheaves; the one we decided to use preserves the metric “flavour” of the classical notion, and it is based on *relative analytic discs and chains* of an analytic stack (see Subsection 4.2).

The proof of the Brody theorem mixes techniques from abstract homotopy theory and complex variables, the interplay of these seemingly different fields being possible by reducing many crucial arguments to a particularly nice model, which we denote $\mathcal{C}[X]$, associated to an analytic stack $X \rightarrow \mathcal{Y}$. There is a set $Q(\mathcal{Y})$ whose elements are classes of an equivalence relation on the atlas X , the geometrization of which is a key step to make all the parts work together. The algebraization of the colimit of a similar diagram derived from algebraic stacks has been an investigated topic (e.g. [8]). It is related to Brody hyperbolicity by means of the main Theorem 7.2. In Section 6 we introduce the metric invariants of an analytic stack necessary to bridge the Kobayashi hyperbolicity with Brody hyperbolicity. We finish the paper outlining the proof of the analogue of the Brody Theorem. Kobayashi hyperbolicity unconditionally implies Brody hyperbolicity; this is almost immediate for complex spaces, but in this context we have the added difficulty of finding two different admissible sections of $\pi_0^{\text{simpl}}(\mathcal{Y}, y)$ whose pseudodistance is zero. In the opposite implication we extensively used the compactness assumption (see beginning of Section 5) and the existence and properties of the complex space structure of $Q(\mathcal{Y})$. We would like to emphasise that the Kobayashi hyperbolicity mandated the introduction of metric structures on analytic stacks which, in turn, revealed the connection between the hyperbolicity of a Deligne-Mumford analytic stack \mathcal{Y} and the complex space $Q(\mathcal{Y})$.

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2. PRELIMINARIES

2.1. Notation and definitions.

- \mathcal{S}_T is the analytic site: the category \mathcal{S} whose objects are complex spaces and coverings those induced by the strong topology.
- Grp is the category of (set theoretic) groupoids and Grp/\mathcal{S} the category of \mathcal{S} -groupoids, \mathcal{S} a category, whose objects are categories fibered in groupoids. By

$\text{Psh}(\text{Grp})$ we will denote the category of presheaves of set-theoretic groupoids. A *stack* is an \mathcal{S} -groupoid satisfying two supplementary conditions (see [10, Definition 3.1]).

- $\Delta^{op} \text{Prsh}_T(\mathcal{S})_J$ is the category of simplicial presheaves of sets on the site \mathcal{S}_T with a topology T and endowed by *local, injective, simplicial model structure* on $\Delta^{op} \text{Prsh}_T(\mathcal{S})$ (cfr. Joyal’s model structure, [7, Section 5.1]). \mathcal{H}_s (respectively $\mathcal{H}_{s\bullet}$) is the homotopy category (respectively the pointed homotopy category) associated.
- Let \mathcal{X} be a groupoid. Then $c_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}\mathcal{X}$ is its stackification (cfr. [10, Lemma 3.2 and Observation 3.2.1 (3)]).
- The morphisms ∂_0 and ∂_1 will denote the face morphisms of a simplicial presheaf \mathcal{X} . or, more frequently, of groupoids $[X]$ and $\mathcal{C}[X]$ from the presheaf in degree 1 to presheaf.

An *analytic stack* is a diagram $p : X \rightarrow \mathcal{Y}$ comprising a complex space $X = \prod_{i=1}^{+\infty} X^{(i)}$, where the $X^{(i)}$ are connected complex spaces, and a stack \mathcal{Y} over the analytic site \mathcal{S}_T , such that the diagonal $\Delta : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$ is representable, separated, quasi compact and with p smooth and surjective. The morphism $p : X \rightarrow \mathcal{Y}$ is also called a *presentation* of the stack \mathcal{Y} and X an *atlas*.

A stack \mathcal{Y} with an étale presentation is called a *Deligne-Mumford analytic stack*, *DM stack* for short.

We recall that an *étale* map $f : X \rightarrow Y$ between complex spaces is a holomorphic map with differential being an invertible linear map at each point. An *étale covering* is a surjective étale holomorphic map such that for each $y \in Y$ there exists a *uniformly covered* open neighborhood, i.e. $y \in U_y$ such that $f^{-1}U_y = \coprod_i V_i$ as topological space and $f|_{V_i} \rightarrow U_y$ is a biholomorphism for all i .

Theorem 1.1 (see also Theorem 3.9) of [7] states that a groupoid \mathcal{F} , seen as a presheaf of groupoids, is a stack if and only if, for any covering $\coprod_i U_i \rightarrow U$, the canonical morphism

$$\mathcal{F}(U) \rightarrow \text{holim}_n \left\{ \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_{ij}) \rightrightarrows \prod \mathcal{F}(U_{ijk}) \dots \right\}$$

is an equivalence of categories for each complex space U , where U_{i_1, \dots, i_k} stands for $U_{i_1} \times \dots \times U_{i_k}$.

2.2. Simplicial presheaves and groupoids. Before being able to state what we think of as a (differently flavoured) hyperbolic groupoid, we will expose the connection between simplicial presheaves and groupoids. The concept of Brody hyperbolicity, in particular, is directly transposed from simplicial presheaves of sets over the site of complex spaces with the strong topology, already introduced in the paper [1]. Such a relation has been investigated in the papers [7] and [6]. While the concept of groupoids in terms of categories fibered in set-theoretic groupoids (\mathcal{S} -groupoids) probably goes back to ideas of Grothendieck, only recently these objects have been related to the homotopy theory of simplicial presheaves of sets (cfr. [7, Theorem 5.4]).

THEOREM 2.1. *There exists a Quillen equivalence between Grp/\mathcal{S} and $(\mathbb{S}^2)^{-1}\Delta^{op} \text{Prsh}_T(\mathcal{S})_J$, the \mathbb{S}^2 -nullification of the model category studied in [1], inducing an isomorphism between the (full) subcategories of stacks and fibrant simplicial presheaves.*

Involved in the definition of this equivalence is the functor N , which will later be used to define holotopy presheaves of groupoids and the hyperbolicity of groupoids (see Definitions 4.1, 4.5). To an \mathcal{S} -groupoid \mathcal{G} , it associates the simplicial presheaf $N\mathcal{G}$ with $(N\mathcal{G})_0 = \text{Ob}(\mathcal{G})$, $(N\mathcal{G})_1 = \text{Mor}(\mathcal{G})$ and $(N\mathcal{G})_i = (N\mathcal{G})_1 \times_{(N\mathcal{G})_0} \cdot^i \cdot \times_{(N\mathcal{G})_0} (N\mathcal{G})_1$ with the following structural face morphisms: $\partial_0, \partial_1 : (N\mathcal{G})_1 \rightarrow (N\mathcal{G})_0$ are the domain and codomain of the isomorphism, respectively; the three morphisms $(N\mathcal{G})_2 \rightarrow (N\mathcal{G})_1$ send (f, g) respectively in f , $g \circ f \in g$; in the general case an n -tuple of composable isomorphisms are sent to $(n - 1)$ -subtuples involving individual isomorphisms and compositions of them, when applicable.

