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Navigating moduli space with complex twists

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Abstract. We discuss a common framework for studying twists of Riemann surfaces coming from earthquakes, Teichmüller theory and Schiffer variations, and use it to analyze geodesics in the moduli space of isoperiodic 1-forms.

1. Introduction

The moduli space of Riemann surfaces can be navigated using Fenchel–Nielsen twists, earthquakes, grafting, the Teichmüller geodesic flow, Schiffer variations and various other geometric constructions.

In this paper we give a common framework for all these operations, and apply it to study the moduli space of holomorphic 1-forms, $\Omega\mathcal{M}_g \to \mathcal{M}_g$. This moduli space has a flat metric along the leaves of its *absolute period foliation* $\mathcal{A}$. By relating Schiffer variations to complex twists, we show that geodesics along the leaves of $\mathcal{A}$ behave well geometrically: they are either complete, or they terminate via a collision of zeros or the creation of a node.

Twists. Let X be a Riemann surface. We begin in §2 by defining the *complex twist* $\delta(C, v)$, a tangent vector to moduli space at X specified by a holomorphic vector field v along an oriented simple loop $C \subset X$. Cotangent vectors to moduli space are specified by integrable holomorphic quadratic differentials $q \in Q(X)$, and we find:

Theorem 1.1. *The natural pairing between a complex twist and a quadratic differential is given by*

$$\langle \delta(C, v), q\rangle = \frac{i}{2}\int_C qv.$$

As a special case we recover Wolpert's formula for the Fenchel–Nielsen twist [Wol1, Lemma 4.2]. More generally, for an earthquake along a lamination λ with transverse measure m we find

$$\langle \mathrm{tw}(\lambda), q\rangle = \frac{i}{2}\int_\lambda qv^2\,|ds|\,dm.$$

Here v is a unit vector field tangent to λ and ds denotes arclength in the hyperbolic metric.

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As another example, let C_p be a small loop around $p \in X$ and choose local coordinates with $z(p) = 0$. We will see the Schiffer variation at p is given by

$$\mu_p = \delta(C, -1/z), \tag{1.1}$$

which implies Gardiner's formula $\langle \mu_p, q \rangle = \pi q(0)$ for all $q \in Q(X)$ (see [Ga, §5]). Here we have used the preferred local coordinate z to write $q = q(z)\, dz^2$.

Moduli of 1-forms. In §3 we turn to the moduli space of holomorphic 1-forms,

$$\pi : \Omega\mathcal{M}_g \to \mathcal{M}_g,$$

an orbifold bundle whose fiber over X consists of the nonzero holomorphic 1-forms $\omega \in \Omega(X)$ modulo the action of the finite group $\mathrm{Aut}(X)$. This space has many interesting features; for example, it carries a natural action of $\mathrm{SL}_2(\mathbb{R})$ with rich topological and measurable dynamics, which in turn are related to polygon billiards (see e.g. [D]).

The moduli space of 1-forms is partitioned into strata $\Omega\mathcal{M}_g(m_i)$ according to the multiplicities of the zeros of ω. The tangent space to a stratum at (X, ω) is naturally identified with the complex cohomology group $H^1(X, Z(\omega))$, where $Z(\omega)$ is the zero set of ω. We begin by computing the derivative of the projection

$$\pi : \Omega\mathcal{M}_g(m_i) \to \mathcal{M}_g.$$

Theorem 1.2. *The projection of a tangent vector $\phi \in H^1(X, Z(\omega))$ to moduli space is given by the complex twist*

$$D\pi(\phi) = \delta(C, 1/\omega),$$

where $C \in H_1(X - Z(\omega))$ is the unique cycle satisfying $\phi(E) = C \cdot E$ for all $E \in H_1(X, Z(\omega))$.

Here $\delta(C, 1/\omega)$ is extended by linearity to complex homology classes.

Twists of 1-forms. Let $\mathrm{tw}(C)$ denote the tangent vector to $\Omega\mathcal{M}_g$ represented by a cycle $[C] \in H_1(X - Z(\omega))$, as above. When C is a simple loop, $\mathrm{tw}(C)$ is given geometrically by cutting (X, ω) open along C, shifting horizontally at unit speed in coordinates where $\omega = dz$, and regluing. If C happens to be a horizontal geodesic for the metric $|\omega|$, then $\mathrm{tw}(C)$ is simply a flat version of the Fenchel–Nielsen twist.

Foliations. The absolute periods of ω define a local submersion of orbifolds

$$\Omega\mathcal{M}_g \supset U \to H^1(X)/\mathrm{Aut}(X),$$

whose fibers are the leaves of the *absolute period foliation* $\mathcal{A}$. This foliation is transverse to the fibers of π and to the orbits of $\mathrm{SL}_2(\mathbb{R})$. Its restriction to a stratum will be denoted by $\mathcal{A}(m_i)$.

Metrics along the leaves. It is easy to see that the tangent space to $\mathcal{A}(m_i)$ at (X, ω) is generated by twists around small loops C_p encircling the zeros of ω, subject to the relation $\sum_{p \in Z(\omega)} \mathrm{tw}(C_p) = 0$. A useful norm on these *relative period deformations* is given by

$$\left\| \sum a_p \, \mathrm{tw}(C_p) \right\|_R = \max |a_p - a_q|. \tag{1.2}$$

Schiffer variations. By (1.1), a twist around C_p comes from a Schiffer variation. If $\omega \in \Omega(X)$ has a simple zero at p, this variation can be carried out geometrically by slitting (X, ω) open along the horizontal and vertical leaves through p, then folding and regluing to obtain a new 1-form (X', ω') (see Figure 4 in §3). This folding operation is very useful in the study of 1-forms, since it patently preserves not only the absolute periods, but also the geometry of (X, ω) away from the slit. In this way it can be compared to a Fenchel–Nielsen twist along a geodesic C, which preserves the hyperbolic metric on X away from C.

The Schiffer folding operation can be carried out simultaneously at different zeros, so long as the slits remain disjoint, which shows:

Theorem 1.3. *The leaf of $\mathcal{A}(m_i)$ through (X, ω) contains an immersed ball of radius r, where $2r$ is the length of the shortest saddle connection on (X, ω).*

Completeness. Now let $\gamma : [0, T) \to \Omega\mathcal{M}_g$ be a smooth path along a leaf of $\mathcal{A}(m_i)$ with $\|\gamma'(t)\|_R \leq 1$. We would like to show that $\gamma(t) = (X_t, \omega_t)$ converges, as $t \to T$, to a suitable form (Y, η) on a stable curve $Y \in \overline{\mathcal{M}_g}$.

This is not quite true; in fact, in §4 we will show:

The Riemann surface $[X_t]$ can oscillate and fail to have a limit in $\overline{\mathcal{M}_g}$ as $t \to T$.

However, this oscillation is eliminated if we form a collapsed moduli space $\Omega\overline{\mathcal{M}_g}/\sim$ by forgetting the components of (Y, η) where η vanishes identically. We then find:

Theorem 1.4. *The limit $(Y, \eta) = \lim_{t \to T} (X_t, \omega_t)$ exists in $\Omega\overline{\mathcal{M}_g}/\sim$, and there is no loss of mass: $\int_{X_t} |\omega_t|^2 \to \int_Y |\eta|^2$.*

Now suppose γ is also a *relative period geodesic*, meaning its direction $\gamma'(t) \in H^1(X, Z(\omega))$ is locally constant. Then γ can be continued past $t = T$ unless $Y \notin \mathcal{M}_g$ or (Y, η) is in a different stratum. This shows:

Corollary 1.5. *A relative period geodesic in a stratum is either complete, or it terminates via a collision of zeros or the creation of a node.*

The proof of Theorem 1.4 uses universal bounds for cylinders, given in the Appendix, to show no mass of ω_t can escape to a node of Y.

