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# On the Components of the Push-out Space with Certain Indices

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ABSTRACT - Given an immersion of a connected, $m$-dimensional manifold $M$ without boundary into the Euclidean $(m+k)$-dimensional space, the idea of the *push - out space* of the immersion under the assumption that immersion has flat normal bundle is introduced in [3]. It is known that the *push-out space* has finitely many path-connected components and each path-connected component can be assigned an integer called the index of the component. In this study, when $M$ is compact, we give some new results on the *push-out space*. Especially it is proved that if the *push-out space* has a component with index 1, then the Euler number of $M$ is 0 and if the immersion has a co-dimension 2, then the number of path-connected components of the *push-out space* with index $(m-1)$ is at most 2.

## 1. Introduction

Throughout we assume $M$ (or $M^m$) is an $m$-dimensional connected smooth ($C^\infty$) manifold without boundary. The tangent space of $M$ at a point $p$ will be denoted by $T_pM$.

$f : M^m \to \mathbb{R}^{m+k}$ will be assumed a smooth immersion or embedding into Euclidean $m+k$ space, i.e. $f$ has *co-dimension* $k$. In this case

$$df_p : T_pM^m \to T_{f(p)}(\mathbb{R}^{m+k}) = \{f(p)\} \times \mathbb{R}^{m+k} \cong \mathbb{R}^{m+k}$$

is an injection. We identify $T_pM$ with $Im\ df_p$, $\forall p \in M$. In this way, we can assume that $f$ is an isometric immersion. There is a standard inner product $\langle , \rangle$ on $\mathbb{R}^{m+k}$. So we can define the normal space at $p$ as the normal complement of $Im\ df_p$. Let $\nu_p(f)$ denote the $k-$ plane which is normal to $f(M)$ at

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$f(p)$. The total space of the normal bundle is defined by

$$N(f) = \{(p, v) \in M \times \mathbb{R}^{m+k} : f(p) + v \in \nu_p(f)\}.$$

Note that $N(f)$ is an $(m+k)$-dimensional smooth manifold.

A normal field on $M$ for $f$ is a smooth map $\xi : M^m \to \mathbb{R}^{m+k}$ where $f(p) + \xi(p) \in \nu_p(f)$ for all $p \in M$.

With this notation, the endpoint map $E : N(f) \to \mathbb{R}^{m+k}$ is defined by $E(p, v) = f(p) + v$, and $E$ is known to be a smooth map.

### 1.1 – *Immersions of manifolds and focal points*

Definition 1. *A point $x \in \mathbb{R}^{m+k}$ is a focal point of $f(M)$ with base $p$ if $E$ is singular at $(p, x - f(p))$, i.e. $(p, x - f(p))$ is a critical point of $E$. The focal point has multiplicity $\mu > 0$ if rank (Jacobian $E$) $= m + k - \mu$ .*

The set of focal points of $f$ (or $f(M)$ ) with base $p$ will be denoted by $F_p(f)$.This is an algebraic variety, that is, it is a set of zeros of a polynomial with degree at most $m$ in $k$ variables in $\nu_p(f)$ and in general it can be quite complicated [8]. In this study, we will be considering the simplest case. We remark that by [6], $x \in F_p(f)$ iff $x \in \nu_p(f)$ and $x = f(p) + \frac{1}{\lambda}\xi(p)$ where $\xi(p) = \dfrac{x - f(p)}{\|x - f(p)\|}$ and $\lambda$ is an eigenvalue of the shape operator $A_{\xi(p)} : T_pM \longrightarrow T_pM$, i.e. $\lambda$ is a principal curvature of $f$ at $f(p)$ in the normal direction $\xi(p)$.

For $x \in \mathbb{R}^{m+k}$ the *distance function* for $f$, $L_x : M^m \to \mathbb{R}$ is defined by $L_x(p) = \|x - f(p)\|^2$. Using [6], the point $p \in M$ is a critical point of $L_x$ if and only if $x \in \nu_p(f)$ and further $p$ is a non-degenerate critical point of $L_x$ if and only if $x$ is not a focal point of $f$ with base $p$. So,

$$F_p(f) = \{x \in \mathbb{R}^{m+k} : p \text{ is a degenerate critical point of } L_x\}.$$

We use this characterisation of $F_p(f)$ to calculate focal points with base $p$. Further, using [6] again, the index of $L_x$ at a non-degenerate critical point $p \in M$ is equal to the number of focal points of $f$ with base $p$ which lie on the line segment from $f(p)$ to $x$, each focal point being counted with its multiplicity.

### 1.2 – *Parallel immersions to a given immersion*

Definition 2. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion and $\{\eta_1, \eta_2, \ldots, \eta_k\}$ be an orthonormal set of normal fields for $f$ in a neigh-*

*bourhood of some point* $p \in M$. *A normal field* $\xi$ *for* $f$ *is said to be a parallel normal field, if* $\left\langle \frac{\partial \xi}{\partial p_i}, \eta_j \right\rangle = 0$ *for all* $p \in M$, *where* $i = 1, \ldots, m$ , $j = 1, \ldots, k$ *and* $p_1, \ldots, p_m$ *is a coordinate system in a neighbourhood of* $p \in M$.

Since we assume $M$ is connected, note that a parallel normal field on $M$ has constant length.

Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion and assume $\xi : M^m \to \mathbb{R}^{m+k}$ is a parallel normal field for $f$ . The map $f_\xi : M^m \to \mathbb{R}^{m+k}$ is defined by

$$f_\xi(p) = f(p) + \xi(p).$$

If $f_\xi$ is an immersion, it is called a *parallel immersion* to $f$ and $\xi$ is said to be *immersive*. We remark that, for all $p \in M$, the normal planes of $f$ and $f_\xi$ at each $p \in M$ are the same.