2.3. Analytic stacks. Let \mathcal{P} be a presheaf of groupoids and $F : \mathcal{P} \rightarrow \mathcal{G}$ be a 1-morphism (functor) to a groupoid \mathcal{G} . We build a groupoid out of it, denoted by $[P.]$. Its objects over a complex space U are the sections in $\mathcal{P}(U)$ and $\text{Hom}_{[P.](U)}(f, g)$ are the sections $\phi \in \mathcal{P}_1(U) := \mathcal{P}(U) \times_{\mathcal{Y}} \mathcal{P}(U)$ such that $\partial_0(\phi) = f$ and $\partial_1(\phi) = g$, where $\partial_i : \mathcal{P}_1 \rightarrow \mathcal{P}$, for $i = 0, 1$, are the projections on the factors, and the fiber product is taken in the category of groupoids. The remaining structure making $[X.]$ a groupoid is inherited by the one of \mathcal{G} and it explained in [10, 2.4.3] and [10, Proposition 3.8]. If $X \rightarrow \mathcal{Y}$ is an analytic stack (see Subsection 2.1), then the objects of $[X.]$ over a complex space U are the holomorphic maps $U \rightarrow X$ and $\partial_i : X_1 \rightarrow X$ are holomorphic maps between complex spaces.

REMARK 2.1. Our notation is slightly different from the one in [10, 2.4.3]: the groupoid $[X.]$ is denoted as $[X.]'$ there. Moreover, through this manuscript, we will identify the \mathcal{S} -espace en groupoides and its associated groupoid.

We recall the following general result (cfr. [10, Prop. 3.8]): let $F : \mathcal{P} \rightarrow \mathcal{Y}$ be a morphism (functor) between a presheaf and a stack. Then the canonical morphism (functor) $[\mathcal{P}.] \rightarrow \mathcal{Y}$ is a monomorphism and is epi if and only if F is.

In the particular case $p : X \rightarrow \mathcal{Y}$ is an analytic stack, we get a simplicial complex space $[X.]$ such that

$$X \times_{\mathcal{Y}} \times \cdot^{i+1} \cdot \times_{\mathcal{Y}} X = X_i = X_1 \times_{\partial_0, X, \partial_1} \cdot^i \cdot \times_{\partial_0, X, \partial_1} X_1.$$

$[X.]$ is a *prestack*, as explained in the example [10, 3.4.3] and is precisely $N([\mathcal{P}.])$ with $\mathcal{P} = X$. To simplify the notation, we will drop the letter N and consider $[X.]$ indifferently as a \mathcal{S} -groupoid or a simplicial complex space, according to the needed properties. Notice that the stackification functor $[X.] \rightarrow \mathcal{C}[X.]$ corresponds to a fibrant resolution of simplicial presheaves. We conclude that

PROPOSITION 2.2. *Let $p : X \rightarrow \mathcal{Y}$ be a analytic stack. Then p induces a groupoid equivalence $p : [X.] \rightarrow \mathcal{Y}$ and a stack equivalence $\mathcal{C}[X.] \rightarrow \mathcal{Y}$.*

An immediate consequence of this proposition is that, to work with a simplicial homotopy invariant, we can indifferently use any presentation and atlas of \mathcal{Y} :

COROLLARY 2.3. *Let $X, Z \rightarrow \mathcal{Y}$ be two presentations of an analytic stack. Then $[X.]$ and $[Z.]$ are equivalent groupoids and $\mathcal{C}[X.]$ and $\mathcal{C}[Z.]$ are equivalent stacks.*

3. SIMPLICIAL PARABOLIC HOLOTOPY PRESHEAVES

We recall that the *parabolic n -th dimensional circle* is the simplicial set $\mathbf{S}_s^1 := \Delta^1 / \partial\Delta^1$, where Δ^1 is the standard 1-dimensional simplex, seen as constant presheaf in the analytic site. As usual, in what follows, we let \wedge be the monoidal structure in $\Delta^{op} \text{Prsh}_T(\mathcal{S})$ and $\mathbf{S}_s^n := \mathbf{S}_s^1 \wedge \cdot^n \wedge \mathbf{S}_s^1$.

Let U be a complex space. Then for any simplicial presheaf \mathcal{X} and any groupoid \mathcal{G} , we set

$$\begin{aligned} \pi_i^{\text{simpl}}(\mathcal{X}, x)(U) &:= \text{Hom}_{\mathcal{H}_s}(\mathbf{S}_s^i \wedge U_+, (\mathcal{X}, x)), \\ \pi_i(\mathcal{G}, g)(U) &:= \pi_i^{\text{simpl}}(N\mathcal{G}, g)(U) \end{aligned}$$

respectively.

By definition, the presheaves π_i^{simpl} induce isomorphisms if applied to local and global weak equivalences or groupoid equivalences. We know already how to compute most of these presheaves for groupoids \mathcal{G} : because of Theorem 2.1, $\pi_i^{\text{simpl}}(\mathcal{G}, g)$ are constant to 0 for all i greater or equal to 2. In general, it is hard to compute π_i^{simpl} of groupoids if $i = 0, 1$. Using Proposition 2.2 we can prove the following: if $X \rightarrow \mathcal{Y}$ is a presentation of an analytic stack, then

$$\pi_i^{\text{simpl}}(\mathcal{Y}, y) \cong \pi_i^{\text{simpl}}([X.], x) \cong \pi_i^{\text{simpl}}(\mathcal{C}[X.], x).$$

which provides the key reduction step of dealing with $\mathcal{C}[X.]$ rather than a general analytic stack \mathcal{Y} . Because of the relevance of the concept in the sequel, we explicitly recall the following. Let \mathcal{G} be a groupoid, U a complex space and $\mathcal{U} = \{U_i\}_i$ a covering of U for the strong topology. Then,

- 1) A relative to \mathcal{U} *descent datum* in \mathcal{G} is a pair $((A_i), (h_{ij}))$, also denoted (A_i, h_{ij}) , with: A_i objects of $\mathcal{G}(U_i)$ and $h_{ij} : A_i|_{U_{ij}} \rightarrow A_j|_{U_{ij}}$ isomorphisms, called *transition morphisms*, satisfying the *cocycle condition* $h_{jk} \circ h_{ij} = h_{ik}$ on U_{ijk} . As always, U_{ij} and U_{ijk} stand for the double and triple intersections of the indicated complex spaces. The set of descent data will be denoted by $\text{Dis}_{\mathcal{G}}(\mathcal{U})$.
- 2) A *descent data morphism* between (A_i, h_{ij}) and (B_i, g_{ij}) in \mathcal{G} and relative to a covering \mathcal{U} is a collection of isomorphisms $\{\phi_i : A_i \rightarrow B_i\}$ respecting the relation $g_{ij} \circ \phi_i = \phi_j \circ f_{ij}$ on U_{ij} for all i, j .

