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Strongly elliptic linear operators without coercive quadratic forms
I. Constant coefficient operators and forms

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Abstract. A family of linear homogeneous fourth order elliptic differential operators L with real constant coefficients, and bounded nonsmooth convex domains Ω are constructed in $\mathbb{R}^6$ so that the L have no constant coefficient coercive integro-differential quadratic forms over the Sobolev spaces $W^{2,2}(\Omega)$.

Keywords. Neumann problem, Rellich identity, Legendre–Hadamard, Korn inequality, Lax–Milgram, sum of squares, null form, indefinite form

1. Introduction

1.1. A nonexistence theorem and consequences

For u and v in the complex Sobolev space $W^{2,2}(\Omega)$ and $\Omega \subset \mathbb{R}^6$ a bounded open set, let

$$A[u, v] = \sum_{|\alpha|\le 2, |\beta|\le 2} \int_\Omega a_{\alpha\beta} \partial^\alpha u \partial^\beta \overline{v}\, dX \tag{1.1}$$

be any constant coefficient *Hermitian bilinear integro-differential form* associated to any one of the elliptic fourth order homogeneous real constant coefficient linear partial differential operators

$$L_\gamma(\partial) = \left(\frac{\partial_1^2 + \partial_2^2}{4} - \partial_3^2\right)^2 + (\partial_3^2 - \gamma(\partial_4^2 + \partial_5^2 + \partial_6^2))^2$$
$$+ (\partial_3\partial_4 - \partial_5\partial_6)^2 + (\partial_3\partial_5 - \partial_6\partial_4)^2 + (\partial_3\partial_6 - \partial_4\partial_5)^2, \quad 0 < \gamma < 1/3. \tag{1.2}$$

In general $\partial_j = \partial/\partial X_j$ for $X = (X_1, \dots, X_n) \in \mathbb{R}^n$, and in (1.1), $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ is multi-index notation with $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the order of α. The $a_{\alpha\beta}$ are complex numbers satisfying no other condition other than that, by definition, the form A is associated to L_γ if and only if

$$L_\gamma = \sum_{|\alpha|,|\beta|\le 2} (-1)^{|\beta|} a_{\alpha\beta} \partial^{\alpha+\beta} = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} \partial^{\alpha+\beta},$$

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the second equality being a necessary condition on the lower order coefficients because of the homogeneity of L_γ.

There are many such bilinear forms associated to a single L_γ. Each is uniquely determined by its corresponding *Hermitian quadratic integro-differential form*

$$A[v] = \sum_{|\alpha|,|\beta| \le 2} \int_\Omega a_{\alpha\beta} \partial^\alpha v \partial^\beta \overline{v} \, dX, \qquad v \in W^{2,2}(\Omega), \tag{1.3}$$

yielding a one-to-one correspondence between (1.1) and (1.3) (see [Aro61, p. 31]).

In this article the following theorem will be proved.

Theorem. *For each fourth order elliptic real constant coefficient operator L_γ there exist bounded convex domains Ω of $\mathbb{R}^6$ in which no constant coefficient Hermitian quadratic integro-differential form $A[v]$ associated to L_γ can be coercive over the Sobolev space $W^{2,2}(\Omega)$.*

Coerciveness fails in the same way over the corresponding real Sobolev space.

In order to make the statements of the theorem as meaningful as possible, Aronszajn's definition of coercive form, specialized to integro-differential Hermitian forms over $W^{m,2}(\Omega)$, m a nonnegative integer, will be invoked. The Sobolev space $W^{m,2}(\Omega)$ of functions with square integrable weak derivatives up to order m is a Hilbert space with inner product $(u, v)_m = \sum_{|\alpha| \le m} \int_\Omega \partial^\alpha u \partial^\alpha \overline{v} \, dX$ and norm $\|v\|_m = \sqrt{(v, v)_m}$. Seminorms are defined by $|v|_j^2 = \sum_{|\alpha|=j} \int_\Omega |\partial^\alpha v|^2 \, dX$ $(j = 1, \dots, m)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let now

$$A[v] = \sum_{|\alpha|,|\beta| \le m} \int_\Omega a_{\alpha\beta}(X) \partial^\alpha v \partial^\beta \overline{v} \, dX, \qquad v \in W^{m,2}(\Omega), \tag{1.4}$$

