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$|Z_{\mathrm{Kup}}| = |Z_{\mathrm{Henn}}|^2$ for lens spaces

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Abstract. M. Hennings and G. Kuperberg defined quantum invariants Z_{Henn} and Z_{Kup} of closed oriented 3-manifolds based on certain Hopf algebras, respectively. We prove that $|Z_{\mathrm{Kup}}| = |Z_{\mathrm{Henn}}|^2$ for lens spaces when both invariants are based on factorizable finite dimensional ribbon Hopf algebras.

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Keywords. Knots, 3-manifolds, Hopf algebras, TQFT.

Contents

1. Introduction

A Turaev–Viro-type topological quantum field theory (TQFT) based on a spherical fusion category $\mathcal{C}$ is equivalent to the Reshetikhin–Turaev-TQFT based on the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$, see [17] and [1]. Consequently, $Z_{\mathrm{TV}}(M) = |Z_{\mathrm{RT}}(M)|^2$ for any closed oriented 3-manifold M. It is known that Hennings invariants are non-semisimple generalizations of Reshetikhin–Turaev-invariants [8], and Kuperberg invariants are non-semisimple generalizations of Turaev–Viro-invariants [2] (in this paper, by Kuperberg invariant, we mean the one from noninvolutory Hopf algebras in [12]). Therefore, a similar relation might exist between the Kuperberg and Hennings invariants as first suggested in [8].

Problem. Establish a generalization of the relation between Turaev–Viro and Reshetikhin–Turaev invariants to Kuperberg and Hennings invariants.

One issue with the above problem is that the Kuperberg invariant Z_{Kup} depends on a combing or framing of the 3-manifold M, while there is no such explicit dependence of combings or framings for the Hennings invariant Z_{Henn}. Ideally, the conjectured relation would follow from a similar relation between two kinds of non-semisimple $(2+1)$-TQFTs. As a first step, we prove the following relation between Kuperberg Z_{Kup} and Hennings Z_{Henn} invariants for the lens spaces $L(p,q)$ with $p, q \in \mathbb{N}, (p,q) = 1$.

Main Theorem. *Let H be a factorizable finite dimensional ribbon Hopf algebra and $L(p,q)$ be an oriented lens space. Then*

$$Z_{\text{Kup}}(L(p,q), f, H) = |Z_{\text{Henn}}(L(p,q), H)|^2$$

for some suitably chosen framing f of $L(p,q)$.

A different choice of framing changes the Kuperberg invariant via a multiplication by a root of unity [12].

Corollary 1.1. $|Z_{\text{Kup}}| = |Z_{\text{Henn}}|^2$ *for lens spaces and factorizable finite dimensional ribbon Hopf algebras.*

The contents of the paper are as follows. In Section 2, we recall the definitions of the Hennings and Kuperberg invariants and set up our notations. Finally in Section 3, we prove the main theorem.

2. Hennings and Kuperberg invariants

2.1. Some facts about Hopf algebras. In this section, we recall some notations and structures on finite dimensional Hopf algebras. Detail can be found in [13], [14], and [11].

Let $H = (m, \Delta, S, 1, \varepsilon)$ be a finite dimensional Hopf algebra over $\mathbb{C}$ with multiplication m, comultiplication Δ, antipode S, unit 1, and counit ε. We also use 1 to denote the identity map id on a Hopf algebra sometimes.

Recall that a Hopf algebra H is *quasitriangular* if there exists an R-matrix $R \in H \otimes H$. Let $R_{ij} \in H \otimes H \otimes H$ be obtained from $R = \sum_k s_k \otimes t_k$ by inserting the unit 1 into the tensor factor labeled by the index in $\{1,2,3\} \backslash \{i, j\}$. In a quasitriangular Hopf algebra with R-matrix $R = \sum_k s_k \otimes t_k$, the special element $u = \sum_k S(t_k) s_k$ satisfies $S^2(x) = uxu^{-1}$ for $x \in H$. We use R^τ to denote $\sum_k t_k \otimes s_k$.

A quasitriangular Hopf algebra is *ribbon* if there exists a central element θ such that

$$\Delta(\theta) = (R^\tau R)^{-1}(\theta \otimes \theta),$$
$$\varepsilon(\theta) = 1,$$

and

$$S(\theta) = \theta.$$

It can be shown that the element $G = u\theta^{-1}$ is group-like and $S^2(x) = GxG^{-1}$ for $x \in H$.

2.1.1. Integrals, cointegrals and unimodular Hopf algebras. A *left integral* λ^L (respectively, a *right integral* λ^R) for H is an element in H^* which satisfies

$$(\mathrm{id} \otimes \lambda^L)\Delta(h) = \lambda^L(h) \cdot 1$$

(respectively, $(\lambda^R \otimes \mathrm{id})\Delta(h) = \lambda^R(h) \cdot 1$) for all $h \in H$. Dually, a *left cointegral* Λ^L (respectively, a *right cointegral* Λ^R) for H is an element in H which satisfies

$$h\Lambda^L = \varepsilon(h)\Lambda^L$$

(respectively, $\Lambda^R h = \varepsilon(h)\Lambda^R$) for all $h \in H$. A Hopf algebra H is called *unimodular* if the space of left cointegrals for H is the same as the space of right cointegrals for H.

For finite dimensional Hopf algebras, the left and right integrals (respectively, left and right cointegrals) are unique up to scalar multiplication, and we may choose a normalization that

$$\lambda^R(\Lambda^L) = \lambda^R(S(\Lambda^L)) = 1.$$

From this, there is an algebra homomorphism $\alpha \in H^*$, called *modulus* of H, independent of the choice of Λ^L, such that $\Lambda^L h = \alpha(h)\Lambda^L$ for all $h \in H$. Likewise, there is a group-like element $g \in H$, called *comodulus* of H, independent of the choice of λ^R, such that

$$(\mathrm{id} \otimes \lambda^R)\Delta(h) = \lambda^R(h)g$$

for all $h \in H$. The elements α and g are of finite order, and

$$\omega = \alpha(g)$$

is a root of unity.

2.1.2. Drinfeld map and factorizable Hopf algebras. Given

$$Q = \sum_i Q_i^{(1)} \otimes Q_i^{(2)} \in H \otimes H,$$

we define a map

$$f_Q\colon H^* \longrightarrow H$$

by

$$f_Q(p) = \sum_i p(Q_i^{(1)}) Q_i^{(2)}, \quad p \in H^*.$$

The conditions for R-matrix imply that f_R is an algebra homomorphism and f_{R^τ} is an algebra anti-homomorphism. The map

$$f_{R^\tau R}\colon H^* \longrightarrow H$$

is called the *Drinfeld map.* If the Drinfeld map for a quasitriangular Hopf algebra H is an isomorphism as a linear map of vector spaces, then H is called *factorizable.*

Proposition 2.1. *If a quasitriangular Hopf algebra H is factorizable, then it is unimodular.*

For a proof, see Proposition 3 on p. 224 of [14].

For a factorizable Hopf algebra H, $f_{R^\tau R}(\lambda^R) = \Lambda^L$ and $\lambda^R(\Lambda^L) = 1$ under some normalization (see [5]). This relates the left cointegral Λ^L with the right integral λ^R. We will use such a pair of related integral and cointegral throughout this paper.

In this paper, we work with factorizable finite dimensional ribbon Hopf algebras. For such Hopf algebras, we use Λ to denote the left and right cointegrals for H. The comodulus α is the counit ε. The right integral for H, denoted by λ, has the following properties for all x and y in H [13]:

(1) $\lambda(xy) = \lambda(S^2(y)x)$;

(2) $\lambda(gx) = \lambda(S(x))$, where g is the comodulus of H.

In particular, we have the following result.

Lemma 2.2. $\lambda(S^{-1}(x)) = \lambda(xg) = \lambda(gx) = \lambda(S(x))$.

Such a λ leads to a trace-like functional with the help of a square root G of the comodulus g, i.e., $G^2 = g$. That is, we have a functional

$$\mathrm{tr}\colon H \longrightarrow \mathbb{C}$$

defined by

$$\mathrm{tr}(x) = \lambda(xG) = \lambda(Gx)$$

such that $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ and $\mathrm{tr}(S(x)) = \mathrm{tr}(x)$, for all $x, y \in H$. The following lemma is important for the proof of the main theorem. Let

$$\Delta^{(n-1)}(x) = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(n)}$$

be in the Sweedler notation for iterated comultiplication. In this paper, we omit the summation symbol, i.e. we write

$$\Delta^{(n-1)}(x) = x_{(1)} \otimes \cdots \otimes x_{(n)}.$$

Lemma 2.3. *For $p \in H^*$ and $n \in \mathbb{N}$, we have*

$$\Delta^{(n-1)}(f_{R^\tau R}(p)) = f_{R^\tau}(p_{(1)}) f_R(p_{(2n-1)}) \otimes f_{R^\tau}(p_{(2)}) f_R(p_{(2n-2)}) \otimes \cdots \otimes f_{R^\tau}(p_{(n-1)}) f_R(p_{(n+1)}) \otimes f_{R^\tau R}(p_{(n)}).$$

In particular, since $f_{R^\tau R}(\lambda) = \Lambda$, we have

$$\Lambda_{(k)} = f_{R^\tau}(\lambda_{(k)}) f_R(\lambda_{(2n-k)}), \quad k = 1, \ldots, n-1$$

and

$$\Lambda_{(n)} = f_{R^\tau R}(\lambda_{(n)}).$$

Proof. Recall that f_R is a coalgebra antihomomorphism and f_{R^τ} is a coalgebra homomorphism, i.e., for $p \in H^*$,

$$\Delta(f_R(p)) = (f_R \otimes f_R)(\Delta(p)) = f_R(p_{(2)}) \otimes f_R(p_{(1)}),$$

$$\Delta(f_{R^\tau}(p)) = (f_{R^\tau} \otimes f_{R^\tau})(\Delta^{\mathrm{op}}(p)) = f_{R^\tau}(p_{(1)}) \otimes f_{R^\tau}(p_{(2)}).$$

These two properties follow from the definition of the R-matrix:

$$(\Delta \otimes \mathrm{id})(R) = R_{13} R_{23}$$

and

$$(\mathrm{id} \otimes \Delta)(R) = R_{13} R_{12}.$$

The proof of the lemma is by induction.

