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Examples of Threefolds with Kodaira Dimension 1 or 2

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ABSTRACT - We construct three nonsingular threefolds X, X' and X'' with vanishing irregularities. X has Kodaira dimension $= \kappa(X) = 1$ and its m-canonical transformation $\varphi_{|mK_X|}$ has the following property: the minimum integer number m_0, such that the dimension of the image $\dim \varphi_{|mK_X|}(X) = \kappa(X) = 1$ for $m \geq m_0$, is given by $m_0 = 32$. X' and X'' have Kodaira dimension $\kappa(X') = \kappa(X'') = 2$ and their m-canonical transformations have the properties: $\dim \varphi_{|mK_{X'}|}(X') = \kappa(X') = 2$ if and only if $m \geq 12$, $\dim \varphi_{|mK_{X''}|}(X'') = \kappa(X'') = 2$ if and only if $m = 9, 10$ or $m \geq 12$.

Introduction.

One of the problems regarding the projective, algebraic, nonsingular variety X, of dimension $\dim X = d$ and of general type, is to establish the finiteness and also the birationality of the m-canonical transformation (improperly called a map) $\varphi_{|mK_X|} : X \dashrightarrow \mathbb{P}^{P_m - 1}$, where K_X is a canonical divisor on X and P_m is the m-genus of X. In other words, the problem is to establish when the dimension of the image of X under $\varphi_{|mK_X|}$ is d and, in addition, when X is birationally equivalent to its image $\varphi_{|mK_X|}(X)$.

We want to generalize the above problem to any variety X with *Kodaira dimension* $= \kappa(X) > 0$. Since "of general type" is equivalent to "$\kappa(X) = d = \dim X$", the new problem is to establish when $\dim \varphi_{|mK_X|}(X) = \kappa(X)$.

We indeed consider the following two problems:

(1) what is the minimum integer μ_0 such that $\dim \varphi_{|mK_X|}(X) = \kappa(X)$ for each $m \geq \mu_0$ and for each threefold X with Kodaira dimension $\kappa(X)$?

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(2) what is the maximum integer M_0 such that there exists a threefold X with Kodaira dimension $\kappa(X)$ and dim $\varphi_{|mK_X|}(X) < \kappa(X)$ for each $m < M_0$?

In particular, we are interested in these problems when X moreover has vanishing irregularities.

First we recall the known answers to the analogous problems for surfaces.
When X is a surface of general type, one has $\mu_0 = M_0 = 5$ (cf. F. Enriques [E], Cap. VIII, § 21; E. Bombieri [B]).

When X is a surface with $\kappa(S) = 1$, it is known that $M_0 = 8$, as recently communicated by I. Dolgachev and proved by F. Catanese via email, and that $\mu_0 = 14$ (cf. F. Enriques [E], pp. 410–412, T. Katsura - K. Ueno [KU], F. Catanese - I. Bauer [CB]).

Consider now varieties of dimension $d \geq 3$.

C. Hacon - J. McKernan [HM], S. Takayama [Ta] and H. Tsuji [Ts] proved the existence of μ_0 (depending only on d) for varieties of general type, but the value of μ_0 is still unknown, even for $d = 3$. M. Chen [Che$_1$, Che$_2$, Che$_3$] obtained upper bounds of μ_0 for threefolds X of general type, under some hypotheses on the geometric genus $p_g = P_1$, or on the m-genus $P_m, m > 1$, of X. In some cases, such limitations are optimal, thanks to examples due to Chen himself [Che$_2$], S. Chiaruttini - R. Gattazzo [CG], S. Chiaruttini [Chi] and C. Hacon considering an example of Reid (cf. [Che$_3$, Re]).

For threefolds X with $\kappa(X) = 1$, it is known that there exists an effectively computable integer m_1 such that the m_1-canonical linear system $|m_1 K_X|$ induces the Iitaka fibration for every threefold X (cf. [FM], Corollary 6.2).

When $\kappa(X) = 2$, J. Kollár conjectures the existence of an integer m_2, independent of X, such that the m-canonical transformation $\varphi_{|mK|}$ gives the Iitaka fibration for $m \geq m_2$ [K, Remark 3.4]; concerning this conjecture cf. [P]. Moreover, Kollár conjectures that the m-genus of X is $P_m(X) > 0$ for some $m < 24492$ and for every X [K, Corollary-Conjecture 3.3].

In the present paper we give three examples of threefolds, X in Chapter 1, X' and X'' in Chapter 2, with vanishing irregularities. X has Kodaira dimension 1 and its m-canonical transformation $\varphi_{|mK_X|}$ has the following property: the minimum integer m_0, such that dim $\varphi_{|mK_X|}(X) = \kappa(X) = 1$ for $m \geq m_0$, is given by $m_0 = 32$. This implies that $\mu_0 \geq 32$ in problem (1) for threefolds with $\kappa = 1$ and vanishing irregularities.

The properties of X (cf. Section 1.8) imply also that $M_0 \geq 20$ in problem (2) for threefolds with $\kappa = 1$ and vanishing irregularities.

X' and X'' have Kodaira dimension 2 and their m-canonical transformations have the properties: $\dim \varphi_{|mK_{X'}|}(X') = \kappa(X') = 2$ if and only if

$m \geq 12$; $\dim \varphi_{|mK_{X''}|}(X'') = \kappa(X'') = 2$ if and only if $m = 9, 10$ or $m \geq 12$. It follows that $\mu_0 \geq M_0 \geq 12$ in problems (1) and (2) for threefolds with $\kappa = 2$ and vanishing irregularities.

The irregularities and the first plurigenera of X, X', X'' are as follows:

TABLE 1. Irregularities and the first plurigenera of X, X' and X''.

	q_1	q_2	p_g	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
X	0	0	0	0	0	1	1	0	0	1	1	1	0	1	1	1
X'	0	0	0	0	1	1	1	1	1	2	2	2	2	3	3	3
X''	0	0	0	0	1	2	1	1	2	3	3	3	3	5	5	5

In our constructions, the ground field $\boldsymbol{k}$ is an algebraically closed field of characteristic 0, which we may assume to be the field of complex numbers $\mathbb{C}$.

1. Construction of X.

1.1 – *Imposing singularities on a degree six hypersurface V in $\mathbb{P}^4$.*

Let us indicate as $f_6(X_0, X_1, X_2, X_3, X_4)$ a form (a homogeneous polynomial) defining a hypersurface of degree six $V \subset \mathbb{P}^4$ with a triple point at each of the five vertices $A_0 = (1,0,0,0,0)$, $A_1 = (0,1,0,0,0)$, $A_2 = (0,0,1,0,0)$, $A_3 = (0,0,0,1,0)$, $A_4 = (0,0,0,0,1)$ of the fundamental pentahedron.

$$V : f_6(X_0, X_1, X_2, X_3, X_4) =$$

$$X_0^3(a_{33000}X_1^3 + \cdots) + X_1^3(a_{23100}X_0^2X_2 + \cdots) + X_2^3(\cdots) + X_3^3(\cdots) + X_4^3(\cdots) + \\ + a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1^2X_2X_3 + \cdots + a_{00222}X_2^2X_3^2X_4^2,$$

where $a_{ijkhl} \in \boldsymbol{k}$ denotes the coefficient of the monomial $X_0^iX_1^jX_2^kX_3^hX_4^l$.

We want to impose an infinitely near double surface $\mathcal{S}_i$ at the point A_i, $i = 0, 1, 2, 3, 4$, in the first neighbourhood. The surface $\mathcal{S}_i$ is locally isomorphic to a plane, according to our hypothesis on the singularities in [S$_1$], Introduction and section 1.

We impose here a double surface $\mathcal{S}_4$ infinitely near A_4 and after this, by means of a permutation of indices and variables, we impose the same singularity at the other A_j, $j < 4$.

The permutations of the indices $ijkhl$ of the coefficient a_{ijkhl} and of variables $X_0, \ldots, X_4$, which appear in $a_{ijkhl}X_0^iX_1^jX_2^kX_3^hX_4^l$, passing from A_4 to A_3, from A_3 to A_2, from A_2 to A_1 and from A_1 to A_0, are as follows.

Permutations of indices and variables

$$A_4 \longmapsto A_3 \longmapsto A_2 \longmapsto A_1 \longmapsto A_0$$

$$ijkhl \longmapsto jiklh \longmapsto jhlki \longmapsto klhji \longmapsto lkhij$$

Let us consider A_4.