REMARK 3.1. In a covering \mathcal{U} associated to a descent datum we will possibly allow $U_i = U_j$ for $i \neq j$.

Given a complex space U , we will denote by $\text{Cov } U$ the set of all the countable, locally finite coverings such that $U_i \subseteq U$ for all i . Any covering of U can be refined to one in $\text{Cov } U$. This set is filtering with respect to the relation $\mathcal{U} \preceq \mathcal{U}'$ if $\mathcal{U}' = \{U_i\}_i$ is finer than \mathcal{U} and $U'_i \subseteq U_{\tau(i)}$, if $\tau: \mathbb{N} \rightarrow \mathbb{N}$ is the refining function. If the groupoid \mathcal{G} is $\mathcal{C}[X]$, the notion of descent data may be expressed in terms of holomorphic maps. Given a covering $\{U_i\}_i = \mathcal{U} \in \text{Cov } U$, a descent datum r (on U) in X relative to the covering \mathcal{U} is a pair $((r_i: U_i \rightarrow X)_i, (f_{ij}: U_{ij} \rightarrow X_1)_{ij})$ with r_i and f_{ij} holomorphic maps such that:

- (\star) $r_i|_{U_{ij}} = \partial_0 \circ f_{ij}, r_j|_{U_{ij}} = \partial_1 \circ f_{ij}$ su U_{ij} ;
- ($\star\star$) $m(f_{ij} \times f_{jk}) = f_{ik}$ su U_{ijk} (cocycle relation)

and, like before,

$$\varinjlim_{\mathcal{U} \in \text{Cov } U} \{((r_i: U_i \rightarrow X)_i, (f_{ij}: U_{ij} \rightarrow X_1)_{ij})\} = \varinjlim_{\mathcal{U} \in \text{Cov } U} \text{Dis}_{[X]}(\mathcal{U}) = \text{Ob}(\mathcal{C}[X])(U)$$

We are ready now to describe the zeroth and first holotopy presheaves of an analytic stack by means of the complex structure of any of its atlases:

THEOREM 3.1. *Let $p: X \rightarrow \mathcal{Y}$ be an analytic stack and U a complex space. Then*

$$\pi_0(\mathcal{Y}, y)(U) \cong \varinjlim_{\mathcal{U} \in \text{Cov } U} \{((r_i: U_i \rightarrow X)_i, (f_{ij}: U_{ij} \rightarrow X_1)_{ij})\} / \sim_0$$

where \sim_0 is the equivalence relation generated by $(r_i, f_{ij}) \sim_0 (s_i, g_{ij})$ if and only if there exist holomorphic maps $\phi_i: U_i \rightarrow X_1$ such that

- 1) $\partial_0 \circ \phi_i = s_i$ e $\partial_1 \circ \phi_i = r_i$ for all i ;
- 2) $m(f_{ij} \times \phi_j) = m(\phi_i \times g_{ij})$.

An isomorphism ϕ between two descent data $r = (r_i, f_{ij})$, $s = (s_i, g_{ij})$ is a collection of holomorphic maps $\phi_i: U_i \rightarrow X_1$ such that

- 1') $\partial_0 \circ \phi_i = r_i$ and $\partial_1 \circ \phi_i = s_i$ for all i ;
- 2') $m(f_{ij}, \phi_j) = m(\phi_i, g_{ij})$ for all i, j .

Each collection $\{\phi_i\}_i$ determines a class in the filtered colimit, over the coverings \mathcal{U} of U , of isomorphisms between the descent data r and s . Representatives of sections of $\pi_1^{\text{simp}}(\mathcal{Y}, y)(U)$ are (classes of) automorphisms $(\phi)_i$ of r , for r ranging in $\varinjlim_{\mathcal{U} \in \text{Cov } U} \text{Dis}_{[X]}(\mathcal{U})$.

THEOREM 3.2. *Let $p : X \rightarrow \mathcal{Y}$ an analytic stack and U be a complex space. Then,*

$$\pi_1(\mathcal{Y}, y)(U) \cong \varinjlim_{\mathcal{U} \in \text{Cov } U} (\phi_i)_i / \sim_1$$

where $\phi = (\phi_i)_i$ is an automorphism of a descent datum r in $[X]$ relative to \mathcal{U} . If ϕ e ψ are automorphisms of descent data r and s , respectively, relative to $\mathcal{U} = \{U_i\}_i$, then \sim_1 is the equivalence relation generated by $\phi \sim_1 \psi$ if and only if there exists an isomorphism between $\partial_0 \circ \phi = s$ and $\partial_0 \circ \psi = r$, i.e. holomorphic maps $H_i : U_i \rightarrow X \times_{\mathcal{Y}} X$ such that $\partial_0 \circ H_i = r_i$ and $\partial_1 \circ H_i = s_i$ satisfying the second of the conditions listed in the Theorem 3.1.

As in the sequel we will mention about “constant” sections of holotopy presheaves, we introduce here such a notion.

DEFINITION 3.3. *Let P be a presheaf on \mathcal{S}_T . A section $\sigma \in P(U)$ is constant if it lies in the image of the map $c^* : P(\text{pt}) \rightarrow P(U)$, where $c : U \rightarrow \text{pt}$.*

4. HYPERBOLICITY

The classical Brody’s Theorem claims that two notions of hyperbolicity for complex spaces coincide. One is rooted in metric aspects of the complex space, the other is defined in terms of certain holomorphic maps.

