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Elemente der Mathematik

On the mean length of the diagonals of an n-gon

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1 Introduction

P. Erdős [1] conjectured that the mean length of the diagonals of a convex n-gon with perimeter L is maximal iff $n/2$ vertices are concentrated in a point A and the remaining $n/2$ vertices in a point B whose distance is $L/2$ in case of $n = 2k$. If $n = 2k + 1$, then k and $k + 1$ vertices are concentrated in A and B, respectively, in the extremal figure. The minimum is attained when $n - 1$ vertices are concentrated in A and a single one in B.

These conjectures have been proved in the author's Msc. thesis [3] and dissertation [4] shortly after they had been stated. In this paper we revisit our solution of the problem.

2 Notations, definitions and a lemma

Notations. $\overline{PQ}$ denotes the distance of two points P and Q.

Vertices of the n-gon are denoted by $P_1, \ldots, P_n$, the whole n-gon by (P).

Bei welcher geschlossenen Kurve gegebener Länge ist die mittlere Länge von Sehnen maximal? Der Autor des vorliegenden Artikels beantwortete diese Frage von Wilhelm Blaschke and Luis Santaló 1965 in seiner Masterarbeit: Es ist der Kreis. In der selben Arbeit betrachtete der Autor ein diskretes Analogon und zeigte: Unter allen gleichseitigen Polygonen zeichnet sich das reguläre Polygon durch maximale mittlere Diagonalenlänge aus. Bereits früher lösten Stephen Vincze und Karl Reinhardt unabhängig voneinander das Problem, unter allen konvexen gleichseitigen Polygonen diejenigen von minimalem Durchmesser zu identifizieren. Dabei treten Reuleaux-Polygone auf. Paul Erdős wollte daraufhin wissen, welche konvexen Polygone von gegebenem Umfang maximale respektive minimale mittlere Länge von Diagonalen besitzen. Diese Frage wird in der vorliegenden Arbeit beantwortet.

V denotes the set $\{P_1, P_2, \dots, P_n\}$.
Multiple points (e.g. $P_1 = P_2$) are allowed. If i is an arbitrary integer lying outside of the interval $[1, n]$, then P_i is defined to be equal to $P_{i'}$, where $i' \equiv i \pmod{n}$.
Vertices of (P) are considered to form a cycle $C = (V, \Gamma)$, where $\Gamma : i \mapsto i + 1$.

Definitions. The anterior vertex of P_i in (P) is the last P_j preceding P_i in C not coincident with it. It is denoted by A_i.
The successor of P_i in (P) is the first vertex P_j following P_i in C not coincident with it. It is denoted by S_i.
The angle α_i at vertex P_i is called the angle between the segments $P_i A_i$ and $P_i S_i$.
P_i is called to be a break-point if $\alpha_i \neq \pi$.

Notations. The length of the diagonal $P_i P_{i+l}$ is denoted by $r_{i,l}$.
l is called the order of $P_i P_{i+l}$.
So the mean length of the diagonals of order l is

$$\varrho_l = \frac{1}{n}\sum_{i=1}^{n} r_{i,l}. \tag{1}$$

For the mean length ϱ of the diagonals of (P) we have obviously

$$\varrho = \begin{cases} \dfrac{1}{k-1}\displaystyle\sum_{l=2}^{k} \varrho_l & \text{if } n = 2k+1, \\ \dfrac{1}{k-2+\frac{1}{2}}\Big(\displaystyle\sum_{l=2}^{k-1} \varrho_l + \dfrac{\varrho_k}{2}\Big) & \text{if } n = 2k. \end{cases}$$

Our results depend on the following

Lemma. *Let l be a natural number from the interval $\big[1, [\frac{n}{2}]\big]$. Then we have*

$$\varrho_l \leq \frac{l}{n} L. \tag{2}$$

Equality occurs in (2) *iff at least l points are concentrated in every break-point of* (P).

3 Statement of the results

Theorem 1. *The inequality*

$$\varrho \leq \sigma L \tag{3}$$

holds, where

$$\sigma = \begin{cases} \dfrac{1}{4}\dfrac{n+3}{n} & \textit{if } n = 2k+1, \\ \dfrac{1}{4}\dfrac{n^2-8}{n(n-3)} & \textit{if } n = 2k, \end{cases}$$

the sign of equality occurs only if at least $\left[\frac{n}{2}\right]$ *points are concentrated at every break-point* P_i *of* (P). *Inequality* (3) *is sharp. If* $n = 2k$, *then equality holds iff* k *points are concentrated in each endpoint of a segment* AB *of length* $L/2$. *If* $n = 2k+1$ *then similarly, the remaining one point lying anywhere on* AB.