Let $\pi_1 : \mathbb{P}_1 \longrightarrow \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ at A_4. Let U_4 be the affine open set $\{X_4 \neq 0\}$ with coordinates $x = \dfrac{X_0}{X_4}, y = \dfrac{X_1}{X_4}, z = \dfrac{X_2}{X_4}$ and $t = \dfrac{X_3}{X_4}$. The polynomial defining $V \cap U_0$ is

$$V \cap U_4 : f_6(x, y, z, t, 1) = a_{33000}x^3y^3 + \cdots + a_{00222}z^2t^2.$$

Locally the blow-up π_1 is given by the formulas:

$$\mathcal{B}_{x_1} : \begin{cases} x = x_1 \\ y = x_1y_1 \\ z = x_1z_1 \\ t = x_1t_1 \end{cases};\ \mathcal{B}_{y_2} : \begin{cases} x = x_2y_2 \\ y = y_2 \\ z = y_2z_2 \\ t = y_2t_2 \end{cases};\ \mathcal{B}_{z_3} : \begin{cases} x = x_3z_3 \\ y = y_3z_3 \\ z = z_3 \\ t = z_3t_3 \end{cases};\ \mathcal{B}_{t_4} : \begin{cases} x = x_4t_4 \\ y = y_4t_4 \\ z = z_4t_4 \\ t = t_4 \end{cases}$$

and we consider, for example, $\mathcal{B}_{x_1}$. The strict (or proper) transform V_{x_1} of $V \cap U_4$ with respect to $\mathcal{B}_{x_1}$ is given by

$$V_{x_1} : \frac{1}{x_1^3} f_6(x_1, x_1y_1, x_1z_1, x_1t_1) = a_{33000}x_1^3y_1^3 + \cdots + a_{00222}x_1z_1^2t_1^2.$$

We impose on V_{x_1} the double surface, better the double plane, $\{x_1 = y_1 = 0\}$ (i.e. we impose that such a plane is a locus of double points on V_{x_1}). Since the coefficients a_{ijklh} are arbitrary, according to Bertini, this means that the plane is (at least) double on every monomial $a_{ijklh}x_4^my_4^nz_4^pt_4^q$. It is consequently very easy to compute the conditions on the coefficients, in order that V_{x_1} has the double plane $\{x_1 = y_1 = 0\}$.

Let us denote by V_1 the strict transform of V with respect to the blow-up π_1. The above conditions on the coefficients impose on V_1 the double surface we called $\mathcal{S}_4$, i.e. $\mathcal{S}_4$ is a double surface on V_1 infinitely near $A_4 = (0, 0, 0, 0, 1)$ in its first neighbourhood.

Now, we impose a double surface $\mathcal{S}_i$ infinitely near A_i, for $i = 0, 1, 2, 3$ by means of the above permutations without repeating the above calculations.

By imposing all the conditions, we obtain for V an equation depending on 26 free coefficients (also called parameters) because some conditions are a duplicate of previous ones. Several of the 26 parameters can be chosen as equal to zero, because they are inessential in the computation of the birational invariants of a desingularization $\sigma_{|_X} : X \longrightarrow V$ of V, as well as in the computation of the dimensions of the images under pluricanonical transformations. The shortest form with the essential coefficients and defining our hypersurface with the abovesaid singularities is given by

$$\begin{aligned} f_6 &= a_{31002}X_0^3X_1X_4^2 + a_{13020}X_0X_1^3X_3^2 + a_{20301}X_0^2X_2^3X_4 + a_{20031}X_0^2X_3^3X_4 \\ &\quad + a_{02013}X_1^2X_3X_4^3 + a_{12210}X_0X_1^2X_2^2X_3 + a_{12120}X_0X_1^2X_2X_3^2 \\ &\quad + a_{11220}X_0X_1X_2^2X_3^2 + a_{11202}X_0X_1X_3^2X_4^2 + a_{10221}X_0X_2^2X_3^2X_4 . \end{aligned}$$

The hypersurface V, obtained for a generic choice of the parameters a_{ijkhl}, will be called a *generic* V. In the sequel, when we shall consider our V, it is understood our generic V.

1.2 – *Imposed and unimposed singularities on V.*

We consider the hypersurface V at the end of section 1.1.

Close to the singularities imposed on V (the triple point A_i having an infinitely near double surface $\mathcal{S}_i$, $i = 0, 1, 2, 3, 4$), new singularities appear on the generic V, either actual or infinitely near. We call *actual singularities* the singularities on V that are not infinitely near. Let us find the unimposed actual singularities on V in the present section.

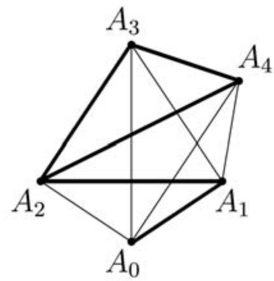
According to Bertini's theorem (characteristic zero), the actual singularities on the generic V belong to the base points of the linear system defining V. It is not difficult to find that the unimposed actual singularities are given by the following five double (straight) lines

$$\{X_0 = X_1 = X_i = 0\},\ i = 2, 3, 4; \qquad \{X_j = X_3 = X_4 = 0\},\ j = 0, 2;$$

and a double plane cubic

$$\{X_0 = a_{13020}X_1^2X_3 + a_{12210}X_1X_2^2 + a_{12120}X_1X_2X_3 + a_{11220}X_2^2X_3 = X_4 = 0\}.$$

In the following picture the five double lines are drawn in bold type.



In particular the unimposed actual singularities have codimension 2, therefore V it is reduced, irreducible and normal.

REMARK 1.1. We note that the double singular curves, actual or infinitely near, do not give conditions of adjointness, that is, they do not give conditions to the hypersurfaces for them to be (any kind of) adjoints (cf. [S1], Remark 17, section 11). In particular, such curves do not affect the birational invariant of a desingularization X such as q_1, q_2, p_g, P_m, as well as the computation of dim $\varphi_{|mK_X|}(X)$. Moreover, we note explicitly that the singularity given by the double plane cubic can be resolved blowing up the plane containing it.

RESOLUTION OF SINGULARITIES OF V

The main purpose of this resolution of singularities of V is to find the infinitely near unimposed singularities and to check that they do not give conditions to any kind of adjoints to V. More precisely we find that the infinitely near unimposed singularities are given by double singular lines and isolated double singular points.

1.3 – *Blowing up the triple point* $A_4 \in U_4$.

According to section 1.1, we consider the affine open set $U_4 = \{X_4 \neq 0\}$ of affine coordinates (x, y, z, t). The actual singularities on V belonging to U_4 are given by the two double lines

$$A_2A_4 \cap U_4, \qquad A_3A_4 \cap U_4.$$

The equation of $V \cap U_4$ is given by

$$f_6(x, y, z, t, 1) = a_{31002}x^3y + \cdots + a_{10221}xz^2t^2 = 0.$$

By using the notations of section 1.1, and denoting by V_{x_1}, V_{y_2}, V_{z_3}, V_{t_4} the strict (or proper) transform of $V \cap U_4$ with respect to $\mathcal{B}_{x_1}$, $\mathcal{B}_{y_2}$, $\mathcal{B}_{z_3}$, $\mathcal{B}_{t_4}$, respectively, we obtain the following results

- V_{x_1} is given by

$$V_{x_1} : \frac{1}{x_1^3}\, f_6(x_1, x_1y_1, x_1z_1, x_1t_1) = a_{31002}x_1y_1 + \cdots + a_{10221}x_1^2z_1^2t_1^2 = 0.$$

On V_{x_1} there is a unique singularity given by the double plane $\{x_1 = y_1 = 0\}$; this means that the double plane is infinitely near A_4 in the first neighbourhood.

- V_{y_2} is nonsingular.
- On V_{z_3} there are two singularities: the double plane $\{y_3 = z_3 = 0\}$ on the exceptional divisor $z_3 = 0$ and the double line $\{x_3 = y_3 = t_3 = 0\}$ outside the exceptional divisor; it is the strict transform of the actual double line $A_2A_4 \cap U_4$.
- On V_{t_4} there are again two singularities: the double plane $\{y_4 = t_4 = 0\}$ on the exceptional divisor $t_4 = 0$ and the double line $\{x_4 = y_4 = z_4 = 0\}$ outside the exceptional divisor; it is the strict transform of the actual double line $A_3A_4 \cap U_4$.

1.4 – *The blow-up* $\pi_2 : \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ *of* $\mathbb{P}_1$ *along the surface* $\mathcal{S}_4$

1.4.1 Let us consider V_{x_1}. On V_{x_1} the surface $\mathcal{S}_4$ is given by the double plane $\{x_1 = y_1 = 0\}$.

Locally the blow-up along this plane is given by the formulas

$$\mathcal{B}_{x_{11}} : \begin{cases} x_1 = x_{11} \\ y_1 = x_{11}y_{11} \\ z_1 = z_{11} \\ t_1 = t_{11} \end{cases} ; \quad \mathcal{B}_{y_{12}} : \begin{cases} x_1 = x_{12}y_{12} \\ y_1 = y_{12} \\ z_1 = z_{12} \\ t_1 = t_{12} \end{cases} .$$

Let us denote by $V_{x_{11}}$, the strict transform of V_{x_1} with respect to $\mathcal{B}_{x_{11}}$ and by $V_{y_{12}}$ the strict transform of V_{x_1} with respect to $\mathcal{B}_{y_{12}}$.