4.1. Brody hyperbolicity. In the paper [1] we have given the following definition: a simplicial presheaf \mathcal{Y} is *Brody hyperbolic* if

- 1) it is simplicially locally fibrant and
- 2) the projection $p_{\mathcal{X}} : \mathbb{C} \times \mathcal{X} \rightarrow \mathcal{X}$ induces set bijections

$$(1) \quad \text{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\cong} \text{Hom}_{\mathcal{H}_s}(\mathbb{C} \times \mathcal{X}, \mathcal{Y})$$

for all $\mathcal{X} \in \text{Prsh}_T(\mathcal{S})$.

Since a groupoid can be seen as a simplicial presheaf by means of the functor N , we will use the same definition:

DEFINITION 4.1. *A groupoid \mathcal{G} is Brody hyperbolic if $N\mathcal{G}$ is a Brody hyperbolic simplicial presheaf.*

Notice that a Brody hyperbolic groupoid is necessarily a stack. This definition can be rephrased in terms of holotopy presheaves:

PROPOSITION 4.2. *Let \mathcal{Y} a locally fibrant simplicial presheaf. The following conditions are equivalent:*

- 1) \mathcal{Y} is Brody hyperbolic;
- 2) $p_{\mathcal{X}}^* : \text{Map}(\mathcal{X}, \mathcal{Y}) \rightarrow \text{Map}(\mathbb{C} \times \mathcal{X}, \mathcal{Y})$ is a weak equivalence of simplicial sets for any $\mathcal{X} \in \Delta^{op} \text{Prsh}_T(\mathcal{S})$, where Map is the simplicial mapping space;
- 3) the simplicial holotopy presheaves are Brody hyperbolic, i.e. the projection $p_U : \mathbb{C} \times U \rightarrow U$ induces isomorphisms

$$p_U^* : \pi_i^{\text{simpl}}(\mathcal{Y}, y)(U) \xrightarrow{\cong} \pi_i^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C} \times U)$$

for each i and complex space U .

The notion of Brody hyperbolicity we will more frequently use is the 3) of the previous proposition.

DEFINITION 4.3. 1) A presheaf \mathcal{Y} is Brody hyperbolic if the projection $p_U : \mathbb{C} \times U \rightarrow U$ induces bijections $p_U^* : \mathcal{Y}(U) \rightarrow \mathcal{Y}(\mathbb{C} \times U)$ for any complex space U .

- 2) A groupoid \mathcal{G} is Brody hyperbolic if the holotopy presheaves $\pi_i^{\text{simpl}}(\mathcal{G}, g)$ (see Section 3) are hyperbolic for all i , hence only for $i = 0, 1$, because of Theorem 2.1.

4.2. Kobayashi hyperbolicity. In the previous subsection we defined Brody hyperbolicity of a groupoid by first giving the same notion for a presheaf and then imposing that condition to the holotopy presheaves of the groupoid. The holotopy presheaves determine whether a groupoid is Kobayashi hyperbolic, as well. Classically, complex spaces Kobayashi hyperbolicity is a notion arising in the attempt to give complex spaces a biholomorphically invariant distance. In general the best that can be done is endowing complex spaces of a biholomorphically pseudodistance. When on a complex space X this happens to be a distance, X is said to be Kobayashi hyperbolic.

The notion of Kobayashi hyperbolicity for groupoids is based upon the concept of relative analytic disc.

Let \mathbb{D} be the unitary open disc in \mathbb{C} . We recall that for a complex space U , we have denoted $\text{Cov}(U)$ the set of countable, locally finite, open coverings $\mathcal{U} = \{U_i\}_i$ of U such that $U_i \subseteq U$ for all i . If $\mathcal{U} \in \text{Cov} U$ and $\mathcal{D} = \{D_a\}_a \in \text{Cov}(\mathbb{D})$ let $\mathcal{D} \times \mathcal{U}$ be the covering $\{D_a \times U_i\}_{ai}$ of $\mathbb{D} \times U$. The set $\text{Cov}(\mathbb{D}) \times \text{Cov}(U)$ is filtering in $\text{Cov}(\mathbb{D} \times U)$.

Let \mathcal{Y} be a presheaf. A relative analytic disc of \mathcal{Y} on a complex space U is an object of $\mathbf{F} \in \mathcal{Y}(\mathbb{D} \times U)$. For any $z \in \mathbb{D}$, the same letter will refer to the inclusion $\{z\} \times U \hookrightarrow \mathbb{D} \times U$. Let $r, s \in \mathcal{Y}(U)$ be two sections and suppose there exists a relative analytic disc \mathbf{F} and two points $z_1, z_2 \in \mathbb{D}$ such that $z_1^* \mathbf{F} = r$ and $z_2^* \mathbf{F} = s$. The sections r and s are then said to be connected by \mathbf{F} . A relative analytic chain on U connecting r to s is the set $\mathbf{C}_{(r,s)}$ of the following data:

- 1) a collection $r_0 = r, \dots, r_k = s$ of sections;
- 2) $2k$ points $a_1, b_1, \dots, a_k, b_k$ in \mathbb{D} ;
- 3) k relative analytic discs $\mathbf{F}_1, \dots, \mathbf{F}_k$ such that the analytic disc \mathbf{F}_i connects the sections r_{i-1} and r_i , i.e. $a_i^* \mathbf{F}_i = r_{i-1}$ and $b_i^* \mathbf{F}_i = r_i$ for all $1 \leq i \leq k$.

If a relative analytic chain $\mathbf{C}_{(r,s)}$ connects the section r with the section s , we call the pair (r, s) *admissible*. If $\mathcal{Y} = Y$ is a complex space, admissibility of all section pairs in $\mathcal{Y}(\text{pt})$ is equivalent to the topological connectedness of Y .

Endow the unitary open disc \mathbb{D} of the Poincaré metric

$$ds^2 = \frac{1}{(1 - |z|^2)^2} dz \otimes d\bar{z}$$

and denote with $\varrho_{\mathbb{D}}(p, q)$ the associated distance function between two points a and b in \mathbb{D} . Then, for every chain $\mathbf{C}_{(r,s)}$ the nonnegative number

$$(2) \quad l(\mathbf{C}_{(r,s)}) = \varrho_{\mathbb{D}}(a_1, b_1) + \dots + \varrho_{\mathbb{D}}(a_k, b_k)$$

is, by definition, the (*Kobayashi*) *length* of the relative analytic chain $\mathbf{C}_{(r,s)}$. If (r, s) is an admissible pair of sections, the nonnegative real number

$$(3) \quad d_{\text{Kob}}^{\mathcal{Y}}(r, s) = \inf_{\mathbf{C}_{(r,s)}} l(\mathbf{C}_{(r,s)})$$

defines a *pseudodistance* function on all the admissible pairs of sections in $\mathcal{Y}(U)$ for all complex spaces U , called *Kobayashi pseudodistance* of \mathcal{Y} .