Our next result is related to the minimum of ϱ.

Theorem 2. *If* (P) *is convex, then we have the relation*

$$\varrho \geq \frac{L}{n}. \tag{4}$$

This inequality is sharp too. The sign of equality occurs iff $n-1$ *vertices are concentrated in one of the endpoints* A *of a segment* AB *and one in* B.

4 Proofs

Proof of the lemma. Obviously

$$r_{i,l} \leq r_{i,1} + r_{i+1,1} + \ldots + r_{i+l-1,1}. \tag{5}$$

Summing up (5) we get (2). The sign of equality occurs in (2) iff the same is true for (5) for every index i. This happens iff P_i, P_{i+1}, …, P_{i+l} are collinear and P_{i+1}, P_{i+2}, …, P_{i+l-1} lie on the segment $[P_i, P_{i+l}]$ for every index i. But this is possible iff at least l vertices are concentrated at every break-point of (P). □

Proof of Theorem 1. First we note that every (P) has at least two break-points – otherwise all the vertices were concentrated in one point – contradicting to the assumption $L > 0$.

Relation (3) follows at once from (2) by summation. Equality occurs in (3) iff we have "=" in (1) for every l $\left(l = 1, \ldots, \left[\frac{n}{2}\right]\right)$.

Regarding the lemma, this case occurs iff at least l break-points are concentrated in every break-point of (P) $\left(l = 1, \ldots, \left[\frac{n}{2}\right]\right)$. Strongest of these conditions is the last one implying that (P) may have at most two break-points.

Let us see which figures (P) satisfy this extremum condition.

If $n = 2k$, all the vertices of (P) have to be concentrated in two break-points A and B lying at distance $L/2$ each from other.

If $n = 2k+1$, then k vertices are concentrated both in A and B, the remaining one may lie anywhere on the segment AB.

In the extremal case we have for $n = 2k+1$

$$\varrho = \frac{2 + \ldots + k}{k-1}\frac{L}{n} = \frac{1}{4}\frac{n+3}{n}L,$$

and for $n = 2k$

$$\varrho = \frac{2 + \ldots + k - 1 + \frac{k}{2}}{k - 2 + \frac{1}{2}}\frac{L}{n} = \frac{(n+2)(n-4)+2n}{4n(n-3)}L. \qquad \square$$

Proof of Theorem 2. Consider the quadrangle $P_i P_{i+1} P_{i+l} P_{i+l+1}$. It is convex in consequence of the convexity of (P) and so we have

$$r_{i,l} + r_{i+1,l} \geq r_{i+1,l-1} + r_{i,l+1}. \tag{6}$$

Summing up (6) we get

$$\varrho_l \geq \frac{\varrho_{l-1} + \varrho_{l+1}}{2} \quad \left(l = 2, 3, \ldots, \left[\frac{n}{2}\right]\right) \tag{7}$$

i.e. the sequence $\{\varrho_l\}_1^n$ is concave. Taking into consideration that obviously

$$\varrho_l = \varrho_{n-l} \quad (l = 1, \ldots, n)$$

(7) implies that

$$\varrho_1 \leq \varrho_2 \leq \cdots \leq \varrho_{[\frac{n}{2}]},$$

i.e.

$$\varrho_l \geq \varrho_1 \quad \left(l = 2, 3, \ldots, \left[\frac{n}{2}\right]\right)$$

implying

$$\varrho \geq \varrho_1 = \frac{L}{n},$$

proving (4). If $n = 2k + 1$, then in the extremal case we have

$$\varrho = \frac{k-1}{k-1}\frac{L}{n} = \frac{L}{n}.$$

In the case $n = 2k$ we obtain

$$\varrho = \frac{k - 2 + \frac{1}{2}}{k - 2 + \frac{1}{2}}\frac{L}{n} = \frac{L}{n}. \qquad \square$$

Remarks

1. Theorem 1 remains valid in higher dimensions too – moreover in any metric space.
2. K. Böröczky gave another proof for Theorem 2 on L. Fejes Tóth's seminar.
3. If the sum of squared lengths of the sides of an n-gon is given, then the sum of squared lengths of its diagonals is maximal in case of the affine images of the regular n-gon [4].
4. Open questions:
   - What n-gons are extremal if the length of every side is given?
   - What can be said about the diameter when the perimeter or every length of side is fixed?
   - What is the minimum of the mean squared length of the diagonals of an n-gon if the mean squared length of the sides is given assuming convexity of the n-gon?

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