- When $n = 2$,

$$\begin{aligned}\Delta(f_{R^\tau R}(p)) &= \Delta(f_{R^\tau}(p_{(1)}) f_R(p_{(2)})) \\ &= \Delta(f_{R^\tau}(p_{(1)})) \Delta(f_R(p_{(2)})) \\ &= f_{R^\tau}(p_{(1)}) f_R(p_{(4)}) \otimes f_{R^\tau}(p_{(2)}) f_R(p_{(3)}) \\ &= f_{R^\tau}(p_{(1)}) f_R(p_{(3)}) \otimes f_{R^\tau R}(p_{(2)}).\end{aligned}$$

- Suppose the lemma is true for $n = k$, then, when $n = k + 1$,

$$\begin{aligned}\Delta^{(k)}(f_{R^\tau R}(p)) &= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta)(\Delta^{(k-1)}(f_{R^\tau R}(p)))\\ &= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta)(f_{R^\tau}(p_{(1)})f_R(p_{(2k-1)})\\ &\qquad \otimes f_{R^\tau}(p_{(2)})f_R(p_{(2k-2)})\\ &\qquad \otimes \cdots \otimes f_{R^\tau}(p_{(k-1)})f_R(p_{(k+1)})\\ &\qquad \otimes f_{R^\tau R}(p_{(k)}))\\ &= f_{R^\tau}(p_{(1)})f_R(p_{(2k-1)}) \otimes f_{R^\tau}(p_{(2)})f_R(p_{(2k-2)})\\ &\quad \otimes \cdots \otimes f_{R^\tau}(p_{(k-1)})f_R(p_{(k+1)}) \otimes \Delta(f_{R^\tau R}(p_{(k)}))\\ &= f_{R^\tau}(p_{(1)})f_R(p_{(2k+1)}) \otimes f_{R^\tau}(p_{(2)})f_R(p_{(2k)})\\ &\quad \otimes \cdots \otimes f_{R^\tau}(p_{(k-1)})f_R(p_{(k+3)})\\ &\quad \otimes f_{R^\tau}(p_{(k)})f_R(p_{(k+2)}) \otimes f_{R^\tau R}(p_{(k+1)}).\end{aligned}$$

Hence, the lemma holds for all $n \in \mathbb{N}$. □

2.1.3. Examples. Factorizable finite dimensional ribbon Hopf algebras include the following important examples.

(1) $U_q\mathrm{sl}(2, \mathbb{C})$ at an odd root of unity. Let q be an l-th primitive root of unity with l an odd integer ≥ 3.

$U_q\mathrm{sl}(2, \mathbb{C})$ is generated by E, F, and K with the following relations:

$$E^l = F^l = 0, \quad K^l = 1$$

and the Hopf algebra structure given by

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = 1,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.$$

It is factorizable and ribbon with the following R-matrix and ribbon element

$$R = \frac{1}{l} \sum_{0\leq m,i,j\leq l-1} \frac{(q - q^{-1})^m}{[m]!} q^{m(m-1)/2+2m(i-j)-2ij} E^m K^i \otimes F^m K^j,$$

and

$$\theta = \frac{1}{l}\Big(\sum_{s=0}^{l-1} q^{s^2}\Big)\Big(\sum_{0\leq m,j\leq l-1} \frac{(q^{-1} - q)^m}{[m]!} q^{-\frac{1}{2}m+mj+\frac{1}{2}(j+1)^2} F^m E^m K^j\Big).$$

Its right integral, two-sided cointegral and comodulus are

$$\lambda(F^m E^n K^j) = \delta_{m,l-1}\delta_{n,l-1}\delta_{j,1}, \quad \Lambda = F^{l-1}E^{l-1}\sum_{j=0}^{l-1} K^j, \quad g = K^2.$$

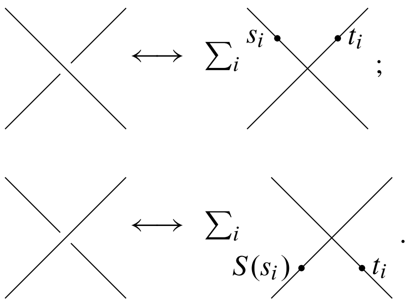
(2) The Drinfeld double $D(H)$ of a finite dimensional Hopf algebra H is factorizable. By [11], it has a ribbon element if and only if

$$S^2(h) = l((\beta^{-1} \otimes \mathrm{id} \otimes \beta)\Delta^2(h))l^{-1} \quad h \in H,$$

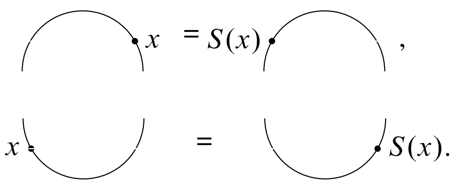
where l and β are group-like elements of H and H^*, respectively, which satisfy $l^2 = g$ and $\beta^2 = \alpha$.

2.2. Hennings invariant. Let (H, R, θ) be a unimodular finite dimensional ribbon Hopf algebra with $\lambda(\theta)\lambda(\theta^{-1}) \neq 0$.

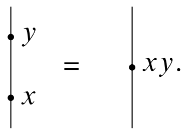
2.2.1. Kauffman–Radford version of the Hennings invariant. We recall now the Kauffman–Radford version of the Hennings invariant [10]. First (H, R, θ) gives rise to a regular isotopy invariant $\mathrm{TR}(L, H)$ for framed links L as follows: given any link diagram L_D of L, decorate each crossing of L_D with the tensor factors from the R-matrix $R = \sum_i s_i \otimes t_i$ as below:



Once all the crossings of L_D have been decorated, let D_L be the labeled diagram immersed in the plane, where all crossings became 4-valent vertices. The Hopf algebra elements on D_L may slide across maxima or minima of D_L on the same component at the expense of the application of the antipode or its inverse. Passing through an extremum in a clockwise direction introduces S^{-1} and passing through an extremum in a counterclockwise direction introduces S as below:



To define $\mathrm{TR}(L, H)$, slide all the Hopf algebra elements on the same component into one vertical portion of the same component. Along a vertical line, all the Hopf algebra elements on the same component of D_L are multiplied together:



The final juxtaposition of labeled elements at the chosen points gives rise to a product $w_i \in H$ for the i-th component of L_D. Let d_i be the Whitney degree of this component obtained by traversing it upward from the chosen vertical portion. The Whitney degree is the total number of turns of the tangent vector as one traverses the curve in the given direction. For example:

Define

$$\mathrm{TR}(L_D, H) = \mathrm{tr}(w_1 G^{d_1}) \ldots \mathrm{tr}(w_{c(L)} G^{d_{c(L)}}),$$

where $c(L)$ denotes the number of components of L. This quantity is invariant under Reidemeister II and III moves, hence is a regular isotopy invariant of the framed link L. Moreover, if $\lambda(\theta)\lambda(\theta^{-1}) \neq 0$, which is always true when H is factorizable [5], then

$$Z_{\mathrm{Henn}}(M(L), H) = [\lambda(\theta)\lambda(\theta^{-1})]^{-\frac{c(L)}{2}} [\lambda(\theta)/\lambda(\theta^{-1})]^{-\frac{\sigma(L)}{2}} \mathrm{TR}(L, H) \qquad (2.2)$$

is an invariant of the closed oriented 3-manifold $M(L)$ obtained from surgery on the framed link L with the blackboard framing, and $\sigma(L)$ denotes the signature of the framing matrix of L.

2.2.2. Properties of Hennings invariant. Given a closed oriented manifold M, the symbol $\bar{M}$ denotes the same manifold with the opposite orientation.

We have the following from [7]:

(1) $Z_{\mathrm{Henn}}(M_1 \# M_2, H) = Z_{\mathrm{Henn}}(M_1, H) Z_{\mathrm{Henn}}(M_2, H)$,

(2) $Z_{\mathrm{Henn}}(\bar{M}, H) = \overline{Z_{\mathrm{Henn}}(M, H)}$.

2.3. Kuperberg invariant. Let H be any finite dimensional Hopf algebra. In the following, we briefly recall some terminologies from [12]. For detail, see Section 2 of [12].

2.3.1. Kuperberg combings. Given a Heegaard diagram of a closed connected oriented 3-manifold M, Kuperberg referred to the attaching curves c_l's of the 2-handles of one handlebody as lower circles and the attaching curves c_u's of the 2-handles of the other handlebody as upper circles. Note that this choice is arbitrary. A Heegaard diagram on a Heegaard surface F of genus g is called *F-minimal* if the Heegaard diagram consists of g lower circles and g upper circles. In the sequel, we will simply call an F-minimal Heegaard diagram a minimal Heegaard diagram. The orientation of M induces an orientation on its Heegaard surface F by appending a normal vector that points from the lower side to the upper side to a positive tangent basis at a point on F which extends to a positive basis for M. Define a *combing* on a minimal Heegaard diagram on surface F to be a vector field on F with $2g$ singularities of index -1, one on each circle, and one singularity of index $+2$ disjoint from all circles. The singularity of index -1 on a given circle, which is called the base point of the circle, should not lie on a crossing and the two outward-pointing vectors should be tangent to the circle. Combings of Heegaard diagrams can be used to represent combings of 3-manifolds due to the following fact.

Proposition 2.4. *Any combing b of a minimal Heegaard diagram of M can be extended to a combing $\bar{b}$ of M. Conversely, any combing of M is homotopic to the Kuperberg extension of some combing of the minimal Heegaard diagram.*

For a proof of the proposition, see Section 2 of [12].

2.3.2. Twist front and rotation numbers. Given a combing b_1 of M, by Proposition 2.4 we may assume it is extended from some combing of a minimal Heegaard diagram D. A framing of M can be obtained from another combing b_2 that is orthogonal to b_1: the third combing b_3 of M is determined by the orientation of M. To describe such a framing (b_1, b_2) of M, where b_2 is an orthogonal combing to b_1, it suffices to give b_1 as a diagram combing and then specify b_2 on the Heegaard surface F and on all upper and lower disks. Kuperberg introduced twist fronts to encode the position of b_2. A *twist front* is an arc along which b_2 is normal to F and points from the lower to the upper handlebody. A twist front is transversely oriented in the direction that b_2 rotates by the right-hand rule relative to b_1 and transverse orientation is presented by the zigzag symbol as in Figure 1.

To define the Kuperberg invariant, orient all Heegaard circles. Let $f = (b_1, b_2)$ be a framing from the minimal Heegaard diagram D. For each point p on some circle c of D with base point o, we define $\psi(p)$ to be the counterclockwise rotation of the tangent to c relative to b_1 from o to p in units of $1 = 360°$. If p is a crossing, then two rotation angles $\psi_l(p)$ and $\psi_u(p)$ are defined. ψ_l and ψ_u are defined to be the total counterclockwise rotation on the lower and upper circles. Let $\varphi(p)$ be the total right-handed twist of b_2 around b_1 from o to p, and similarly define $\varphi_l(p)$ and $\varphi_u(p)$. Using twist fronts, we can compute $\varphi(p)$ as the total signs of all fronts crossed from o to p, not counting the front that terminates at o itself.

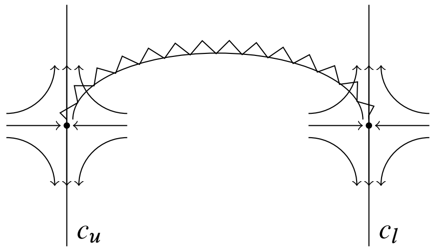


Figure 1. Twist front.

