

Riccati Differential Equation for Hypergeometric Differential Equation

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ABSTRACT - In this paper, we study the solutions of the Riccati differential equation corresponding to a p -adic differential equation which is solvable on the generic disc. As an application, we consider the Grothendieck conjecture for Riccati differential equations. We see that the Riccati differential equation for some globally nilpotent differential equation with coefficients in $\mathbb{Q}(t)$ have, for almost all prime number, a solution in $\mathbb{F}_p(t)$, but do not have any algebraic solutions.

1. Introduction

Let p be a prime and K a complete non-archimedean valued field with mixed characteristics. In this paper, we consider the solutions of the Riccati differential equation for a p -adic differential equation L which is solvable on the generic disc $D(t, 1^-)$ (see the Definition 2.5).

We attach the Riccati differential equation $C(L)_1$ for each linear ordinary differential operator L in section 3. Let E be the completion of $K(t)$ under the Gauss norm. We see that a solution of the Riccati differential equation $C(L)_1$ for L on E is connected with a bounded solution of L at the generic point.

Let $\text{bddKer}_i(L)$ be the vector space consisting of the bounded solutions of L . If its dimension is one, then $C(L)_1$ has a unique solution on E . In particular if we can reduce L modulo p , then $C(L)_1$ has a solution on the rational function field $\bar{K}(t)$ over the residue field \bar{K} of K .

Let L be an n -th order linear ordinary differential operator with coefficients in $\mathbb{Q}(t)$. The Grothendieck conjecture (in its simplest form) asserts the following statements are equivalent (see [Put]):

- (i) the equation $L(y) = 0$ has n linear independent (over \mathbb{Q}^{alg}) solutions which are algebraic over $\mathbb{Q}(t)$.
- (ii) For almost all p , the equation $L(y) = 0$ has n linear independent solutions over the field $\mathbb{F}_p(t^p)$ in the field $\mathbb{F}_p(t)$.

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Van der Put studied the Grothendieck conjecture for non-linear differential equations. He proved that the differential equation $y' = (1/t^2)y + 1$ has no algebraic solutions but there exists a solution in $\mathbb{F}_p(t)$ for all primes p . He also proved that the conjecture is true for some Risch equations [Put], [Put2].

We will prove the following theorem.

THEOREM 1.1. *Let L be a monic differential operator with coefficients in $\mathbb{Q}(t)$. We assume that when we regard L as a p -adic differential equation, L is solvable on the generic disc $D(t, 1)$ for almost all prime p , and the dimension of the vector space which consists of the bounded solutions of L is one. If L has no algebraic solutions, then $C_1(L)$ modulo p has a solution on $\mathbb{F}_p(t)$ for almost all prime p and no algebraic solutions.*

In particular some hypergeometric differential equations satisfy the condition of Theorem 1.1.

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2. Notation

2.1 – Analytic elements

Let K be a complete non-archimedean valued field with mixed characteristics and \bar{K} the residue field of K . Let $K(t)$ be the field of rational functions. For a polynomial $f = \sum a_i t^i$ with coefficients in K , we set

$$(2.1) \quad |f|_0 = \max |a_i|.$$

The field of rational functions $K(t)$ has the Gauss norm $|\cdot|_0$ defined by $|f|_0 = |g|_0/|h|_0$ if $f = g/h$ and $g, h \in K[t]$. Let E be a completion of $K(t)$ under the Gauss norm. An element of E is called an analytic element. The derivation d/dt of $K(t)$ is extended to E . We regard $(E, d/dt)$ as a differential field.

For $a \in E$ and $r \in \mathbb{R}_{>0}$, let

$$(2.2) \quad D(a, r^-) = \{z \in E \mid |z - a|_0 < r\}$$

and

$$(2.3) \quad D(a, r) = \{z \in E \mid |z - a|_0 \leq r\}.$$

We define the ring of analytic functions on the disc $D(a, r^-)$ to be

$$\mathcal{A}(a, r) = \left\{ \sum_{n=0}^{\infty} a_n(x-a)^n \in E[[x-a]] \mid a_n \in E, \lim_{n \rightarrow \infty} |a_n|s^n = 0 \text{ for all } 0 \leq s < r \right\}.$$