- • $V_{x_{11}}$ is nonsingular; its equation is given by

$$V_{x_{11}} : \frac{1}{x_{11}^5}\, f_6(x_{11}, x_{11}^2y_{11}, x_{11}z_{11}, x_{11}t_{11}, 1) = a_{31002}y_{11} + \cdots + a_{10221}z_{11}^2t_{11}^2 = 0.$$

- • $V_{y_{12}}$ is nonsingular.

1.4.2 Let us consider V_{z_3}. On V_{z_3} the surface $\mathcal{S}_4$ is given by the double plane $\{y_3 = z_3 = 0\}$. Blowing up this plane, we find that there are no singularities infinitely near it.

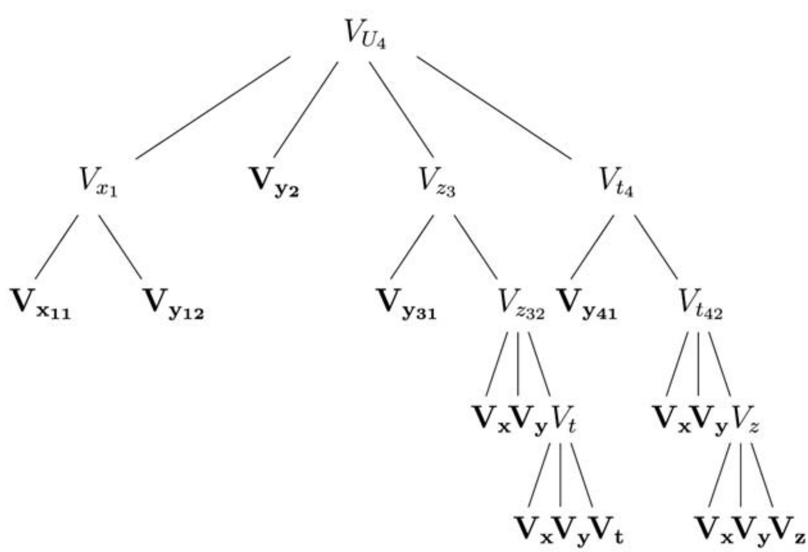
1.4.3 Let us consider V_{t_4}. On V_{t_4} the surface $\mathcal{S}_4$ is given by the double plane $\{y_4 = t_4 = 0\}$. Blowing up this plane, again we find that there are no singularities infinitely near it.

1.5 – *The blow-ups along the double lines that are strict transforms of the double lines $A_2A_4 \cap U_4$ and $A_3A_4 \cap U_4$.*

Blowing up the strict transforms of $A_2A_4 \cap U_4$ and $A_3A_4 \cap U_4$, it is not difficult to see that infinitely near each of these double lines there is another double line and infinitely near the last double line there are no singularities.

The tree of the blow-ups resolving the singularities on $V \cap U_4$ is described below.

Where the nonsingular threefolds are drawn in bold type.



In the above sections 1.3–1.5, we gave, as an example, the blow-ups resolving the singularities of $V \cap U_4$ and we wrote the equations of the strict transforms that we need in the sequel. We calculated also the other similar desingularizations but we do not reproduce them here: we consider them as being achieved, as well as the complete desingularization of V.

1.6 – *The m-canonical adjoints to* $V \subset \mathbb{P}^4$.

Let

$$\mathbb{P}_r \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}_0 = \mathbb{P}^4$$

be a sequence of blow-ups resolving the singularities of V.

If we call $V_i \subset \mathbb{P}_i$ the *strict transform* of V_{i-1} with respect to π_i, then we obtain from the above sequence

$$V_r \xrightarrow{\pi'_r} \cdots \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = V,$$

where $\pi'_i = \pi_{i|V_i} : V_i \longrightarrow V_{i-1}$ and $\sigma_{|_X} : X \longrightarrow V$, $\sigma = \pi_r \circ \cdots \circ \pi_1$, is a desingularization of $V \subset \mathbb{P}^4$.

Let us assume that π_i is a blow-up along a subvariety Y_{i-1} of $\mathbb{P}_{i-1}$, of dimension j_{i-1}, which can be either a singular or a nonsingular subvariety of $V_{i-1} \subset \mathbb{P}_{i-1}$ (i.e. Y_{i-1} is a locus of singular or simple points of V_{i-1}). Let m_{i-1} be the multiplicity of the variety Y_{i-1} on V_{i-1}.

Let us set $n_{i-1} = -3 + j_{i-1} + m_{i-1}$, for $i = 1, \ldots, r$ and $\deg(V) = d$.

A hypersurface $\Phi_{m(d-5)}$ of degree $m(d-5)$ in $\mathbb{P}^4$ is an *m-canonical adjoint* to V (with respect to the sequence of blow-ups $\pi_1, \ldots, \pi_r$) if the restriction to X of the divisor

$$D_m = \pi_r^*\{\pi_{r-1}^*[\cdots \pi_1^*(\Phi_{m(d-5)}) - mn_0E_1 \cdots] - mn_{r-2}E_{r-1}\} - mn_{r-1}E_r$$

is effective, i.e. $D_{m|_X} \geq 0$, where $E_i = \pi^{-1}(Y_{i-1})$ is the exceptional divisor of π_i and $\pi_i^* : Div(\mathbb{P}_{i-1}) \longrightarrow Div(\mathbb{P}_i)$ is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [S_1], sections 1,2).

An m-canonical adjoint $\Phi_{m(d-5)}$ is a *global* m-canonical adjoint to V (with respect to $\pi_1, \ldots, \pi_r$) if the divisor D_m is effective on $\mathbb{P}_r$, i.e. $D_m \geq 0$ (loc. cit.).

Note that, if $\Phi_{m(d-5)}$ is an m-canonical adjoint to V, then $D_{m|_X} \equiv mK$, where '$\equiv$' denotes linear equivalence and K denotes a canonical divisor on X.

In our above example, an order can be established in the sequence of blow-ups, e.g. let us assume that the blow-up π_1 is the blow-up at the 3-ple point A_4, π_2 is the blow-up along the double surface $\mathcal{S}_4$ infinitely near A_4 (see also section 1.1, 1.3 and 1.4), π_3 is the blow-up at the triple point A_3, π_4 is the blow-up along the double surface $\mathcal{S}_3$ infinitely near A_3, π_5 is the blow-up at the triple point A_2, π_6 is the blow-up along the double surface $\mathcal{S}_2$ infinitely near A_2, π_7 is the blow-up at the triple point A_1, π_8 is the blow-up along the double surface $\mathcal{S}_4$ infinitely near A_1, and π_9 is the blow-up at the triple point A_0, π_{10} is the blow-up along the double surface $\mathcal{S}_0$ infinitely near A_0.

The hypersurface V has degree $d = 6$ and D_m is given by:

$$(^\diamond)\quad D_m = \pi_r^* \dots \{\pi_2^*[\pi_1^*(\Phi_m) - mE_2] - mE_4\} - mE_6 - mE_8 - mE_{10} + \sum mE,$$

where E_i is the exceptional divisor of the blow-up π_i and, to be more specific, E_2 is the exceptional divisor of the blow-up π_2 along $\mathcal{S}_4, \dots, E_{10}$ is the exceptional divisor of the blow-up π_{10} along $\mathcal{S}_0$.

No other exceptional divisors are subtracted in D_m because, as we said, the unimposed singularities are either actual or infinitely near double singular curves or isolated double points on our (generic) V. We note that the exceptional divisors of the blow-ups at double isolated points appear with coefficient $n_h = -1$, we have indicated these divisors as $\sum mE$. From here on, we omit writing $\sum mE$, because they are not essential in the computation of the birational invariants, as well as in the computation of dim $\varphi_{|mK_X|}(X)$, that we shall consider.

1.7 – *The global and non-global m-canonical adjoints to $V \subset \mathbb{P}^4$.*

PROPOSITION 1. *If we consider a non-global m-canonical adjoint to V*

$$\Phi_m : F_m(X_0, X_1, X_2, X_3, X_4) = \sum_{i+j+k+h+l=m} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = 0,$$

where $b_{ijkhl} \in \boldsymbol{k}$, then a form $A = A(X_0, X_1, X_2, X_3, X_4)$ exists such that $\Phi_m^\star : F_m - Af_6 = 0$ is a global m-canonical adjoint to V. In other words, the following equality holds

$$\Phi_{m|_V} = \Phi^\star_{m|_V}.$$

Proposition 1 holds for the three constructions in the present paper (see Proposition 2 in the Appendix). The three proofs are similar, so, to avoid unnecessary repetitions, we produce only one proof in the Appendix. There are two ways to prove these Propositions: the first way is contained in [S$_2$], cf. the proof of Lemma 5, section 18, p. 1177; the second way is due to Maria Cristina Ronconi. We reproduce in the Appendix the proof of Mrs. Ronconi.

LEMMA 1. *The global m-canonical adjoints to V are given by*

$$\Psi_m : \sum_{i+j+k+h+l=m} c_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = 0,$$

where $c_{ijkhl} \in \boldsymbol{k}$ and where $l \geq i \geq h \geq j \geq l$, $i \geq k$, i.e. $l = i = h = j$ and $i \geq k$ (for all monomials).