2.3.3. The Kuperberg invariant. First we construct some special elements from the integral and cointegral. For any integer n, define $\lambda_{n-\frac{1}{2}} \in H^*$ such that

$$\lambda_{n-\frac{1}{2}}(x) = \lambda^R(xg^n), \quad x \in H,$$

and

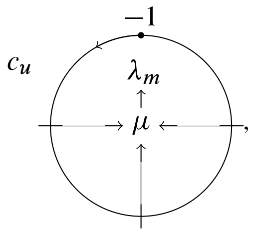
$$\Lambda_{n-\frac{1}{2}} = (\mathrm{id} \otimes \alpha^n)\Delta(\Lambda^R) \in H.$$

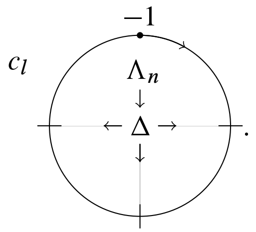
Also define the tilt map T to be

$$T(x) = (\alpha \otimes \mathrm{id} \otimes \alpha^{-1})\Delta^2(S^{-2}(x)), \quad x \in H,$$

where g (or α) are the comodulus (or modulus) of H.

In Kuperberg's tensor notation, the algorithm for his invariant $Z_{\mathrm{Kup}}(M, f, H)$ is as follows: replace each upper circle c_u with the multiplication tensor μ with one inward arrow for each crossing and the outward arrow with λ_m at the base point, with the arrows ordered as indicated. Here $m = -\psi(c_u)$. Replace each lower circle c_l with the comultiplication tensor Δ with an outward arrow for each crossing and the inward arrow with Λ_n at the base point, with the arrow ordered as indicated. Here $n = \psi(c_l)$. Replace each crossing by the tensor $\to S^a T^b \to$ where $a = 2(\psi_l(p) - \psi_u(p)) - \frac{1}{2}$, $b = \varphi_l(p) - \varphi_u(p)$, and p is the crossing point:





Finally, contract all tensor corresponding to circles and crossings according to incidence. The Kuperberg invariant is then a big summation:

$$Z_{\mathrm{Kup}}(M, f, H) = \sum_{(\Lambda)} \prod_{\substack{\text{upper}\\ \text{circles}}} \lambda(\ldots S^{a_i} T^{b_i}(\Lambda_{(i)}) \ldots g^m).$$

Here the order for multiplication and comultiplication follows the orientations of the upper and lower circles.

For a factorizable finite dimensional ribbon Hopf algebra H, we have $\alpha = \varepsilon$. So $\Lambda_{n-\frac{1}{2}} = \Lambda$ for all integer n and $T = S^{-2}$. Thus, the Kuperberg invariant is of the following form:

$$Z_{\mathrm{Kup}}(M, f, H) = \sum_{(\Lambda)} \prod_{\substack{\text{upper}\\ \text{circles}}} \lambda(\ldots S^{a_i - 2b_i}(\Lambda_{(i)}) \ldots g^m). \tag{2.3}$$

2.3.4. Basic properties of the Kuperberg invariant. With suitable choices of framings, we have [12]:

(1) $Z_{\mathrm{Kup}}(M_1 \# M_2, H) = Z_{\mathrm{Kup}}(M_1, H) Z_{\mathrm{Kup}}(M_2, H)$,

(2) $Z_{\mathrm{Kup}}(M, H^*) = Z_{\mathrm{Kup}}(\bar{M}, H^{\mathrm{op}}) = Z_{\mathrm{Kup}}(\bar{M}, H^{\mathrm{cop}}) = Z_{\mathrm{Kup}}(M, H)$.

3. A relation between Kuperberg and Hennings invariants

In this section, we prove our main theorem.

Theorem 3.1. *Let H be a factorizable finite dimensional ribbon Hopf algebra and $L(p, q)$ be an oriented lens space, then*

$$Z_{\mathrm{Kup}}(L(p, q), f, H) = Z_{\mathrm{Henn}}(L(p, q) \# \overline{L(p, q)}, H)$$

for some suitably chosen framing f of $L(p, q)$.

Using $Z_{\mathrm{Henn}}(M_1 \# M_2, H) = Z_{\mathrm{Henn}}(M_1, H) Z_{\mathrm{Henn}}(M_2, H)$ and $Z_{\mathrm{Henn}}(\bar{M}, H) = \overline{Z_{\mathrm{Henn}}(M, H)}$, we can deduce the version of our main theorem in the introduction.

We will calculate $Z_{\mathrm{Kup}}(L(p, q), f, H)$ and $Z_{\mathrm{Henn}}(L(p, q) \# \overline{L(p, q)}, H)$ through the framed Heegaard diagram and the chain-mail link, respectively. Since $L(p, q)$ is homeomorphic to $L(p, q + kp)$ for any integer k, it suffices to prove the theorem for the case $p > q > 0$.

3.1. Chain-mail links. Let M be a closed oriented connected 3-manifold. We can turn a Heegaard diagram of M into a surgery diagram of $M \# \bar{M}$ using the chain-mail link introduced in [15]. Let (F, H_1, H_2) be a Heegaard decomposition of M with a F-minimal Heegaard diagram. We will refer to H_1 as the lower handlebody, so H_2 would be the upper handlebody. Push the upper circles c_u's into H_1 slightly, then the upper circles and the lower circles form a link in H_1. All circles are framed by thickening them into thin bands parallel to the Heegaard surface F. This results in a so-called chain-mail link $C(M) \subseteq H_1$, which is in fact a surgery presentation for $M \# \bar{M}$ ([15]). Figure 2 and Figure 3 are the Heegaard diagram and the corresponding chain-mail link for the lens space $L(5, 2)$, respectively.

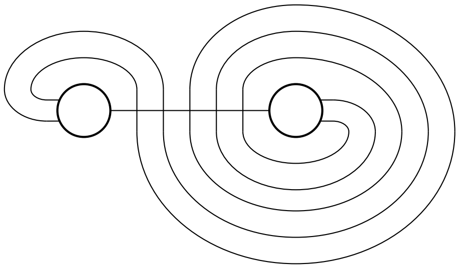


Figure 2. Heegaard diagram of $L(5,2)$.

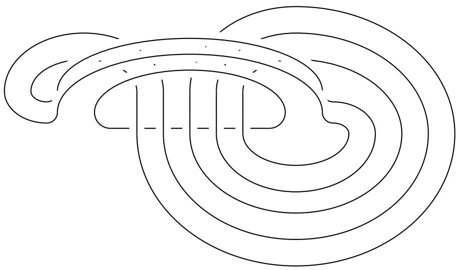


Figure 3. Chain-mail link of $L(5,2)$.

In Figure 2, a 1-handle (not drawn) is attached to the two round circles in the 2-sphere S^2 regarded as the plane together with the point at infinity. In general, Figure 4 gives us a minimal Heegaard diagram for $L(p,q)$ with $p > q > 0$, where r is the remainder, i.e., $r = p - [\frac{p}{q}]q$ and $0 < r < q$. Here $[x]$ means the integral part of x. In the picture, the horizontal line represents the lower circle c_l (note that our lower circles are above the plane and the part of the circle over the 1-handle is not drawn). The upper circle c_u has q strands coming out from the right circle and then going clockwise around the right circle for $[\frac{p}{q}] - 1$ times until they meet the q strands from the left circle. To make p intersections with c_l, we let the first r strands of the left q strands go around counterclockwise to match the r strands of the right q strands from above. Figure 5 is the corresponding chain-mail link.

3.2. $Z_{\mathrm{Kup}}(L(p,q), f, H)$.

We calculate the Kuperberg invariant for $L(p,q)$ with some framing. Since the Kuperberg invariants depend on framings of 3-manifolds, we need to choose a particular framing for $L(p,q)$ in order to match them with the Hennings invariants. The choice of framing depends on the values of p and q. First, let

$$N_1 = \begin{cases} \dfrac{q+1}{2} & \text{if } q \text{ is odd,} \\ \dfrac{p+q+1}{2} & \text{if } q \text{ is even.} \end{cases}$$

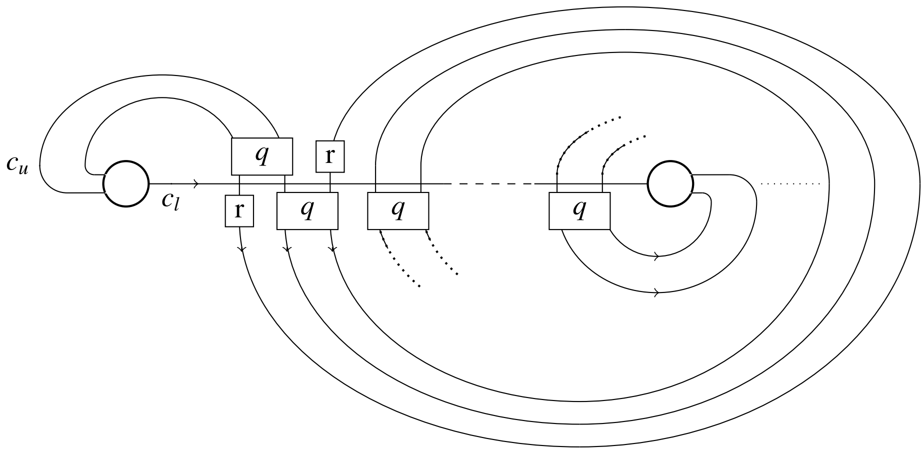


Figure 4. Heegaard diagram of $L(p, q)$.

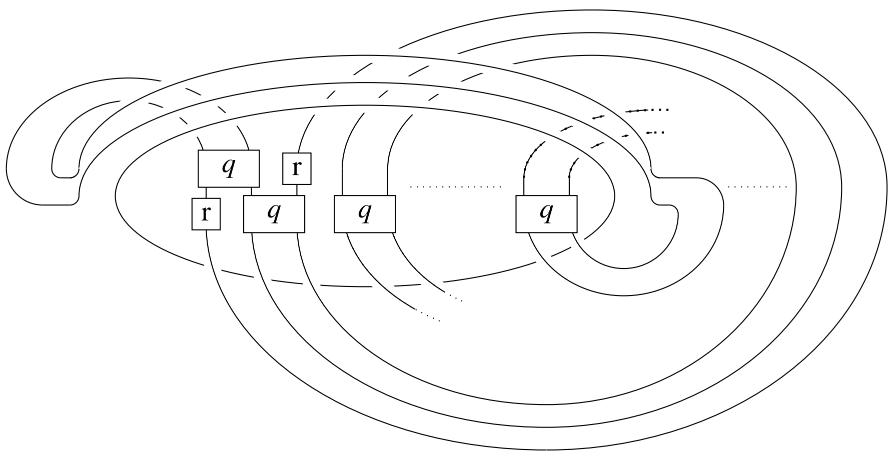


Figure 5. Chain-mail link of $L(p, q)$.

Since p and q are relatively prime, N_1 is always a natural number. Let

$$N_j \equiv N_1 + (j-1)q \pmod p$$

such that $N_j \in \{1, \dots, p\}$, for $j = 1, \dots, p$. We order the intersection points between the lower and upper circles as 1 to p from left to right in the plane, then $k_1, \dots, k_q$ is the order of the point starting from N_1 to visit the first q points following the orientation of the upper circle, i.e., the sequence $k_1, \dots, k_q$ is determined by $N_{k_i} = i$ for $i = 1, \dots, q$.