where the $a_{\alpha\beta}$ are complex valued bounded measurable functions. Thus A is a *bounded quadratic form*, $|A[v]| \le C_A \|v\|_m^2$, with C_A depending only on the coefficient bounds. Let $K[v]$ denote any *completely continuous* Hermitian quadratic form on $W^{m,2}(\Omega)$. This means K is bounded and, given any $\epsilon > 0$, there exist a bounded Hermitian quadratic form K_ϵ on $W^{m,2}(\Omega)$ and a subspace $V_\epsilon \subset W^{m,2}(\Omega)$ of finite codimension such that

$$\begin{aligned} &\text{(a) } |K_\epsilon[v]| \le \epsilon \|v\|_m^2 \text{ on } W^{m,2}(\Omega), \\ &\text{(b) } K[v] - K_\epsilon[v] \text{ vanishes identically on } V_\epsilon. \end{aligned} \tag{1.5}$$

The standard example of a completely continuous quadratic form on $W^{m,2}(\Omega)$ is the square of the $L^2(\Omega)$ norm (see 1 of Section 2).

Definition 1.1 ([Aro61, p. 38]). The quadratic form $A[v]$ (1.4) with bounded measurable coefficients is *coercive over* $W^{m,2}(\Omega)$ if there exist a completely continuous quadratic form K and a number $c > 0$ such that

$$|A[v]| + K[v] \ge c\|v\|_m^2 \qquad \text{for all } v \in W^{m,2}(\Omega). \tag{1.6}$$

As will be seen in 3 of Section 2, replacing the real valued completely continuous K in (1.6) with $|K[v]|$, for K a complex valued completely continuous quadratic form, does not enlarge the set of coercive forms A in Aronszajn's definition. Moreover, in 4 of Section 2, (1.6) will be replaced with the equivalent condition

$$|A[v]| + c_0 \int_\Omega |v|^2 \, dX \geq c\|v\|_m^2, \qquad c > 0 \text{ and } c_0 \in \mathbb{R}, \tag{1.7}$$

whenever Rellich's compactness theorem holds—for example, in any bounded Lipschitz domain, hence any bounded convex domain, or in any bounded domain with the segment property (see, for example, [Agm65, p. 30] for Rellich's theorem).

1.1.1. Nonexistence of Hilbert space methods for Neumann problems. Conditions (1.6) or (1.7) together with boundedness imply that the corresponding Hermitian bilinear form

$$A[u, v] = \sum_{|\alpha|,|\beta| \leq m} \int_\Omega a_{\alpha\beta}(X) \partial^\alpha u \partial^\beta \overline{v} \, dX \tag{1.8}$$

satisfies the hypotheses of the Lax–Milgram theorem on a subspace $V \subset W^{m,2}(\Omega)$ of finite codimension. In general a *Hermitian bilinear form* B on a Hilbert space H with norm $\| \ \|$ is any functional $H \times H \to \mathbb{C}$ that is linear in its first variable and anti-linear in its second, $B[\alpha x, y] = \alpha B[x, y] = B[x, \overline{\alpha} y]$ for all $\alpha \in \mathbb{C}$. The bilinear form is *bounded* if there is a constant C_B so that $|B[x, y]| \leq C_B \|x\| \, \|y\|$. It is *strongly coercive* if $|B[x, x]| \geq c\|x\|^2$ for some constant $c > 0$.

Theorem 1.2 (Lax–Milgram). *Let $B[x, y]$ be a bounded Hermitian bilinear form on a Hilbert space H with norm $\| \ \|$. If there is a constant $c > 0$ such that $|B[x, x]| \geq c\|x\|^2$ for all $x \in H$ and if $F : H \to \mathbb{C}$ is a bounded linear functional, then there exist unique u and v such that $F(x) = B[x, v] = \overline{B[u, x]}$ for all $x \in H$.*

Given the coerciveness estimate (1.6) one chooses $\epsilon = c/2$ in (1.5) and deduces the strong coerciveness estimate $|A[v]| \geq (c/2)\|v\|_m^2$ for all v in the Hilbert space $V_{c/2}$ of finite codimension in $W^{m,2}(\Omega)$. Lax–Milgram then shows that unique solutions, to a *variational Neumann problem* determined by A and for the linear differential operator formally associated with A, exist in $V_{c/2}$ for all Neumann data derived from $W^{m,2}(\Omega)$ modulo a finite-dimensional subspace of possible data.