We define the subring $E[[x-a]]_0$ of the ring of analytic functions $\mathcal{A}(a, 1)$ by

$$E[[x-a]]_0 = \left\{ \sum_{n=0}^{\infty} a_n(x-a)^n \in E[[x-a]] \mid \sup_{0 \leq n < \infty} |a_n| < \infty \right\}.$$

PROPOSITION 2.1 ([Christol, 2.5.1]). *We define a ring homomorphism $\tau : E \rightarrow E[[x-t]]_0$ so that, for $f \in E$,*

$$\tau : E \rightarrow E[[x-t]]_0 \quad f \mapsto \sum_{n=0}^{\infty} \frac{(d/dt)^n(f)}{n!} (x-t)^n.$$

Then τ is an isometric isomorphism from E to the subfield of $E[[x-t]]_0$ consisting of elements f which satisfy:

$$(2.4) \quad \tau((d/dt)(f)) = (d/dx)(\tau(f)).$$

2.2 – Differential modules

Let (H, D) be a differential field of characteristic zero and C_H the constant field of H . We say differential field (L, D') is a *differential extension* of (H, D) if $L \supset H$ and $D'|_H = D$. An H -differential module (M, ∇) is a free H -module M of finite type endowed with a C_H -linear homomorphism $\nabla(D)$ which satisfies $\nabla(D)(am) = Da.m + a\nabla(D)(m)$ for $a \in H$ and $m \in M$.

Let (M, ∇) be an H -differential module and L a differential extension of H . An H -linear map $s : M \rightarrow L$ is said *horizontal* if $s(\nabla m) = D(s(m))$ for $m \in M$. We denote the C_H -vector space by $\text{Hom}_{\nabla}(M, L) := \{s \in \text{Hom}_{C_H}(M, L) \mid s \text{ is horizontal}\}$.

PROPOSITION 2.2. *Let (M, ∇) be an H -differential module. For any differential extension L/H ,*

$$(2.5) \quad \dim_{C_H}(\text{Hom}_{\nabla}(M, L)) \leq \dim_H(M)$$

REMARK 2.3. By the existence of Picard-Vessiot field of $H \otimes_{C_H} C_H^{alg}$, there exists a differential extension L/H such that equality (2.5) holds (see [PS]).

PROPOSITION 2.4. Let G be a matrix in $M_n(E)$. There exists a unique matrix U of $GL_n(E[[x-t]])$ such that

$$(2.6) \quad U(t) = I_n, \quad (d/dx)(U) = \tau(G)U.$$

Let $\{G_i\}_{i \in \mathbb{N}}$ be the sequence of matrix in $M_n(E)$ defined by

$$G_0 = I_n, \quad G_{i+1} = (d/dt)(G_i) + G_i G.$$

Then we have

$$U = \sum_{i=0}^{\infty} G_i \frac{(x-t)^i}{i!}.$$

DEFINITION 2.5. Let (M, ∇) be an E -differential module of rank n . Let e_1, \dots, e_n be a basis of M and $G = (g_{ij})$ the matrix such that $\nabla(D)e_i = \sum_{j=0}^n g_{ij}e_j$. We say (M, ∇) is solvable over $D(t, r)$ if the matrix U which is defined by (2.6) is an element of $GL_n(\mathcal{A}(t, r))$.

For a differential field (H, D) which satisfies $K(t) \subset H \subset E$, we call an H -differential module (M, ∇) is solvable over $D(t, r)$ if the E -differential module $(M \otimes_H E, \nabla)$ is solvable over $D(t, r)$.

3. Riccati differential equation

Let (H, D) be a differential field of characteristic zero and C_H a constant field of H . Let $L = D^n - g_n D^{n-1} - \dots - g_1$ be a differential operator with coefficients in H . Then $(M, \nabla) = H[D]/H[D]L$ is an H -differential module of rank n . Let G be the matrix of representation of $\nabla(D)$ with respect to the basis $1, \dots, D^{n-1}$ of (M, ∇) , i.e,

$$(3.1) \quad G = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & & & \cdots & 1 \\ g_1 & g_2 & & \cdots & g_n \end{pmatrix}.$$

We suppose that L has a decomposition

$$L = L_1 \circ R, \quad L_1, R \in H[D] \text{ and } R \text{ is monic.}$$

We write $R = D^k - c_k D^{k-1} - \dots - c_1$. Since the canonical morphism $M = H[D]/H[D]L \rightarrow H[D]/H[D]R$ is surjective, the induced morphism $(H[D]/H[D]R)^\vee \rightarrow M^\vee$ is injective.