If $f_\xi$ is an immersion, then the *index* of $f_\xi$, *ind* $f_\xi$, is defined to be the total multiplicity of the focal points of $f$ with base $p$ on the line segment between $f(p)$ and $f_\xi(p)$, this index is shown to be constant over $M$ by the following well-known fact. We call this number as the *index* of the immersive parallel normal field $\xi$ as well.

LEMMA 1. *Let* $f : M^m \to \mathbb{R}^{m+k}$ *be an immersion and let* $\xi : M^m \to \mathbb{R}^{m+k}$ *be a parallel normal field for* $f$. *Then the following are satisfied.*

(i) $f_\xi$ *is an immersion if and only if for all* $p \in M$, $f_\xi(p)$ *is not a focal point of* $f$ *with base* $p$

(ii) $x \in \mathbb{R}^{m+k}$ *is a focal point of* $f_\xi$ *with base* $p$ *if and only if* $x$ *is a focal point of* $f$ *with base* $p$. *So,* $F_p(f_\xi) = F_p(f)$ *for all* $p \in M$.

### 1.3 – *The push-out space of immersions with flat normal bundle*

Let $M$ be a connected, $m$-dimensional manifold and $f : M^m \to \mathbb{R}^{m+k}$ be an immersion. If for all $p \in M$, there exists a neighbourhood $U \subset M$ of $p$ and a parallel normal frame field for $f$ on $U$, then it is said that $f$ has locally flat normal bundle. The normal bundle $N(f)$ is flat (or globally flat) if there exists a global parallel normal frame on $M$.

If the immersion $f$ has locally flat normal bundle, then at each base point $p \in M$, the focal set on $\nu_p(f)$ is a union of at most $m$ hyperplanes (which is the simplest set that can occur as focal set, if non empty) where each plane is counted with its proper multiplicity [8, pp. 69-70]. A generalisation and the converse of this result can be derived from [4].

First, we assume that the normal bundle of $f$ is globally flat. So there exists an orthonormal set of parallel normal fields $\xi_1, \ldots, \xi_k : M^m \to \mathbb{R}^{m+k}$ for $f$. For each $p \in M$, a map $\varphi_p : \nu_p(f) \to \mathbb{R}^k$ can be defined by $\varphi_p\left(f(p) + \sum_{i=1}^{k} a_i \xi_i(p)\right) = (a_1, \ldots, a_k)$. For each $p \in M$, we denote $\Omega_p = \mathbb{R}^k \setminus \varphi_p(F_p(f))$. Then, the *push-out space* of the immersion $f$ is defined by

$$\Omega(f) = \bigcap_{p \in M} \Omega_p.$$

This set is essentially defined and many properties of it are studied in [3]. For example, $\Omega(f)$ has finitely many path-connected components with each component convex and each component can be assigned an integer called as index. Further, if $M$ is compact, each component is open. The definition of $\Omega(f)$ depends on the choice of $\xi_1, \ldots, \xi_k$, but, it is shown in [3] that different choices produces an isometric set. We are going to study some properties of $\Omega(f)$ which are related to number of path-connected components of $\Omega(f)$ with certain indices and some relations with the Euler characteristic of $M$ (when $M$ is compact).

As pointed out in [3] we can next consider an immersion $f$ of $m$ -dimensional manifold $M$ which has locally flat normal bundle but the normal holonomy group is nontrivial. In this case we can take the simply connected covering space $\tilde{M}$ of $M$ with covering map $\pi : \tilde{M}^m \to M^m$ and work with the immersion $\tilde{f} = f \circ \pi : \tilde{M}^m \to \mathbb{R}^{m+k}$ which has globally flat normal bundle with trivial normal holonomy. We know that $f$ and $\tilde{f}$ have the same focal set:

Proposition 1. *With the notation above, $F_p(f) = F_{\tilde{p}}(\tilde{f})$ for all $p \in M$ and $\tilde{p} \in \tilde{M}$ with $\pi(\tilde{p}) = p$, where $\pi : \tilde{M} \longrightarrow M$ is the covering map.*

Proof. Let $x \in \mathbb{R}^{m+k}$ and define $\tilde{L}_x : \tilde{M}^m \to \mathbb{R}$ (distance function for the immersion $\tilde{f}$) by

$$\tilde{L}_x(\tilde{p}) = \|x - \tilde{f}(\tilde{p})\|^2 = L_x \circ \pi(\tilde{p}),$$

where $L_x : M^m \to \mathbb{R}$ is the usual distance function for $f$. Since $\pi$ is an immersion, $\tilde{p}$ is a degenerate critical point of $\tilde{L}_x$ if and only if $\pi(\tilde{p})$ is a degenerate critical point of $L_x$. Therefore $F_p(f) = F_{\tilde{p}}(\tilde{f})$ for all $\tilde{p} \in \tilde{M}$ and $p \in M$ with $\pi(\tilde{p}) = p$. $\square$

So this result allows $\Omega(f)$ to be defined by $\Omega(f) = \Omega(f \circ \pi) = \Omega(\tilde{f})$. This is useful especially when we have an immersion of a nonorientable manifold with locally flat normal bundle where obviously the normal holonomy group is nontrivial. Consequently, by replacing $f$ with $\tilde{f}$ if necessary, we may assume that $f$ has globally flat normal bundle with trivial normal holonomy group. Remark that $\tilde{M}$ may fail to be compact again even $M$ is compact, but we can use critical point theory of distance function thorough the immersion of $M$ to deduce some results on $\Omega(\tilde{f})$.