PROOF OF LEMMA 1. Let us consider a global m-canonical adjoint to V

$$\Psi_m : \sum_{i+j+k+h+l=m} c_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l = 0.$$

The total transform Ψ^* of $\Psi_m \cap U_4$ with respect to $\mathcal{B}_{x_1}$ (section 1.1) is given by

$$\Psi^* = \mathcal{B}_{x_1}^*(\Psi_m \cap U_4) : \sum_{i+j+k+h+l=m} c_{ijkhl}(x_1)^i (x_1 y_1)^j (x_1 z_1)^k (x_1 t_1)^h = 0,$$

The double surface $\mathcal{S}_4$ infinitely near A_4 in affine coordinates (x_1, y_1, z_1, t_1) is given by $\{x_1 = y_1 = 0\}$ and the blow-up π_2 along $\mathcal{S}_4$ is given locally by the formulas $\mathcal{B}_{x_{11}}$ and $\mathcal{B}_{y_{12}}$ (see section 1.4.1).

The total transform Ψ^{**} of $\Psi^* = \mathcal{B}_{x_1}^*(\Psi_m \cap U_4)$ with respect to $\mathcal{B}_{x_{11}}$ is given by

$$\Psi^{**} = \mathcal{B}_{x_{11}}^*(\Psi^*) : \sum_{i+j+k+h+l=m} c_{ijkhl} x_{11}^i (x_{11}^2 y_{11})^j (x_{11} z_{11})^k (x_{11} t_{11})^h =$$

$$\sum_{i+j+k+h+l=m} c_{ijkhl} x_{11}^{i+2j+k+h} y_{11}^j z_{11}^k t_{11}^h = 0.$$

Since Ψ_m is a global m-canonical adjoint to V, by definition in ($^\diamond$), section 1.6, we have $D_m \geq 0$.

We note that $\mathcal{B}_{x_{11}} \circ \mathcal{B}_{x_1}$ coincides, up to isomorphisms, with the desingularization $\sigma_{|_X}$ on the affine open set $V_{x_{11}}$. In fact, $V_{x_{11}}$ is nonsingular (see the tree of blow-ups, section 1.5) and then it is isomorphic to a Zariski open set on the desingularization X of V. The above coincidence and the inequality $D_m \geq 0$ imply the inequality $\pi_2^*[\pi_1^*(\Phi_m) - mE_2] \geq 0$ and the following inequality between divisors (of rational functions) on the affine open set $U_{x_{11}}$ of (affine) coordinates $(x_{11}, y_{11}, z_{11}, t_{11})$

$$\left(\frac{\Psi^{**}}{x_{11}^m}\right) = \left(\frac{1}{x_{11}^m}\right)\left(\sum_{i+j+k+h+l=m} c_{ijkhl} x_{11}^{i+2j+k+h} y_{11}^j z_{11}^k t_{11}^h = 0\right) \geq 0.$$

Since, on the affine open set $U_{x_{11}}$, $x_{11} = 0$ is the local equation, of the exceptional divisor E_2 of the blow-up π_2, this last inequality is equivalent to

$$i + 2j + k + h - m \geq 0, \text{ i.e. } j \geq l.$$

We have proved the above inequality for a particular sequence of open sets appearing in the blow-ups. If we consider all the other sequences of open sets, we find the same inequality.

This proves the last inequality in the sequence of inequalities $l \geq i \geq h \geq j \geq l$ in the statement of Lemma 1.

If we consider the singular points A_3, A_2, A_1 and A_0 and we go on with the blow-ups $\pi_3, \ldots, \pi_{10}$, then all the other inequalities follow in a similar way. More precisely, if we consider the point A_3, then we obtain $i \geq h$; considering A_2 we obtain $i \geq k$; A_1 leads to $h \geq j$ and A_0 to $l \geq i$.

We note that, up to isomorphisms, we can start by blowing up A_i first, with $i \neq 4$, and in this case, we repeat for A_i what we did for A_4 obtaining in the same way all the inequalities.

So, Lemma 1 is proved.

1.8 – *Computing the plurigenera of X.*

Now, we consider [S_1], Corollary 8, section 3: if V is normal, there is an isomorphism of projective spaces for any $m \geq 1$

$$\begin{array}{ccc} \begin{pmatrix} \textit{linear system of} \\ m - \textit{canonical adjoints to } V \end{pmatrix}_{|_V} & \longrightarrow & |mK_X| \\ \\ \Phi_{m|_V} & \longmapsto & D_{m|_X}. \end{array}$$

D_m is defined in ($^\diamondsuit$), section 1.6.

Bearing in mind that our purpose is to compute the m-canonical genus $P_m = \dim |mK_X| + 1 = \dim \big(\textit{linear system of } m - \textit{canonical adjoints}\big)_{|_V} + 1$, we can substitute Φ_m with Φ'_m if $\Phi'_{m|_V} = \Phi_{m|_V}$.

Next, the Proposition 1 in section 1.7 tells us that in order to compute the m-genus P_m, we can restrict ourselves to consider global m-canonical adjoints to V and Lemma 1 in the same section tells us that the global m-canonical adjoints are given by

$$\Psi_m : \sum_{4s+v=m} c_{4s+v}\, X_0^s X_1^s X_2^v X_3^s X_4^s = 0,$$

where $c_{4s+v} \in \boldsymbol{k}$ and $v \leq s,\ s > 0$.

By doing the easy calculations, we obtain

$p_g = P_1 = P_2 = P_3 = 0$;
$P_4 = 1$, the global 4-canonical adjoint is defined by $X_0X_1X_3X_4$;
$P_5 = 1$, the global 5-canonical adjoint is defined by $X_0X_1X_2X_3X_4$;

$P_6 = P_7 = 0$;
$P_8 = 1$, the global 8-canonical adjoint is defined by $X_0^2 X_1^2 X_3^2 X_4^2$;
$P_9 = 1$, the global 9-canonical adjoint is defined by $X_0^2 X_1^2 X_2 X_3^2 X_4^2$;
$P_{10} = 1$, the global 10-canonical adjoint is defined by $X_0^2 X_1^2 X_2^2 X_3^2 X_4^2$;
$P_{11} = 0$;
$P_{12} = P_{13} = P_{14} = P_{15} = P_{16} = P_{17} = P_{18} = P_{19} = 1$;
$P_{20} = 2$ the global 20-canonical adjoints are defined by $\lambda_1 X_0^5 X_1^5 X_3^5 X_4^5 + \lambda_2 X_0^4 X_1^4 X_2^4 X_3^4 X_4^4 = X_0^4 X_1^4 X_3^4 X_4^4 (\lambda_1 X_0 X_1 X_3 X_4 + \lambda_2 X_2^4)$, $\lambda_i \in \boldsymbol{k}$; and so on.

REMARK 1.2. 1) We have seen that the first integer m such that $P_m = 2$ is $m = 20$ and the 20-canonical adjoints are given, up to fixed components, by the pencil $\lambda_1 X_2^4 + \lambda_2 X_0 X_1 X_3 X_4 = 0$.

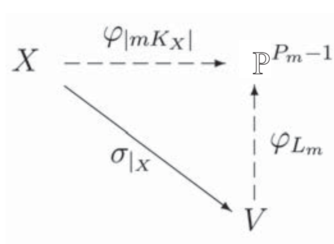
Continuing the above list, we obtain

2) the minimum integer m_0 such that $P_m \geq 2$ for $m \geq m_0$ is given by $m_0 = 32$;

3) the first integer m such that $P_m = 3$ is $m = 40$ and the global 40-canonical adjoints are defined by $\mu_1 X_0^8 X_1^8 X_2^8 X_3^8 X_4^8 + \mu_2 X_0^9 X_1^9 X_2^4 X_3^9 X_4^9 + \mu_3 X_0^{10} X_1^{10} X_3^{10} X_4^{10} = X_0^8 X_1^8 X_3^8 X_4^8 (\mu_1 X_2^8 + \mu_2 X_0 X_1 X_2^4 X_3 X_4 + \mu_3 X_0^2 X_1^2 X_3^2 X_4^2)$, $\mu_i \in \boldsymbol{k}$;

1.9 – *The m-canonical transformation* $\varphi_{|mK_X|}$.

Let us consider the following triangle



where $\sigma_{|_X} : X \longrightarrow V$, with $\sigma = \pi_r \circ \cdots \circ \pi_1$, denotes our desingularization of V, where L_m denotes the (incomplete) linear system of m-canonical adjoints to V restricted to V and φ_{L_m} the rational transformation defined by the linear system L_m.

The above triangle is commutative. This follows from the fact that the divisors of the linear system $|mK_X|$ on X are the divisors $D_{m|_X}$, where $D_m = \pi_r^*\{\pi_{r-1}^*[\cdots \pi_1^*(\Phi_m) - mn_0E_1 \cdots] - mn_{r-2}E_{r-1}\} - mn_{r-1}E_r$, with Φ_m varying in the linear system of m-canonical adjoints to V.