Let f_1, \dots, f_n (resp. l_1, \dots, l_k) be the dual basis of the basis $1, D, \dots, D^{n-1}$ of M (resp. $(H[D]/H[D]R)$). If we regard $(H[D]/H[D]R)^\vee$ as a submodule of M^\vee then each l_i is a linear combination of f_i . In fact, by the definition of dual basis, we have

$$(3.2) \quad \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_k \end{pmatrix} = C \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

such that $C = \{c_{i,j}\}$ is the $k \times n$ matrix defined by

$$(3.3) \quad D^{m-1} = c_{k,m} D^{k-1} + c_{k-1,m} D^{k-2} + \dots + c_{1,m} \pmod{R}.$$

In particular $c_{j,k+1} = c_j$ and if $j \leq k$ then $c_{i,j} = \delta_{i,j}$.

Let F/H be a differential extension such that $\dim_{C_H}(\text{Hom}_\nabla(M, F)) = \dim_H(M)$ (see Remark 2.3). Let s_1, \dots, s_n be a basis of $\text{Hom}_\nabla(M, F)$. By the equality (3.2), we have

$$(3.4) \quad \begin{pmatrix} s_1(l_1) & s_2(l_1) & \dots & s_n(l_1) \\ s_1(l_2) & s_2(l_2) & \dots & s_n(l_2) \\ \vdots & & & \vdots \\ s_1(l_k) & s_2(l_k) & \dots & s_n(l_k) \end{pmatrix} = C \begin{pmatrix} s_1(f_1) & s_2(f_1) & \dots & s_n(f_1) \\ s_1(f_2) & s_2(f_2) & \dots & s_n(f_2) \\ \vdots & & & \vdots \\ s_1(f_n) & s_2(f_n) & \dots & s_n(f_n) \end{pmatrix}.$$

Operating D on both sides the equation (3.4), we have

$$\begin{aligned} & - \begin{pmatrix} 0 & \dots & c_1 \\ 1 & & \vdots \\ & & 1 & c_k \end{pmatrix} \begin{pmatrix} s_1(l_1) & s_2(l_1) & \dots & s_n(l_1) \\ s_1(l_2) & s_2(l_2) & \dots & s_n(l_2) \\ \vdots & & & \vdots \\ s_1(l_k) & s_2(l_k) & \dots & s_n(l_k) \end{pmatrix} \\ & = \left(D(C) - C \begin{pmatrix} 0 & \dots & g_1 \\ 1 & \dots & g_2 \\ \vdots & & \vdots \\ 0 & \dots & 1 & g_n \end{pmatrix} \right) \begin{pmatrix} s_1(f_1) & s_2(f_1) & \dots & s_n(f_1) \\ s_1(f_2) & s_2(f_2) & \dots & s_n(f_2) \\ \vdots & & & \vdots \\ s_1(f_n) & s_2(f_n) & \dots & s_n(f_n) \end{pmatrix} \end{aligned}$$

Since the matrix $(s_i(f_j))_{i,j}$ is invertible, we have

$$-\begin{pmatrix} 0 & \cdots & c_1 \\ 1 & & \vdots \\ & & 1 & c_k \end{pmatrix} C = D(C) - C \begin{pmatrix} 0 & \cdots & g_1 \\ 1 & \cdots & g_2 \\ \vdots & & \vdots \\ 0 & \cdots & 1 & g_n \end{pmatrix}.$$

Therefore we have the equations

$$(3.5) \quad \begin{aligned} c_1 c_{k,n} + D(c_{1,n}) - g_1 - c_{1,k+1} g_{k+1} - \cdots - c_{1,n} g_n &= 0, \\ c_{1,n} + c_2 c_{k,n} + D(c_{2,n}) - g_2 - c_{2,k+1} g_{k+1} - \cdots - c_{2,n} g_n &= 0, \\ &\vdots \\ c_{k-1,n} + c_k c_{k,n} + D(c_{k,n}) - g_k - c_{k,k+1} g_{k+1} - \cdots - c_{k,n} g_n &= 0. \end{aligned}$$