It is immediately seen that morphisms of presheaves decrease the Kobayashi pseudodistance.

DEFINITION 4.4. *A presheaf \mathcal{Y} is said to be Kobayashi hyperbolic if its Kobayashi pseudodistance is indeed a distance, hence if and only if $d_{\text{Kob}}^{\mathcal{Y}}(r, s) \neq 0$ for all admissible pairs $(r, s) \in \mathcal{Y}(U)$ with $r \neq s$ and all complex spaces U .*

The Kobayashi hyperbolicity for a groupoid is defined as follows:

DEFINITION 4.5. *A groupoid \mathcal{G} is Kobayashi hyperbolic if the holotopy presheaves $\pi_0^{\text{simpl}}(\mathcal{G}, g)$ and $\pi_1^{\text{simpl}}(\mathcal{G}, g)$ are Kobayashi hyperbolic.*

These notions of Brody and Kobayashi hyperbolicity extend the classical ones in the case the groupoid \mathcal{G} is a complex space.

5. THE AUXILIARY QUOTIENT SPACE

Although the set $\varinjlim_{\mathcal{Y} \in \text{Cov } U} \text{Dis}_{[X]}(\mathcal{Y})$ is closely tied to the complex structure of an atlas X of an analytic stack \mathcal{Y} , it is unclear how to metrize it in a usable way and arguments employing complex variables theory, such as those necessary to prove Brody theorem, seem not possible. As we will see, this is possible for DM stacks (see Subsection 5.1).

Let $p : X = \coprod_{i=1}^N X^{(i)} \rightarrow \mathcal{Y}$ be a presentation of an analytic stack \mathcal{Y} . We say that \mathcal{Y} is *connected* if given $f, g \in \mathcal{Y}(\text{pt})$ with $f(\text{pt}) \in X^i, g(\text{pt}) \in X^j$ there exist $\alpha, \beta \in \mathcal{Y}(\text{pt})$ with $\alpha(\text{pt}) \in X^i, \beta(\text{pt}) \in X^j$ such that $p \circ \alpha = p \circ \beta$. This property does not depend on the atlas.

Two sections $r, s \in \mathcal{Y}(\text{pt})$ of a connected analytic stack \mathcal{Y} are always admissible.

An analytic stack \mathcal{Y} is said to be *compact* if there exists two presentations $p_X : X = \coprod_{i=1}^N X^{(i)} \rightarrow \mathcal{Y}$, $p_Z : Z = \coprod_{i=1}^N Z^{(i)} \rightarrow \mathcal{Y}$ with $Z^{(i)}$ connected, such that for each $i = 1, \dots, N$ there exists an embedding $\phi_i : X^{(i)} \subseteq Z^{(i)}$ with relatively compact image in $Z^{(i)}$ and such that $p_Z \circ \phi_i = p_{X^{(i)}}$.

If p_Z is étale, X and $X \times_{\mathcal{Y}} X$ are embedded as open, relatively compact subspaces of Z and $Z \times_{\mathcal{Y}} Z$, respectively, ($X \subseteq Z$, $X \times_{\mathcal{Y}} X \subseteq Z \times_{\mathcal{Y}} Z$) and the structural morphisms

$$(4) \quad X \times_{\mathcal{Y}} X \begin{matrix} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{matrix} X$$

are étale and restrictions of their counterparts

$$(5) \quad Z \times_{\mathcal{Y}} Z \begin{matrix} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{matrix} Z.$$

Notice that in a compact DM analytic stack, the fibres of the maps ∂_0 and ∂_1 are equifinite.

For a DM stack \mathcal{Y} the following properties are equivalent:

- 1) \mathcal{Y} is compact;
- 2) every atlas X has a finite subatlas, i.e. X has finitely many connected components.

Compactness implies the following: given a sequence $\{x_\nu\}$ of an atlas X of \mathcal{Y} with no limit points, there exists a sequence $\{w_\mu\}$ in $X_1 = X \times_{\mathcal{Y}} X$ such that $\{\partial_0(w_\mu)\}$ is a subsequence of $\{x_\nu\}$ and $\{\partial_1(w_\mu)\}$ is convergent in X . The same conclusion we have when exchanging ∂_0 with ∂_1 .

5.1. Existence of the complex structure. Given an analytic stack $p : X \rightarrow \mathcal{Y}$ we are interested in finding a complex space X' and a holomorphic map q making the diagram

$$X_1 = X \times_{\mathcal{Y}} X \begin{matrix} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{matrix} X \xrightarrow{q} X'$$

commutative and preserving as much information about the diagram as possible.

Consider the following relation on the points of X :

$$(6) \quad x \sim y \text{ if and only if there exists } a \in X_1 \text{ such that } \partial_0(a) = x \text{ and } \partial_1(a) = y.$$

The existence of a groupoid structure on $[X, \sim]$ implies that this is a set theoretic equivalence relation on X . Notice that the diagram

$$(7) \quad X \times_{\mathcal{Y}} X \begin{matrix} \xrightarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{matrix} X$$

is not a *categorical* equivalence relation, hence we cannot apply a well known result on the existence of its colimit in the category of complex spaces (see, for instance, [10, Proposition 1.2]).

Denote by X/\sim the quotient set by the equivalence relation (6). When a fixed presentation $p : X \rightarrow \mathcal{Y}$ is understood, for cosmetic reasons, we will denote X/\sim as $Q(\mathcal{Y})$ and $q : X \rightarrow Q(\mathcal{Y})$ its projection. We list some noticeable properties of the diagram $X \rightarrow Q(\mathcal{Y})$ for a compact DM stack $p : X \rightarrow \mathcal{Y}$.

Given $x \in X$, let $[x]$ be the set of points in X equivalent to x : $[x] = \partial_1(\partial_0^{-1}(x))$ is a finite set. Let

$$A = \{x \in X : \partial_0^{-1}(x) \cap \partial_1^{-1}(x) \neq \emptyset\};$$

A is the projection through ∂_0 or ∂_1 of the closed analytic set

$$C = \{w \in X_1 : \partial_0(w) = \partial_1(w)\}.$$

and we also let $B = q(A)$.

The following properties hold true:

- i) A is a closed analytic subset of X .
- ii) $q : X \rightarrow Q(\mathcal{Y})$ is a continuous open map with finite fibers.
- iii) $Q(\mathcal{Y})$ is a compact Hausdorff space and B is a closed subset of $Q(\mathcal{Y})$.
- iv) $q : X \rightarrow Q(\mathcal{Y})$ is locally proper, that is given x , there exist an open neighborhoods $N = N(x)$ and $V = V(q(x))$ of x and $q(x)$, respectively, such that q is proper from N to V .