For example, take $m = 2$ in (1.8) with only second order derivatives and the $a_{\alpha\beta}$ constant. Let Ω be a bounded Lipschitz domain and let ds denote surface measure on $\partial\Omega$. Then for every $v \in W^{2,2}(\Omega)$ the array of derivatives $\dot{v} = \{v, \partial_1 v, \ldots, \partial_n v\}$ can be *strictly defined* at a.e.(ds) point of $\partial\Omega$ and the set of all such *traces* is the Besov space $B_{3/2}^{2,2}(\partial\Omega)$ (see [JW84, pp. 206, 208]). The space of Neumann data for the equation $Lu = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} \partial^{\alpha+\beta} u = 0$ is determined by $V_{c/2}$ to be a subspace of finite codimension in the dual space of $B_{3/2}^{2,2}(\partial\Omega)$ in the following way. By 5 of Section 2 there is a subspace W such that $W^{2,2}(\Omega) = V_{c/2} \oplus W$ is a direct sum and such that $w \in W$ implies

$$\sum_{|\alpha|=|\beta|=2} \int_\Omega a_{\alpha\beta} \partial^\alpha u \partial^\beta \overline{w} \, dX = 0 \qquad \text{for all } u \in V_{c/2}. \tag{1.9}$$

Take the space of Neumann data to be the finite-codimensional subspace of all bounded linear functionals F in the dual of $B^{2,2}_{3/2}(\partial\Omega)$ that vanish on the traces of the finite-dimensional W. By imbedding theorems [JW84, p. 208] every bounded linear functional on $B^{2,2}_{3/2}(\partial\Omega)$ is also bounded on $W^{2,2}(\Omega)$. Therefore by the coercive estimate and Lax–Milgram each Neumann data F yields a unique $u = u_F \in V_{c/2}$ such that $\overline{F(\dot{v})} = A[u, v]$ for all $v \in V_{c/2}$. Every $\varphi \in C_0^\infty(\Omega)$ can be written according to the direct sum as $v + w$. By $F(\dot{w}) = 0$ and (1.9),

$$0 = \overline{F(\dot{\varphi})} = \overline{F(\dot{v})} = A[u, v] = A[u, \varphi] = \int_\Omega uL\overline{\varphi}\, dX$$

so that $Lu_F = 0$ in the sense of distributions. By regularity theory [Agm65] we have $u_F \in C^\infty(\Omega)$. Now a concrete representation of the Neumann data F can be obtained by taking smooth interior approximating domains $\Omega_j \uparrow \Omega$ [Neč62] and letting ∂_ν denote the derivative with respect to the outer unit normal vector to a boundary. By the Gauss divergence theorem,

$$\overline{F(\dot{v})} = \lim_j \sum_{|\alpha|=|\beta|=2} \int_{\Omega_j} a_{\alpha\beta}\partial^\alpha u \partial^\beta \overline{v}\, dX = \lim_j \int_{\partial\Omega_j} (Mu\partial_\nu\overline{v} - Ku\overline{v})\, ds \tag{1.10}$$

represents F as a sequence of Neumann data $F_j = (Mu, Ku)_{|\partial\Omega_j}$ acting on the Dirichlet data of each $v \in W^{2,2}(\Omega)$, where M and K are the compositions of differential and multiplier operators,

$$M = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}\nu^\beta\partial^\alpha \tag{1.11}$$

$$K = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}\Big(\nu^{\beta'}\partial^{\beta''} + \sum_{k=1}^n (\nu_k\partial^{\beta'} - \nu^{\beta'}\partial_k)\nu_k\nu^{\beta''}\Big)\partial^\alpha, \tag{1.12}$$

that have been fixed by the coefficients of the bilinear form. Here β is the sum of some choice of first order indices β', β''; the choice does not affect the action of Ku on v. Moreover, for each F the F_j are bounded linear functionals on $W^{2,2}(\Omega)$ that converge to F in the norm of the dual of $W^{2,2}(\Omega)$, as (1.10) and $\sum_{|\alpha|=2}\int_{\Omega\setminus\Omega_j} |\partial^\alpha u_F|^2 \to 0$ show. By the extension theorem [JW84, p. 208] the F_j also converge to F in the norm of the Besov space dual.