Let $a = (a_1, a_2, \ldots, a_k) \in \Omega(f)$. As in [3], define $\xi(a) : M^m \to \mathbb{R}^{m+k}$ by $\xi(a)(p) = \sum_{i=1}^{k} a_i \xi_i(p)$, where $\xi_1, \xi_2, \ldots, \xi_k$ are unit parallel normal fields on $M$ forming a basis for the normal $k$-plane at $f(p)$ for all $p \in M$. Then it is easy to check that $\xi(a)$ is an immersive parallel normal field for $f$ on $M$. With this notation $\Omega(f)$ can be defined as

$$\Omega(f) = \{a \in \mathbb{R}^k : f(p) + \xi(a)(p) \text{ is not a focal point of } f \text{ with base } p,\ \forall p \in M\}.$$

DEFINITION 3. *Let $a \in \Omega(f)$. The index of $a$, ind $a$, is defined to be the index of the immersion $f_{\xi(a)}$.*

We know by [3] that if $A$ is a path-connected component of $\Omega(f)$ and if $a, b \in A$, then *ind* $a =$ *ind* $b$. Then the *index* of $A$ is defined to be *ind* $a$ for some $a \in A$ which is constant over $A$. So each path-connected component of $\Omega(f)$ can be assigned a number, called its *index*. We will denote the union of the path-connected components of $\Omega(f)$ with index $\mu$ by $\Omega^\mu$. So $\Omega(f) = \Omega^0 \cup \Omega^1 \cup \cdots \cup \Omega^m$. Note that $\Omega^0$ is always non empty and the others may be empty or not.

## 2. Path-connected components and their respective indices.

In this section, firstly, we give an example to illustrate the $\Omega(f)$ for a given embedding $f$ with flat normal bundle and then we prove some general results on $\Omega(f)$.

EXAMPLE 1. Let $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^3 \subset \mathbb{R}^{2+2}$ be given by

$$\tilde{f}(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi),$$

then $\tilde{f}$ induces an embedding $f$ of $\mathbb{S}^1 \times \mathbb{S}^1$ into $\mathbb{S}^3 \subset \mathbb{R}^{2+2}$ by taking $\theta$ mod $2\pi$, $\phi$ mod $2\pi$ and also $\Omega(f) = \Omega(\tilde{f})$. Now,

$$\xi_1(\theta, \phi) = \tilde{f}(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi),$$
$$\xi_2(\theta, \phi) = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\phi, \sin\phi)$$

are unit parallel normal fields to $\tilde{f}$ and form a basis for the normal planes for all $(\theta, \phi) \in \mathbb{R} \times \mathbb{R}$. Put $\xi(\theta, \phi) = t\xi_1(\theta, \phi) + s\xi_2(\theta, \phi)$, for some $t, s \in \mathbb{R}$, then

$$\begin{aligned}\tilde{f}_\xi(\theta, \phi) &= \tilde{f}(\theta, \phi) + t\xi_1(\theta, \phi) + s\xi_2(\theta, \phi). \\ &= \frac{1}{\sqrt{2}}((1+t-s)\cos\theta,\ (1+t-s)\sin\theta,\ (1+t+s)\cos\phi,\ (1+t+s)\sin\phi).\end{aligned}$$

Using the distance function $L_x(\theta, \phi) = \|x - \tilde{f}(\theta, \phi)\|^2$ for $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, we get

$$\frac{\partial L_x}{\partial \theta} = \frac{2}{\sqrt{2}}(x_1 \sin\theta - x_2 \cos\theta), \quad \frac{\partial L_x}{\partial \phi} = \frac{2}{\sqrt{2}}(x_3 \sin\phi - x_4 \cos\phi),$$
$$\frac{\partial^2 L_x}{\partial \theta^2} = \frac{2}{\sqrt{2}}(x_1 \cos\theta + x_2 \sin\theta), \quad \frac{\partial^2 L_x}{\partial \phi^2} = \frac{2}{\sqrt{2}}(x_3 \cos\theta + x_4 \sin\theta),$$
$$\frac{\partial^2 L_x}{\partial \phi \partial \theta} = \frac{\partial^2 L_x}{\partial \theta \partial \phi} = 0.$$

Then

$$Hess(L_x) = H = \begin{bmatrix} \frac{\partial^2 L_x}{\partial \theta^2} & 0 \\ 0 & \frac{\partial^2 L_x}{\partial \phi^2} \end{bmatrix}.$$

So $\tilde{f}_\xi(\theta, \phi)$ is a focal point of $\tilde{f}$ at $(\theta, \phi) \Longleftrightarrow (\theta, \phi)$ is a degenerate critical point of $L_x$. From the equations $\frac{\partial L_x}{\partial \theta} = 0 = \frac{\partial L_x}{\partial \phi}$, we obtain, for each $(\theta, \phi) \in \mathbb{R}^2$, $x = \tilde{f}_\xi(\theta, \phi)$ for some $t, s \in \mathbb{R}$. By replacing $x$ by $\tilde{f}_\xi(\theta, \phi)$ and using $det\ H = 0$

we get

$$det\begin{bmatrix} \frac{2}{\sqrt{2}}(1+t-s) & 0 \\ 0 & \frac{2}{\sqrt{2}}(1+t+s) \end{bmatrix} = 0 \Longleftrightarrow (1+t+s)(1+t-s)=0$$
$$\Longleftrightarrow (1+t)^2 - s^2 = 0$$
$$\Longleftrightarrow s = \pm(1+t).$$