In the following, we shall write $\Lambda_j = \Lambda_{(N_j)}$ as defined in Lemma 1 for $j = 1, \dots, p$ for short. In other words, we rename $\Lambda_{(j)}$'s along the direction of the upper circle. In the picture, it means the corresponding Λ_{k_i}'s are labeled on the leftmost strands.

The following technical lemma collects some symmetry properties of the indices we have just introduced.

Lemma 3.2. (1) *For $i, j \in \{1, \dots, p\}$ such that $i + j = p + 1$, we have $N_i + N_j = p + 1$. As a consequence, for the two-sided cointegral Λ, we have*

$$\sum S^{\pm 1}(\Lambda_p) \otimes \cdots \otimes S^{\pm 1}(\Lambda_{p+1-j}) \otimes \cdots \otimes S^{\pm 1}(\Lambda_1)$$
$$= \sum \Lambda_1 \otimes \cdots \otimes \Lambda_j \otimes \cdots \otimes \Lambda_p.$$

(2) *When q is odd, $k_i = 1 + \left[\frac{p}{q}(i-1) + \frac{q-1}{2q}\right]$ and $k_i + k_{q+2-i} = p + 2$ for $i = 2, \dots, q$.*

(3) *When q is even, $k_j = 1 + \left[\frac{p}{q}(j - \frac{1}{2}) + \frac{q-1}{2q}\right]$ and $k_j + k_{q+1-j} = p + 2$ for $j = 1, \dots, q$.*

Proof. Proof of 1). By the definition of N_i, if $i + j = p + 1$, then

$$N_i + N_j \equiv 2N_1 + (i + j - 2)q \equiv 1 \pmod p$$

Note that $1 \le N_i, N_j \le p$, we have $N_i + N_j = p + 1$. The identity for cointegral results from the unimodular property that $S(\Lambda) = \Lambda$. Indeed, this implies

$$\sum S(\Lambda_{(p)}) \otimes \cdots \otimes S(\Lambda_{(p+1-j)}) \otimes \cdots \otimes S(\Lambda_{(1)})$$
$$= \sum \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(j)} \otimes \cdots \otimes \Lambda_{(p)}.$$

Replace $\Lambda_{(N_j)}$ by Λ_j and rearrange the order of tensor product factors, then we obtain the identity in (1).

Proof of 2). Since

$$1 + (i-1)p \leq N_{k_i} = N_1 + (k_i - 1)q \leq q + (i-1)p,$$

then

$$\frac{(i-1)p}{q} \leq \frac{N_1}{q} + (k_i - 1) \leq 1 + \frac{(i-1)p}{q}.$$

So

$$k_i = 1 + \Big[\frac{p}{q}(i-1) + 1 - \frac{N_1}{q}\Big] = 1 + \Big[\frac{p}{q}(i-1) + \frac{q-1}{2q}\Big].$$

For $i = 2, \dots, q$, set

$$x_i = \frac{p}{q}(i-1) + \frac{q-1}{2q}.$$

Then

$$k_i + k_{q+2-i} = 2 + [x_i] + [x_{q+2-i}].$$

Note that

$$x_i + x_{q+2-i} = p + 1 - \frac{1}{q}.$$

So

$$[x_i] + [x_{q+2-i}] = [x_i] + \Big[p + 1 - \frac{1}{q} - x_i\Big] = p + [x_i] + \Big[1 - \frac{1}{q} - x_i\Big].$$

We claim

$$[x_i] + \Big[1 - \frac{1}{q} - x_i\Big] = 0.$$

For this, let us first study the function $f(x) = [x] + [\beta - x]$ where $\beta \in [0,1)$ is a constant. It has period $T = 1$ since $[x+n] = [x]$ for $n \in \mathbb{Z}$. It suffice to study it on the interval $[0,1)$.

- If $0 \leq x \leq \beta$, then we have $[x] + [\beta - x] = 0 + 0 = 0$.
- If $\beta < x < 1$, then we have $[x] + [\beta - x] = 0 + (-1) = -1$.

Hence, for $x \in \mathbb{R}$,

$$[x] + [\beta - x] = \begin{cases} 0 & \text{for } x \in [n, n+\beta],\ n \in \mathbb{Z}, \\ -1 & \text{for } x \in (n+\beta, n+1),\ n \in \mathbb{Z}. \end{cases}$$

Let

$$\{x\} = x - [x]$$

be the fractional part of x. Then $[x] + [\beta - x] = 0$ if and only if $\{x\} \leq \beta$. Thus our claim is equivalent to $\{x_i\} \leq 1 - \frac{1}{q}$. In the following, we calculate $\{x_i\}$ case by case.

Let

$$r = (i-1)p - q\Big[\frac{(i-1)p}{q}\Big]$$

be the remainder. Then

$$\{x_i\} = \Big\{\Big[\frac{(i-1)p}{q}\Big] + \frac{r}{q} + \frac{q-1}{2q}\Big\} = \Big\{\frac{r}{q} + \frac{q-1}{2q}\Big\}.$$

Case 1: $1 \leq r \leq \frac{q-1}{2}$. Because

$$\frac{r}{q} + \frac{q-1}{2q} \leq \frac{q-1}{2q} + \frac{q-1}{2q} = 1 - \frac{1}{q} \leq 1,$$

we have

$$\Big\{\frac{r}{q} + \frac{q-1}{2q}\Big\} = \frac{r}{q} + \frac{q-1}{2q} \leq 1 - \frac{1}{q}.$$

Case 2: $\frac{q+1}{2} \leq r \leq q-1$. Because

$$2 > \frac{r}{q} + \frac{q-1}{2q} \geq \frac{q+1}{2q} + \frac{q-1}{2q} = 1,$$

we have

$$\Big\{\frac{r}{q} + \frac{q-1}{2q}\Big\} = \frac{r}{q} + \frac{q-1}{2q} - 1 < \frac{q}{q} + \frac{q-1}{2q} - 1 = \frac{1}{2}\Big(1 - \frac{1}{q}\Big) \leq 1 - \frac{1}{q}.$$

Proof of 3). Since

$$1 + jp \le N_{k_j} = N_1 + (k_j - 1)q \le q + jp,$$

then

$$\frac{jp}{q} \le \frac{N_1}{q} + (k_j - 1) \le 1 + \frac{jp}{q}.$$

So

$$k_j = 1 + \Big[\frac{jp}{q} + 1 - \frac{N_1}{q}\Big] = 1 + \Big[\frac{p}{q}\Big(j - \frac{1}{2}\Big) + \frac{q-1}{2q}\Big].$$

Similarly as above, we set

$$y_j = \frac{p}{q}\Big(j - \frac{1}{2}\Big) + \frac{q-1}{2q}, \quad j = 1, \dots, q.$$

Then

$$y_j + y_{q+1-j} = p + 1 - \frac{1}{q}.$$

Thus $k_j + k_{q+1-j} = p + 2$ is equivalent to

$$[y_j] + \Big[1 - \frac{1}{q} - y_j\Big] = 0.$$

Further more, this is equivalent to $\{y_j\} \le 1 - \frac{1}{q}$.

Let

$$r = (2i-1)p - 2q\Big[\frac{(2i-1)p}{2q}\Big]$$

be the remainder. Note that $r \ne q$ for $(p,q) = 1$. Then

$$\{y_j\} = \Big\{\Big[\frac{(2i-1)p}{2q}\Big] + \frac{r}{2q} + \frac{q-1}{2q}\Big\} = \Big\{\frac{r}{2q} + \frac{q-1}{2q}\Big\}.$$

Case 1: $1 \le r \le q-1$. Because

$$\frac{r}{2q} + \frac{q-1}{2q} \le \frac{q-1}{2q} + \frac{q-1}{2q} = 1 - \frac{1}{q} < 1,$$

we have

$$\Big\{\frac{r}{q} + \frac{q-1}{2q}\Big\} = \frac{r}{2q} + \frac{q-1}{2q} \le 1 - \frac{1}{q}.$$

Case 2: $q+1 \le r \le 2q-1$. Because

$$2 > \frac{r}{2q} + \frac{q-1}{2q} \ge \frac{q+1}{2q} + \frac{q-1}{2q} = 1,$$

we have

$$\begin{aligned}\left\{\frac{r}{2q} + \frac{q-1}{2q}\right\} &= \frac{r}{2q} + \frac{q-1}{2q} - 1 \\ &< \frac{2q}{2q} + \frac{q-1}{2q} - 1 \\ &= \frac{1}{2}\Big(1 - \frac{1}{q}\Big) \\ &< 1 - \frac{1}{q}.\end{aligned}$$

□

In addition, we define $k_0 = 1$ and $k_{q+1} = p+1$ for future use.

We set up a framed Heegaard diagram for $L(p,q)$ shown in Figure 8 for q odd, and Figure 15 for q even. The framing f is represented by the dashed flows and the twist front. The twist fronts vary depending on whether q is odd or even. Two index -1 singularities are located at the two ends of the twist front on the horizontal line. The right -1 singularity is at the N_1-th intersection of the horizontal line and the upper circle c_u. The lower circle c_l is represented by the horizontal line with the base point the left -1 singularity and oriented towards right from the base point. To avoid the right singularity, we make c_l turn around slightly when it meets this singularity. Likewise, the upper circle c_u with the base point the right singularity is oriented towards down from its base point. The index $+2$ singularity is located at the infinity. The orientation of circles is shown in Figure 4.

3.3. Examples. Before we turn to the general calculation, it is helpful to examine two concrete examples.

3.3.1. $L(2,1)$. We choose a framing for $L(2,1)$ as shown in Figure 6 then go downwards from $\Lambda_{(1)}$. The power for S for $\Lambda_{(1)}$ and $\Lambda_{(2)}$ are, respectively,

$$2\Big(-\frac{1}{4} - 0\Big) - \frac{1}{2} = -1,$$

$$2\Big(-\frac{1}{2} - \Big(-\frac{1}{4}\Big)\Big) - \frac{1}{2} = -1.$$

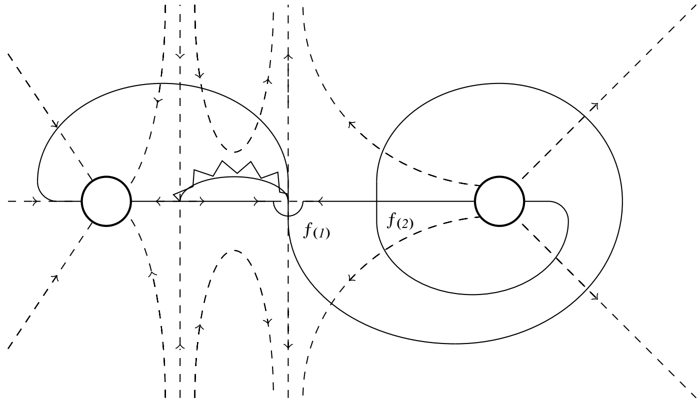


Figure 6. Heegaard diagram of $L(2, 1)$.