We define the differential system $C(L)_k$ in k -variable y_1, \dots, y_k by

$$(3.6) \quad \begin{aligned} y_1 y_{k,n} + D(y_{1,n}) - g_1 - y_{1,k+1} g_{k+1} - \cdots - y_{1,n} g_n &= 0, \\ y_{1,n} + y_2 y_{k,n} + D(y_{2,n}) - g_2 - y_{2,k+1} g_{k+1} - \cdots - y_{2,n} g_n &= 0, \\ &\vdots \\ y_{k-1,n} + y_k y_{k,n} + D(y_{k,n}) - g_k - y_{k,k+1} g_{k+1} - \cdots - y_{k,n} g_n &= 0 \end{aligned}$$

where $y_{i,j}$ ($1 \leq i \leq k$, $j \geq 1$) is defined inductively by

$$y_{1,1} = 1, y_{j,1} = 0 \quad (2 \leq j \leq k),$$

$$y_{1,m+1} = D(y_{1,m}) + y_{k,m} \cdot y_1, \quad y_{j,m+1} = D(y_{j,m}) + y_{k,m} \cdot y_j + y_{j-1,m} \quad (2 \leq j \leq k).$$

By definition, the vector (c_1, \dots, c_k) is solutions of differential system $C(L)_k$.

In particular, $C(L)_1$ is a differential equation

$$(3.7) \quad y_1 y_{1,n} + D(y_{1,n}) - g_1 - y_{1,2} g_2 - \cdots - y_{1,n} g_n = 0$$

where $y_{1,j}$ is defined inductively by

$$y_{1,1} = 1, \quad y_{1,m+1} = D(y_{1,m}) + y_{1,m} \cdot y_1.$$

Conversely we assume differential system $C(L)_k$ has a solution $(c_1, \dots, c_k) \in H^k$. Let F/H be a differential extension which satisfies $\dim_{C_H}(\{y \in F \mid (D^k - c_k D^{k-1} - \cdots - c_1)y = 0\}) = k$ (see Remark 2.3). If $y \in F$ satisfies the equality $(D^k - c_k D^{k-1} - \cdots - c_1)y = 0$ then we have

$$\begin{aligned}
 G \begin{pmatrix} y \\ Dy \\ \vdots \\ D^{n-1}y \end{pmatrix} &= G \cdot {}^t C \begin{pmatrix} y \\ Dy \\ \vdots \\ D^{k-1}y \end{pmatrix} \\
 &= D({}^t C) \begin{pmatrix} y \\ Dy \\ \vdots \\ D^{k-1}y \end{pmatrix} + {}^t C \cdot \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & 0 & 1 & 0 \\ \vdots & & & \\ c_1 & c_2 & & c_k \end{pmatrix} \begin{pmatrix} y \\ Dy \\ \vdots \\ D^{k-1}y \end{pmatrix} \\
 &= D({}^t C) \begin{pmatrix} y \\ Dy \\ \vdots \\ D^{k-1}y \end{pmatrix} + {}^t C \cdot \begin{pmatrix} Dy \\ D^2y \\ \vdots \\ D^k y \end{pmatrix} = \begin{pmatrix} Dy \\ D^2y \\ \vdots \\ D^n y \end{pmatrix}.
 \end{aligned}$$

Therefore y satisfies a solution of differential equation

$$(3.8) \quad D^n y = g_n D^{n-1} y + g_{n-1} D^{n-2} y + \cdots + g_1 y.$$

By proposition 2.2, we have a decomposition

$$(3.9) \quad D^n - g_n D^{n-1} - \cdots - g_1 \in H[D](D^k - c_k D^{k-1} - \cdots - c_1).$$

As a result, we have the following proposition.

PROPOSITION 3.1. *Let H be a differential field such that the characteristic of H is zero. Let $L = D^n - g_n D^{n-1} - \cdots - g_1$ be a differential polynomial with coefficients in H .*

There is a one to one correspondence between a k -th order differential operator $R \in H[D]$ which satisfies $L \in H[D]R$ and a solution of differential system $C(L)_k$ in H^k .

If there exists a differential polynomial $R = D^k - c_k D^{k-1} - \cdots - c_1$ such that $L \in H[D] \cdot R$ then $(c_1, c_2, \dots, c_k) \in H^k$ is a solution of $C(L)_k$. Conversely, if $(c_1, c_2, \dots, c_k) \in H^k$ is a solution of $C(L)_k$ then the differential polynomial $R = D^k - c_k D^{k-1} - \cdots - c_1$ satisfies $L \in H[D] \cdot R$.