In the present section, we want to find the minimum integer m_0 such that the m-canonical transformation $\varphi_{|mK_X|}$ enjoys the property: dim $\varphi_{|mK_X|}(X) = 1$, for $m \geq m_0$. The values of m such that dim $\varphi_{|mK_X|}(X) = 1$ are exactly given by the values for which $P_m \geq 2$. This is a consequence of the following facts:

a) restricting the global m-canonical adjoints to V there is no identifications;

b) the commutativity of the above triangle;

c) 4), 5) in Remark 3.

So, thanks to 2) in Remark 3, the above value of m_0 is given by $m_0 = 32$.

Therefore, the Kodaira dimension of X is $\kappa(X) = 1$ *and the minimum integer* m_0 *such that dim* $\varphi_{|mK_X|}(X) = 1$, *for* $m \geq m_0$, *is given by* $m_0 = 32$.

Moreover, we note that the generic fiber of the rational transformation $\varphi_{|mK_X|}$ is irreducible (by Bézout theorem) for those values of m for which $P_m = 2$, whereas such generic fiber is reducible for those values of m for which $P_m \geq 3$.

1.10 – *Computing the irregularities of X.*

It remains to prove that $q_i = \dim_{\boldsymbol{k}} H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_{\boldsymbol{k}} H^1(X, \mathcal{O}_X) = q(S_r) = \dim_{\boldsymbol{k}} H^1(S_r, \mathcal{O}_{S_r})$, where $S_r \subset X$ is the strict transform of a generic hyperplane section S of V (cf. [S_1], section 4, for instance). S has several isolated (actual or infinitely near) double points and no other singularities. This follows from the fact that the hypersurface V, outside the points A_0, A_1, A_2, A_3 and A_4, only has actual or infinitely near double curves or isolated double points. So, $q_1 = 0$.

To prove that $q_2 = 0$, we use the formula (36), section 4 in [S_1], which states that

$$q_2 = p_g(X) + p_g(S_r) - \dim_{\boldsymbol{k}}(W_2),$$

where W_2 is the vector space of the degree 2 forms defining global adjoints Φ_2 to V, i.e. defining hyperquadrics Φ_2 such that

$$\pi_r^* \cdots \pi_2^*[\pi_1^*(\Phi_2)] - E_2 - E_4 - E_6 - E_8 - E_{10} \geq 0,$$

(cf. the expression of D_m in ($^\diamond$), section 1.6). So the above hyperquadrics Φ_2 are those passing through the points A_0, A_1, A_2, A_3 and A_4. Thus, we have $\dim_{\boldsymbol{k}}(W_2) = 15 - 5 = 10$. It follows from $p_g(S_r) = 10$ and $p_g(X) = 0$ (cf. section 1.8), that $q_2 = 0$.

2. Construction of X' and of X''.

In this chapter we construct two threefolds X' and X'', with the properties described in the Introduction, as desingularizations of two hypersurfaces V' and V'' of degree six in $\mathbb{P}^4$. Using the same method of Chapter 1, we impose sextic hypersurfaces to have triple points at the coordinate points A_λ, $\lambda = 0, \dots 4$, each one with infinitely near a double plane, namely a double surface whose local equations are linear, obtained with slight modifications of the permutations described in Chapter 1, § 1.1.

Indeed, the singularities of V', V'' are of the same type as the singularities of V in Chapter 1. The difference with V (and between V' and V'') is mainly given by the position of the double surface infinitely near to A_4, which will imply the difference in the birational equivalence classes of V, V' and V''.

The explicit equation of V' is

$$\begin{aligned} f_6' = &\, a_{30102}X_0^3X_2X_4^2 + a_{13020}X_0X_1^3X_3^2 + a_{20301}X_0^2X_2^3X_4 \\ &+ a_{20031}X_0^2X_3^3X_4 + a_{10203}X_0X_2^2X_4^3 + a_{12210}X_0X_1^2X_2^2X_3 \\ &+ a_{11220}X_0X_1X_2^2X_3^2 + a_{02112}X_1^2X_2X_3X_4^2 = 0, \end{aligned}$$

while the equation of V'' is

$$\begin{aligned} f_6'' = &\, a_{30012}X_0^3X_3X_4^2 + a_{13020}X_0X_1^3X_3^2 + a_{20301}X_0^2X_2^3X_4 + a_{20031}X_0^2X_3^3X_4 \\ &+ a_{10023}X_0X_3^2X_4^3 + a_{22011}X_0^2X_1^2X_3X_4 + a_{12210}X_0X_1^2X_2^2X_3 \\ &+ a_{12012}X_0X_1^2X_3X_4^2 + a_{11220}X_0X_1X_2^2X_3^2 + a_{10221}X_0X_2^2X_3^2X_4 \\ &+ a_{10212}X_0X_2^2X_3X_4^2 + a_{02112}X_1^2X_2X_3X_4^2 = 0, \end{aligned}$$

where the coefficients $a_{ijkhl} \in \boldsymbol{k}$ are sufficiently general. (Actually, one can construct such a threefold X'' from a hypersurface V'' depending on 28 — instead of 12 — parameters, but we used only 12 of them for brevity.)

Reasoning like in Chapter 1, one sees that V', V'' are normal, they have triple points at A_λ, $0 \le \lambda \le 4$, they are double along the lines A_0A_1, A_1A_2, A_1A_4, A_2A_3 and along the rational cubic plane curve:

$$X_0 = X_4 = a_{13020}X_1^2X_3 + a_{12210}X_1X_2^2 + a_{11220}X_2^2X_3 = 0.$$

The further actual singularity of V' is the double line A_3A_4, while the further actual singularities of V'' are the double line A_2A_4 and two double rational cubic plane curves:

$$X_0 = X_1 = a_{10221}X_2^2X_3 + a_{10212}X_2^2X_4 + a_{10023}X_3X_4^2 = 0,$$
$$X_2 = X_3 = a_{30012}X_0^2X_4 + a_{22011}X_0X_1^2 + a_{12012}X_1^2X_4 = 0.$$

Setting $\mathbb{P}_0 = \mathbb{P}^4$, we perform the blow-ups $\pi_i : \mathbb{P}_i \longrightarrow \mathbb{P}_{i-1}$, $i = 1, \ldots, 9$, where $\pi_{2\lambda+1}$, $\lambda = 0, \ldots, 4$, is the blow-up at $A_\lambda \in U_\lambda = \{X_\lambda \neq 0\}$ and $\pi_{2\lambda+2}$, $\lambda = 0, \ldots, 3$, is the blow-up along the double surface $\mathcal{S}_\lambda$ infinitely near to A_λ, which is the same for V, V' and V''.

With the usual affine coordinates x, y, z, t in U_λ, $\lambda = 0, \ldots, 4$, each blow up $\pi_{2\lambda+1}$ is given locally by the formulas $\mathcal{B}_{x_1}$, $\mathcal{B}_{y_2}$, $\mathcal{B}_{z_3}$, $\mathcal{B}_{t_4}$ written in § 1.1. With respect to $\mathcal{B}_{y_2}$, we see that the local equation of the double surface $\mathcal{S}_0$, infinitely near to A_0, is $y_2 = t_2 = 0$; the local equation of $\mathcal{S}_1$ is $y_2 = z_2 = 0$; those of $\mathcal{S}_2$ and of $\mathcal{S}_3$ are $x_2 = y_2 = 0$.

Finally, for V', let $\pi'_{10} : \mathbb{P}'_{10} \longrightarrow \mathbb{P}_9$ be the blow up along the surface $\mathcal{S}'_4$ infinitely near to A_4, with local equation $y_2 = z_2 = 0$ with respect to $\mathcal{B}_{y_2}$. For V'', let $\pi''_{10} : \mathbb{P}''_{10} \longrightarrow \mathbb{P}_9$ be the blow up along the surface $\mathcal{S}''_4$ infinitely near to A_4, with local equation $y_2 = t_2 = 0$ with respect to $\mathcal{B}_{y_2}$.

We then checked that the strict transforms of V' in $\mathbb{P}'_{10}$ and of V'' in $\mathbb{P}''_{10}$ are singular along double curves only, and that no further essential singularity appears in the resolution process. In other words, the double surfaces infinitely near to the coordinate triple points are the only essential singularities of V' and of V''. Therefore we may compute global m-canonical adjoints to V' and to V'' in the same way we did in Chapter 1.

LEMMA 2. *Let* $F = \sum b_{ijkhl}X_0^iX_1^jX_2^kX_3^hX_4^l$, *be a homogeneous polynomial of degree* m, *i.e. with* $i+j+k+h+l = m$ *and* $b_{ijkhl} \in \boldsymbol{k}$. *Then* $\Phi_m = \{F = 0\} \subset \mathbb{P}^4$ *is a global* m*-canonical adjoint to* V' *[resp. to* V''*] if and only if* $j \leq h \leq i = k = l$ *[resp.* $j \leq i = h = l \geq k$*] for each monomial in* F.