The point of view of this article then is that

- *There are bounded convex domains and real constant coefficient linear elliptic operators L for which no associated constant coefficient Hermitian bilinear integro-differential form will work to solve a variational Neumann problem for L by the Hilbert space methods outlined above.*

Moreover, the collection of associated forms for each L includes a formally positive quadratic form, and includes indefinite forms that are coercive on domains more regular than convex (see Section 3 below).

1.1.2. Rellich identities. Consider again (1.8) with only mth order derivatives and constant coefficients that in addition are Hermitian symmetric, $\overline{a_{\alpha\beta}} = a_{\beta\alpha}$. Then the corresponding quadratic forms can be placed on Lipschitz boundaries and used to obtain a priori estimates for *solutions* $Lu = 0$ in *boundary Sobolev spaces* $W^{m,2}(\partial\Omega)$.

Let ς be a smooth vector field transverse to the boundary of Ω, i.e. $\varsigma \cdot \nu \geq c_\Omega > 0$ a.e. (ds). Integration by parts yields

$$\sum_{|\alpha|=|\beta|=m} \int_{\partial\Omega} a_{\alpha\beta}\partial^\alpha u \partial^\beta \overline{u}\; \varsigma \cdot \nu\, ds = 2\,\mathrm{Re} \sum_{|\alpha|=|\beta|=m} \int_{\partial\Omega} a_{\alpha\beta}\partial^\alpha u \partial^\beta (\varsigma \cdot \nabla \overline{u})\, ds + B[u] \tag{1.13}$$

where B is a Hermitian quadratic form (1.4) over the domain Ω consisting only of terms that have at least one derivative applied to the smooth vector field. By Gagliardo–Nirenberg estimates, $B[u]$ can be bounded by a constant depending only on L, ς and the Lipschitz character of Ω, times $|u|_m^2 + |u|_0^2$. Let Ω' be an interior approximating domain. Because u is a solution, estimates in [PV95] show that the integral of $|u|_m^2$ over $\Omega \setminus \Omega'$ is bounded by the maximum distance of Ω' to $\partial\Omega$, times a constant depending only on L, ς and the Lipschitz character of Ω, times the boundary Sobolev square seminorm $\sum_{|\alpha|=m} \int_{\partial\Omega} |\partial^\alpha u|^2\, ds$. Because u is a solution the rest of $|u|_m^2$ can be bounded by the canonical $c_1 \int_\Omega |u|^2\, dX$ using interior estimates. Solution representations by spherical means (see [Joh55, pp. 153–154]) or interior L^2 regularity [Agm65, Theorem 6.3] can be used for this. Also because u is a solution, the integral on the right of (1.13) can be written in terms of the Neumann data for u and the Dirichlet data of $v = \varsigma \cdot \nabla u$ as above. For example, in the $m = 2$ case,

$$\begin{aligned}&\sum_{|\alpha|=|\beta|=2} \int_{\partial\Omega} a_{\alpha\beta}\partial^\alpha u \partial^\beta \overline{u} \varsigma \cdot \nu\, ds \\ &\qquad = 2\,\mathrm{Re} \sum_{|\alpha|=|\beta|=2} \int_{\partial\Omega} \big(Mu \partial_\nu (\varsigma \cdot \nabla \overline{u}) - Ku \varsigma \cdot \nabla \overline{u}\big)\, ds + B[u]\end{aligned} \tag{1.14}$$

where the data pairings are now $L^2(\partial\Omega)$ with $L^2(\partial\Omega)$ and $W^{-1,2}(\partial\Omega)$ with $W^{1,2}(\partial\Omega)$. If now there is a *boundary coerciveness estimate for solutions*

$$c \sum_{|\alpha|=m} \int_{\partial\Omega} |\partial^\alpha u|^2\, ds \leq \sum_{|\alpha|=|\beta|=m} \int_{\partial\Omega} a_{\alpha\beta}\partial^\alpha u \partial^\beta \overline{u} \varsigma \cdot \nu\, ds + c_0 \int_\Omega |u|^2\, dX \tag{1.15}$$