By induction, we have the following lemma.

LEMMA 3.2. *We assume $H = E$. Let $L = D^n - g_n D^{n-1} - \cdots - g_1$ be a differential polynomial with coefficients in E . If $|g_j|_0 \leq 1$ for $j = 1, \dots, n$, then, for any solution $y \in E$ of $C_1(L)$, we have $|y|_0 \leq 1$.*

EXAMPLE 3.3. For a differential operator $L = D^2 - g_2D - g_1$ of rank 2, $C(L)_1$ is a Riccati differential equation $c^2 + c' - g_1 - g_2c = 0$. Young considered in this case [Young].

4. Analytic element solution of Riccati differential equation

Let L be a differential operator with coefficients in E and $D = d/dt$. Let $C(L)_k$ be the differential system defined by L . We suppose that L is solvable on $D(t, 1)$ (i.e., $E[D]/E[D]L$ is solvable on $D(t, 1)$). We denote the vector space which is generated by the solutions of (resp. the bounded solutions) of L by

$$\begin{aligned} \text{Ker}_t(L) &= \{y \in \mathcal{A}(t, 1) \mid \tau(L)y = 0\}, \\ (\text{resp. Kerbdd}_t(L) &= \{y \in E[[x - t]]_0 \mid \tau(L)y = 0\}). \end{aligned}$$

In general, since L is solvable on $D(t, 1)$, we have $\dim_K(\text{Kerbdd}_t(L)) \geq 1$ by [Robba, Theorem 3.5]. If $\dim_K(\text{Kerbdd}_t(L)) \leq n - 1$, then there exists a monic differential operator $R \in E[D]$ such that $L \in E[D]R$ and $\text{Ker}_t(R) = \text{Kerbdd}_t(L)$ by [Robba, Theorem 2.6].

PROPOSITION 4.1. *Let $L \in E[D]$ be a monic n -th order differential operator which is solvable on $D(t, 1)$.*

- (i) *If $k = \dim_K(\text{Kerbdd}_t(L)) \leq n - 1$, then $C(L)_k$ has a solution on E .*
- (ii) *If $1 = \dim_K(\text{Kerbdd}_t(L))$, $C(L)_1$ has a unique solution on E .*
- (iii) *If there exists fundamental system of solutions y_1, \dots, y_n of L at t such that y_i is exactly of log-growth $i - 1$ (see the definition of log-growth in [C-T]) then, for each j , $C(L)_j$ has a unique solution over E .*

PROOF. There exists a monic differential operator $R \in E[D]$ such that $L \in E[D]R$ and $\text{Ker}_t(R) = \{y \in E[[x - t]] \mid \tau(L)y = 0 \text{ and } y \text{ is log-growth } i\}$ by [Robba, Theorem 2.6]. By Proposition 3.1, there exists a solution of $C(L)_k$ on E where $k = \deg(R)$. The uniqueness of case (ii),(iii) is confirmed by [Dwork1, Theorem 1]. □

By [D-R, Theorem 2.4.3 (iii)], we have the following proposition.

PROPOSITION 4.2. *Let $L \in E[D]$ be a monic n -th differential operator which is solvable on $D(t, 1)$. Assume $k = \dim_K(\text{Kerbdd}_t(L)) \leq n - 1$. Let S be the set $\{\bar{a} \in \bar{K} \mid L \text{ has no pole on } D(a, 1^-)\}$. Then there exists a solution*

$(y_1, \dots, y_k) \in E^k$ of $C(L)_k$ such that

$$\{\bar{a} \in S \mid \dim(\text{Ker}b_{dd_i}(L)) > \dim(\text{Ker}b_{dd_a}(L))\} \subseteq$$

$$\{\bar{a} \in S \mid y_i \text{ is not analytic on } D(a, 1^-) \text{ for some } i\}$$

5. Hypergeometric differential equation and Riccati differential equation mod p

Let p be a prime number and $D = d/dx$. For $a, b, c \in \mathbb{Q} \cap \mathbb{Z}_p$, we set the hypergeometric differential operator

$$(5.1) \quad L(a, b, c) = D^2 + \frac{c - (1 + a + b)x}{x(1 - x)} D - \frac{ab}{x(1 - x)}.$$

We assume the following condition:

$$(5.2) \quad c - a, c - b, b, a \text{ all lie outside } \mathbb{Z}.$$

$$(5.3) \quad c \in \mathbb{Q} - \{0, -1, -2, \dots\}.$$

By [Dwork2, p. 11, Remark], we see the Riccati differential equation

$$(5.4) \quad y^2 + y' + \frac{c - (1 + a + b)x}{x(1 - x)} y - \frac{ab}{x(1 - x)} = 0$$

has no rational solutions.