PROOF. Following the proof of Lemma 1 in Chapter 1, we compute the conditions imposed by the double surfaces infinitely near to A_0 [A$_1, A_2, A_3$, resp.] and we find out that $l \geq i$ [$h \geq j$, $i \geq k$, $i \geq h$, resp.]. Concerning A_4, we find that $k \geq l$ for V' and that $h \geq l$ for V''. It follows that $i = k = l$ for V' and that $i = h = l$ for V'', which conclude the proof.

By Proposition 2 in the Appendix, global m-canonical adjoints to V' and to V'' are enough to compute the plurigenera $P_m(X'), P_m(X'')$ of X' and X'', and their pluricanonical transformations, which we study now.

2.1 – *Canonical adjoints to V', pluricanonical transformations of X'.*

By Lemma 2, the global m-canonical adjoints to V' are given by

$$\Phi_m : \sum_{5j+4u+3v=m,\, j\geq 0,\, u\geq j,\, v\geq k} b_{juv} X_1^j X_3^{j+u} (X_0X_2X_4)^{j+u+v} = 0,$$

where $b_{juv} \in \boldsymbol{k}$. Denote by $\mathcal{A}'_m$ the $\boldsymbol{k}$-vector space of the polynomials defining global m-canonical adjoints to V'. Setting $Y = X_0X_2X_4$, it follows that

$$\begin{aligned}
&\mathcal{A}'_1 = \mathcal{A}'_2 = \{0\}, && \mathcal{A}'_3 = \langle Y\rangle, && \mathcal{A}'_4 = \langle X_3Y\rangle,\\
&\mathcal{A}'_5 = \langle X_1X_3Y\rangle, && \mathcal{A}'_6 = \langle Y^2\rangle, && \mathcal{A}'_7 = \langle X_3Y^2\rangle,\\
&\mathcal{A}'_8 = \langle X_1, X_3\rangle X_3Y^2, && \mathcal{A}'_9 = \langle X_1X_3^2, Y\rangle Y^2, && \mathcal{A}'_{10} = \langle X_1^2X_3, Y\rangle X_3Y^2,\\
&\mathcal{A}'_{11} = \langle X_1, X_3\rangle X_3Y^3, && \mathcal{A}'_{12} = \langle X_1X_3^2, X_3^3, Y\rangle Y^3,\\
&\mathcal{A}'_{13} = \langle X_1^2X_3, X_1X_3^2, Y\rangle X_3Y^3, && \mathcal{A}'_{14} = \langle X_1^2X_3^2, X_1Y, X_3Y\rangle X_3Y^3,
\end{aligned}$$

and so on, which give the $P_m(X')$'s written in Table 1 in the Introduction.

We now study the m-canonical transformations $\varphi'_m = \varphi_{|mK_{X'}|}$ of X'. We will show that X' has Kodaira dimension 2 and that $\varphi'_m(X')$ has dimension 2 if and only if $m \geq 12$. Clearly, $\dim \varphi'_m(X') < 2$ if $m < 12$.

Then, it is easy to check that $\varphi'_m(X') = \mathbb{P}^2$ for $m = 12, 13, 14$. Since $P_3(X') = 1$, it follows that $\dim \varphi'_m(X') \geq 2$ also for each $m \geq 15$.

An upper bound to $P_m(X')$ is given by the function $\nu\colon \mathbb{N} \to \mathbb{N}$,

$$\nu(m) = \sharp\{(j, u, v) \in \mathbb{N}^3 \mid 5j + 4u + 3v = m\} \geq P_m(X'),$$

where $\mathbb{N}$ is the set of non-negative integers. We then see that

$$P_m(X') \leq \nu(m) \leq \left(\frac{m}{3} + 1\right)\left(\frac{m}{4} + 1\right),$$

thus the Kodaira dimension of X' is 2 and hence $\dim \varphi'_m(X') = 2$ for $m \geq 12$.

2.2 – *Canonical adjoints to V'', pluricanonical transformations of X''.*

By Lemma 2, the global m-canonical adjoints to V'' are given by

$$\Phi_m : \sum_{3i+j+k=m,\, i\geq 0,\, i\geq j,\, i\geq k} b_{ijk} X_1^j X_2^k (X_0X_3X_4)^i = 0,$$

where $b_{ijk} \in \boldsymbol{k}$. Denote by $\mathcal{A}''_m$ the $\boldsymbol{k}$-vector space of the polynomials defining global m-canonical adjoints of X''. Setting $Y = X_0X_3X_4$, it follows that

$$\begin{aligned}
&\mathcal{A}''_1 = \mathcal{A}''_2 = \{0\}, \qquad \mathcal{A}''_3 = \langle Y\rangle, \qquad \mathcal{A}''_4 = \langle X_1, X_2\rangle Y,\\
&\mathcal{A}''_5 = \langle X_1X_2Y\rangle, \qquad \mathcal{A}''_6 = \langle Y^2\rangle, \qquad \mathcal{A}''_7 = \langle X_1, X_2\rangle Y^2,\\
&\mathcal{A}''_8 = \langle X_1^2, X_1X_2, X_2^2\rangle Y^2, \qquad \mathcal{A}''_9 = \langle X_1^2X_2, X_1X_2^2, Y\rangle Y^2,\\
&\mathcal{A}''_{10} = \langle X_1^2X_2^2, X_1Y, X_2Y\rangle Y^2, \qquad \mathcal{A}''_{11} = \langle X_1^2, X_1X_2, X_2^2\rangle Y^3,\\
&\mathcal{A}''_{12} = \langle X_1^3, X_1^2X_2, X_1X_2^2, X_2^3, Y\rangle Y^3,\\
&\mathcal{A}''_{13} = \langle X_1^3X_2, X_1^2X_2^2, X_1X_2^3, X_1Y, X_2Y\rangle Y^3,\\
&\mathcal{A}''_{14} = \langle X_1^3X_2^2, X_1^2X_2^3, X_1^2Y, X_1X_2Y, X_2^2Y\rangle Y^3,
\end{aligned}$$

and so on, which give the $P_m(X'')$'s written in Table 1 in the Introduction.

We next describe the m-canonical transformations $\varphi''_m = \varphi_{|mK_{X''}|}$ of X''. We claim that X'' has Kodaira dimension 2 and that $\varphi''_m(X'')$ has dimension 2 if and only if $m = 9, 10$, or $m \geq 12$. Clearly, $\dim \varphi''_m(X'') < 2$ if $m < 8$.

Setting $W_0 = X_1^2Y^2$, $W_1 = X_1X_2Y^2$ and $W_2 = X_2^2Y^2$, we see that $\varphi''_8(X'')$ is the plane conic $W_1^2 = W_0W_2$ in $\mathbb{P}^2$ with coordinates W_0, W_1, W_2. One sees that $\varphi''_{11}(X'')$ is a plane conic too, hence $\dim \varphi''_8(X'') = \dim \varphi''_{11}(X'') = 1$.

Moreover, we easily see that $\varphi''_9(X'') = \varphi''_{10}(X'') = \mathbb{P}^2$ and that $\varphi''_{12}(X'')$ is a surface scroll in $\mathbb{P}^4$, namely a cone over a rational normal curve in $\mathbb{P}^3$. In coordinates $Z_0 = X_1^3Y^3$, $Z_1 = X_1^2X_2Y^3$, $Z_2 = X_1X_2^2Y^3$, $Z_3 = X_2^3Y^3$, $Z_4 = Y^4$, the equations of $\varphi''_{12}(X'')$ are indeed

$$\varphi_{12}(X'') : \operatorname{rank}\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix} = 1.$$

One similarly sees that $\varphi''_{13}(X'')$ and $\varphi''_{14}(X'')$ are surface scrolls in $\mathbb{P}^4$, hence $\dim \varphi''_m(X'') = 2$ if $m = 9, 10,\ 12, 13,$ or 14. Since $P_3 = 1$, it follows that $\dim \varphi''_m(X'') \geq 2$ for each $m \geq 15$.

An upper bound to $P_m(X'')$ is given by the function $\mu\colon \mathbb{N} \to \mathbb{N}$,

$$\mu(m) = \sharp\{(i, j, k) \in \mathbb{N}^3 \mid 3i + j + k = m\} \geq P_m(X''),$$

and, setting $m = 3n + \varepsilon$, with $\varepsilon \in \{0, 1, 2\}$, we see that

$$P_m(X'') \leq \mu(m) = \sum_{i=0}^{n} (3(n-i) + 1 + \varepsilon) = (n+1)\left(\frac{3}{2}\, n + 1 + \varepsilon\right).$$

Thus the Kodaira dimension of X'' is 2 and $\dim \varphi''_m(X'') = 2$ for $m \geq 12$.

Finally, the same proof as in the case of the threefold X, cf. § 1.10, shows that the irregularities of X' and that of X'' are $q_1 = q_2 = 0$. This concludes the proof that X' and X'' have the properties described in the Introduction.

Appendix. Non-global canonical adjoints can be made global.