The total rotation along the upper circle is

$$-\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{1}{2}$$

and the power of g is

$$-\frac{1}{2} + \frac{1}{2} = 0.$$

So the Kuperberg invariant for $L(2, 1)$ is

$$Z_{\text{Kup}}(L(2,1)) = \lambda(S^{-1}(\Lambda_{(1)})S^{-1}(\Lambda_{(2)})) = \lambda(\Lambda_{(2)}\Lambda_{(1)}) = \text{Tr}(S^{-1}),$$

where Tr is the trace for vector spaces. The last equality follows from a trace formula in terms of integrals in [13].

3.3.2. $L(5, 2)$. In Figure 7, a framing is set up for $L(5, 2)$. We start from $\Lambda_{(4)}$ and go downwards. The power for S for $\Lambda_{(4)}, \Lambda_{(1)}, \Lambda_{(3)}, \Lambda_{(5)}$ and $\Lambda_{(2)}$ are, respectively,

$$2\Big(-\frac{1}{4} - 0\Big) - \frac{1}{2} = -1,$$

$$2\Big(0 - \Big(-\frac{1}{4} + \frac{1}{2}\Big)\Big) - \frac{1}{2} = -1,$$

$$2\Big(0 - \Big(-\frac{1}{4} + \frac{1}{2} - 1\Big)\Big) - \frac{1}{2} = 1,$$

$$2\Big(-\frac{1}{2} - \Big(-\frac{1}{4} + \frac{1}{2} - 1 - \frac{1}{2}\Big)\Big) - \frac{1}{2} - 2 = 3,$$

$$2\Big(0 - \Big(-\frac{1}{4} + \frac{1}{2} - 1 - \frac{1}{2} + \frac{1}{2}\Big)\Big) - \frac{1}{2} - 2 = 3.$$

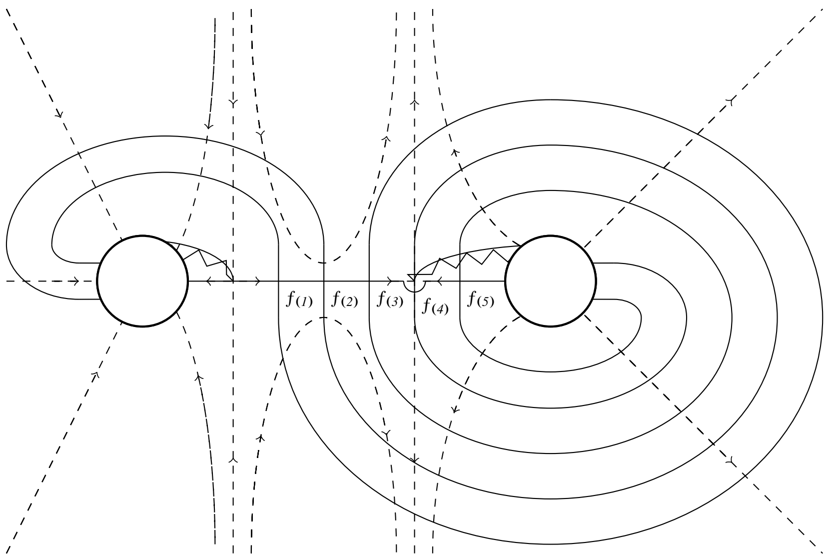


Figure 7. Heegaard diagram of $L(5,2)$.

In the last two equations, the -2's result from crossing the twist front before $\Lambda_{(5)}$ and $\Lambda_{(2)}$. The total rotation along the upper circle is $-\frac{1}{4}+\frac{1}{2}-1-\frac{1}{2}+\frac{1}{2}-\frac{3}{4}=-\frac{3}{2}$ and the power of g is $\frac{3}{2}+\frac{1}{2}=2$. From this data, the Kuperberg invariant for $L(5,2)$ is

$$\begin{aligned} Z_{\mathrm{Kup}}(L(5,2)) &= \lambda(S^{-1}(\Lambda_{(4)})S^{-1}(\Lambda_{(1)})S(\Lambda_{(3)})S^{3}(\Lambda_{(5)})S^{3}(\Lambda_{(2)})g^{2}) \\ &= \lambda(\Lambda_{(2)}\Lambda_{(5)}S^{2}(\Lambda_{(3)})S^{4}(\Lambda_{(1)})S^{4}(\Lambda_{(4)})g^{2}) \\ &= \lambda(S^{-4}(\Lambda_{(2)})S^{-4}(\Lambda_{(5)})S^{-2}(\Lambda_{(3)})\Lambda_{(1)}\Lambda_{(4)}g^{2}). \end{aligned}$$

In this case, $N_1=4$ and Λ_1, Λ_2, Λ_3, Λ_4, Λ_5 are, respectively, $\Lambda_{(4)}$, $\Lambda_{(1)}$, $\Lambda_{(3)}$, $\Lambda_{(5)}$, $\Lambda_{(2)}$. Therefore, $k_1=2$ and $k_2=5$.

3.4. General calculation for $Z_{\mathrm{Kup}}(L(p,q), f, H)$

3.4.1. $Z_{\mathrm{Kup}}(L(p,q), f, H)$ when q is odd. We choose a framed Heegaard diagram for $L(p,q)$ as shown in Figure 8. In this case, $k_1=1$. Let us first analyze the pattern of powers of the antipode S in the product in Eq. (2.3). For Λ_{k_1}, which is the starting point to do the multiplication along c_u, the power of S is

$$2(\psi_l(k_1)-\psi_u(k_1))-\frac{1}{2}=2\Big(-\frac{1}{4}-0\Big)-\frac{1}{2}=-1.$$

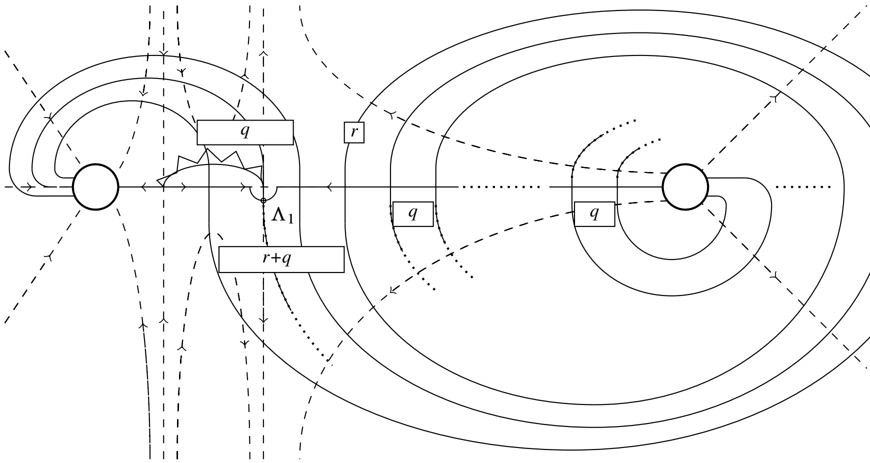


Figure 8. Framed Heegaard diagram when q is odd.

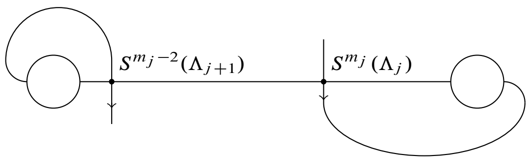


Figure 9. Power of S changing when q is odd.

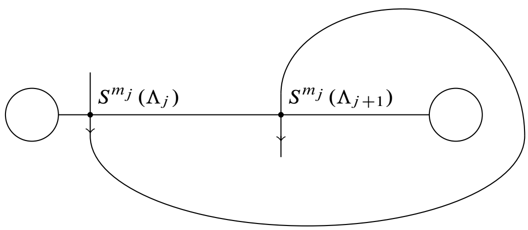


Figure 10. Power of S changing when q is odd.

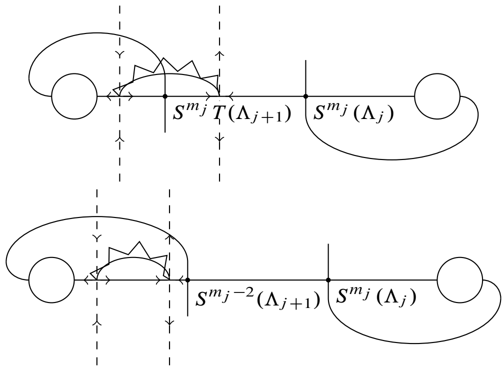


Figure 11. Power of S changing when q is odd: case 1 and 2.

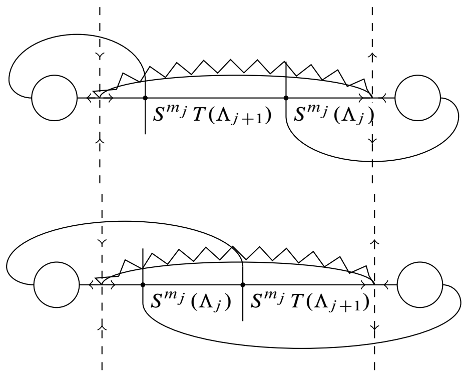


Figure 12. Power of S changing when q is odd: case 3 and 4.

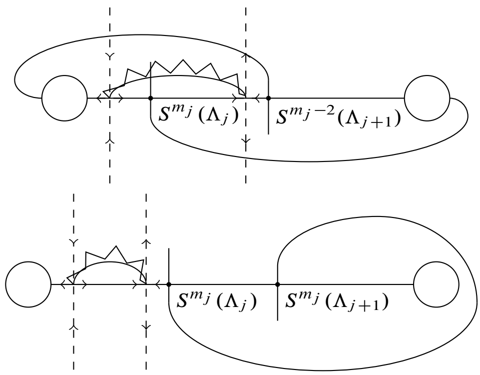


Figure 13. Power of S changing when q is odd: case 5 and 6.

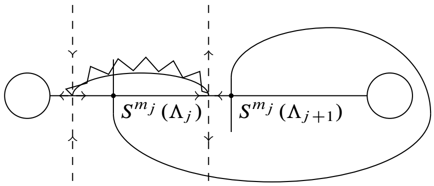


Figure 14. Power of S changing when q is odd: case 7.

Lemma 3.3. *The powers of S change as shown in Figure 11 to Figure 14. Namely, for $j = 1, \ldots, p$,*

(1) *the power of S from Λ_j to the next factor Λ_{j+1} decreases by 2 when travelling along the 1-handle from right to left;*

(2) *the power of S from Λ_j to the next factor Λ_{j+1} remains the same when travelling along the upper circle counterclockwise around the right disk.*

Proof. Suppose the j-th term in the summation is $S^{m_j}(\Lambda_j)$, we calculate the difference $m_{j+1} - m_j$ case by case, which is

$$m_{j+1} - m_j = 2(\psi_l(\Lambda_{j+1}) - \psi_l(\Lambda_j)) - 2(\psi_u(\Lambda_{j+1}) - \psi_u(\Lambda_j)) + 2(\varphi_u(\Lambda_{j+1}) - \varphi_u(\Lambda_j)).$$

Here $2(\varphi_l(\Lambda_{j+1}) - \varphi_l(\Lambda_j))$ makes no contribution since the lower circle does not intersect with the twist fronts. Note that $T = S^{-2}$ for factorizable Hopf algebras.