Application: Hilbert modular surfaces. The absolute period foliation and the natural metric along its leaves have been studied, for eigenforms of genus two, in [Mc3] and [Ba].

Let $\mathcal{O}_D$ denote the real quadratic order of discriminant $D > 0$. Let

$$\Omega\overline{E_D} \subset \Omega\overline{\mathcal{M}_2}$$

denote the space of finite mass stable eigenforms (Y, η) for real multiplication by $\mathcal{O}_D$. This locus is saturated by the leaves of $\mathcal{A}$ (cf. [Mc2, Cor. 5.6]). The mass condition implies Y is either a smooth curve of genus 2 or a pair of curves of genus 1 joined at a node. In either case, $\mathrm{Jac}(Y)$ is compact.

Every principally polarized Abelian variety with real multiplication by $\mathcal{O}_D$ arises as such a Jacobian, so $\Omega\overline{E_D}$ forms a $\mathbb{C}^*$ bundle over the Hilbert modular surface

$$X_D = (\mathbb{H} \times \mathbb{H})/\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee).$$

The points of X_D correspond to eigenforms $(Y, [\eta])$ up to scale. On the universal cover of X_D, we can normalize at (s, s') so the absolute periods of η are given by $\mathcal{O}_D \oplus \mathcal{O}_D^\vee s$. The result is a map

$$p : \mathbb{H} \times \mathbb{H} \to \Omega\overline{E_D}$$

commuting with projection to X_D.

The absolute periods of $(Y, \eta) = p(s, s')$ are constant along any slice $\mathbb{H}_s = \{s\} \times \mathbb{H}$. The remaining relative period, given locally by

$$\rho(Y, \eta) = \int_{p_1}^{p_2} \eta$$

where $Z(\eta) = \{p_1, p_2\}$, determines a globally well-defined *meromorphic* quadratic differential $q = (d\rho)^2$ on $\mathbb{H}_s$. This differential has simple poles where Y is singular, and simple zeros where $p_1 = p_2$ (see [Mc3, §8]).

Using Theorem 1.4, we can now show:

Theorem 1.6. *Any slice $\mathbb{H}_s$ of the universal cover of X_D is complete in the metric $|q|$ determined by the relative period differential.*

Proof. Let $\gamma : [0, T) \to \mathbb{H}_s$ be a path of finite length in the metric $|q|$. Consider the diagram

$$\begin{tikzcd}
{[0,T)} \arrow[r, "\gamma"] & \mathbb{H}_s \arrow[r, hook] & \mathbb{H} \times \mathbb{H} \arrow[r, "p"] \arrow[dr] & \Omega\overline{E_D} \arrow[d] \\
 & & & X_D.
\end{tikzcd}$$

We wish to show $\lim_{t \to T} \gamma(t)$ exists in $\mathbb{H}_s$. To this end, first perturb γ slightly so it avoids the zeros and poles of q. Then $p(\gamma(t))$ takes values in $\Omega E_D(1, 1)$. Since s is fixed, its image lies on a single leaf of $\mathcal{A}(1, 1)$. Since $\int_\gamma |q|$ is finite, this path also has finite length in the relative period metric (1.2). So by Theorem 1.4, $p(\gamma(t))$ converges as $t \to T$ to a point $[(Y, \eta)] \in \Omega\overline{\mathcal{M}_2}/\sim$. Since we have not changed the absolute periods, we have $(Y, \eta) \in \Omega\overline{E_D}$. But an eigenform always has full support, so in fact $p(\gamma(t)) \to (Y, \eta) \in \Omega\overline{E_D}$. Projecting to X_D and then lifting to its universal cover, we conclude that the corresponding paths $[0, T) \to X_D$ and $[0, T) \to \mathbb{H} \times \mathbb{H}$ also converge. But $\mathbb{H}_s$ is closed in $\mathbb{H} \times \mathbb{H}$, so the original path $\gamma : [0, T) \to \mathbb{H}_s$ converges as well. □

Notes and references. A general completeness result for the leaves of $\mathcal{A}(m_i)$ (with suitable stable forms adjoined) is established in a sequel to this paper [Mc4], which also treats the case of genus two in detail. The question of completeness was raised by Hooper, Smillie and Weiss.

For useful perspectives on the moduli spaces we consider here and the degeneration of Riemann surfaces, see e.g. [KZ], [Mas], [Mo], [Wol3] and [ACG].

Notation. The notation $A \asymp B$ means A/B and B/A are both bounded above by an implicit constant C. Absolute and relative cohomology groups with coefficients in $\mathbb{C}$ will be denoted by $H^*(X)$ and $H^*(X, Z)$ respectively; sheaf cohomology groups will be denoted $H^*(X; \mathcal{F})$.

2. Complex twists

In this section we define the complex twist deformation $\delta(C, v)$ and show:

Theorem 2.1. *For any holomorphic quadratic differential $q \in Q(X)$ we have*

$$\langle \delta(C, v), q \rangle = \frac{i}{2} \int_C qv. \tag{2.1}$$

Note that qv is naturally a holomorphic 1-form, and the integral above gives its period around C. Thus $\delta(C, v)$ only depends on the homology class of C in the domain of v.

We will show several other deformations are special cases of the complex twist, and obtain estimates for their size in the Teichmüller metric. For related discussions, see [Wol2] and [SS, §7.8].

Teichmüller theory. Let X be any Riemann surface. Let $Q(X)$ and $M(X)$ denote the spaces of holomorphic quadratic differentials q and measurable Beltrami differentials μ on X, respectively. We require that $\|q\|_1 = \int_X |q| < \infty$ and $\|\mu\|_\infty < \infty$; then we have a natural pairing

$$\langle \mu, q \rangle = \int_X \mu q,$$

making $M(X)/Q(X)^\perp$ into the dual of $Q(X)$.

The Teichmüller space of X parameterizes marked Riemann surfaces quasiconformally equivalent to X. Its tangent space at X is naturally identified with $M(X)/Q(X)^\perp$, and the quotient norm

$$\|\mu\|_T = \sup\{|\langle \mu, q \rangle| : q \in Q(X), \|q\|_1 = 1\} \tag{2.2}$$

gives the Teichmüller metric.

Definition of a complex twist. Let C be a smooth closed loop on a Riemann surface X and v a holomorphic vector field defined near C. Let Θ be the sheaf of holomorphic vector fields on X. We will define the deformation $\delta(C, v)$ as a (suitably bounded) element of the Čech cohomology group $H^1(X; \Theta)$.

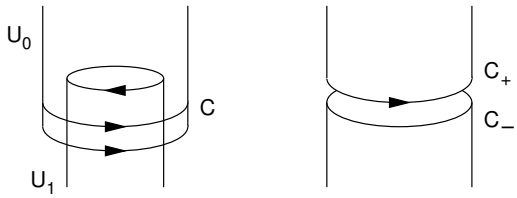

Fig. 1. Orientations.