where $c > 0$ *depends only on the quadratic form,* ς *and the Lipschitz character of* Ω, then the term $B[u]$ of (1.13) and (1.14) can be replaced by the canonical completely continuous form and estimates follow from, for example,

$$\begin{aligned}&\frac{c}{2} \sum_{|\alpha|=2} \int_{\partial\Omega} |\partial^\alpha u|^2\, ds \\ &\qquad \leq 2\,\mathrm{Re} \sum_{|\alpha|=|\beta|=2} \int_{\partial\Omega} \big(Mu \partial_\nu (\varsigma \cdot \nabla \overline{u}) - Ku\; \varsigma \cdot \nabla \overline{u}\big)\, ds + c_2 \int_\Omega |u|^2\, dX.\end{aligned} \tag{1.16}$$

Identities like (1.14) have come to be known as *Rellich identities*, while (1.16) is an example of a *Rellich estimate*. For second order operators ($m = 1$) they were introduced into the scale invariant analysis of boundary values, taken in the strong pointwise sense, for solutions in Lipschitz domains by Jerison–Kenig [JK81]. Earlier uses were made by Nečas [Neč67], Rellich, Payne–Weinberger, Pohozaev and Morawetz. See [Ken94, p. 112]. The identities and estimates can be used in various ways to solve boundary value problems. One of these ways, since the boundedness of Calderón–Zygmund singular integrals on Lipschitz graphs was proved by Coifman–McIntosh–Meyer [CMM82], is to prove invertibility of classical *layer potentials* which have the virtue of simultaneously solving Neumann and Dirichlet problems. Beginning with Laplace's equation [Ver84], an incomplete list of applications of this kind would be the heat equation [Bro89], linearized Stokes system [FKV88], harmonic transmission [EFV92], nonstationary Stokes system [She91], Dirac operator systems [MM99], the Hilbert problem and spectral properties of the rotation operator [Axe03], [Axe04], Navier–Stokes on Riemannian manifolds [MT01], the harmonic Neumann problem in non-Lipschitz polyhedra [Ver01], [VV06], electromagnetic scattering [Mit95], [MMP97], spectral properties of electromagnetic layer potentials [MO10], mixed problems for the Stokes system [BMMW10], a priori layer potential solvability of the harmonic mixed problem [Ven12], homogenization problems for elliptic systems [KS11], divergence form bounded measurable complex coefficient equations [AAAHK11]. The only higher order result of this kind has been for the biharmonic operator [Ver05].

The boundary coercive estimate (1.15) is an automatic consequence of ellipticity when $m = 1$. Some of the uses for Rellich identities listed above are for second order systems for which boundary coerciveness can be problematic depending on the particular quadratic form used, i.e. the particular Neumann problem being solved for the given system. For example, coerciveness for the *traction* Neumann problem [DKV88] required proof of a boundary Korn inequality for solutions. The same is true for higher order scalar valued equations. In [Ver05] boundary coerciveness either holds or is disproved by counterexample depending on the quadratic form.

Of interest here is that the boundary coercive estimate for solutions is stronger than the classical interior coercive estimate (1.6)–(1.7) for general $v \in W^{2,2}(\Omega)$. The completion under the Sobolev norm of the $C_0^\infty(\Omega)$ functions is denoted $W_0^{m,2}(\Omega)$.

Lemma 1.3. *For L with a homogeneous Hermitian symmetric bilinear form let boundary coerciveness (1.15) hold for solutions $Lu = 0$, $u \in C^m(\overline{\Omega})$, with $c > 0$ and c_0 depending only on ς and the Lipschitz character of Ω. Let A denote the corresponding quadratic form (1.4). There are then constants $c' > 0$ and c_0' such that*

$$c'|v|_m^2 \le A[v] + c_0' \int_\Omega |v|^2\, dX \quad \text{for all } v \in W^{m,2}(\Omega). \tag{1.17}$$