Then, using Cartan's Theorem [5] we prove a result analogue to the one in [8] but for DM analytic stacks

THEOREM 5.1. *Let $p : X \rightarrow \mathcal{Y}$ be a compact DM stack. Then,*

- 1) *there exists a complex structure on $Q(\mathcal{Y})$ making it a compact complex space of the same dimension of X such that $q : X \rightarrow Q(\mathcal{Y})$ is a holomorphic map;*
- 2) *$B = q(A)$ is a closed analytic subspace, $q^{-1}(B) = A$ and $q : X \setminus A \rightarrow Q(\mathcal{Y}) \setminus B$ is étale;*
- 3) *two presentations $p_X : X \rightarrow \mathcal{Y}$, $p_{X'} : X' \rightarrow \mathcal{Y}$ of \mathcal{Y} determine isomorphic complex spaces $Q(\mathcal{Y})$, $Q'(\mathcal{Y})$, respectively.*

Finally, we point out that a DM stack \mathcal{Y} is connected if and only $Q(\mathcal{Y})$ is connected.

6. TOPOLOGICAL AND METRIC STRUCTURES

6.1. Distances. For the time being, we will only consider connected, compact, DM analytic stacks.

Let \mathcal{Y} one such analytic stack with atlases $X \subseteq Z$, $X = \coprod_{i=1}^N X^{(i)}$, $Z = \coprod_{i=1}^N Z^{(i)}$ (cfr. Section 5). We can assume X_i and Z_i are Stein.

Fix a differentiable length function H on the quotient space $Q(\mathcal{Y})$ and let $d : Q(\mathcal{Y}) \times Q(\mathcal{Y}) \rightarrow \mathbb{R}_{\geq 0}$ be the distance determined by H . Let $q = q_X$ be the projection $X \rightarrow Q(\mathcal{Y})$. Even though q^*H is only a pseudolength function, a distance is associated to q^*H on any connected component of X : this is because q is locally proper and equifinite fibres. This is the same distance induced by the restriction of q^*H to X . These distances on the connected components can be assembled together to a unique distance d_X on all X in the following way. Fix two points $x \neq y \in X$; a (piecewise differentiable) path γ through x and y is a set $\{\gamma_1, \gamma_2, \dots, \gamma_m\}$ of paths $[0, 1] \rightarrow X$ such that:

- (1) $\gamma_1(0) = x$ and $\gamma_m(1) = y$;
- (2) $q(\gamma_{i+1}(0)) = q(\gamma_i(1))$ for all $1 \leq i \leq m - 1$.

If $l(\gamma_i)$ denotes the length of γ_i with respect to the distance on the connected component of X in which the image of γ_i lies, the positive real number $l(\gamma) = \sum_{i=1}^m l(\gamma_i)$ is the length of γ by definition. We then set

$$d_X(x, y) = \inf_{\gamma} l(\gamma).$$

We proceed likewise for the complex space $X_1 = X \times_{\mathcal{Y}} X$, considering on X_1 the length functions $(q \circ \partial_0)^*H, (q \circ \partial_1)^*H$ which give rise to the same distance making ∂_0 and ∂_1 into local isometries, since $q \circ \partial_0 = q \circ \partial_1$. A similar argument applies to $X_2 := X_1 \times_{\partial_0, X, \partial_1} X_1$ and to the multiplication $m : X_2 \rightarrow X_1$ that becomes a local isometry, as well.

The distance just introduced allows to metrize in a natural way the sets $\text{Dis}_{[X]}(U), \pi_0(\mathcal{Y}, y)(U) \in \pi_1(\mathcal{Y}, y)(U)$ as follows. Let $r = (r_i, f_{ij}) \neq s = (s_i, g_{ij})$ be two descent data in $\text{Dis}_{[X]}(\mathcal{U})$, with $\mathcal{U} \in \text{Cov } U$; define

$$(8) \quad \delta(r, s) = \sup_{\substack{u \in U_i \\ i \in \mathbb{N}}} \{d_X(r_i(u), s_i(u))\} + \sup_{\substack{u \in U_{ij} \\ i, j \in \mathbb{N}}} \{d_{X_1}(f_{ij}(u), g_{ij}(u))\}.$$

The distance function $\delta(r, s)$ is invariant by restriction, that is $\delta(r, s) = \delta(r|_{\mathcal{U}'}, s|_{\mathcal{U}'})$ for any $\mathcal{U}' \succeq \mathcal{U}$, thus it is defined for pairs of objects in $\text{Ob}(\mathcal{C}[X](U))$. In turn it induces a distance function $\delta_{\pi_0, U}$ on $\pi_0^{\text{simpl}}(\mathcal{Y}, y)(U)$: for $\alpha, \beta \in \pi_0^{\text{simpl}}(\mathcal{Y}, y)(U)$

$$(9) \quad \delta_{\pi_0, U}(\alpha, \beta) = \inf_{r \in \alpha, s \in \beta} \delta(r, s).$$

For $\pi_1^{\text{simpl}}(\mathcal{Y}, y)(U)$ we proceed similarly. Given $\mathcal{U} = \{U_i\}_i \in \text{Cov } U$, the elements of $\pi_1^{\text{simpl}}(\mathcal{Y}, y)(U)$ are represented by pairs $[r, \phi]$ comprising a descent datum r and $\phi = \{\phi_i\}_i$ an automorphism of r (cfr. Theorem 3.2). The distance δ between two such pairs (r, ϕ) and (s, ψ) is defined as

$$(10) \quad \delta((r, \phi), (s, \psi)) = \delta(r, s) + \sup_{\substack{u \in U_i \\ i \in \mathbb{N}}} \{d_{X_1}(\phi_i(u), \psi_i(u))\}$$

and the distance between two classes $\alpha, \beta \in \pi_1^{\text{simpl}}(\mathcal{Y}, y)(U)$ is

$$(11) \quad \delta_{\pi_1, U}(\alpha, \beta) = \inf_{\substack{(r, \phi) \in \alpha \\ (s, \psi) \in \beta}} \delta((r, \phi), (s, \psi)).$$

PROPOSITION 6.1. $\delta_{\pi_0, U}$ e $\delta_{\pi_1, U}$ are distances.