For a rational number $a \in \mathbb{Q} \cap \mathbb{Z}_p$, we set $\mu_a \in [0, p - 1]$ such that $\mu_a = -a \pmod{p}$. According to Dwork, we say $(a, b, c) \in \mathbb{Q}$ is *split* (resp. *unsplit*) if $\mu_c < \min\{\mu_a, \mu_b\}$ or $\mu_c > \max\{\mu_a, \mu_b\}$ (resp. $\mu_a < \mu_c < \mu_b$ or $\mu_b < \mu_c < \mu_a$). There exists an integer m such that

$$(5.5) \quad F := p^m \sum_{n=0}^{p-1} \frac{(a, n)(b, n)}{(c, n)n!} x^n \in \mathbb{Z}_p[x] - p\mathbb{Z}_p[x],$$

where (a, n) is defined

$$(a, n) = \begin{cases} 1 & (n = 0) \\ a(a + 1) \cdots (a + n - 1) & (n \geq 1). \end{cases}$$

If (a, b, c) is split, then F is uniquely determined. Since F is a solution of hypergeometric differential equation

$$(5.6) \quad \left(D^2 + \frac{c - (1 + a + b)x}{x(1 - x)} D - \frac{ab}{x(1 - x)} \right) y = 0 \pmod{p},$$

by Lemma 5.2, F'/F is a solution of the Riccati differential equation

$$(5.7) \quad y^2 + y' + \frac{c - (1 + a + b)x}{x(1 - x)}y - \frac{ab}{x(1 - x)} = 0 \pmod{p}.$$

Under the special condition, Dwork proved that if (a, b, c) is split, then $L(a, b, c)$ has a unbounded solution and if (a, b, c) is unsplit, then $L(a, b, c)$ has two independent bounded solutions (see [Dwork2, Theorem 6.6, 9.4]). We consider solutions of the Riccati differential equation (5.7).

LEMMA 5.1. *We assume that $(a, b, c) \in (\mathbb{Q} \cap \mathbb{Z}_p)^3$ is unsplit. Then the Riccati differential equation (5.7) has infinite solutions.*

PROOF. In this case, the hypergeometric differential equation (5.6) has solution

$$(5.8) \quad u \sum_{n=0}^{\min\{\mu_a, \mu_b\}} \frac{(a, n)(b, n)}{(c, n)n!} x^n + vp^m \sum_{n=\mu_c+1}^{\max\{\mu_a, \mu_b\}} \frac{(a, n)(b, n)}{(c, n)n!} x^n \quad (u, v \in \mathbb{F}_p(x^p))$$

with $m \in \mathbb{Z}$ which satisfies $p^m \sum_{n=\mu_c+1}^{\max\{\mu_a, \mu_b\}} \frac{(a, n)(b, n)}{(c, n)n!} x^n \in \mathbb{Z}_p[x] - p\mathbb{Z}_p[x]$. By Lemma 5.2, the Riccati differential equation (5.7) has infinite solutions. \square

LEMMA 5.2. *Let $R = D^n - g_n D^{n-1} - g_{n-1} D^{n-2} - \dots - g_1$ be a differential operator with coefficients $\mathbb{F}_p(x)$. Then R has a decomposition $R = Q \cdot (D - c)$ where $Q \in \mathbb{F}_p(x)[D]$ and $c \in \mathbb{F}_p(x)$ if and only if $c \in \mathbb{F}_p(x)$ is a solution of the Riccati differential equation $C(R)_1$.*