The patterns shown in Figure 9 include five cases.

(1) This case is shown in the first picture in Figure 11:

$$m_{j+1} - m_j = 2\Big(0 - \Big(-\frac{1}{2}\Big)\Big) - 2\Big(\frac{1}{2}\Big) + 2(-1) = -2.$$

(2) This case is shown in the second picture in Figure 11:

$$m_{j+1} - m_j = 2\Big(\Big(-\frac{1}{2}\Big) - \Big(-\frac{1}{2}\Big)\Big) - 2\Big(\frac{1}{2} + \frac{1}{2}\Big) + 2(0) = -2.$$

(3) and (4) These two cases are shown in the first picture in Figure 12 and they share the same calculation:

$$m_{j+1} - m_j = 2(0 - 0) - 2\Big(-\frac{1}{2} + \frac{1}{2}\Big) + 2(-1) = -2.$$

(5) This case is shown in the first picture in Figure 13:

$$m_{j+1} - m_j = 2\Big(-\frac{1}{2} - 0\Big) - 2\Big(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\Big) + 2(0) = -2.$$

The patterns shown in Figure 10 include the following two cases.

(6) This case is shown in the second picture in Figure 13:

$$m_{j+1} - m_j = 2\Big(\Big(-\frac{1}{2}\Big) - \Big(-\frac{1}{2}\Big)\Big) - 2(0) + 2(0) = 0.$$

(7) This case is shown in the second picture in Figure 14:

$$m_{j+1} - m_j = 2\Big(\Big(-\frac{1}{2}\Big) - 0\Big) - 2\Big(-\frac{1}{2}\Big)\Big) + 2(0) = 0. \qquad \square$$

Two more values to write down the Kuperberg invariant are $\psi(c_l)$ and $\psi(c_u)$. It is easy to see that

$$\psi(c_l) = -\frac{1}{2}$$

and

$$\psi(c_u) = -\frac{1}{4} + (N_1 - 1)\Big(\frac{1}{2} - \frac{1}{2}\Big) + (q - N_1)\Big(\frac{1}{2} + \frac{1}{2}\Big) + \frac{1}{2} + \frac{1}{4} = \frac{q}{2}.$$

It follows that the power of g is $-\psi(c_u) + \frac{1}{2} = \frac{-q+1}{2}$.

Now we can write down the Kuperberg invariant $Z_{\text{Kup}}(L(p,q), f, H)$:

$$\begin{aligned}
Z_{\text{Kup}} &= \prod_{m=1}^{q} \prod_{n=k_m}^{k_{m+1}-1} S^{-2m+1}(\Lambda_n) \\
&= \lambda(S^{-1}(\Lambda_{k_1})S^{-1}(\Lambda_{k_1+1}) \dots S^{-1}(\Lambda_{k_2-1}) \\
&\qquad S^{-3}(\Lambda_{k_2}) \dots S^{-3}(\Lambda_{k_3-1}) \dots\dots \\
&\qquad S^{-2q+1}(\Lambda_{k_q}) \dots S^{-2q+1}(\Lambda_p) g^{\frac{-q+1}{2}}) \\
&= \lambda(S^{2q-1}(\Lambda_{k_1})S^{2q-1}(\Lambda_{k_1+1}) \dots S^{2q-1}(\Lambda_{k_2-1}) \\
&\qquad S^{2q-3}(\Lambda_{k_2}) \dots S^{2q-3}(\Lambda_{k_3-1}) \dots\dots \\
&\qquad S(\Lambda_{k_q}) \dots S(\Lambda_p) g^{\frac{-q+1}{2}}) \\
&= \lambda(S^{2q-2}(\Lambda_p)S^{2q-2}(\Lambda_{p-1}) \dots S^{2q-2}(\Lambda_{p+2-k_2}) \\
&\qquad S^{2q-4}(\Lambda_{p+1-k_2}) \dots S^{2q-4}(\Lambda_{p+2-k_3}) \dots\dots \\
&\qquad S^2(\Lambda_{p+1-k_{q-1}}) \dots S^2(\Lambda_{p+2-k_q})\Lambda_{p+1-k_q} \dots \Lambda_{k_1+1}\Lambda_{k_1} g^{\frac{-q+1}{2}}) \\
&= \lambda(S^{2q-2}(\Lambda_p)S^{2q-2}(\Lambda_{p-1}) \dots S^{2q-2}(\Lambda_{k_q}) \\
&\qquad S^{2q-4}(\Lambda_{k_q-1}) \dots S^{2q-4}(\Lambda_{k_{q-1}}) \dots\dots \\
&S^2(\Lambda_{k_3-1}) \dots S^2(\Lambda_{k_2})\Lambda_{k_2-1} \dots \Lambda_{k_1+1}\Lambda_{k_1} g^{\frac{-q+1}{2}}).
\end{aligned}$$

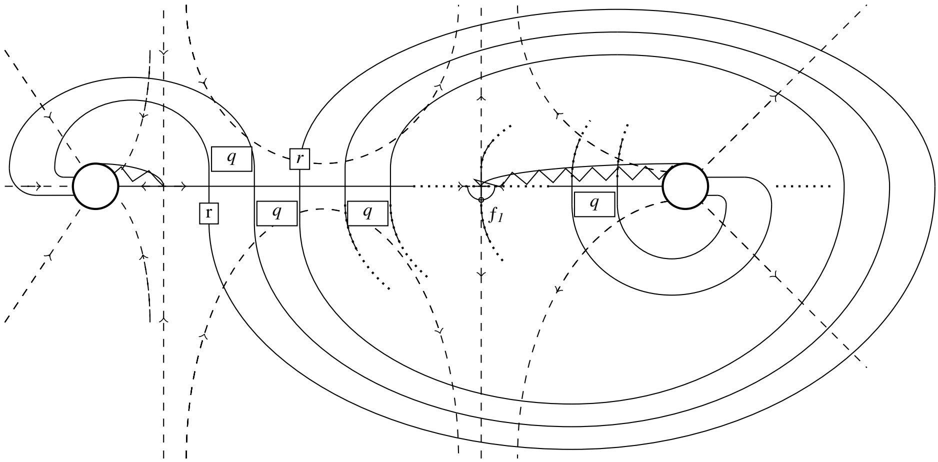


Figure 15. Framed Heegaard diagram when q is even.

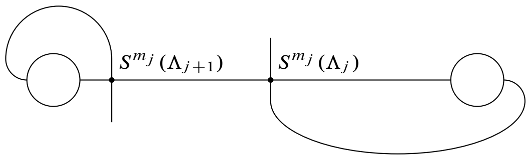


Figure 16. Power of S changing when q is even.

Here we have used the unimodular property that $S(\Lambda) = \Lambda$ and so

$$\sum S(\Lambda_{(p)}) \otimes \cdots \otimes S(\Lambda_{(p+1-j)}) \otimes \cdots \otimes S(\Lambda_{(1)})$$
$$= \sum \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(j)} \otimes \cdots \otimes \Lambda_{(p)}$$

and by symmetry $k_j + k_{q+2-j} = p + 2$ for $j = 2, \ldots, q$.

3.4.2. $Z_{\text{Kup}}(L(p,q), f, H)$ when q is even. Figure 15 is a framed Heegaard diagram for $L(p,q)$ when q is even. In this case, the twist front is different from the case when q is odd, so the pattern of the power changes of S is also different.

Lemma 3.4. *As shown in Figure 16 and Figure 17, for $j = 1, \ldots, p$,*

(1) *the power of S from Λ_j to the next factor Λ_{j+1} remains the same when travelling along the upper circle counterclockwise around the right disk;*

(2) *the power of S from Λ_j to the next factor Λ_{j+1} increases by 2 when travelling along the 1-handle from right to left.*

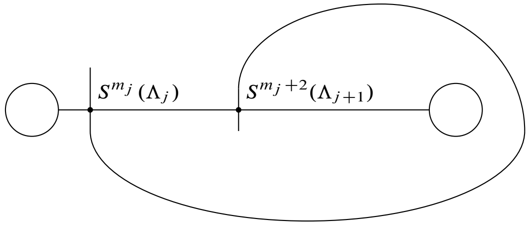


Figure 17. Power of S changing when q is even.

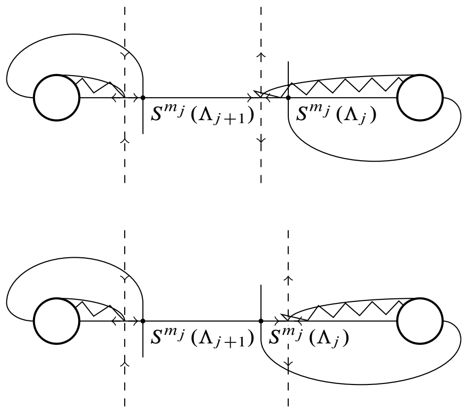


Figure 18. Power of S changing when q is even: case 1 and 2.

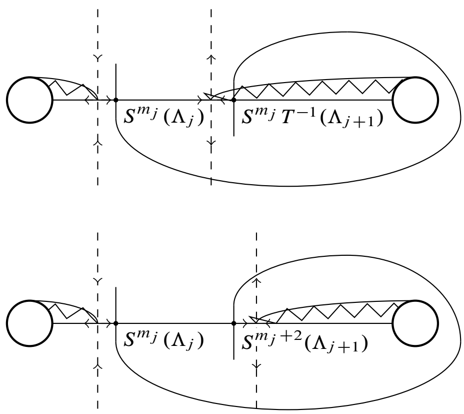


Figure 19. Power of S changing when q is even: case 3 and 4.

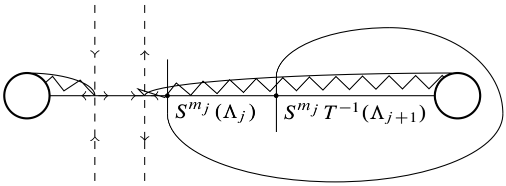


Figure 20. Power of S changing when q is even: case 5.

Proof. Similar to the q odd case, we calculate the change $m_{j+1} - m_j$ of the powers of S case by case shown from Figure 18 to Figure 20.