The proof, rather involved, is based on the following

LEMMA 6.2. Given two descent data $r = (r_i, f_{ij}) \neq s = (s_i, g_{ij})$ relative to a covering $\mathcal{U}' = \{U'_i\}_i$ of a complex space U , such that $q \circ r = q \circ s$, there is a refinement $\mathcal{U} = \{U_i\}_i \succeq \mathcal{U}'$ and collection of holomorphic maps $\phi_i : U_i \rightarrow X_1 = X \times_{\mathcal{Y}} X$ related to r and s by the equations $\partial_0 \circ \phi_i = r_i$ and $\partial_1 \circ \phi_i = s_i$.

Notice that the collection $(\phi_i)_i$ does not need to be an isomorphism between the descent data r and s , the classifying stacks $\mathcal{B}G$ providing an example of this.

6.2. The function $c(\mathcal{Y})$. Let U be a complex space and

$$F = F_{\mathcal{U} \times \mathcal{U}} = (F_{ai} : D_a \times U_i \rightarrow \mathbb{D} \times U, F_{abj} : D_{ab} \times U_{ij} \rightarrow X_1)$$

be a relative analytic disc of \mathcal{Y} on U . We associate a function $\mathbb{D} \rightarrow \mathbb{R}_{\geq 0}$ to F as follows. Take $z \in \mathbb{D}$ and a vector $v \in T\mathbb{D}$, the holomorphic tangent bundle on \mathbb{D} , and consider $D_a \times U_i$ with $z \in D_a$. Then, denoting with dF_{ai} the differential of the holomorphic map F_{ai} , $dF_{ai}(z, \xi)v$ is a vector field tangent to X along the points of the image of $(F_{ai})_{z \times U_i}$, ξ ranging in U_i . Since the maps ∂_0 , ∂_1 and m are local isometries, by differentiating with respect of the variable z the structural equations of F , (\star) and $(\star\star)$ of Section 3, we notice that the real number $|dF_{ai}(z, \xi)v| := q^*H(dF_{ai}(z, \xi)v)$ only depends on ξ , z , v and not on the open subspaces $D_a \times U_i^1$, so $(z, \xi, v) \mapsto |dF_{ai}(z, \xi)v|$ is a well defined real valued continuous function which will be occasionally written as $|dF(z, \xi)v|$. Moreover, for any $(z, \xi) \in D_{ab} \times U_{ij}$ we have that

$$(12) \quad |dF_{ai, bj}(z, \xi)v| = |dF_{ai}(z, \xi)v|.$$

Let

$$(13) \quad |dF(z)|_U = \sup_{\xi \in U} \sup_{\substack{v \in T_z \mathbb{D} \\ v \neq 0}} \frac{|dF(z, \xi)v|}{|v|_{\text{hyp}}},$$

where $|v|_{\text{hyp}}$ is the length induced by the Poincaré metric on \mathbb{D} , and

$$(14) \quad c(\mathcal{Y}; U) = \sup_{F \in \mathcal{Y}(\mathbb{D} \times U)} \sup_{z \in \mathbb{D}} |dF(z)|_U = \sup_{F \in \mathcal{Y}(\mathbb{D} \times U)} |dF(0)|_U.$$

¹We recall that H denotes a fixed length function on $Q(\mathcal{Y})$.

because of the transitivity of the action of $\text{Aut}(\mathbb{D})$ and the invariance of the Poincaré metric.

By considering the natural restriction maps $\mathcal{Y}(\mathbb{D} \times U) \rightarrow \mathbf{F} \in \mathcal{Y}(\mathbb{D} \times \{u\})$, $u \in U$, we see that $c(\mathcal{Y}; U)$ is actually independent of U so we define

$$(15) \quad c(\mathcal{Y}) = c(\mathcal{Y}; \text{pt}).$$

Since ∂_0 and ∂_1 are local isometries, if the relative analytic disc

$$(16) \quad \mathbf{G} = \mathbf{G}_{\mathcal{G} \times \mathcal{U}} = (G_{ai} : D_a \times U_i \rightarrow \mathbb{D} \times \mathbb{C}, G_{aibj} : D_{ab} \times U_{ij} \rightarrow X_1)$$

is equivalent to \mathbf{F} , i.e. they have the same class in $\pi_0^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{D} \times U)$, the functions $|\text{dF}(z, \xi)v|$ and $|\text{dG}(z, \xi)v|$ coincide.

Given a pair $[\mathbf{F}, \Phi]$ representing a class of $\pi_1^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{D} \times U)$, where \mathbf{F} is a relative analytic disc on U , and $\Phi = \{\Phi_{ai}\}_{ai}$ is an automorphism of \mathbf{F} (see Section 3) we have well defined functions $|\text{dF}(z)|_U$ and $|\text{d}\Phi(z)|_U$, as in equation (13). Keeping in mind the Theorem 3.2, we deduce that $|\text{dF}(z)|_U = |\text{d}\Phi(z)|_U$. If $[\mathbf{G}, \Psi]$ is a pair equivalent to $[\mathbf{F}, \Phi]$, that is their images coincide in $\pi_1^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{D} \times U)$, then

$$(17) \quad |\text{dF}(z)|_U = |\text{d}\Phi(z)|_U = |\text{d}\Psi(z)|_U = |\text{dG}(z)|_U.$$

REMARK 6.1. Because of the last observations, the vanishing of the H -norm of the derivative of a descent data or of one of their isomorphisms along the z -direction is equivalent to their relevant classes in π_0^{simpl} or π_1^{simpl} being constant (see Section 3).

Under the same notation as in the Subsection 6.1, we prove the following fundamental

LEMMA 6.3. *Let $\alpha_1, \alpha_2 \in \pi_i^{\text{simpl}}(\mathcal{Y}, y)(U)$, for $i = 0, 1$ and $\mathbf{C}_{(\alpha_1, \alpha_2)}$ be an analytic chain through α_1 and α_2 . Then*

$$(18) \quad \delta_{\pi_i, U}(\alpha_1, \alpha_2) \leq 2c(\mathcal{Y})l(\mathbf{C}_{(\alpha_1, \alpha_2)})$$

(see the equation (2) for the definition of length of an analytic chain). In particular,

$$(19) \quad \text{d}_{\text{Kob}}(\alpha_1, \alpha_2) \geq \frac{\delta_{\pi_0, U}(\alpha_1, \alpha_2)}{2c(\mathcal{Y})}.$$

7. BRODY’S THEOREM

7.1. Kobayashi hyperbolicity implies Brody hyperbolicity. We can now proceed with the proof of the Brody theorem for stacks. Classically, that theorem refers

to the implication “compactness and Brody hyperbolicity imply Kobayashi hyperbolicity”, the converse being a simple consequence of non Kobayashi hyperbolicity of \mathbb{C} . This is true for general analytic stacks too, even if the proof now is not entirely obvious due to the difficulty of determining two different admissible sections in π_0^{simpl} and π_1^{simpl} :