To do this, choose an open covering (U_i) of X such that $A = U_0 \cap U_1$ is an annular neighborhood of C on which v is defined, and $U_0 \cap U_1 \cap U_j = \emptyset$ for $j > 1$. Assuming ∂U_i is smooth and with the induced orientations, we have

$$[A \cap \partial U_0] = -[A \cap \partial U_1] = [C] \quad \text{in } H_1(A). \tag{2.3}$$

(See Figure 1 (left).) We then define

$$[v_{ij}] \in H^1(X; \Theta)$$

by $v_{01} = v \in \Theta(A)$, $v_{10} = -v_{01}$, and $v_{ij} = 0$ in all other cases. Note that $\delta(C, v)$ is a complex linear function of $v \in \Theta(A)$, and $\delta(-C, -v) = \delta(C, v)$.

The cocycle $[v_{ij}]$ determines a family of Riemann surfaces X_t for small t by gluing $z \in U_i$ to $z + tv_{ij} + O(t^2) \in U_j$. Hence it determines a tangent vector

$$\delta(C, v) = \left.\frac{dX_t}{dt}\right|_{t=0}$$

to Teichmüller space at X. (In fact, when X is compact, we have a natural isomorphism $H^1(X; \Theta) \cong M(X)/Q(X)^{\perp}$.)

Proof of Theorem 2.1. Let ϕ be a compactly supported smooth function on $\overline{U_1}$, equal to one on A and vanishing outside the domain of v. Then $w_1 = \phi v$ extended by zero is a smooth vector field on U_1. Defining $w_i = 0$ for $i \neq 1$, we have

$$v_{ij} = w_j - w_i \tag{2.4}$$

for all i, j. In other words, $[v_{ij}]$ is the coboundary of the 0-cycle $[w_i]$. Geometrically, this means for small t we have a smooth map $f_t : X_0 \to X_t$ given by $f_t(z) = z + tw_i + O(t^2)$ on U_i. Condition (2.4) ensures that the diagram

$$\begin{tikzcd} U_i \arrow[rr, "z+tv_{ij}"] & & U_j \\ & U_i \cap U_j \arrow[ul, "z+tw_i"] \arrow[ur, "z+tw_j"'] & \end{tikzcd}$$

commutes to first order in t.

Since $w_j - w_i$ is holomorphic, there is a unique Beltrami differential $\mu \in M(X)$ such that $\mu = \overline{\partial} w_i$ on U_i. This μ represents the tangent vector $\delta(C, v)$ to X in Teichmüller space. To complete the proof, we now compute:

$$\begin{aligned}\langle \delta(C, v), q\rangle &= \int_X q\mu = \int_{U_1} q\overline{\partial}(\phi v) = \frac{i}{2}\int_{U_1} q(z)v(z)\, dz \wedge d\phi \\ &= -\frac{i}{2}\int_{U_1} d(\phi q v) = -\frac{i}{2}\int_{\partial U_1} qv = \frac{i}{2}\int_C qv.\end{aligned}$$

The factor $i/2$ arises from replacing $|dz|^2$ with $(i/2)\, dz \wedge d\overline{z}$, and the last change of sign comes from (2.3). □

Real twisting. Assume v is tangent to C and nowhere vanishing. In this case C is a real analytic curve, and we can use v to twist along C for any time $t \in \mathbb{R}$ to obtain a new Riemann surface X_t.

Indeed, the tangency assumption implies there is a real-analytic flow $f_t : C \to C$ satisfying $f_0(z) = z$ and $df_t/dt = v$. Cutting X open along C, we obtain a pair of curves $C_+ \cong C \cong C_-$, where the induced orientation of C_+ agrees with that of C (see Figure 1 (right)). To obtain X_t, we simply reglue using the map $f_t : C_+ \to C_-$. The direction of regluing is chosen so that

$$\delta(C, v) = \left.\frac{dX_t}{dt}\right|_{t=0}. \tag{2.5}$$

Fenchel–Nielsen. Now suppose C is an *unoriented* closed geodesic on a hyperbolic Riemann surface C. Let v be a continuous vector field of length 1 tangent to C, and orient C in the same direction as v. Then v locally extends to a holomorphic vector field, and we obtain the *Fenchel–Nielsen twist deformation*

$$\mathrm{tw}(C) = \delta(C, v).$$

It gives a right twist at unit speed along C that can be defined for all time. By the results above, we have

$$\langle \mathrm{tw}(C), q\rangle = \frac{i}{2}\int_C qv^2\, |ds|. \tag{2.6}$$

Here qv^2 is naturally a function, and $|ds|$ denotes arclength along C. Compare [Wol1, Lemma 4.2].

Earthquakes. Since the formula above no longer involves the orientation of C, we can pass to the limit and obtain a similar result for the tangent vector defined by a *right earthquake* along a measured lamination λ on X [Ker], [Th], [GL, §18]. Namely, we have

$$\langle \mathrm{tw}(\lambda), q\rangle = \frac{i}{2}\int_\lambda qv^2\, |ds|\, dm, \tag{2.7}$$

where dm denotes the transverse measure to λ, and v is locally a unit vector field tangent to λ.

Grafting. One can also cut open along C and insert a Euclidean annulus of width t. The resulting *grafting deformation* is given by $\mathrm{gr}(C) = \delta(C, iv) = i\delta(C, v)$. The combination of twisting and grafting yields *complex earthquakes disks* in Teichmüller space [Mc1].

Twisting quadratic differentials. A quadratic differential q_0 determines a pair of vertical and horizontal measured foliations on X. In a local chart where $q_0 = dz^2$, their leaves are vertical and horizontal lines.

Let $\mathrm{tw}(C, q_0)$ denote a unit speed right twist on a smooth closed leaf of the horizontal foliation of q_0, with respect to the metric $|q_0|^{1/2}$. We then have

$$\langle \mathrm{tw}(C, q_0), q\rangle = \delta(C, v) = \frac{i}{2}\int_C \frac{q}{q_0}|q_0|^{1/2}.$$

This equation is parallel to (2.6) with $v^2 = 1/q_0$.

As in the case of an earthquake, we can also consider twisting along the horizontal measured foliation $\mathcal{F}$. This is equivalent to the shearing deformation $f_t : z \mapsto z + t\,\mathrm{Im}(z)$ in local coordinates where $q_0 = dz^2$, and we find

$$\langle \mathrm{tw}(\mathcal{F}, q_0), q\rangle = \frac{i}{2}\int_X \frac{q}{q_0}|q_0|.$$

This is analogous to the earthquake formula (2.7), with $|ds|dm = |dq|$. As a consistency check, note that the Beltrami coefficient

$$\mu = \overline{\partial}(df/dt) = \overline{\partial}(\mathrm{Im}(z)d/dz) = (i/2)d\overline{z}/dz = (i/2)|q_0|/q_0$$

represents the same deformation as $\mathrm{tw}(\mathcal{F}, q_0)$, and that the integral above coincides with the usual pairing between μ and q.

For vertical closed leaves and for the vertical foliation, the corresponding formulas acquire a minus sign.

The Teichmüller flow. A quadratic differential q_0 also determines a unit speed Teichmüller geodesic X_t. The extremal map $f_t : X_0 \to X_t$ has the form

$$f_t(x + iy) = e^t x + ie^{-t} y$$

in local coordinates where $q_0 = dz^2$, $z = x + iy$, and thus $[\mu] = dX_t/dt|_{t=0} = |q_0|/q_0$.

Note that $[\mu] = -2i\,\mathrm{tw}(\mathcal{F}, q_0)$. Thus X_t is obtained by grafting along the vertical foliation of q_0.

The Schiffer variation. We now turn to a folding deformation supported near a single point $p \in X$, introduced by Schiffer [Sch], [Ah, §7], [IT, §A].