(1) As shown in the first picture in Figure 18,

$$m_{j+1} - m_j = 2\Big(0 - \Big(-\frac{1}{2}\Big)\Big) - 2\Big(\frac{1}{2}\Big) + 2(0) = 0.$$

(2) As shown in the second picture in Figure 18,

$$m_{j+1} - m_j = 2(0 - 0) - 2\Big(-\frac{1}{2} + \frac{1}{2}\Big) + 2(0) = 0.$$

(3) As shown in the first picture in Figure 19,

$$m_{j+1} - m_j = 2\Big(\Big(-\frac{1}{2}\Big) - 0\Big) - 2\Big(-\frac{1}{2}\Big) + 2(1) = 2.$$

(4) As shown in the second picture in Figure 19,

$$m_{j+1} - m_j = 2(0 - 0) - 2\Big(-\frac{1}{2} - \frac{1}{2}\Big) + 2(0) = 2.$$

(5) As shown in the second picture in Figure 20,

$$\begin{aligned} m_{j+1} - m_j &= 2\Big(\Big(-\frac{1}{2}\Big) - \Big(-\frac{1}{2}\Big)\Big) - 2(0 - 0) + 2(1) \\ &= 2. \end{aligned} \tag{1}$$

□

For $\psi(c_u)$, we have

$$\begin{aligned} \psi(c_u) &= -\frac{1}{4} + (N_1 - 1 - q)\Big(-\frac{1}{2} - \frac{1}{2}\Big) - \frac{1}{4} \\ &= \frac{-p + q}{2} \end{aligned}$$

and then the power of g is

$$-\psi(c_u) + \frac{1}{2} = \frac{p-q+1}{2}.$$

Thus the Kuperberg invariant is

$$\begin{aligned}
Z_{\text{Kup}} &= \prod_{m=0}^{q} \prod_{n=k_m}^{k_{m+1}-1} S^{2n-2m-3}(\Lambda_n) \\
&= \lambda(S^{-1}(\Lambda_1)S(\Lambda_2)\dots S^{2k_1-5}(\Lambda_{k_1-1})S^{2k_1-5}(\Lambda_{k_1}) \\
&\quad S^{2k_1-3}(\Lambda_{k_1+1})\dots S^{2k_2-7}(\Lambda_{k_2-1})\dots\dots \\
&\quad S^{2k_q-2q-3}(\Lambda_{k_q})S^{2k_q-2q-1}(\Lambda_{k_q+1})\dots S^{2p-2q-3}(\Lambda_p)g^{\frac{p-q+1}{2}}) \\
&= \lambda(\Lambda_p S^2(\Lambda_{p-1})\dots S^{2k_1-4}(\Lambda_{p+2-k_1})S^{2k_1-4}(\Lambda_{p+1-k_1}) \\
&\quad S^{2k_1-2}(\Lambda_{p-k_1})\dots S^{2k_2-6}(\Lambda_{p-k_2+1})\dots\dots \\
&\quad S^{2k_q-2q-2}(\Lambda_{p+1-k_q})S^{2k_q-2q}(\Lambda_{p-k_q})\dots S^{2p-2q-2}(\Lambda_1)g^{\frac{p-q+1}{2}}) \\
&= \lambda(S^{-2p+2q+2}(\Lambda_p)S^{-2p+2q+4}(\Lambda_{p-1})\dots \\
&\quad S^{-2p+2q+2k_1-2}(\Lambda_{p+2-k_1})S^{-2p+2q+2k_1-2}(\Lambda_{p+1-k_1}) \\
&\quad S^{-2p+2q+2k_1}(\Lambda_{p-k_1})\dots S^{-2p+2q+2k_2-4}(\Lambda_{p-k_2+1})\dots\dots \\
&\quad S^{-2p+2k_q}(\Lambda_{p+2-k_q})S^{-2p+2k_q}(\Lambda_{p+1-k_q}) \\
&\quad S^{-2p+2k_q+2}(\Lambda_{p-k_q})\dots S^{-2}(\Lambda_2)\Lambda_1 g^{\frac{p-q+1}{2}}) \\
&= \lambda(S^{-2p+2q+2}(\Lambda_p)S^{-2p+2q+4}(\Lambda_{p-1})\dots \\
&\quad S^{-2k_q+2q+2}(\Lambda_{k_q})S^{-2k_q+2q+2}(\Lambda_{k_q-1}) \\
&\quad S^{-2k_q+2q+4}(\Lambda_{k_q-2})\dots S^{-2k_{q-1}+2q}(\Lambda_{k_{q-1}-1})\dots\dots \\
&\quad S^{-2k_1+4}(\Lambda_{k_1})S^{-2k_1+4}(\Lambda_{k_1-1}) \\
&\quad S^{-2k_1+6}(\Lambda_{k_1-2})\dots S^{-2}(\Lambda_2)\Lambda_1 g^{\frac{p-q+1}{2}}).
\end{aligned}$$

Here we have used that

$$\begin{aligned}
&\sum S^{-1}(\Lambda_{(p)}) \otimes \cdots \otimes S^{-1}(\Lambda_{(p+1-j)}) \otimes \cdots \otimes S^{-1}(\Lambda_{(1)}) \\
&\quad = \sum \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(j)} \otimes \cdots \otimes \Lambda_{(p)}
\end{aligned}$$

and

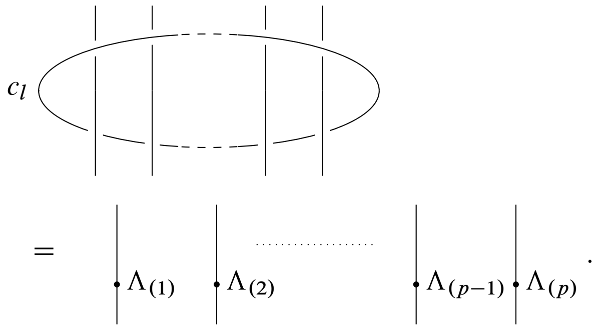
$$k_j + k_{q+1-j} = p + 2, \quad j = 1, \dots, q.$$

3.5. $Z_{\mathrm{Henn}}(L(p,q)\#\overline{L(p,q)}, H)$. We use the chain-mail link L in Figure 5 to calculate the Hennings invariant for $L(p,q)\#\overline{L(p,q)}$. Since the signature $\sigma(L)$ of the framing matrix of the chain-mail link is zero, so the normalization factor

$$[\lambda(\theta)\lambda(\theta^{-1})]^{-\frac{c(L)}{2}}[\lambda(\theta)/\lambda(\theta^{-1})]^{-\frac{\sigma(L)}{2}} = 1$$

because $\lambda(\theta)\lambda(\theta^{-1}) = 1$ for a factorizable ribbon Hopf algebra (see [5]). Hence it is sufficient to find the link invariant $\mathrm{TR}(L, H)$. First, the contribution of the lower circle c_l to the Hennings invariant is equivalent to decorating the upper circle c_u with cointegrals. That is we get the following lemma.

Lemma 3.5. *We have*



Proof. By Kauffman and Radford's algorithm, we label the crossings with components of the R-matrix $R = \sum s \otimes t$, where $\sum s^1 \otimes t^1 = \cdots = \sum s^{2p} \otimes t^{2p}$ are copies of the R-matrix, and obtain the immersed diagram in Figure 21. Then we can separate the circle from the rest and push all the decorated elements to one side as shown in Figure 22. This diagram gives us the following tensor element. Here the last equality results from Lemma 2.3:

$$\begin{aligned}
&\sum \lambda(t^{2p}t^{2p-1}\dots t^{p+2}t^{p+1}s^p s^{p-1}\dots s^2 s^1)s^{2p}t^1 \otimes s^{2p-1}t^2 \otimes \dots \\
&\qquad \otimes s^{p+2}t^{p-1} \otimes s^{p+1}t^p \\
&= \sum \lambda_{(1)}(t^{2p})\lambda_{(2p)}(s^1)s^{2p}t^1 \otimes \lambda_{(2)}(t^{2p-1})\lambda_{(2p-2)}(s^2)s^{2p-1}t^2 \otimes \dots \\
&\qquad \otimes \lambda_{(p-1)}(t^{p+2})\lambda_{(p+1)}(s^{p-1})s^{p+2}t^{p-1} \otimes \lambda_{(p)}(t^{p+1}s^p)s^{p+1}t^p \\
&= f_{R^\tau}(\lambda_{(1)})f_R(\lambda_{(2p)}) \otimes f_{R^\tau}(\lambda_{(2)})f_R(\lambda_{(2p-2)}) \otimes \dots \\
&\qquad \otimes f_{R^\tau}(\lambda_{(p-1)})f_R(\lambda_{(p+1)}) \otimes f_{R^\tau R}(\lambda_{(p)}) \\
&= \sum \Lambda_{(1)} \otimes \dots \otimes \Lambda_{(p)}.
\end{aligned}$$

□

The next step is to resolve the crossings where the upper circle c_u crosses itself. A typical crossing in the chain-mail is shown in Figure 23.

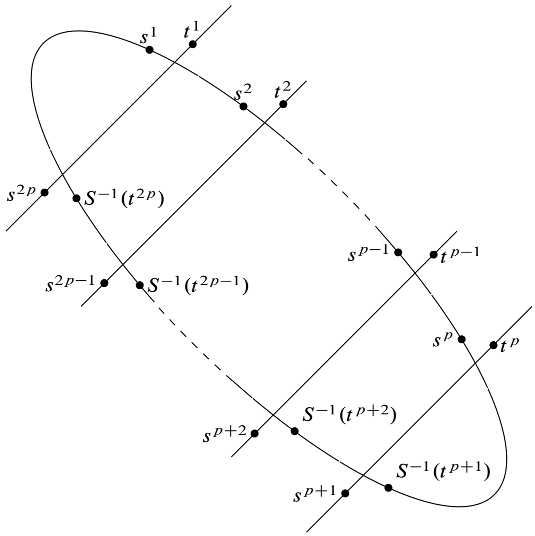


Figure 21. Immersion diagram of lower circle in chain-mail link.

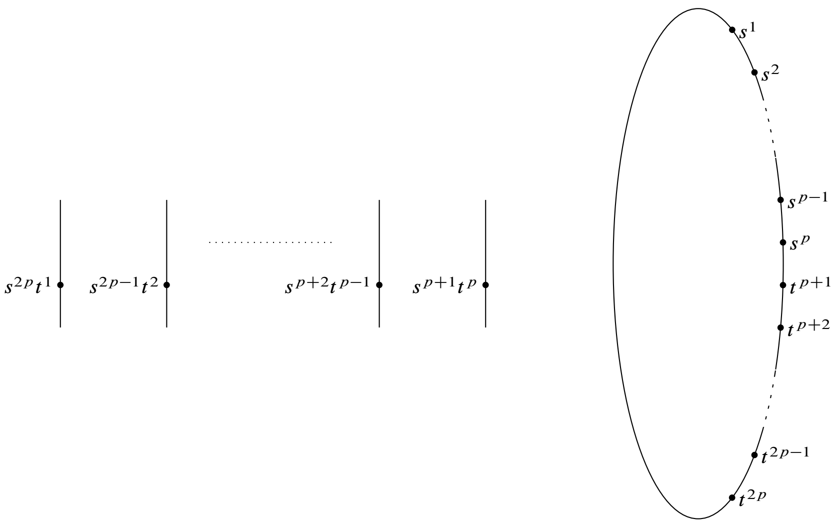
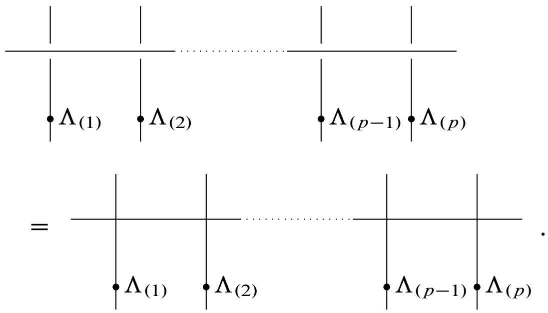
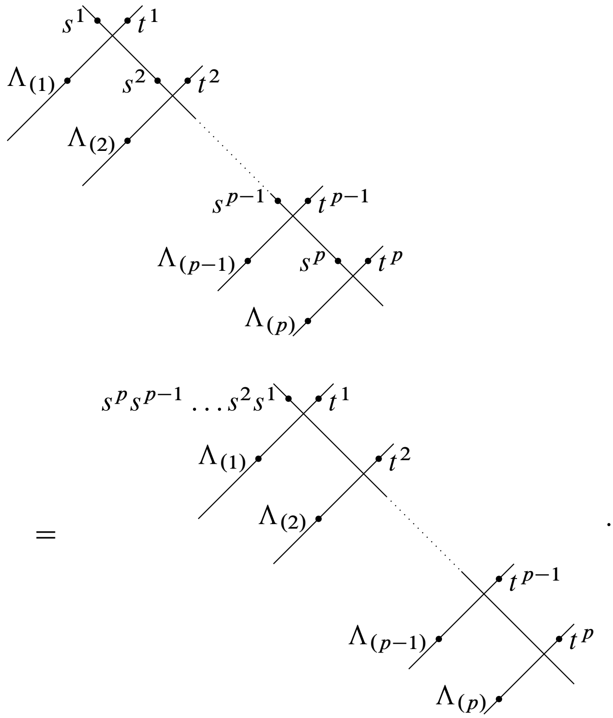