Therefore the focal set of $\tilde{f}$ with base $(\theta, \phi) \in \mathbb{R} \times \mathbb{R}$ is a pair of lines perpendicular to one another which is the same for all base points $(\theta, \phi)$. Consequently

$$\Omega(\tilde{f}) = \Omega_{(\theta,\phi)}(\tilde{f}) = \Omega_{(\theta',\phi')}(\tilde{f}), \quad \forall(\theta,\phi),(\theta',\phi') \in \mathbb{R} \times \mathbb{R}.$$

Then, $\Omega(\tilde{f})$ has four path-connected components since each $\Omega_{(\theta,\phi)}(\tilde{f})$ has four path-connected components; one of index 0, two of index 1, and one of index 2. Hence the same is true for $\Omega(f)$, as $\Omega(f) = \Omega(\tilde{f})$, see the Figure 1. Then $\Omega(f) = \Omega^0 \cup \Omega^1 \cup \Omega^2$ and in the Figure 1, we put $\Omega^1 = A \cup B$.

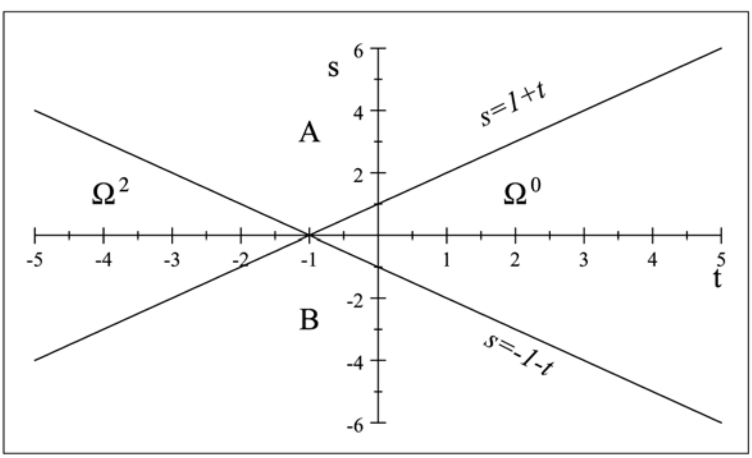


Figure 1

We know by [3], $\Omega(f)$ can have at most one component with index $m$ and if there is such a component, it is unbounded. Note that, we have examples of immersions such that $\Omega^0$ is bounded.

THEOREM 1. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion of a compact, m-dimensional manifold such that $f$ has flat normal bundle. If $\Omega(f)$ has a component with index m, then $\Omega^0$ is unbounded.*

PROOF. Let $a \in \Omega^m$ and take the immersive parallel normal field $\xi(a)$. Then $\varphi_p^{-1}(a) = f_{\xi(a)}(p), \forall p \in M$ and *index* $f_{\xi(a)} = m$. So for all $p \in M$, the total multiplicity of focal points with base $p$ on the line segment from $f(p)$ to $f_{\xi(a)}(p)$ is $m$. Therefore there are no focal points on the rays

$$R_p = \{f(p) + t\xi(a)(p) \in \nu_p(f) : \ t \geq 1\}$$

$$Q_p = \{f(p) + t\xi(a)(p) \in \nu_p(f) : \ t \leq 0\}.$$

Also $\forall p \in M, \ \varphi_p(Q_p) = \{ta : t \leq 0\} \subset \Omega_p$ and so

$$\{ta : t \leq 0\} \subset \bigcap \{\Omega_p : p \in M\} = \Omega(f).$$

Hence $\{ta : t \leq 0\} \subset \Omega^0$ and $\Omega^0$ is unbounded. □

PROPOSITION 2. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with flat normal bundle and assume $\xi, \eta : M^m \to \mathbb{R}^{m+k}$ are immersive parallel normal fields for $f$ with indices $\lambda$ and $\mu$ respectively. Then, the number of focal points with base $p$ on the line segment from $f_\xi(p)$ to $f_\eta(p)$ is constant for all $p \in M$ and it is $\lambda + \mu - 2l$ for some $l \in \mathbb{N}$ where* $\max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\}$.

PROOF. As in Lemma 4.4 of [3], since $f_\xi + (\eta - \xi) = f_\eta$, so $\eta - \xi$ is an immersive parallel normal field for the immersion $f_\xi$ and we need to find its index for $f_\xi$ which is constant. Here we try to formulate this constant.

Let $p \in M$ and put $x = f_\xi(p)$ and $y = f_\eta(p)$. If $f(p), x, y$ are collinear, then the number of focal points with base $p$ on the line segment from $f_\xi(p)$ to $f_\eta(p)$ is $\lambda + \mu$ or $|\lambda - \mu|$ with respect to positioning of $f(p)$ and we can take $l = 0$ or $l = \lambda$ or $l = \mu$ . Otherwise, take the triangle on $\nu_p(f)$ with vertices $f(p), x, y$, and consider the 2 -plane say $Q(p)$ which contains this triangle. We know that $Q(p) \cap F_p(f)$ is a union of at most $m$ lines if it is non empty, since $F_p(f)$ is a union of at most $m$ hyperplanes on $\nu_p(f)$ [8].