Let ω be a locally defined holomorphic 1-form on X with a zero at p. Choose local coordinates so that $z(p) = 0$ and $\omega = (z/2)dz$. For small $t \geq 0$, let $A_t = i[-2u, 2u] \subset X$ denote a vertical arc in these coordinates with $u^2 = t$. Then $\int_{A_t} |\omega| = 2t$. Slit X open along A_t, and fold each of the two resulting arcs so that z is identified with $-z$. (For example, $2iu$ is glued to $-2iu$; see Figure 2.)

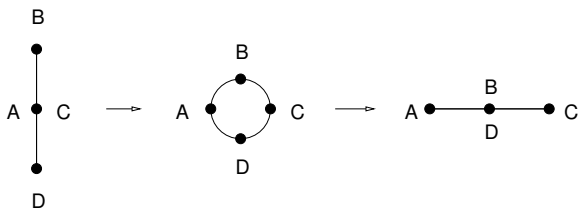

Fig. 2. The Schiffer variation.

The result is a new Riemann surface X_t with a horizontal arc B_t, and a natural holomorphic map $f_t : X - A_t \to X_t - B_t$. More concretely, in local coordinates on X_t we can write $f_t(z) = g_+ \circ g_-^{-1}(z)$, where

$$g_\pm(z) = z \pm \frac{t}{z}.$$

The images of the circle $|z| = u$ under g_- and g_+ are then A_t and $B_t = [-2u, 2u]$ respectively. (The normalization $\omega = (z/2)dz$ was chosen to put $g_\pm(z)$ into this simple form.)

Since $g_-^*(z\,dz) = g_+^*(z\,dz)$, there is a natural holomorphic 1-form ω_t defined near B_t such that

$$f_t^*\omega_t = \omega. \tag{2.8}$$

If ω is globally defined, so is ω_t.

The *Schiffer variation* of X is defined by $\mathrm{Sch}(\omega, p) = dX_t/dt|_{t=0}$.

Theorem 2.2. *The Schiffer variation satisfies* $\mathrm{Sch}(\omega, p) = \delta(C, -1/\omega)$ *for any small, positively oriented loop around* p. *Thus*

$$\langle \mathrm{Sch}(\omega, p), q\rangle = \pi\,\mathrm{Res}_p(q/\omega) \tag{2.9}$$

for any $q \in Q(X)$.

Proof. Let $X = U_0 \cup U_1$ be an open cover such that U_0 is a small disk containing p and C, and $U_0 \cap U_1$ is an annular neighborhood of C disjoint from p. Then X_t is obtained by gluing U_0 to U_1 with f_t^{-1}. Since $f_t(z) = z + 2t/z + O(t^2)$, the corresponding deformation is given by $\delta(C, v)$ where $v = v_{01} = -(2/z)(d/dz) = -1/\omega$. Applying Theorem 2.1 and the residue theorem, we obtain equation (2.9). □

The traditional Schiffer variation. It is also common to define the Schiffer variation by simply collapsing the circle $|z| = t$ to a horizontal slit to obtain X_t. This definition gives the variation

$$\mu_p = \left.\frac{dX_t}{dt}\right|_{t=0} = \mathrm{Sch}(z\,dz, p) = \frac{1}{2}\,\mathrm{Sch}(\omega, p).$$

If we write q locally near p as $q(z)\,dz^2$, then $\langle \mu_p, q\rangle = \pi q(0)$ by (2.9). A similar calculation appears in [Ga, §5].

The definition we have chosen, which begins and ends with slits, is better adapted to the geometry of holomorphic 1-forms, as we will see in §3.

The size of deformations. We conclude with some estimates on the size of folding and twisting deformations, using (2.2). The first estimate shows the Schiffer variation at the center of a large disk is small.

Theorem 2.3. *Suppose $\omega = z\,dz$ in a local chart $U \cong \Delta(r)$ on X, and $z(p) = 0$. Then the Schiffer variation satisfies*

$$\|\mathrm{Sch}(\omega, p)\|_T \leq 1/r^2.$$

Proof. For any unit norm $q \in Q(X)$ given locally by $q = q(z)\,dz^2$, we have

$$|\langle \mathrm{Sch}(\omega, p), q\rangle| = \pi|q(0)| \leq \pi(\pi r^2)^{-1} \int_{\Delta(r)} |q| \leq r^{-2} \int_X |q| = r^{-2}. \qquad \square$$

To make an estimate for twists, suppose X contains an annulus given in local coordinates by

$$A = \{z : 0 < \mathrm{Im}(z) < h\}/\langle z \mapsto z + c\rangle.$$

Then A is a right cylinder of height h and circumference c in the metric $|dz|$; hence $\mathrm{mod}(A) = h/c$. Let $C \subset A$ be a horizontal closed geodesic, oriented so $L(C) = \int_C dz = c$, and let $v = d/dz$.

Theorem 2.4. *The unit speed twist about the core curve C of A satisfies*

$$\|\delta(C, v)\|_T \leq \frac{1}{2\,\mathrm{mod}(A)L(C)}.$$

Proof. Locally writing $q = q(z)\,dz^2$, we have

$$|\langle \delta(C, v), q\rangle| = \frac{1}{h}\int_0^h \frac{1}{2}\left|\int_0^c q(x+iy)\,dx\right| dy \leq (2h)^{-1}\int_A |q| \leq (2h)^{-1}\|q\|_1. \qquad \square$$

Using the collar lemma in hyperbolic geometry, we obtain:

Corollary 2.5. *The Fenchel–Nielsen twist around a geodesic C satisfies $\|\mathrm{tw}(C)\|_T = O(1)$ when C is short, and $\|\mathrm{tw}(C)\|_T = O(e^{L(C)/2})$ when C is long.*

These estimates can also be demonstrated by constructing explicit shearing deformations supported in a collar.

3. Navigating the space of 1-forms

In this section we discuss the moduli space $\Omega\mathcal{M}_g$ of holomorphic 1-forms, its strata and its foliations, and the use of Schiffer variations to control the geometry of a 1-form as its relative periods vary. For more on the space $\Omega\mathcal{M}_g$, see e.g. [KZ], [Mas].

Moduli spaces, strata and periods. Let $\Omega(X) \cong H^{1,0}(X)$ denote the g-dimensional vector space of holomorphic 1-forms on X. Assume $\omega \neq 0$, and let $Z(\omega) \subset X$ denote its zero set.

The form ω determines a natural flat metric $|\omega|$ on X with singularities of negative curvature at its zeros. Geodesics for this metric are straight lines in local coordinates where $\omega = dz$; they may bend at the zeros.

The *relative periods* of ω are recorded by the cohomology class

$$[\omega] \in H^1(X, Z(\omega)) = H_1(X, Z(\omega))^*$$

defined on integral 1-cycles by $\langle[\omega], E\rangle = \int_E \omega$.

The set of pairs (X, ω) with $\omega \neq 0$ forms the moduli space of 1-forms,

$$\Omega\mathcal{M}_g \to \mathcal{M}_g.$$

This orbifold is a union of strata $\Omega\mathcal{M}_g(m_1, \ldots, m_s)$ labeled by partitions of $2g - 2$. A form (X, ω) lies in the given stratum iff ω has s distinct zeros with multiplicities $m_1, \ldots, m_s$.

The cohomology groups $H^1(X, Z(\omega))$ form a flat bundle over any stratum. Thus we have a locally defined *period map*

$$\pi : \Omega\mathcal{M}_g(m_i) \to H^1(X_0, Z(\omega_0))/\mathrm{Aut}(X_0, \omega_0)$$

given by $\pi(X, \omega) = [\omega] \in H^1(X, Z(\omega)) \cong H^1(X_0, Z(\omega_0))$. By [V] and [MS, Lemma 1.1], the period map is a holomorphic local homeomorphism of orbifolds.