PROOF. Assume that $R = (D^{n-1} + b_{n-1}D^{n-2} + \dots + b_1)(D - c)$ where $b_1, \dots, b_{n-1}, c \in \mathbb{F}_p(x)$. Comparing the both sides, we have the equalities

$$\begin{aligned} g_n &= c - b_{n-1} \\ g_{n-1} &= \binom{n-1}{1} c^{(1)} + cb_{n-1} - b_{n-2} \\ &\vdots \\ g_i &= \binom{n-1}{n-i} c^{(n-i)} + \binom{n-2}{n-i-1} c^{(n-i-1)} b_{n-1} + \dots + c' \binom{i}{1} b_{i+1} + cb_i - b_{i-1} \\ &\vdots \\ g_2 &= \binom{n-1}{n-2} c^{(n-2)} + \binom{n-2}{n-3} c^{(n-3)} b_{n-1} + \dots + c' \binom{2}{1} b_3 + cb_2 - b_1 \\ g_1 &= c^{(n-1)} + c^{(n-2)} b_{n-1} + \dots + c^{(1)} b_2 + cb_1. \end{aligned}$$

We set c_m, C_m by

$$c_{m+1} = c'_m + cc_m, \quad c_1 = 1$$

$$C_m = \sum_{i=1}^m \binom{m-1}{i-1} c_{m-i+1} c^{(i-1)} - c_{m+1}.$$

We have the equality

$$-g_1 - c_2g_2 - \cdots - c_n g_n = c_{n+1} + C_n + \sum_{i=1}^{n-1} C_i b_i.$$

To prove that c is a solution of $C(R)_1$, we have only to prove $C_m = 0$ for all m . We use induction on m . For $m = 1$, $C_1 = c_1 c^{(0)} - c_2 = 0$. Therefore the case $m = 1$ is true.

We assume that the assertion is true for $m = k - 1$.

$$\begin{aligned} c'_k &= \left(\sum_{i=1}^{k-1} \binom{k-2}{i-1} c_{k-i} c^{(i-1)} \right)' \\ &= \sum_{i=1}^{k-1} \binom{k-2}{i-1} (c'_{k-i} c^{(i-1)} + c_{k-i} c^{(i)}) \\ &= \sum_{i=1}^{k-1} \binom{k-2}{i-1} ((c_{k-i+1} - cc_{k-i}) c^{(i-1)} + c_{k-i} c^{(i)}) \\ &= \sum_{i=1}^{k-1} \binom{k-2}{i-1} (c_{k-i+1} c^{(i-1)} + c_{k-i} c^{(i)}) - cc_k \\ &= \sum_{i=2}^{k-1} \left(\binom{k-2}{i-1} + \binom{k-2}{i-2} \right) c_{k-i+1} c^{(i-1)} + c^{(k-1)} \\ &= \sum_{i=2}^{k-1} \binom{k-1}{i-1} c_{k-i+1} c^{(i-1)} + c^{(k-1)} = C_k - cc_k + c_{k+1} \end{aligned}$$

This completes the induction.

Conversely let $c \in \mathbb{F}_p(x)$ be a solution of $C(R)_1$. We define $b_1, \dots, b_{n-1} \in \mathbb{F}_p(x)$ as above. Then we have the decomposition $R = (D^{n-1} + b_{n-1}D^{n-2} + \cdots + b_1)(D - c)$. □

LEMMA 5.3. *We assume that $(a, b, c) \in (\mathbb{Q} \cap \mathbb{Z}_p)^3$ is split. The Riccati differential equation (5.7) has a unique solution if and only if*

$$(5.9) \quad \left(\frac{x^{\mu_c}(1-x)^{-1+\mu_a+\mu_b-\mu_c}}{F^2} \right)^{(p-1)} \neq 0 \pmod{p}$$

PROOF. Substituting $y = z + F'/F$ for y in the Riccati differential equation (5.7), we have

$$(5.10) \quad z^2 + z' + \left(2 \frac{F'}{F} + \frac{c - (a+b+1)x}{x(1-x)} \right) z = 0 \pmod{p}.$$

Substituting $z = hx^{\mu_c}(1-x)^{-1+\mu_a+\mu_b-\mu_c}/F^2$ for z in the Riccati differential equation (5.7), we have

$$(5.11) \quad x^{\mu_c}(1-x)^{-1+\mu_a+\mu_b-\mu_c} \frac{h^2}{F^2} + h' = 0 \pmod{p}.$$

We suppose $h \neq 0$. By this equality,

$$(5.12) \quad \left(\frac{1}{h} \right)' = \frac{x^{\mu_c}(1-x)^{-1+\mu_a+\mu_b-\mu_c}}{F^2} \pmod{p}.$$