If $u, v \in \nu_p(f)$, then the notation $\overline{uv}$ denotes the line segment from $u$ to $v$. We know the total multiplicity of focal points on $\overline{f(p)x}$ is $\lambda$ and on $\overline{f(p)y}$ is $\mu$. Now, let $l(p) \geq 0$ be an integer and assume that $l(p)$ lines(counting multiplicities) meet both of the edges $\overline{f(p)x}$ and $\overline{f(p)y}$. Clearly $0 \leq l(p) \leq \min\{\lambda, \mu\}$. Then the remaining $\lambda - l(p)$ lines intersecting $\overline{f(p)x}$ must intersect $\overline{xy}$. And similarly the remaining $\mu - l(p)$ lines intersecting $\overline{f(p)y}$ must intersect $\overline{xy}$. So we get the total multiplicity on $\overline{xy}$ is exactly

$\lambda - l(p) + \mu - l(p) = \lambda + \mu - 2l(p)$. So we deduce the total multiplicity of focal points on the line segment from $f_\xi(p)$ to $f_\eta(p)$ is $\lambda + \mu - 2l(p)$. But the index of the parallel immersion $(f_\xi)_{(\eta-\xi)}$ to $f_\xi$ is a constant number, so $l(p)$ is constant for all $p \in M$.

Put $l = l(p)$. Since there exist at most $m$ lines on $Q(p)$, then $\lambda + \mu - m \leq l$. Then $\max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\}$. □

COROLLARY 1. *If $\lambda = \mu = 1$ in Proposition 2, then, for all $p \in M$, the number of focal points with base $p$ on the line segment from $f_\xi(p)$ to $f_\eta(p)$ is 2 (where $l = 0$).*

THEOREM 2. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with flat normal bundle and assume $\xi : M^m \to \mathbb{R}^{m+k}$ is an immersive parallel normal field for $f$.*

(i) *There exists a $w \in \mathbb{R}^k$ such that $\Omega(f_\xi) = \Omega(f) - w$, where $\Omega(f) - w = \{a - w : a \in \Omega(f)\}$,*

(ii) *if $A$ is a path-connected component of $\Omega(f)$ with index $\mu$ and if the index of $f_\xi$ is $\lambda$, then there exists an $l \in \mathbb{N}$ such that $A - w$ is a path-connected component of $\Omega(f_\xi)$ with index $\lambda + \mu - 2l$, where* $\max\{0, \lambda + \mu - m\} \leq l \leq \min\{\lambda, \mu\}$.

PROOF. (*i*) Since $f$ has flat normal bundle, there exists a set of orthonormal parallel normal fields $\{\xi_1, \xi_2, \cdots, \xi_k\}$ forming a basis of the normal space at each base point $p \in M$. We will use this basis to define $\Omega(f)$ and $\Omega(f_\xi)$. We can write

$$\xi = w_1\xi_1 + \cdots + w_k\xi_k$$

for some constants $w_1, \ldots, w_k \in \mathbb{R}$ and put $w = (w_1, \ldots, w_k) \in \mathbb{R}^k$. Then we can easily see that $a \in \Omega(f) \Longleftrightarrow a - w \in \Omega(f_\xi)$. In fact, let $a = (a_1, \ldots, a_k) \in \Omega(f)$. We know $F_p(f) = F_p(f_\xi)$ for all $p \in M$ by Lemma 1 (ii). Then, for all $p \in M$

$$\begin{aligned} f(p) + a_1\xi_1 + \cdots + a_k\xi_k \notin F_p(f) &\Longleftrightarrow f + \xi + a_1\xi_1 + \cdots + a_k\xi_k - \xi \\ &= f_\xi + (a_1 - w_1)\xi_1 + \cdots + (a_k - w_k)\xi_k \notin F_p(f_\xi). \end{aligned}$$

So $a - w \in \Omega(f_\xi)$ and therefore $\Omega(f_\xi) = \Omega(f) - w$.

(*ii*) Let $a \in A$, then clearly $a - w \in \Omega(f_\xi)$ by Theorem 2 (i), hence $A - w$ is a path-connected component of $\Omega(f_\xi)$. Since $A$ is a path-connected component of $\Omega(f)$ with index $\mu$, there exists an immersive parallel normal field $\eta$ for $f$ with index $\mu$ and $\varphi_p(f(p) + \eta(p)) = a$ for all $p \in M$. As in

Proposition 2, $f_\xi + (\eta - \xi) = f_\eta$, so $\eta - \xi$ is an immersive parallel normal field for $f_\xi$ and its index for $f_\xi$ is $\lambda + \mu - 2l$ for some $l \in \mathbb{N}$ where $\max\{0, \lambda + \mu - m\} \le l \le \min\{\lambda, \mu\}$. □

The following result concerns the positioning of the path-connected components of $\Omega(f)$ in $\mathbb{R}^k$.

THEOREM 3. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with flat normal bundle. Let $A, B$ be path-connected components of $\Omega(f)$ with index $\lambda$ and $\mu$ respectively. If $\lambda + \mu > m$, then there exists a hyperplane in $\mathbb{R}^k$ such that $A$ and $B$ lie on one side of the hyperplane and $\Omega^0$ lies on the opposite side of the hyperplane.*

PROOF. Let $A, B$ be path-connected components of $\Omega(f)$ with index $\lambda$ and $\mu$ respectively and $a \in A$, $b \in B$. Then there exist immersive parallel normal fields $\xi, \eta$ for $f$ such that *index* $f_\xi = \lambda$, *index* $f_\eta = \mu$ and also for all $p \in M$, $\varphi_p^{-1}(a) = f_\xi(p)$ and $\varphi_p^{-1}(b) = f_\eta(p)$. Now consider the normal plane $v_p(f)$ for a fixed $p \in M$ and the focal hyperplanes $\Pi_1, \dots, \Pi_s$ on $v_p(f)$ with their respective multiplicity $w_i$ where $1 \le i \le s$, $s \le m$ and $w_1 + \dots + w_s \le m$.