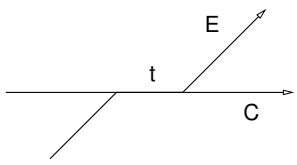

Fig. 3. A right twist on C adds t to the period of E.

Twists. In a local orbifold chart, the tangent space to the stratum of $\Omega\mathcal{M}_g$ through (X, ω) can be naturally identified with $H_1(X, Z(\omega))^*$. The orbifold structure is recorded by the action of the finite group $\mathrm{Aut}(X, \omega)$. Similarly, the tangent space to $\mathcal{M}_g$ at X in an orbifold chart is given by $H^1(X; \Theta)$, equipped with the action of $\mathrm{Aut}(X)$. The projection $\Omega\mathcal{M}_g \to \mathcal{M}_g$ is a smooth map of orbifolds, compatible with the inclusion $\mathrm{Aut}(X, \omega) \subset \mathrm{Aut}(X)$.

Any smooth, simple loop C in $X - Z(\omega)$ determines an element in the dual of $H_1(X, Z(\omega))$ by

$$\langle \mathrm{tw}(C), E\rangle = C \cdot E;$$

thus C also determines a tangent vector $\mathrm{tw}(C)$ to $\Omega\mathcal{M}_g$ at (X, ω), in the sense of orbifolds. The vector $\mathrm{tw}(C)$ depends only on the homology class $[C] \in H_1(X - Z(\omega))$.

For a more geometric picture, consider the case where C is a closed horizontal geodesic on $(X, |\omega|)$. Then we can cut X open along C, twist distance t to the right and reglue to obtain a new holomorphic form (X_t, ω_t). The periods of the twisted form ω_t are given by

$$\int_E \omega_t = \int_E \omega_0 + tC \cdot E, \tag{3.1}$$

as can be seen from Figure 3. A similar twist can be carried out along an oriented closed geodesic of slope θ, and we find:

Twisting to the right along C defines an integral curve (X_t, ω_t) of the vector field $e^{i\theta}\,\mathrm{tw}(C)$ for all $t \in \mathbb{R}$.

To treat the case of general C, let $\Theta^\omega \subset \Theta$ denote the sheaf of *translation vector fields* on X. By definition, a section of Θ^ω is a holomorphic vector field v satisfying $d(\omega(v)) = 0$. This condition means that $v = a(d/dz)$ in local coordinates where $\omega = dz$, and $v = 0$ at the zeros of ω. The stalks of Θ^ω are 1-dimensional except over the zeros of ω, where they are 0-dimensional.

There is a unique translation vector field v defined near C satisfying $\omega(v) = 1$. Thus C and ω determine a cocycle

$$\Delta(C, v) \in H^1(X; \Theta^\omega),$$

lifting $\delta(C, v) \in H^1(X; \Theta)$ to a deformation of the translation structure defined by ω. This cocycle also represents $\mathrm{tw}(C)$, which shows:

Theorem 3.1. *The projection $\Omega\mathcal{M}_g \to \mathcal{M}_g$ sends the tangent vector $\mathrm{tw}(C) = \Delta(C, 1/\omega)$ to $\delta(C, 1/\omega)$.*

Corollary 3.2. *The derivative of the projection map $\pi : \Omega\mathcal{M}_g(m_i) \to \mathcal{M}_g$ satisfies*

$$\langle \pi_*(\mathrm{tw}(C)), q\rangle = \frac{i}{2}\int_C q\omega$$

for any $[C] \in H_1(X - Z(\omega))$ and $q \in Q(X)$.

This follows from Theorem 2.1.

Remark. There is a natural isomorphism

$$H^1(X; \Theta^\omega) \cong H^1(X, Z(\omega)) \cong T_{(X,\omega)}\Omega\mathcal{M}_g(m_i)$$

coming from the exact sequence of sheaves

$$0 \to \Theta^\omega \to \mathbb{C} \to \bigoplus_{P \in Z(\omega)} \mathbb{C}_P \to 0.$$

(The second arrow is given by $v \mapsto v/\omega$.) In these terms, the derivative of the projection $\Omega\mathcal{M}_g(m_i) \to \mathcal{M}_g$ can also be identified with the natural map $H^1(X; \Theta^\omega) \to H^1(X; \Theta)$.

The Schiffer variation revisited. In Section 2 we saw that a simple zero p of ω determines a deformation $\mathrm{Sch}(\omega, p)$ of X supported near p. This construction also lifts to a deformation of the 1-form (X, ω).

Indeed, by (2.8) the data (p, ω) determines a path of 1-forms (X_t, ω_t), as well as holomorphic maps

$$f_t : X - A_t \to X_t - B_t$$

such that $f_t^*\omega_t = \omega$. Geometrically, the new form (X_t, ω_t) is obtained from (X, ω) by opening the vertical arc A_t to obtain a square with sides of length t, and then folding its sides together to form a horizontal arc B_t (see Figure 4).

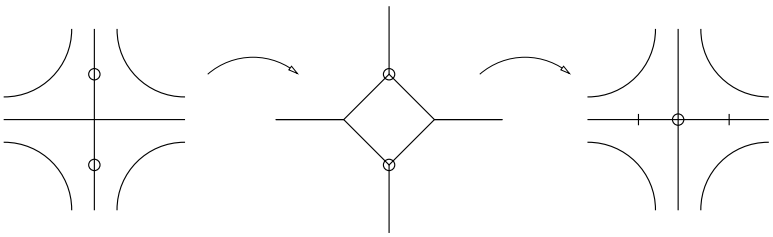

Fig. 4. Folding at a zero of ω.

There is an analogous folding operation when p is a zero of order $n > 1$. In this case A_t is a union of $n + 1$ geodesic rays $[p, e_i]$ with $\int_p^{e_i} \omega_0 = t$. Cutting along A_t yields a $2n + 2$-gon, which we refold to obtain B_t.

Let $C_p \subset X - Z(\omega)$ be a positively oriented loop around p. The following observations will be useful:

- The path (X_t, ω_t) is an integral curve of the vector field $-\mathrm{tw}(C_p)$. (For simple zeros this follows from Theorem 2.2; the generalization to zeros of higher order is straightforward.)
- The path (X_t, ω_t) can be defined for all $t \in [0, T]$, provided A_T embeds in X.
- Folds at different zeros $p, p', p'', \ldots$ can be carried out simultaneously up to time T, provided the arcs $A_T, A'_T, A''_T, \ldots$ are disjoint.

Saddle connections. A geodesic segment $C = [p, p']$ on $(X, |\omega|)$ is a *saddle connection* if p and p' are zeros of ω, and there no zeros between them on C. (We allow the case $p = p'$.) The length R of the shortest saddle connection on (X, ω) is a useful measure of its 'injectivity radius'; for example, the open balls of radius $R/2$ about different zeros of ω are embedded and disjoint.

The absolute period foliation. The *absolute period map* is the locally defined submersion of orbifolds

$$\Omega\mathcal{M}_g \supset U \to H^1(X_0)/\mathrm{Aut}(X_0),$$

which sends a form (X, ω) with X near X_0 to the cohomology class $[\omega] \in H^1(X) \cong H^1(X_0)$. Its fibers form the leaves of the *absolute period foliation* $\mathcal{A}$.