Figure 22. Immersion diagram of lower circle in chain-mail link.

Lemma 3.6. *We have:*



Proof. The corresponding immersed diagram is



From this immersed diagram, we obtain the tensor element

$$\sum (s^p s^{p-1} \dots s^2 s^1) \otimes \Lambda_{(1)} t^1 \otimes \Lambda_{(2)} t^2 \otimes \cdots \otimes \Lambda_{(p-1)} t^{p-1} \otimes \Lambda_{(p)} t^p$$
$$= \sum s \otimes \Lambda_{(1)} t^{(1)} \otimes \Lambda_{(2)} t^{(2)} \otimes \cdots \otimes \Lambda_{(p-1)} t^{(p-1)} \otimes \Lambda_{(p)} t^{(p)}$$
$$= \sum \varepsilon(t) s \otimes \Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(p-1)} \otimes \Lambda_{(p)}$$

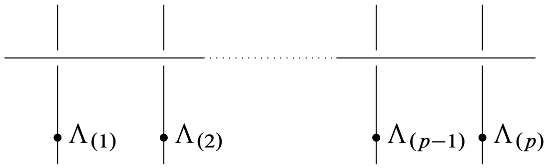


Figure 23. Typical crossing in upper circle.

$$= \sum 1 \otimes \Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(p-1)} \otimes \Lambda_{(p)}$$
$$= \sum \Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(p-1)} \otimes \Lambda_{(p)}.$$

Here we have used the property of the R-matrix that

$$(\varepsilon \otimes \mathrm{id})(R) = 1$$

and

$$(\mathrm{id} \otimes \Delta^{(p-1)})(R) = R_{1,p+1} \ldots R_{12}$$

which comes from $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$. □

The last step is to push all the labels $\Lambda_{(i)}$'s in Figure 24 to where $\Lambda_{(N_1)}$ is located and then do the evaluation by λ to get the Hennings invariants.

In the example of $L(2,1)$, we push all the labels to $\Lambda_{(1)}$, then

$$Z_{\mathrm{Henn}}(L(2,1)\#\overline{L(2,1)}) = \lambda(S^{-2}(\Lambda_{(2)})\Lambda_{(1)}G^2).$$

To compare with $Z_{\mathrm{Kup}}(L(2,1))$, we use

$$G^{-1}S^2(x) = xG^{-1}, \quad x \in H$$

and

$$\Lambda G^{-1} = \Lambda;$$

then obtain

$$\begin{aligned} Z_{\mathrm{Henn}}(L(2,1)\#\overline{L(2,1)}) &= \lambda(S^{-2}(\Lambda_{(2)})\Lambda_{(1)}G^2) \\ &= \lambda(S^{-2}(\Lambda_{(2)})G^{-1}\Lambda_{(1)}G^{-1}G^2) \\ &= \lambda(G^{-1}\Lambda_{(2)}\Lambda_{(1)}G) \\ &= \lambda(\Lambda_{(2)}\Lambda_{(1)}). \end{aligned}$$

That is equal to $Z_{\mathrm{Kup}}(L(2,1))$.

For $L(5,2)$, we push all the labels to $\Lambda_{(4)}$, then

$$Z_{\mathrm{Henn}}(L(5,2)\#\overline{L(5,2)}) = \lambda(S^{-4}(\Lambda_{(2)})S^{-4}(\Lambda_{(5)})S^{-2}(\Lambda_{(3)})\Lambda_{(1)}\Lambda_{(4)}G^4).$$

which is the same as $Z_{\mathrm{Kup}}(L(5,2))$.

The general case will be done according to whether q is odd or even.

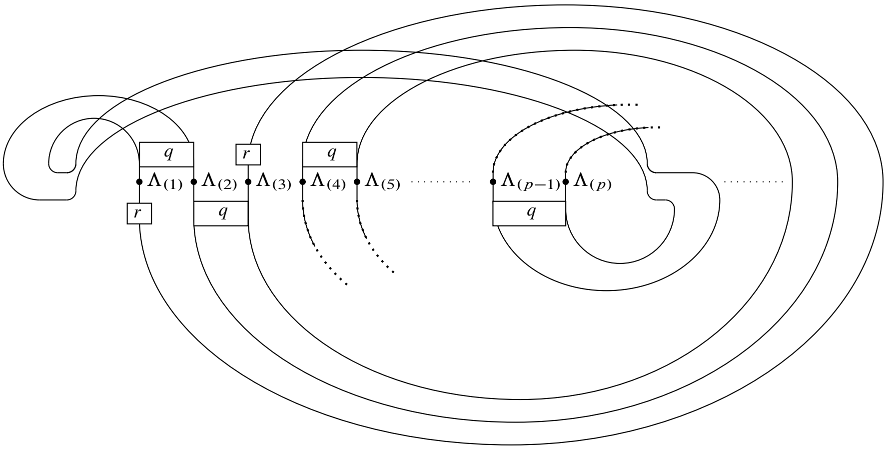


Figure 24. Chain-mail link decorated with cointegrals.

3.5.1. $Z_{\mathrm{Henn}}(L(p,q)\#\overline{L(p,q)}, H)$ when q is odd. In this case, we push all the labels $\Lambda_{(i)}$'s to $\Lambda_{(N_0)}$ along the upper circle and write down the following equality for $Z_{\mathrm{Henn}}(L(p,q)\#\overline{L(p,q)}, H)$:

$$\begin{aligned}
Z_{\mathrm{Henn}} &= \lambda(S^{-2p+2q}(\Lambda_p)\dots S^{-2k_q+2q}(\Lambda_{k_q})\dots\dots \\
&\qquad S^{-2k_3+6}(\Lambda_{k_3-1})\dots S^{-2k_2+4}(\Lambda_{k_2}) \\
&\qquad S^{-2k_2+4}(\Lambda_{k_2-1})\dots S^{-2}(\Lambda_{k_1+1})\Lambda_{k_1}G^{p-q+1}) \\
&= \lambda(S^{-2p+2q}(\Lambda_p)G^{-1}\dots S^{-2k_q+2q}(\Lambda_{k_q})G^{-1}\dots\dots \\
&\qquad S^{-2k_3+6}(\Lambda_{k_3-1})G^{-1}\dots \\
&\qquad S^{-2k_2+4}(\Lambda_{k_2})G^{-1}S^{-2k_2+4}(\Lambda_{k_2-1})G^{-1}\dots \\
&\qquad S^{-2}(\Lambda_{k_1+1})G^{-1}\Lambda_{k_1}G^{-1}G^{p-q+1}) \\
&= \lambda(S^{2q-2}(\Lambda_p)\dots S^{2q-2}(\Lambda_{k_q})\dots\dots S^{2}(\Lambda_{k_3-1})\dots S^{2}(\Lambda_{k_2}) \\
&\qquad \Lambda_{k_2-1}\dots\Lambda_{k_1+1}\Lambda_{k_1}G^{-q+1}).
\end{aligned}$$

Here we have used the fact that G is group-like and

$$G^{-1}S^2(x) = xG^{-1}$$

and

$$\Lambda G^{-1} = \Lambda.$$

Note that $G^2 = g$. Hence we obtain that when q is odd,

$$Z_{\mathrm{Kup}}(L(p,q), f, H) = Z_{\mathrm{Henn}}(L(p,q)\#\overline{L(p,q)}, H).$$

3.5.2. $Z_{\mathrm{Henn}}(L(p,q)\#\overline{L(p,q)}, H)$ when q is even. Now, we push all the labels $\Lambda_{(i)}$'s to Λ_1 along the upper circle and obtain

$$\begin{aligned} Z_{\mathrm{Henn}} = \lambda(&S^{-2p+2q+2}(\Lambda_p)S^{-2p+2q+4}(\Lambda_{p-1})\dots S^{-2k_q+2q+2}(\Lambda_{k_q})\\ &S^{-2k_q+2q+2}(\Lambda_{k_q-1})S^{-2k_q+2q+4}(\Lambda_{k_q-2})\dots\\ &S^{-2k_{q-1}+2q}(\Lambda_{k_{q-1}-1})\dots\dots\\ &S^{-2k_1+4}(\Lambda_{k_1})S^{-2k_1+4}(\Lambda_{k_1-1})\dots S^{-2}(\Lambda_2)\Lambda_1 G^{p-q+1}). \end{aligned}$$

Thus $Z_{\mathrm{Kup}}(L(p,q), f, H) = Z_{\mathrm{Henn}}(L(p,q)\#\overline{L(p,q)}, H)$ when q is even.

3.6. Remarks on the general case. It is natural to conjecture that the relation $|Z_{\mathrm{Kup}}(M, f, H)| = |Z_{\mathrm{Henn}}(M, H)|^2$ always holds for any closed oriented 3-manifold M and factorizable finite dimensional ribbon Hopf algebra H. For a general closed oriented 3-manifold M, inspired by the result in [3], one strategy to prove the conjecture is to divide the problem into two cases:

(1) if $H_1(M, \mathbb{Q}) \neq 0$, then both invariants of M are 0;

(2) if $H_1(M, \mathbb{Q}) = 0$, then a similar comparison can be carried out.

Unfortunately, the choice of a suitable framing for the Kuperberg invariant which allows a direct comparison with the Hennings invariant is extremely hard to come by. Even for the lens spaces, we are lucky to find the suitable framings. Some other choices that we tried led to expressions that were hard to compare the two invariants. Ideas on fermionic TQFTs in [6] might be relevant for a conceptual approach to the conjecture.

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