Since $f_\xi$ has index $\lambda$, the line segment joining $f(p)$ to $f_\xi(p)$ must cross $\Pi_{\alpha_1}, \dots, \Pi_{\alpha_l}$ where $l \le \lambda$, $w_{\alpha_1} + \dots + w_{\alpha_l} = \lambda$ and similarly the line segment joining $f(p)$ to $f_\eta(p)$ must cross $\Pi_{\beta_1}, \dots, \Pi_{\beta_d}$ where $d \le \mu$, $w_{\beta_1} + \dots + w_{\beta_d} = \mu$.

Here, $\Pi_{\alpha_1}, \dots, \Pi_{\alpha_l}, \Pi_{\beta_1}, \dots, \Pi_{\beta_d}$ are not all distinct since $\lambda + \mu > m$ . So let $\Pi \in \{\Pi_{\alpha_1}, \dots, \Pi_{\alpha_l}\} \cap \{\Pi_{\beta_1}, \dots, \Pi_{\beta_d}\}$. Then we claim that $A, B$ stay on one side of the hyperplane $\varphi_p(\Pi) = \Lambda$ in $\mathbb{R}^k$. Set $\varphi_p(\Pi_i) = \Lambda_i$, $1 \le i \le s$ . Since each $\Lambda_i$ divides $\mathbb{R}^k$ into two open connected regions, we identify them by writing $\Lambda_i^-$ for the region including the origin and $\Lambda_i^+$ for the other part.

Then, $\Omega^0 \subset \Lambda_i^-$ for all $1 \le i \le s$, $A \subset \Lambda_{\alpha_1}^+ \cap \dots \cap \Lambda_{\alpha_l}^+$ and $B \subset \Lambda_{\beta_1}^+ \cap \dots \cap \Lambda_{\beta_d}^+$. Therefore $A$ and $B$ stay in $\Lambda^+$, and hence $\Lambda$ is the hyperplane we are seeking. □

## 3. Number of path-connected components of $\Omega(f)$ with certain indices

It is interesting to know the number of path-connected components of $\Omega(f)$ with their respective indices for an immersion $f$ of $M$ as it includes some information on the geometry and the topology of the $m$ -dimensional compact manifold $M$. Here, we prove that if we have a path-connected component of $\Omega(f)$ with index 1, then the Euler characteristic of $M$ is 0. Secondly, we prove that the number of path-connected components of $\Omega(f)$ with index $(m-1)$ is at most 2 for a co-dimension 2 immersion.

THEOREM 4. *Let $f : M^m \longrightarrow \mathbb{R}^{m+k}$ be an immersion of compact manifold with flat normal bundle and let $\chi(M) \neq 0$ where $m$ is an even number. Then $\Omega^1 = \emptyset$.*

PROOF. If $\Omega^1 \neq \emptyset$, then there exists a unit parallel normal field $\xi$ for $f$ such that $f_{s\xi} = f + s\xi$ is an immersion with index 1 for some $s > 0$. So $\forall\, p \in M$, there exists only one focal point $c(p)$ of multiplicity 1 on the line segment from $f(p)$ to $f_{s\xi}(p)$ such that $c : M^m \longrightarrow \mathbb{R}^{m+k}, \ \ p \rightarrow c(p)$ is continuous. Define $\lambda : M^m \longrightarrow \mathbb{R}$ by

$$\lambda(p) = \frac{1}{\|f(p) - c(p)\|}.$$

Then $\lambda$ is continuous as it is the principal curvature function of $f$ in the unit normal direction $\xi$. Also $\lambda$ is smooth since it is of constant multiplicity 1 on $M$ [7]. So the principal direction corresponding to the principal curvature $\lambda(p)$ defines a nonzero smooth tangent vector field on $M$ which has no zeros. So considering that $M$ is compact, $\chi(M) = 0$ by the Poincaré-Hopf Theorem in [5]. But this contradicts $\chi(M) \neq 0$. Therefore $\Omega^1 = \emptyset$. □

A generalisation of this theorem to any odd indexed component is proved in [1] by a different method. Present method here may not be generalized, because respective vector field can fail to be smooth.

DEFINITION 4. *Let $f : M^m \longrightarrow \mathbb{R}^{m+k}$ be an immersion with flat normal bundle, then $d(f)$ is defined to be the total number of the path-connected components of $\Omega(f)$.*

It was proved in [3] that $d(f) \leq \alpha(m,k)$ where $\alpha(m,k)$ is the number of path-connected regions in the complement of $m$ hyperplanes in general position in $\mathbb{R}^k$ as

$$\alpha(m,k) = \begin{cases} 2^m & \text{if } m \leq \ k \\ \displaystyle\sum_{i=0}^{k} \binom{m}{i} & \text{if } m > k \end{cases}.$$

Corollary 2. *Let* $f : M^2 \longrightarrow \mathbb{R}^{2+k}$ *be an immersion with flat (or locally flat) normal bundle of a compact surface for some* $k \geq 1$ *and let* $\chi(M) \neq 0$. *Then* $\Omega^1 = \emptyset$ *and so* $d(f) \leq 2$.

Proof. By Theorem 4, $\Omega^1 = \emptyset$ and also $\Omega^0$, $\Omega^2$ are connected [3], hence $d(f) \leq 2$. Of course $\Omega^2$ can occur, definitely when $f$ is spherical, [2]. □

Example 2. For the homology groups of real projective space $\mathbb{R}P^m$, we know that $H_i(\mathbb{R}P^m, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $i = 1, 2, \ldots, m$. Then

$$\chi(\mathbb{R}P^m) = \begin{cases} 0\ , & \text{if } m \text{ is odd} \\ 1\ , & \text{if } m \text{ is even} \end{cases}.$$

So, by Theorem 4, if $f : \mathbb{R}P^m \longrightarrow \mathbb{R}^{m+k}$ is any immersion with locally flat normal bundle, we have $\Omega^1 = \emptyset$ for $m$ is even.