THEOREM 7.1. *Let \mathcal{Y} be a Kobayashi hyperbolic analytic stack. Then \mathcal{Y} is Brody hyperbolic.*

Suppose that $\pi_0^{\text{simpl}}(\mathcal{Y}, y)$ is not Brody hyperbolic; then there exists a section $s \in \pi_0^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C} \times U)$ not in the image of

$$p^* : \pi_0^{\text{simpl}}(\mathcal{Y}, y)(U) \rightarrow \pi_0^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C} \times U),$$

p being the projection. We wish to construct two sections, or objects, $r_1 \not\cong r_2 \in \text{Ob}(\mathcal{C}[X.](V))$ for some complex space V , whose Kobayashi pseudodistance is zero. Take $V = \mathbb{C} \times U$ and consider the two sections: $r_1 = s$ and $r_2 = p^*(i_0^*(s))$, where $i_0 : U \rightarrow \mathbb{C} \times U$ is the embedding in zero. By assumption, $r_1 \not\cong r_2$. To show that the Kobayashi pseudodistance between r_1 and r_2 is zero, we construct relative analytic chains s_n^λ between them for $n \in \mathbb{N}$, where $\lambda \in \mathbb{D}_2 = \{z \in \mathbb{C} : |z| < 2\}$. For each n , s_n^λ is, in fact, a relative analytic disc and its length tends to zero as n tends to infinity.

7.2. Compactness and Brody hyperbolicity imply Kobayashi hyperbolicity. Brody and Kobayashi hyperbolicity of an analytic stack $p : X \rightarrow \mathcal{Y}$ are statements concerning the holotopy presheaves $\pi_0^{\text{simpl}}(\mathcal{Y}, y)$ and $\pi_1^{\text{simpl}}(\mathcal{Y}, y)$, as conceived in definitions 4.3 and 4.5. Surprisingly, it turns out that for DM (connected) compact stacks, the Brody hyperbolicity content in the holotopy presheaf π_0^{simpl} absorbed the one of π_1^{simpl} , to the point of making the latter irrelevant, when dealing with the hyperbolicity of compact DM stacks.

THEOREM 7.2. *Let \mathcal{Y} be a compact, DM stack. If $\pi_0^{\text{simpl}}(\mathcal{Y}, y)$ is Brody hyperbolic, then*

- i) $c(\mathcal{Y}) < +\infty$;
- ii) $\pi_i^{\text{simpl}}(\mathcal{Y}, y)$ are Kobayashi hyperbolic, for $i = 0, 1$.

The statement i) implies statement ii). Indeed, let $\alpha_1 \neq \alpha_2 \in \pi_i^{\text{simpl}}(\mathcal{Y}, y)(U)$ admissible sections. From i) and Lemma 6.3, we have

$$d_{\text{Kob}}(\alpha_1, \alpha_2) \geq \frac{\delta_{\pi_i, U}(\alpha_1, \alpha_2)}{c(\mathcal{Y})} > 0$$

hence $\pi_i^{\text{simpl}}(\mathcal{Y}, y)(U)$ are Kobayashi hyperbolic.

The proof of the first assertion is rather long and comprises several technical constructions, thus we restrict ourselves to underline its main points.

Assume by contradiction that $c(\mathcal{Y}) = +\infty$. Then there exists a sequence $\{\mathcal{Y}_v\}_v$ of analytic discs $\{\mathbb{F}^v\}_v$, i.e. descent data, over \mathbb{D} that $\lim_{v \rightarrow +\infty} |d\mathbb{F}^v(0)| = +\infty$ (see (14), (15)). This sequence descends to a sequence $\{f^v\}_v$ of holomorphic maps $f^v : \mathbb{D} \rightarrow \mathcal{Q}(\mathcal{Y})$ (see Section 5) such that $\lim_{v \rightarrow +\infty} |df^v(0)| = +\infty$. Indeed, for every complex space U there exists a commutative diagram

$$(20) \quad \begin{array}{ccc} \mathcal{C}[X.](U) & \longrightarrow & \pi_0(\mathcal{Y})(U) \\ & \searrow \phi & \swarrow \phi_Q \\ & \text{Hol}(U, \mathcal{Q}(\mathcal{Y})) & \end{array}$$

where the application ϕ is defined by associating to a descent datum $r = (r_i, f_{ij})$ on $U = \bigcup_{i \in \mathbb{N}} U_i$ the holomorphic map $f_r : U \rightarrow \mathcal{Q}(\mathcal{Y})$ defined for $u \in U_i$ as $f_r(u) = q(r_i(u))$.

Since $\mathcal{Q}(\mathcal{Y})$ is a compact complex space (see Theorem 5.1) the sequence of the maps $f^v : \mathbb{D} \rightarrow \mathcal{Q}(\mathcal{Y})$ may be reparametrized, by means of the classic “reparametrization Lemma” (cfr. [3]), to get a sequence of maps $\tilde{f}^v : \mathbb{D}_v \rightarrow \mathcal{Q}(\mathcal{Y})$, where $\mathbb{D}_v = \{|z| < v\}$ and $|d\tilde{f}^v(0)| = 1$ for all v . By Ascoli-Arzelà Theorem, there exists a subsequence $\{\tilde{f}^{\mu}\}$ uniformly convergent on compacts to a holomorphic map $f : \mathbb{C} \rightarrow \mathcal{Q}(\mathcal{Y})$, which is not constant since $|df(0)| = 1$ (cfr. [3]). Then we get a contradiction by proving that f lifts to a non constant descent datum $\psi(f) \in \pi_0(\mathcal{Y})(\mathbb{C})$.

The proof of the Theorem 7.2 has highlighted the connection between hyperbolicity of $\mathcal{Q}(\mathcal{Y})$ as a complex space and \mathcal{Y} as DM analytic stack:

COROLLARY 7.3. *Let $X \rightarrow \mathcal{Y}$ be a compact DM analytic stack. Then*

- 1) *if $\mathcal{Q}(\mathcal{Y})$ is hyperbolic \mathcal{Y} is hyperbolic;*
- 2) *\mathcal{Y} is hyperbolic if and only if the presheaf $\pi_0^{\text{simpl}}(\mathcal{Y}, y)$ is hyperbolic if and only if $\text{Ob}(\mathcal{C}[X.])(-)$ is an hyperbolic presheaf.*

In a work in progress, we prove that the first assertion in the corollary is, in fact, an equivalence.

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