Proof. By the geometric formula in 6 of Section 2 there is a continuum of interior Lipschitz domains Ω_t such that the main hypothesis can be applied as

$$\sum_{|\alpha|=m} \int_{\Omega\setminus\Omega_1} |\partial^\alpha u|^2\, dX$$
$$= \int_0^1 dt \sum_{|\alpha|=m} \int_{\partial\Omega_t} |\partial^\alpha u|^2 \varsigma \cdot \nu_t\, ds_t \le \|\varsigma\|_\infty \int_0^1 dt \sum_{|\alpha|=m} \int_{\partial\Omega_t} |\partial^\alpha u|^2\, ds_t$$

$$\leq \frac{2\|\varsigma\|_\infty}{c}\int_0^1 dt \sum_{|\alpha|=|\beta|=m}\int_{\partial\Omega_t} a_{\alpha\beta}\partial^\alpha u\partial^\beta\overline{u}\varsigma\cdot\nu_t\,ds_t + \frac{2c_0}{c}\int_\Omega |u|^2\,dX$$
$$= \frac{2\|\varsigma\|_\infty}{c}\sum_{|\alpha|=|\beta|=m}\int_{\Omega\backslash\Omega_1} a_{\alpha\beta}\partial^\alpha u\partial^\beta\overline{u}\,dX + \frac{2c_0}{c}\int_\Omega |u|^2\,dX.$$

Now by interior estimates, (1.17) follows for solutions u in place of v.

The coerciveness estimate (1.17) is also known to hold for all $w \in W_0^{m,2}(\Omega)$ (Gårding's inequality), which, in addition, gives the solution to the Dirichlet problem via Lax–Milgram in the classical Hilbert space sense. Modulo finite-dimensional subspaces, given any $v \in W^{m,2}(\Omega)$ (data) there is a unique $w_v \in W_0^{m,2}(\Omega)$ such that $u = v - w_v$ is a weak solution to $Lu = 0$. The map $S : v \mapsto w_v$ is a bounded linear operator on $W^{m,2}(\Omega)$, and $A[u] + A[w_v] = A[u + w_v]$ because u is a solution while $w_v \in W_0^{m,2}(\Omega)$. Interior regularity shows $u \in C^\infty(\Omega)$, justifying the a priori boundary coerciveness estimates. Altogether it follows that

$$\frac{c'}{2}|v|_m^2 \leq c'|u|_m^2 + c'|w_v|_m^2 \leq A[v] + c_0'\int_\Omega (|u|^2 + |w_v|^2)$$
$$\leq A[v] + 2c_0'\int_\Omega |v|^2 + 3c_0'\int_\Omega |Sv|^2,$$

and (1.17) follows since the last integral is also a completely continuous form on $v \in W^{m,2}(\Omega)$ (see 1, 3 and 4 of Section 2). □

By this lemma and the main theorem it follows that

- *For each of the constant coefficient elliptic operators L_γ there are bounded convex domains for which every associated constant coefficient Hermitian symmetric quadratic integro-differential form fails to give a coercive Rellich identity.*

1.1.3. Second order Legendre–Hadamard systems. Letting subscripts denote derivatives and letters dependent variables, the symmetric Legendre–Hadamard systems,

$$\triangle u - 2w_{13} - x_{14} - y_{15} - z_{16} = 0,$$
$$\triangle v - 2w_{23} - x_{24} - y_{25} - z_{26} = 0,$$
$$-2u_{13} - 2v_{23} + (2\triangle - \partial_1^2 - \partial_2^2)w - (\gamma+1)x_{34} - x_{56} - (\gamma+1)y_{35} - y_{46} - (\gamma+1)z_{36} - z_{45} = 0,$$
$$-u_{14} - v_{24} - (\gamma+1)w_{34} - w_{56} + \triangle x - (1-\gamma^2)x_{44} + x_{55} - (1-\gamma^2)y_{45} - (1-\gamma^2)z_{46} = 0,$$
$$-u_{15} - v_{25} - (\gamma+1)w_{35} - w_{46} - (1-\gamma^2)x_{45} + \triangle y - (1-\gamma^2)y_{55} + y_{66} - (1-\gamma^2)z_{56} = 0,$$
$$-u_{16} - v_{26} - (\gamma+1)w_{36} - w_{45} - (1-\gamma^2)x_{46} - (1-\gamma^2)y_{56} + \triangle z + z_{44} - (1-\gamma^2)z_{66} = 0.$$