Since $\Omega(X) \to H^1(X)$ is injective, $\mathcal{A}$ is transverse to the fibers of the projection $\pi : \Omega\mathcal{M}_g \to \mathcal{M}_g$. By the same reasoning, $\mathcal{A}$ is transverse to the orbits of the $\mathrm{SL}_2(\mathbb{R})$ acting on $\Omega\mathcal{M}_g$.

Foliated strata. Let $\mathcal{A}(m_i)$ denote the restriction of $\mathcal{A}$ to a stratum $\Omega\mathcal{M}_g(m_i)$. The tangent space to a leaf of $\mathcal{A}(m_i)$ is simply the kernel of the map

$$H^1(X, Z(\omega)) \to H^1(X).$$

It is generated by the deformations $\mathrm{tw}(C_p)$, $p \in Z(\omega)$, subject to the relation $\sum \mathrm{tw}(C_p) = 0$. In particular, the leaves of $\mathcal{A}(m_i)$ have dimension $s - 1$. A natural metric along the leaves is given by

$$\left\| \sum a_p \,\mathrm{tw}(C_p) \right\|_R = \max |a_p - a_q|. \tag{3.2}$$

It is convenient to set

$$\|\mathrm{tw}(C)\|_R = \infty \tag{3.3}$$

whenever $\mathrm{tw}(C)$ is not tangent to $\mathcal{A}$.

Geometry of $\mathcal{A}$. Let $\mathbb{X} = \mathbb{C}^s/\mathbb{C}(1, \dots, 1)$. For any open disk U in a leaf of $\mathcal{A}(m_i)$, the *relative period map* $\rho : U \to \mathbb{X}$ is defined by

$$\rho(X, \omega) = \left(\int_q^{p_1} \omega, \dots, \int_q^{p_s} \omega \right).$$

Here $(p_1, \dots, p_s)$ are the zeros of ω. Since we have taken the quotient of $\mathbb{C}^s$ by the constant vector, this map does not depend on q. It does depend on the ordering of the zeros and the choice of paths from q to p_i; but since the absolute periods are constant, changing these choices only translates $\rho(X, \omega)$ by a constant and permutes its coordinates.

Thus the leaves of $\mathcal{A}(m_i)$ are naturally $(G, \mathbb{X})$ manifolds, where $G = \mathrm{Sym}(m_i) \ltimes \mathbb{C}^{s-1}$ and $\mathrm{Sym}(m_i)$ is the group of permutations σ satisfying $m_{\sigma i} = m_i$. In the case $(m_i) = (g-1, g-1)$, the leaves of $\mathcal{A}(m_i)$ are Riemann surfaces and this $(G, \mathbb{X})$ structure is given by a holomorphic quadratic differential on each leaf. See [Mc3] and [Ba] for examples in genus two, where one obtains a foliated Hilbert surface.

Geodesics. A *relative path* is a smooth map $\gamma : [a, b] \to \Omega\mathcal{M}(m_i)$ whose length

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_R \, dt$$

in the metric (3.2) is finite. Our convention (3.3) implies that γ lies in a leaf of the absolute period foliation.

A relative period such as $\int_{p_1}^{p_2} \omega$ can change by at most $L(\gamma)$ along a relative path. The same is true of the length of the shortest saddle connection.

A *geodesic* is a path in a stratum such that $\gamma'(t) \in H^1(X, Z(\omega))$ is locally constant. If γ also lies in a leaf of $\mathcal{A}$, we say it is a *relative period geodesic*.

Theorem 3.3. *Let R be the length of the shortest saddle connection on (X, ω). Then the relative period geodesic (X_t, ω_t) tangent to a unit vector $\sum a_p \operatorname{tw}(C_p)$ at (X, ω) is defined for $|t| < R/2$. Moreover we have a continuous family of holomorphic embeddings*

$$f_t : (X - D_t, \omega) \to (X_t, \omega_t),$$

where D_t is the $|t|$-neighborhood of $Z(\omega)$.

Proof. By the definition (3.2) of a unit vector, we may assume $|a_p| \leq 1$ for all p. Then so long as $|t|$ is less than half the length of the shortest saddle connection, the rays of length $t|a_p|$ at different zeros p are disjoint, embedded and lie within distance t of p. Thus the folding construction is defined simultaneously at each zero, and it does not change the form outside of D_t. □

Corollary 3.4. *Every leaf of $\mathcal{A}(m_i)$ contains an immersed ball of radius $R(X, \omega)/2$ centered at (X, ω).*

4. Relative period geodesics

In this section we use the space of 1-forms on stable curves to analyze the behavior of relative period geodesics in $\Omega\mathcal{M}_g$. For more on $\Omega\overline{\mathcal{M}_g}$, see e.g. [HM], [Ba, Ch. 5].

Stable curves. A *stable curve* Y is a connected 1-dimensional complex variety with only nodal singularities, such that its smooth locus Y^* has a finite automorphism group.

Let $\Omega(Y)$ denote the space of holomorphic 1-forms on Y^* with at worst simple poles, of opposite residues, at any pair of punctures forming a node. The *arithmetic genus* of Y (so long as it is singular) is given by

$$g = \dim \Omega(Y) = \sum_{i=1}^{s} g_i - s + 2|N(Y)|$$

where $(g_1, \ldots, g_s)$ are the genera of the components of Y^*, and $N(Y) \subset Y$ is the set of nodes. The moduli space $\overline{\mathcal{M}_g}$ of stable curves of genus g provides a natural compactification of $\mathcal{M}_g$. Moreover, the bundle $\Omega\mathcal{M}_g$ extends to $\overline{\mathcal{M}_g}$ with fibers (Y, η), $0 \neq \eta \in \Omega(Y)$. Thus the corresponding bundle of projective spaces,

$$\mathbb{P}\Omega\overline{\mathcal{M}_g} \to \overline{\mathcal{M}_g},$$

is compact.

Sketch of the proof. This compactness can be seen analytically as follows. Let (X_n, ω_n) be a sequence of forms in $\Omega\mathcal{M}_g$. Pass to a subsequence such that X_n converges to a stable

curve Y. Then $X_n \to Y$ geometrically with respect to their hyperbolic metrics ρ_n and ρ. Scale ω_n so $|\omega_n|/\rho_n \leq 1$ in the hyperbolic thick part of X_n. By compactness of bounded analytic functions, we can assume ω_n converges to a limiting form ω on the thick part of Y.

Now any component of the thin part of X_n is isomorphic to an annulus $A(r_n) = \{z : r_n \leq |z| \leq 1\}$, and $\rho_n \asymp |dz|/|z|$ near the boundary of $A(r_n)$. Thus $\omega_n = \omega_n(z)dz/z$, where $\omega_n(z)$ is uniformly bounded in $A_n(r_n)$ and converging near S^1. Assume $r_n \to 0$, so Y acquires a cusp; then $\omega_n(z)$ converges to a bounded analytic function $\omega(z)$ with a removable singularity on the punctured disk. Consequently, $\omega = \omega(z)dz/z$ has at worst a simple pole. The condition of opposite residues follows by considering $\int_{|z|=1} \omega_n$. □

For more details, see e.g. [Wol3].

Compactness and mass. Let $m(X, \omega) = \int_X |\omega|^2$ denote the total area or *mass* of a given 1-form. Let $\Omega_{[a,b]}\overline{\mathcal{M}_g}$ denote the space of stable forms with $m(X, \omega) \in [a, b]$. (Such forms have no poles. We allow $\omega = 0$ if $a = 0$.) By Fatou's lemma, if $(X_n, \omega_n) \to (Y, \eta)$ then we have

$$m(Y, \eta) \leq \liminf m(X_n, \omega_n);$$

thus $\Omega_{[0,1]}\overline{\mathcal{M}_g}$ is compact.