In this appendix we show that, in order to compute m-canonical adjoints to V, V' and V'', it suffices to compute global m-canonical adjoints.

We point out that the key ideas of these proofs are due to Maria Cristina Ronconi (cf. her approach in [Ro], § 4).

PROPOSITION 2. *Let* $F = \sum_{i+j+k+h+l=m} b_{ijkhl} X_0^i X_1^j X_2^k X_3^h X_4^l$ *be a homogeneous polynomial of degree* m*, with* $b_{ijkhl} \in \boldsymbol{k}$*, defining a non-global* m*-canonical adjoint to* V *[*V'*,* V''*, resp.]. Then there is a homogeneous polynomial* $A(X_0, \dots, X_4)$ *of degree* $m-6$ *such that* $F - Af_6$ *[*$F - Af_6'$*,* $F - Af_6''$*, resp.] defines a global* m*-canonical adjoint to* V *[*V'*,* V''*, resp.].*

We first write the proof of Proposition 2 for V. We will then show what to change for V' and V''. We begin with some definitions and a lemma.

Let us define 5 functions $\sigma_\lambda : \mathbb{N}^5 \to \mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers), $0 \le \lambda \le 4$, as follows: for each $\underline{\alpha} = (i, j, k, h, l) \in \mathbb{N}^5$, we set

$$\sigma_0(\underline{\alpha}) = |\underline{\alpha}| + l - i, \quad \sigma_1(\underline{\alpha}) = |\underline{\alpha}| + h - j, \quad \sigma_2(\underline{\alpha}) = |\underline{\alpha}| + i - k,$$
$$\sigma_3(\underline{\alpha}) = |\underline{\alpha}| + i - h, \quad \sigma_4(\underline{\alpha}) = |\underline{\alpha}| + j - l,$$

where $|\underline{\alpha}| = i + j + k + h + l$. For brevity, we write $\underline{X}^{\underline{\alpha}} = X_0^i X_1^j X_2^k X_3^h X_4^l$.

The function σ_λ will help to understand what happens to a monomial $\underline{X}^{\underline{\alpha}}$ appearing in the equation of a canonical adjoint when blowing up the point A_λ and the surface $\mathcal{S}_\lambda$ infinitely near to A_λ. Roughly speaking, in our situation the equation of an exceptional divisor corresponding to $\mathcal{S}_\lambda$ is given by just a coordinate variable and σ_λ counts how many times that variable appears in a monomial (cf. the proof of Lemma 1 in Chapter 1).

Fix λ, $0 \le \lambda \le 4$. For any homogeneous polynomial $G = \sum_{|\underline{\alpha}|=m} c_{\underline{\alpha}} \underline{X}^{\underline{\alpha}} \in K[\underline{X}] = K[X_0, \dots, X_4]$ of degree $m > 0$, we define the integer

$$r_\lambda(G) = \min\{\sigma_\lambda(\underline{\alpha}) : c_{\underline{\alpha}} \neq 0\},$$

and the polynomial

$$G^{(\lambda)} = \sum_{\underline{\alpha}:\, \sigma_\lambda(\underline{\alpha}) = r_\lambda(G)} c_{\underline{\alpha}} \underline{X}^{\underline{\alpha}},$$

which is the part of G with monomials $\underline{X}^{\underline{\alpha}}$ such that $\sigma_\lambda(\underline{\alpha}) = r_\lambda(G)$. The definition implies that, if $G^{(\lambda)} \neq G$, then

$$r_\lambda(G - G^{(\lambda)}) \geq r_\lambda(G) + 1.$$

Roughly speaking, $r_\lambda(G)$ counts how many times the variable defining the exceptional divisor corresponding to $\mathcal{S}_\lambda$ factorizes all monomials of G.

For each $0 \leq \lambda \leq 4$, we see that $r_\lambda(f_6) = 5$ and we set $p_\lambda = f_6^{(\lambda)}$, i.e.

$$\begin{aligned} p_0 = {} & a_{31002}X_0^3X_1X_4^2 + a_{13020}X_0X_1^3X_3^2 + a_{20301}X_0^2X_2^3X_4 + a_{20031}X_0^2X_3^3X_4 \\ & + a_{12210}X_0X_1^2X_2^2X_3 + a_{12120}X_0X_1^2X_2X_3^2 + a_{11220}X_0X_1X_2^2X_3^2, \end{aligned}$$

$$\begin{aligned} p_1 = {} & a_{31002}X_0^3X_1X_4^2 + a_{13020}X_0X_1^3X_3^2 \\ & + a_{02013}X_1^2X_3X_4^3 + a_{12210}X_0X_1^2X_2^2X_3, \end{aligned}$$

$$\begin{aligned} p_2 = {} & a_{20301}X_0^2X_2^3X_4 + a_{12210}X_0X_1^2X_2^2X_3 \\ & + a_{11220}X_0X_1X_2^2X_3^2 + a_{10221}X_0X_2^2X_3^2X_4, \end{aligned}$$

$$\begin{aligned} p_3 = {} & a_{13020}X_0X_1^3X_3^2 + a_{20031}X_0^2X_3^3X_4 + a_{02013}X_1^2X_3X_4^3 + a_{12120}X_0X_1^2X_2X_3^2 \\ & + a_{11220}X_0X_1X_2^2X_3^2 + a_{11202}X_0X_1X_3^2X_4^2 + a_{10221}X_0X_2^2X_3^2X_4, \end{aligned}$$

$$\begin{aligned} p_4 = {} & a_{31002}X_0^3X_1X_4^2 + a_{20301}X_0^2X_2^3X_4 + a_{20031}X_0^2X_3^3X_4 \\ & + a_{02013}X_1^2X_3X_4^3 + a_{11202}X_0X_1X_3^2X_4^2 + a_{10221}X_0X_2^2X_3^2X_4. \end{aligned}$$

LEMMA 3. *Fix λ, $0 \leq \lambda \leq 4$. Assume that the homogeneous polynomial $G = \sum_{|\underline{\alpha}|=m} c_{\underline{\alpha}} \underline{X}^{\underline{\alpha}}$ of degree m defines an m-canonical adjoint to V. If $r_\lambda(G) < m$, then there exists a homogeneous polynomial B_λ of degree $m-6$ such that $G^{(\lambda)} = B_\lambda p_\lambda$ and $B_\lambda^{(\lambda)} = B_\lambda$. Moreover, we have $r_\lambda(B_\lambda) = r_\lambda(G) - 5$ and $r_\lambda(G - B_\lambda f_6) \geq r_\lambda(G) + 1$.*

PROOF. We show in detail the case $\lambda = 4$, namely we see what happens when we blow up $A_4 \in U_4 = \{X_4 \neq 0\}$ and the double surface $\mathcal{S}_4$ infinitely near to A_4. We leave the other similar cases $0 \leq \lambda \leq 3$ to the reader.

On U_4 with coordinates x, y, z, t, we consider the composition $\mathcal{B}_{x_{11}} \circ \mathcal{B}_{x_1}$ (cf. the proof of Lemma 1 in Chapter 1). We set $\xi = (x_{11}, y_{11}, z_{11}, t_{11})$ and $\upsilon = (x_{11}, x_{11}^2 y_{11}, x_{11}z_{11}, x_{11}t_{11}, 1)$.

Since $G = \sum_{|\underline{\alpha}|=m} c_{\underline{\alpha}} \underline{X}^{\underline{\alpha}}$ defines an m-canonical adjoint to V, there exists a nonzero polynomial $A'(\xi) \in K[\xi] = K[x_{11}, y_{11}, z_{11}, t_{11}]$ such that

$$G(\upsilon) - A'(\xi)\frac{f_6(\upsilon)}{x_{11}^5} \in (x_{11}^m) \subset K[\xi]. \tag{1}$$

On the other hand, we have

$$G(\upsilon) = \sum_{i+j+k+h\leq m} c'_{ijkh} x_{11}^{i+2j+k+h} y_{11}^{j} z_{11}^{k} t_{11}^{h},$$

where $c'_{ijkh} = c_{ijkhl}$ with $l = m - (i+j+k+h)$. Setting $B'_4(\xi)$ the part of $A'(\xi)$ with the lowest degree in x_{11}, the assumption $r_4(G) < m$, the definition of $G^{(4)}(\underline{X})$ given in the previous pages and formula (1) above imply

$$(2) \qquad G^{(4)}(\upsilon) = \sum_{i+2j+k+h=r_4(G)} c'_{ijkh} x_{11}^{i+2j+k+h} y_{11}^{j} z_{11}^{k} t_{11}^{h} = B'(\xi)\,\frac{p_4(\upsilon)}{x_{11}^5},$$

since $\sigma_4(\underline{\alpha}) = i + 2j + k + h$. Going back to U_4 via $x_{11} = x$, $y_{11} = y/x^2$, $z_{11} = z/x$, $t_{11} = t/x$, and then to the original coordinates $\underline{X}$, formula (2) becomes

$$G^{(4)}(\underline{X}) = \frac{B_4(\underline{X})p_4(\underline{X})}{X_0^n},$$

for some $n \geq 0$ and a polynomial $B_4(\underline{X})$. Since $G^{(4)}(\underline{X})$ is a homogeneous polynomial and X_0 does not factorize $p_4(\underline{X})$, we see that X_0^n has to factorize $B_4(\underline{X})$, thus we may assume that $n = 0$ and B_4 is the homogeneous polynomial of degree $m - 6$ we were looking for. The final assertions of the lemma follow from the fact that $r_4(p_4) = 5$, $p_4^{(4)} = p_4$ and the definition of $B'(\xi)$.