can be derived from the examples here after a scaling of the independent variables X_1 and X_2. Here $\triangle$ denotes the Laplacian of $\mathbb{R}^6$. The systems can be shown to have no constant coefficient quadratic forms that are coercive over the vector valued Sobolev spaces $W^{1,2}(\Omega)$ when the Ω's are taken to be the corresponding scalings in X_1 and X_2 of the convex domains constructed in this article. No constant coefficient form leads to a Korn inequality even though, like the L_γ, there is an associated formally positive quadratic form (see Subsection 1.3 below) for each system,

$$\int_\Omega \big(|u_1+v_2-w_3|^2+|w_3-\gamma x_4-\gamma y_5-\gamma z_6|^2+|w_4-y_6|^2+|w_5-z_4|^2+|w_6-x_5|^2 \\ +|z_4-x_6|^2+|x_5-y_4|^2+|y_6-z_5|^2+|w_4-x_3|^2+|w_5-y_3|^2+|w_6-z_3|^2 \\ +|v_1-u_2|^2+|w_1-u_3|^2+|x_1-u_4|^2+|y_1-u_5|^2+|z_1-u_6|^2+|w_2-v_3|^2 \\ +|x_2-v_4|^2+|y_2-v_5|^2+|z_2-v_6|^2\big)\,dX.$$

A complete proof [Ver12a] will appear elsewhere.

1.2. *Reduction to real homogeneous forms*

In order to prove the main theorem and its consequences for the variational Neumann problem, it is important that the coerciveness condition (1.6) be replaced, without loss of generality, by

$$\mathrm{Re}(e^{i\theta}A[v])+K[v]\ge c\|v\|_m^2 \tag{1.18}$$

where θ is some fixed angle and K is again completely continuous. Once this is done it will be shown here that the noncoerciveness of the forms (1.3) for the operators L_γ follows from the failure to achieve a coerciveness estimate that is equivalent to (1.6), viz.

$$A[v]+c_0\int_\Omega |v|^2\,dX\ge c\|v\|_2^2 \tag{1.19}$$

whenever A is any real symmetric quadratic form associated to the L_γ. The failure to achieve (1.19) is the statement of Theorem 1.5 below. Its proof is the greater part of this article.

Aronszajn proves that (1.6) and (1.18) are equivalent conditions for coerciveness. Condition (1.18) implies condition (1.6) directly. For the converse (see 7 of Section 2)

Theorem 1.4 ([Aro61, pp. 38–39]). *If the integro-differential quadratic form with bounded measurable coefficients $A[v]$ of (1.4) is coercive over $W^{m,2}(\Omega)$, then there are a constant $c>0$ and a completely continuous quadratic form K such that (1.18) holds for all $v\in W^{m,2}(\Omega)$ and for all θ in an interval $\theta_0<\theta<\theta_1$. Moreover K can be taken to have finite rank.*

A quadratic form has *finite rank* if it vanishes identically on a subspace of finite codimension.

One immediate consequence of Theorem 1.4 is that any quadratic form with bounded measurable coefficients that satisfies Aronszajn's definition of coercive in a bounded open set must then be a *uniformly strongly elliptic quadratic form* after suitable rotation of its range in the complex plane, i.e. satisfy

$$\mathrm{Re}\Big(e^{i\theta}\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(X)\xi^{\alpha+\beta}\Big)\ge E|\xi|^{2m} \tag{1.20}$$

for a.e. $X\in\Omega$, all $\xi\in\mathbb{R}^n$ and for some ellipticity constant $E>0$. See, for example, [Agm65, p. 87 line (7.18)] and compare Theorem 1.4 with lines (7.20) and (7.21).

Another consequence of Theorem 1.4 is

- *The quadratic form* $A[v]$ (1.4) *is coercive only if the real valued quadratic forms*

$$\widetilde{A}_\theta[v] := \frac{1}{2} \sum_{|\alpha|,|\beta|\le m} \int_\Omega \big(e^{i\theta} a_{\alpha\beta}(X) + e^{-i\theta}\overline{a_{\beta\alpha}(X)}\big)\partial^\alpha v \partial^\beta \overline{v}\, dX \qquad (1.21)$$

are coercive, i.e. satisfy condition (1.18) *which can now be written* $\widetilde{A}_\theta[v] + K[v] \ge c\|v\|_m^2$.