However, the locus $\Omega_1\overline{\mathcal{M}_g}$ where $m(X, \omega) = 1$ is not compact. For example, we can find forms with $m(X_n, \omega_n) = 1$ such that $(X_n, \omega_n) \to (Y, \eta)$ and Y is a union of curves of genus zero; then necessarily $\eta = 0$.

Collapsing. A pair of forms $(Y_i, \eta_i) \in \Omega\overline{\mathcal{M}_g}$, $i = 1, 2$, are *equivalent* if they are isomorphic after removing the components of Y_i^* where η_i is identically zero. By collapsing equivalent forms to a single point, we obtain a quotient space $\Omega\overline{\mathcal{M}_g}/\sim$.

We now show that relative paths of finite length have well-defined limits, after collapsing.

Theorem 4.1. *Let $\gamma(t) = (X_t, \omega_t) \in \Omega\mathcal{M}_g(m_i)$ be a relative path of finite length, defined for $t \in [0, 1)$. Then the limit*

$$(Y, \eta) = \lim_{t \to 1} (X_t, \omega_t)$$

exists in $\Omega\overline{\mathcal{M}_g}/\sim$, and there is no loss of mass:

$$m(Y, \eta) = \lim_{t \to 1} m(X_t, \omega_t).$$

Oscillating Riemann surfaces. We begin the discussion of Theorem 4.1 by showing the result is false without collapsing.

Theorem 4.2. *There exists a path $\gamma(t) = (X_t, \omega_t)$ as above such that $[X_t]$ has no limit in $\overline{\mathcal{M}_g}$ as $t \to 1$.*

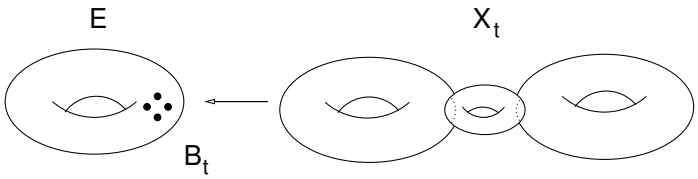

Fig. 5. A degree two branched cover.

Proof. Let $E = \mathbb{C}/\mathbb{Z}[i]$ be the square torus. Let $D \subset E$ be the ball $|z| < 1/2$, and for each $t \in [0, 1)$ let $B_t \subset D$ be a set of four distinct points, $p_i(t)$, $i = 1, \ldots, 4$. Assume $|dp_i/dt| < 1/2$ and $p_i(t) \to 0$ as $t \to 1$.

Let $\pi_t : X_t \to E$ be the unique degree two covering space branched over B_t such that $E - D$ lifts to X_t. There are two such lifts (see Figure 5). Let $\omega_t = \pi_t^*(dz)$. Then X_t has genus three, and $\gamma(t) = (X_t, \omega_t)$ is a path in $\Omega\mathcal{M}_3(1, 1, 1, 1)$ with constant absolute periods and with $\|\gamma'(t)\|_R \leq 1$.

Since the branch locus collapses to a single point as $t \to 1$, X_t is close to a stable curve made up of two copies of E together with a third elliptic curve F_t. But the location of F_t in $\mathcal{M}_1$ depends on the cross-ratio of the points of B_t. Since we can arrange that this cross-ratio oscillates as $t \to 1$, we can also arrange that $[X_t]$ has no limit in $\overline{\mathcal{M}_3}$.

On the other hand, if we pass to a subsequence $t_n \to 1$ such that $(Y, \eta) = \lim \gamma(t_n)$ exists, then (Y, η) consists of two copies of (E, dz) and one additional elliptic curve (F, ξ). The limiting projection $\pi : Y \to E$ is constant on F, so $\xi = 0$. (This also follows from the fact that $\lim m(X_t, \omega_t) = 2 = \int_{Y-F} |\eta|^2$.) Thus all such limits (Y, η) are equivalent. □

Conservation of mass. We now turn to the proof of Theorem 4.1. Reparameterizing, we may assume $\|\gamma'(t)\|_R \leq 1$. We may also assume $m(X_t, \omega_t) = 1$ for all t (recall that $\int |\omega|^2$ depends only on its absolute periods). Since $\Omega_{[0,1]}\overline{\mathcal{M}_g}$ is compact, the forms (X_t, ω_t) have *some* accumulation point (Y, η).

Lemma 4.3. *Any accumulation point satisfies* $m(Y, \eta) = 1$.

Proof. Choose a sequence $t_n \to 1$ such that $(X_n, \omega_n) = \gamma(t_n) \to (Y, \eta)$. Passing to a further subsequence, we can assume that the probability measures $|\omega_n|^2$ on X_n converge to a measure

$$|\eta|^2 + \sum_{p \in N(Y)} a_p \delta_p$$

on Y, where $a_p \geq 0$ and $m(Y, \eta) + \sum a_p = 1$. Our goal is to show $a_p = 0$ for all p, i.e. no mass escapes to the nodes of Y. Note that η has no poles, since its mass is finite.

Suppose to the contrary that $a_p > 0$. Choose any $\epsilon > 0$ that is much less than a_p. Choose a neighborhood A of p in Y consisting of two disks meeting at the node, and small enough that $\int_{\partial A} |\eta| < \epsilon$. (This is possible since η is holomorphic at p.) Then for all $n \gg 0$, there is a corresponding annulus $A_n \subset X_n$ such that $\int_{\partial A_n} |\omega_n| < \epsilon$.

Since $\limsup \int_{A_n} |\omega_n|^2 \geq a_p$, we can find a large value $n = N$ and a Euclidean cylinder $C_N \subset (A_N, \omega_N)$ such that $\int_{C_N} |\omega_N|^2 \geq a_p/2$. (See Theorem A.2 of the Appendix.)

Let h and c denote the height and circumference of C_N. Note that c is the absolute value of the period of ω_N around the core of C_N. Since this period can also be computed by integrating around a boundary component of A_N, we have $c < \epsilon$. Thus $h \geq a_p/(2\epsilon) \asymp 1/\epsilon$.

Shrinking C_N slightly, we can assume C_N is at distance at least one from $Z(\omega_N)$ in the metric $|\omega_N|$. Since the relative periods of (X_t, ω_t) change at most by one, Theorem 3.3 provides a continuous family of isomorphic embeddings

$$f_t : (C_N, \omega_N) \to (X_t, \omega_t),$$

defined for all t. In particular, (X_0, ω_0) contains a Euclidean cylinder of height $h \asymp 1/\epsilon$. Since the diameter of $(X_0, |\omega_0|)$ is bounded, this is a contradiction. □

Proof of Theorem 4.1. It remains only to show that as $t \to 1$, any two accumulation points (Y_1, η_1) and (Y_2, η_2) of $\gamma(t)$ are equivalent. Fix $\epsilon > 0$. Let $K_1 \subset Y_1^* - Z(\eta_1)$ be a compact subsurface with area at least $1 - \epsilon$ and boundary of length less than ϵ, in the $|\eta_1|$ metric. Let $\delta > 0$ denote the minimum distance between K_1 and $N(Y_1) \cup Z(\eta_1)$.