Let $f : M^2 \longrightarrow \mathbb{R}^{2+k}$ be an immersion of a 2 -dimensional manifold $M$ with flat normal bundle. Then for any point $p \in M$, $F_p(f)$ is a union of at most 2 hyperplanes in $\nu_p(f)$. So $F_p(f)$ can divide $\nu_p(f)$ into at most 4 path-connected regions, and the number of path-connected components of $\Omega(f)$ with index 1 can be at most 2 for any $k \geq 2$.

In the following theorems we generalize this and prove a result concerning the number of path-connected components of $\Omega(f)$ with index $(m-1)$ where $m \geq 3$.

Theorem 5. *Let* $m \geq 2$ *and* $f : M^m \longrightarrow \mathbb{R}^{m+k}$ *be an immersion with flat normal bundle. Assume* $A, B$ *are two different path-connected components of* $\Omega(f)$ *both with index* $(m-1)$ *and* $a \in A$, $b \in B$. *Then for each* $p \in M$, *all the focal hyperplanes in* $\nu_p(f)$ *meet the triangle* $\triangle$ *with vertices* $f(p)$, $\varphi_p^{-1}(a)$, $\varphi_p^{-1}(b)$, *and moreover the total number of focal points on the line segment from* $\varphi_p^{-1}(a)$ *to* $\varphi_p^{-1}(b)$ *is exactly* 2.

Proof. Let $a \in A$, $b \in B$. Then there are corresponding parallel normal fields $\xi = \xi(a)$ and $\eta = \xi(b)$ say, such that *index* $f_\xi =$ *index* $f_\eta = m - 1$. By Proposition 2, for all $p \in M$, we have total number of focal points between $f_\xi(p) = \varphi_p^{-1}(a)$ and $f_\eta(p) = \varphi_p^{-1}(b)$ is $2(m-1) - 2l$ for some $l \in \mathbb{N}$ where $m - 2 \leq l \leq m - 1$. Since $a$ and $b$ are in different components, there must be at least one focal point between $f_\xi(p)$ and $f_\eta(p)$ for all $p \in M$. So $l = m - 2$.

Let $Q \subset \nu_p(f)$ be the plane including the triangle $\triangle$ with vertices $f(p)$, $f_\xi(p)$, $f_\eta(p)$. Since $l = m - 2$, we have proved that the total multiplicity of focal points on $\overline{f_\xi(p) f_\eta(p)}$ is exactly 2 for all $p \in M$ and hence there are exactly $m$ focal lines meeting with the triangle $\triangle$ as required. Since there are $m$ lines in $Q$, this implies that all focal hyperplanes on $\nu_p(f)$ meet with the triangle $\triangle$, for all $p \in M$. □

THEOREM 6. *Let $f: M^m \longrightarrow \mathbb{R}^{m+2}$ be an immersion of a compact manifold such that $f$ has flat normal bundle, where $m \geq 3$. Then the number of path-connected components of $\Omega(f)$ with index $(m-1)$ is at most 2.*

PROOF. Assume there exist at least three path-connected components of $\Omega(f)$ with index $(m-1)$, say $A, B, C$. Take $a \in A$, $b \in B$, $c \in C$. Let $p \in M$ be an arbitrary point and consider the points $x = \varphi_p^{-1}(a)$, $y = \varphi_p^{-1}(b)$, $z = \varphi_p^{-1}(c)$ on $\nu_p(f)$. Clearly $x, y, z$ are nonfocal distinct points, since $a, b, c$ are in different components.

Since $a, b, c$ are in different components there is at least one focal point on each line segment $\overline{xy}, \overline{yz}, \overline{zx}$. So we can check that the points $x, y, f(p)$ cannot be collinear. Assume they lie on a line $\ell$ say. If $f(p)$ is on $\overline{xy}$ then the total multiplicity of focal points on $\ell$ is at least $(2m-2)$ which is not possible for $m \geq 3$ , since $2m - 2 > m$. If $f(p)$ is not on $\overline{xy}$ we get the total multiplicity of focal points on $\overline{f(p)x}$ or $\overline{f(p)y}$ is at least $m$ depending on the positioning of $f(p)$ on $\ell$ with respect to the points $x, y$. This contradicts the hypothesis that this number is $(m-1)$ . By a similar discussion we get the points $x, z, f(p)$ or $y, z, f(p)$ or $x, y, z, f(p)$ cannot be collinear.

By Theorem 5, all of the focal lines must meet the triangle with vertices $x, y, f(p)$ and further the total multiplicity of focal points on $\overline{xy}$ is exactly 2. Similarly we get the same result considering the triangles with vertices $y, z, f(p)$ and $x, z, f(p)$.

We next show $x, y, z$ are not collinear. For if $x, y, z$ all lie on a line then by the above argument the total multiplicity of focal points on each line segment $\overline{xy}, \overline{yz}, \overline{zx}$ is exactly 2. Without loss of generality we can assume $y$ is on $\overline{xz}$. Then we obtain the total multiplicity of focal points on $\overline{xz}$ is $2 + 2 = 4$ which is a contradiction.

Now consider the triangle with vertices $x, y, z$. There are 3 cases to be considered.