Proof of Proposition 2. By Lemma 1 in Chapter 1, a homogeneous polynomial G of degree m defines a global m-canonical adjoint to V if and only if $r_\lambda(G) \geq m$ for each $0 \leq \lambda \leq 4$. Therefore there is λ such that $r_\lambda(F) < m$.

If $r_0(F) < m$, Lemma 4 implies that there is a homogeneous polynomial $B'_0(\underline{X})$ of degree $m - 6$ such that $F^{(0)} = B'_0 p_0$ and $r_0(F - B'_0 f_6) \geq r_0(F) + 1$. Repeating the same argument for the value of r_0 on the polynomial $F - B'_0 f_6$, by induction it follows that there exists a homogeneous polynomial $B_0(X)$ of degree $m - 6$ such that $r_0(F - B_0 f_6) \geq m$, and we set $F_0 = F - B_0 f_6$.

If instead $r_0(F) \geq m$, we set $F_0 = F$ and $B_0 = 0$.

If $r_1(F_0) < m$, Lemma 4 again implies that there is a homogeneous polynomial B'_1 of degree $m - 6$ such that $F_0^{(1)} = B'_1 p_1$ and $r_1(F_0 - B'_1 f_6) \geq r_1(F_0) + 1$. If instead $r_1(F_0) \geq m$, we set $F_1 = F_0$ and $B_1 = 0$.

We claim that we still have $r_0(F_0 - B_1' f_6) \geq m$. Since $F_0^{(1)} = B_1' p_1$ is part of F_0, we have $r_0(B_1' p_1) \geq m$. Recall that $r_0(p_1) = 5$ and note that p_0 and p_1 share monomials $\underline{X}^{\underline{\alpha}}$ with $\sigma_0(\underline{\alpha}) = \sigma_1(\underline{\alpha}) = 5$, e.g. $X_0 X_1^2 X_2^2 X_3$. Hence $r_0(B_1') \geq m - 5$ and $r_0(B_1' f_6) \geq m$, which implies our claim.

Repeating the same arguments for the value of r_1, by induction it follows that there is a homogeneous polynomial B_1 of degree $m - 6$ such that $r_\lambda(F_0 - B_1 f_6) \geq m$, $\lambda = 0, 1$, and we set $F_1 = F_0 - B_1 f_6 = F - (B_0 + B_1) f_6$.

If $r_2(F_1) < m$, we follow the same steps. Since p_2 share with both p_0, p_1 the monomial $X_0 X_1^2 X_2^2 X_3$ (which has $\sigma_\lambda(\underline{\alpha}) = 5$, $\lambda = 0, 1, 2$), we see that there is a homogeneous polynomial B_2 of degree $m - 6$ such that $r_\lambda(F_1 - B_2 f_6) \geq m$, $\lambda = 0, 1, 2$, and we set $F_2 = F_1 - B_2 f_6 = F - (B_0 + B_1 + B_2) f_6$.

If instead $r_2(F_1) \geq m$, we set $F_2 = F_1$ and $B_2 = 0$.

The same arguments apply for $\lambda = 3$ and then for $\lambda = 4$, noting that p_3 and p_4 share with p_0 the monomial $X_0^2 X_3^3 X_4$, with p_1 the monomial $X_1^2 X_3 X_4^3$, and with p_2 the monomial $X_0 X_2^2 X_3^2 X_4$. Therefore, by following the same steps, we find out homogeneous polyomials B_3, B_4 such that, setting $A = B_0 + B_1 + \cdots + B_4$, we have $r_\lambda(F - A f_6) \geq m$, $0 \leq \lambda \leq 4$, which is the assertion of Proposition 2 for V.

In case of V', we just replace f_6 by f_6' and σ_4 by the function $\underline{\alpha} \mapsto |\underline{\alpha}| + k - l$, hence $p_0, \ldots, p_4$ become

$$\begin{aligned}
p_0 &= a_{30102} X_0^3 X_2 X_4^2 + a_{13020} X_0 X_1^3 X_3^2 + a_{20301} X_0^2 X_2^3 X_4 \\
&\quad + a_{20031} X_0^2 X_3^3 X_4 + a_{12210} X_0 X_1^2 X_2^2 X_3 + a_{11220} X_0 X_1 X_2^2 X_3^2,
\end{aligned}$$

$$p_1 = a_{13020} X_0 X_1^3 X_3^2 + a_{12210} X_0 X_1^2 X_2^2 X_3 + a_{02112} X_1^2 X_2 X_3 X_4^2,$$

$$\begin{aligned}
p_2 &= a_{20301} X_0^2 X_2^3 X_4 + a_{10203} X_0 X_2^2 X_4^3 + a_{12210} X_0 X_1^2 X_2^2 X_3 \\
&\quad + a_{11220} X_0 X_1 X_2^2 X_3^2 + a_{02112} X_1^2 X_2 X_3 X_4^2,
\end{aligned}$$

$$\begin{aligned}
p_3 &= a_{13020} X_0 X_1^3 X_3^2 + a_{20031} X_0^2 X_3^3 X_4 \\
&\quad + a_{11220} X_0 X_1 X_2^2 X_3^2 + a_{02112} X_1^2 X_2 X_3 X_4^2,
\end{aligned}$$

$$\begin{aligned}
p_4 &= a_{30102} X_0^3 X_2 X_4^2 + a_{20031} X_0^2 X_3^3 X_4 + \\
&\quad + a_{10203} X_0 X_2^2 X_4^3 + a_{02112} X_1^2 X_2 X_3 X_4^2,
\end{aligned}$$

and we note that $p_1, \ldots, p_4$ share the monomial $X_1^2 X_2 X_3 X_4^2$; p_0, p_1, p_2 share the monomial $X_0 X_1^2 X_2^2 X_3$ and finally p_0, p_3, p_4 share the monomial $X_0^2 X_3^3 X_4$.

In case of V'', we replace f_6 by f_6'' and σ_4 by the function $\underline{\alpha} \mapsto |\underline{\alpha}| + h - l$, thus $p_0, \ldots, p_4$ now become

$$\begin{aligned} p_0 = &a_{30012}X_0^3X_3X_4^2 + a_{13020}X_0X_1^3X_3^2 + a_{20301}X_0^2X_2^3X_4 + a_{20031}X_0^2X_3^3X_4 \\ &+ a_{22011}X_0^2X_1^2X_3X_4 + a_{12210}X_0X_1^2X_2^2X_3 + a_{11220}X_0X_1X_2^2X_3^2, \end{aligned}$$

$$\begin{aligned} p_1 = &a_{13020}X_0X_1^3X_3^2 + a_{22011}X_0^2X_1^2X_3X_4 + a_{12210}X_0X_1^2X_2^2X_3 \\ &+ a_{12012}X_0X_1^2X_3X_4^2 + a_{02112}X_1^2X_2X_3X_4^2, \end{aligned}$$

$$\begin{aligned} p_2 = &a_{20301}X_0^2X_2^3X_4 + a_{12210}X_0X_1^2X_2^2X_3 + a_{11220}X_0X_1X_2^2X_3^2 \\ &+ a_{10221}X_0X_2^2X_3^2X_4 + a_{10212}X_0X_2^2X_3X_4^2 + a_{02112}X_1^2X_2X_3X_4^2, \end{aligned}$$

$$\begin{aligned} p_3 = &a_{13020}X_0X_1^3X_3^2 + a_{20031}X_0^2X_3^3X_4 + a_{10023}X_0X_3^2X_4^3 + a_{11220}X_0X_1X_2^2X_3^2 \\ &+ a_{10221}X_0X_2^2X_3^2X_4 + a_{02112}X_1^2X_2X_3X_4^2, \end{aligned}$$

$$\begin{aligned} p_4 = &a_{30012}X_0^3X_3X_4^2 + a_{20301}X_0^2X_2^3X_4 + a_{12012}X_0X_1^2X_3X_4^2 \\ &+ a_{10023}X_0X_3^2X_4^3 + a_{10212}X_0X_2^2X_3X_4^2 + a_{02112}X_1^2X_2X_3X_4^2. \end{aligned}$$

and we note that $p_1, \ldots, p_4$ share the monomial $X_1^2X_2X_3X_4^2$; p_0, p_1, p_2 share the monomial $X_0X_1^2X_2^2X_3$; p_0, p_3 share $X_0X_1^3X_3^2$ and p_0, p_4 share $X_0^3X_3X_4^2$. This concludes the proof of Proposition 2 in all cases.

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