We claim (K_1, η_1) embeds isomorphically in (X_t, ω_t) for all t close enough to one. Indeed, the fact that $\gamma(t)$ accumulates on (Y_1, η_1) means that we have a sequence $t_n \to 1$ and smooth maps $f_n : K_1 \to X_n$, $(X_n, \omega_n) = \gamma(t_n)$, such that

$$f_n^* \omega_n \to \eta_1$$

as C^∞ forms on K_1. We also have $d(f_n(K_1), Z(\omega_n)) > \delta/2$ for all n sufficiently large. By Theorem 3.3, this implies X_t contains a slightly distorted copy of (K_1, η_1) for all $t \in (1 - \delta/4, 1)$. Passing to the limit as $n \to \infty$, we obtain an undistorted copy as well.

Now consider a second sequence $t_n' \to 1$ such that $\gamma(t_n') \to (Y_2, \eta_2)$. Then the maps above also yield, in the limit, a holomorphic embedding $(K_1, \eta_1) \to (Y_2, \eta_2)$. Letting ϵ tend to zero, a normal families argument further implies we have an embedding

$$f : Y_1^* - Z(\eta_1) \to Y_2^*$$

sending η_1 to η_2. Since both forms have total mass one, the image of f is $Y_2^* - Z(\eta_2)$. It is routine to check that f extends to the closure of its domain in Y_1, and thus (Y_1, η_1) and (Y_2, η_2) are equivalent. □

Examples. Let $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz)$ be a pair of forms of genus one. We define $T = \min(\lambda \in \Lambda_1 \cup \Lambda_2 : \lambda > 0)$. Then the connected sum

$$\gamma(t) = (X_t, \omega_t) = (E_1, \omega_1) \underset{[0,t]}{\#} (E_2, \omega_2)$$

(cf. [Mc2, §7]) gives a relative period geodesic

$$\gamma : (0, T) \to \Omega\mathcal{M}_2(1, 1).$$

As $t \to 0$, $\gamma(t)$ terminates via the creation of a node; the limit (Y, η) is obtained by simply identifying E_1 and E_2 at $z = 0$.

The behavior as $t \to T$ is more interesting. If $T \in \Lambda_1$ but $T \notin \Lambda_2$, then (X_T, ω_T) lies in $\Omega\mathcal{M}_2(2)$ and $\gamma(t)$ terminates via a collision of the zeros of ω_t. On the other hand,

if T belongs to both Λ_1 and Λ_2, then we can write these lattices $\Lambda_i = \mathbb{Z}T \oplus \mathbb{Z}\tau_i$ with $\tau_i \in \mathbb{H}$. In this case $\gamma(t)$ terminates at the stable form (Y, η) obtained from

$$(E, \omega) = (\mathbb{C}/\Lambda, dz), \quad \Lambda = \mathbb{Z}T \oplus \mathbb{Z}(\tau_1 + \tau_2),$$

by gluing $z = 0$ to $z = \tau_1$ to form a node.

Finally, if $T = \infty$ we obtain a complete geodesic ray in $\Omega\mathcal{M}_2$. It would be interesting to understand the asymptotic distribution of this ray and other period geodesics, including those transverse to $\mathcal{A}$.

Appendix: Flat metrics and cylinders

In this section we collect some basic length-area bounds for disks and cylinders. These bounds are used in §4.

Proposition A.1. *Let ω be a holomorphic 1-form on a disk D. Then the area of D in the metric $|\omega|$ is controlled by the length of its boundary:*

$$\int_X |\omega|^2 \leq \frac{1}{4}\left(\int_{\partial X} |\omega|\right)^2.$$

Note that ω is allowed to have zeros in D.

Proof. Let $L = \int_{\partial X} |\omega|$ and let $\omega = df$. If we normalize so that $f(p) = 0$ for some $p \in \partial D$, then we have $\max_{\partial D} |f| \leq L/2$. Thus

$$\int_X |\omega|^2 = \frac{i}{2}\int_X d(f\overline{\omega}) \leq \frac{1}{2}\int_{\partial D} |f\omega| \leq \frac{L^2}{4}. \tag{A.1}$$

□

The *Euclidean cylinder* (C, dz) of height h and circumference c is given by

$$C = \{0 < \mathrm{Im}(z) < h\}/\langle z \mapsto z + h\rangle.$$

Theorem A.2. *Let (A, ω) be an annulus equipped with a holomorphic 1-form of total area one such that $L = \int_{\partial A} |\omega|$ is small. Then A contains a Euclidean cylinder of nearly the same area. More precisely, there exists an inclusion of a Euclidean cylinder (C, dz) into (A, ω) such that*

- *$\omega|C = e^{i\theta}dz$;*
- *$\int_{A-C} |\omega|^2 \leq 1 - O(L^2)$; and*
- *the inclusion gives an isomorphism $\pi_1(C) \cong \pi_1(A)$.*

Proof. First suppose $\omega = df$ is exact. Normalize so that $f : A \to \mathbb{C}$ vanishes at some point on ∂A. By assumption, the length of $f(\partial A)$ is $O(L)$. By the open mapping theorem, $f(\partial A)$ is either a connected set or it consists of two immersed loops in $\mathbb{C}$, one of which

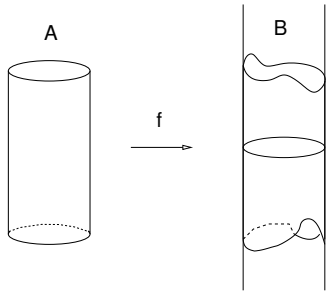

Fig. 6

encloses the other. In either case we find $|f| = O(L)$ on ∂A. A variant of (A.1) then implies $1 = \int_A |\omega|^2 = O(L^2)$. Since L is small, we may assume this case does not occur.

Now let $c = \int_\gamma \omega$ be the period of ω, where γ generates $\pi_1(A)$. We may assume $c > 0$, and certainly $c < L$. Let B be the infinite Euclidean cylinder $\mathbb{C}/\langle z \mapsto z + c\rangle$, equipped with the metric dz.

By integrating ω we obtain a map $f : A \to B$, inducing an isomorphism $\pi_1(A) \cong \pi_1(B)$, such that $\omega = f^*(dz)$. (See Figure 6.) The image of ∂A under f gives a pair of immersed loops $\gamma_1, \gamma_2 \subset B$ each of length less than L in the Euclidean metric $|dz|$.

We may assume the minimum of $\operatorname{Im} f(z)$ on A is 0; let its maximum be h. By considering the preimage of the vertical segment $i[0, h] \subset B$, we obtain an arc α joining the ends of A with $\int_\alpha |\omega| \le h$. Cutting along α to obtain a disk and applying Proposition A.1, we find $1 = \int_A |\omega|^2 \le h^2 + O(hL + L^2)$. Since L is small, we may assume $h > 1/2 \gg L$. Consequently,

$$C = \{z : \operatorname{Im}(z) \in [L, h - L]\} \subset B$$

is a nontrivial cylinder disjoint from ∂A.

Let $C' = f^{-1}(C)$. Since $f : C' \to C$ is proper, every component of C' has at least two boundary components. But every component of $A - C'$ must meet ∂A, by the maximum principle for $\pm \operatorname{Im}(f)$. Thus C' consists of a single annulus and $f : C' \to C$ is an isomorphism. Using Proposition A.1 again, we find the integral of $|\omega|^2$ over the preimage of $B - C$ is $O(L^2)$, and hence $f^{-1} : C \to A$ gives the required embedded cylinder. □

Remark. The same result can be proved more analytically by normalizing so $A = \{z : 1 < |z| < R\}$ and then studying $\omega = \sum a_n z^n \, dz$ as an element of unit norm in $L^2(A)$.

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