The differential equation (5.12) has a solution if and only if

$$(5.13) \quad \frac{x^{\mu_c}(1-x)^{-1+\mu_a+\mu_b-\mu_c}}{F^2} \in \mathbb{F}_p(x^p) \oplus x\mathbb{F}_p(x^p) \oplus \dots \oplus x^{p-2}\mathbb{F}_p(x^p)$$

i.e.

$$(5.14) \quad \left(\frac{x^{\mu_c}(1-x)^{-1+\mu_a+\mu_b-\mu_c}}{F^2} \right)^{(p-1)} = 0 \pmod{p}. \quad \square$$

PROPOSITION 5.4. *We assume that $\mu_c = p - 1$ and (a, b, c) is split. Then the Riccati differential equation (5.7) has a unique solution F'/F .*

PROOF. Let v_x be a valuation of $\mathbb{F}_p(x)$ with respect to a prime ideal $x\mathbb{F}[x]$ of $\mathbb{F}[x]$. Since $p - 1 = \mu_c > \max\{\mu_a, \mu_b\}$ by assumption, $v_x(F) = 0$. Therefore we have

$$(5.15) \quad v_x \left(\left(\frac{1}{h} \right)' \right) = v_x \left(\frac{x^{\mu_c}(1-x)^{-1+\mu_a+\mu_b-\mu_c}}{F^2} \right) = p - 1.$$

Since there exists no $h \in \mathbb{F}(x)$ which satisfies (5.15), we have $h = 0$. \square

6. Algebraic solution of Riccati differential equation

Let L be a monic differential operator with coefficients in $\mathbb{Q}(t)$ and $D = d/dt$. For almost all prime p , we can reduce L modulo p . If we can reduce L modulo p , then we denote L modulo p by L_p .

THEOREM 6.1. (i) *If L has an algebraic solution, then $C(L)_1$ has an algebraic solution.*

(ii) *We assume that for almost all prime p , L is solvable on the generic disc $D(t, 1)$ as a p -adic differential equation. If $C(L)_1$ has a rational solution then L has an algebraic solution.*

(iii) *We assume that for almost all prime p , L is solvable on the generic disc $D(t, 1)$ and $\dim(\text{bddKer}_t(L)) = 1$ as a p -adic differential equation. Then for almost all prime p , $C(L)_1$ has a solution in $\mathbb{F}_p(t)$. And if $C(L)_1$ has an algebraic solution, then L has an algebraic solution.*

PROOF. If L has an algebraic solution then there exists an algebraic function h such that $L \in \mathbb{Q}(t)^{alg}[D](D - h)$. Hence the Proposition 3.1 implies (i).

Suppose that $C(L)_1$ has a rational solution y . By Proposition 3.1, we have $L \in \mathbb{Q}(t)[D](D - y)$. Let p be a prime such that when we regard L as a p -adic differential equation, L is solvable on the generic disc $D(t, 1)$ and that we can reduce L and $D - y$ modulo p . By [DGS, 3. Proposition 5.1], L_p is nilpotent. And by [DGS, 3. Theorem 2.1], $(D - y)_p$ is also nilpotent and therefore has a solution of $\mathbb{F}_p(t)$. Since for all almost prime p , $(D - y)_p$ has a solution of $\mathbb{F}_p(t)$, by Honda's theorem([Put2, Proposition 2.2]), $D - y$ has an algebraic solution. This implies (ii).

We will prove (iii). By Lemma 3.2 and Proposition 4.1(ii), for all almost prime p , $C(L)_1$ has a solution on $\mathbb{F}_p(t)$. We assume that $C(L)_1$ has an algebraic solution y which is not a rational function. Let G be the Galois group which corresponds to a finite Galois extension of $\mathbb{Q}(t)$ contains y . For all $\sigma \in G$, $\sigma(y)$ is a solution of $C(L)_1$. Since y is not a rational function, $C(L)_1$ has at least two algebraic solutions. By Proposition 3.1, this implies $\dim(\text{bddKer}_t(L)) > 1$. This is a contradiction. \square

COROLLARY 6.2. *Let L be a monic differential operator with coefficients in $\mathbb{Q}(t)$. We assume that for almost all prime p , when we regard L as a p -adic differential equation, L is solvable on the generic disc $D(t, 1)$ and $\dim(\text{bddKer}_t(L)) = 1$. If L has no algebraic solutions, then $C(L)_1$ has a solution on $\mathbb{F}_p(t)$ for almost all prime p and no algebraic solutions.*

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