CASE 1. Assume $f(p)$ is in the region I bounded by the triangle with vertices $x, y, z$ as shown in Figure 2. By Theorem 5 there exists at least one focal line meeting with $\overline{xf(p)}$ and $\overline{zf(p)}$ considering the triangle with vertices $x, f(p), z$. Similarly there exists one focal line meeting with $\overline{xf(p)}$ and $\overline{yf(p)}$ considering the triangle with vertices $x, f(p), y$. And also there exists one focal line meeting with $\overline{yf(p)}$ and $\overline{zf(p)}$ considering the triangle with vertices $y, f(p), z$. These focal lines are necessarily all different and together bound $f(p)$. This implies that $f(p)$ is in a bounded region of the complement of the focal lines on $\nu_p(f)$.

CASE 2. Assume $f(p)$ is in the region II as shown in Figure 2. Consider the triangle with vertices $z, f(p), y$. By Theorem 5, there must be a focal line meeting with $\overline{f(p)z}$ and $\overline{zy}$ and this line must necessarily meet $\overline{xf(p)}$ and $\overline{xy}$. Similarly by considering the triangle with vertices $x, f(p), z$, there must be a focal line meeting with $\overline{f(p)z}$ and $\overline{zx}$ and this line must necessarily meet $\overline{f(p)y}$ and $\overline{xy}$. Now we get at least 2 focal points on $\overline{xy}$. But again by Theorem 5 and considering the triangle with vertices $x, f(p), y$, it is exactly 2. So there are no more focal lines meeting with $\overline{xy}$. So far we have one focal line meeting both $\overline{xf(p)}$ and $\overline{zf(p)}$. By Theorem 5 and considering the triangle with vertices $x, f(p), z$, we need $(m-3)$ more focal lines meeting with $\overline{xf(p)}$ and $\overline{zf(p)}$ which must necessarily meet with $\overline{yf(p)}$. And one more focal line meeting both $\overline{xf(p)}$ and $\overline{xz}$ which must necessarily meet with $\overline{xy}$ or $\overline{zy}$. We know there are no more focal lines meeting with $\overline{xy}$. So the focal line meeting both $\overline{xf(p)}$ and $\overline{xz}$ must necessarily meet with $\overline{zy}$. This implies that for all $p \in M$, $z = f_{\xi(a)}(p)$ is bounded by focal lines on $\nu_p(f)$ where the immersive parallel normal field $\xi(a)$ is corresponding to $a$.

CASE 3. Now assume $f(p)$ is in the region III as shown in Figure 2. Then we know every focal line must meet the triangle with vertices $f(p), x, y$. But there must be a focal line meeting with $\overline{f(p)z}$ and $\overline{zx}$ simultaneously. So this line cannot meet the triangle with vertices $f(p), x, y$. This gives a contradiction by Theorem 5. So we deduce that Case 3 cannot occur.

Since $p$ is an arbitrary point in $M$ and ${\varphi_p}^{-1}$ is an isometry, then either Case 1 holds for all $p \in M$ or Case 2 holds for all $p \in M$ i.e. either $f(p)$ is bounded by focal lines on $\nu_p(f)$ or $f_{\xi(a)}(p)$ is bounded by focal lines on $\nu_p(f)$ for all $p \in M$.

Now, for some $w \in \mathbb{R}^{m+2}$, take the distance function $L_w$ for $f$. Since $M$ is compact, there is a critical point of $L_w$ with index $m$. So the total number of focal points with base $p$ on the line segment from $w$ to $f(p)$ is $m$ and so there is no focal point with base $p$ on the ray $\{f(p) + t(w - f(p)) \mid t \leq 0\} \subset \nu_p(f)$. This implies that for some $p \in M$, $f(p)$ is not bounded by focal hyperplanes on $\nu_p(f)$ and a similar statement is true for $f_{\xi(a)}(q)$ considering the immersion

$f_{\xi(a)}$ for some $q \in M$. So there cannot be such path-connected components $A, B, C$ of $\Omega(f)$. Therefore $\Omega(f)$ can have at most two path-connected components with index $(m-1)$.

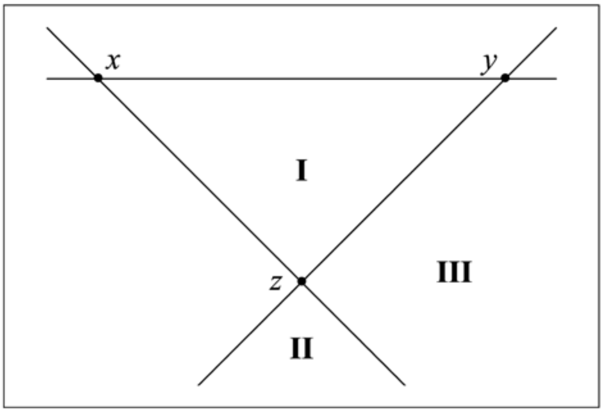


Figure 2

□

REMARK 1. Theorem 6 is not true for an immersion with co-dimension $k > 2$. We can see that by taking product immersions. In Example 1, we have an embedding $f$ of $\mathbb{T}^2$ into $\mathbb{S}^3 \subset \mathbb{R}^4$ such that $\Omega(f)$ has two path-connected components with index 1. Now take

$$f \times f : \mathbb{T}^2 \times \mathbb{T}^2 \longrightarrow \mathbb{R}^{4+4}$$

by $(f \times f)(p, q) = (f(p), f(q))$ where $p, q \in \mathbb{T}^2$. Note that, by Theorem 4.2 of [3], $f \times f$ has flat normal bundle and $\Omega(f \times f) = \Omega(f) \times \Omega(f)$, since $f$ has flat normal bundle. Then, we can easily check that $\Omega(f \times f)$ has 4 path-connected components with index 3.

Consequently, for $m > 2$ and $k > 2$, it is a considerable question to ask what is the maximum number of path-connected components of $\Omega(f)$ with index $(m-1)$. This might be at most $k$.

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