Define $\widetilde{a}_{\alpha\beta}(\theta) = \frac{1}{2}(e^{i\theta}a_{\alpha\beta} + e^{-i\theta}\overline{a_{\beta\alpha}})$. Unlike the general Hermitian quadratic forms (1.4) the forms (1.21) have coefficients satisfying $\overline{\widetilde{a}_{\alpha\beta}(\theta)} = \widetilde{a}_{\beta\alpha}(\theta)$ and will be called *Hermitian symmetric quadratic forms*. The unique Hermitian bilinear form for each $\widetilde{A}_\theta[v]$ is $\widetilde{A}_\theta[u, v] = \sum_{|\alpha|,|\beta|\le m} \int_\Omega \widetilde{a}_{\alpha\beta}(\theta)\partial^\alpha u \partial^\beta \overline{v}\, dX$ and satisfies $\overline{\widetilde{A}_\theta[u, v]} = \widetilde{A}_\theta[v, u]$.

Denote by L any of the elliptic real constant coefficient operators L_γ from (1.2). By (1.21), if any quadratic form $A[v]$ (1.3) associated to L is coercive, then so too must its corresponding Hermitian symmetric form $\widetilde{A}_\theta[v]$. Since $L = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}\partial^{\alpha+\beta}$ is real, letting v be a test function in the bilinear form $\widetilde{A}_\theta[u, v] = \sum_{|\alpha|,|\beta|\le 2} \int_\Omega \widetilde{a}_{\alpha\beta}(\theta)\partial^\alpha u \partial^\beta \overline{v}\, dX$ and integrating by parts shows that $\widetilde{A}_\theta$ is associated with the differential operator $\cos(\theta)L$. By the ellipticity of L and (1.20), $\cos(\theta) > 0$. Consequently:

- *If* L_γ (1.2) *is associated to a coercive constant coefficient form* (1.3)*, then* L_γ *is associated to a coercive Hermitian symmetric constant coefficient form.*

Suppose then that $L = L_\gamma$ has the coercive quadratic form $A[v]$ (1.3), now with Hermitian symmetric coefficients, $\overline{a_{\alpha\beta}} = a_{\beta\alpha}$. Another real valued quadratic form associated to L is

$$A_{\text{real}}[v] = \sum_{|\alpha|\le 2,|\beta|\le 2} \int_\Omega (\operatorname{Re} a_{\alpha\beta})\partial^\alpha v \partial^\beta \overline{v}\, dX = A_{\text{real}}[\operatorname{Re} v] + A_{\text{real}}[\operatorname{Im} v].$$

Whenever v is real valued, $A[v] = A_{\text{real}}[v]$. Consequently, it can now be shown that

- *If a Hermitian symmetric form* $A[v]$ *is coercive, then the corresponding* $A_{\text{real}}[v]$ *is coercive.*

To show this, one argues using the contrapositive lemma from 8 of Section 2. If A_{real} is not coercive, there is a sequence of nonzero v_j such that

$$A_{\text{real}}[\operatorname{Re} v_j] + A_{\text{real}}[\operatorname{Im} v_j] + j\int_\Omega \big((\operatorname{Re} v_j)^2 + (\operatorname{Im} v_j)^2\big)\, dX \le \frac{1}{j}\|\operatorname{Re} v_j\|_m^2 + \frac{1}{j}\|\operatorname{Im} v_j\|_m^2. \qquad (1.22)$$

The inequalities must also hold for a subsequence of $\operatorname{Re} v_j$ or $\operatorname{Im} v_j$. Since A and A_{real} are identical on either of these, and sequences of inequalities like (1.22) imply noncoerciveness, neither can A be coercive.

In contrast, when v is complex valued $A[v] \ne A_{\text{real}}[v]$ in general, and A_{real} can be coercive while A is not. For example, this is true for the Hermitian symmetric form associated to the Laplacian in the plane, $\int_\Omega |(\partial_x + i\partial_y)v|^2$ (see [Fol95, p. 242]).

Continuing, the forms $A_{\text{real}}[v]$ have *real symmetric coefficients*

$$\sum_{|\alpha|\leq 2, |\beta|\leq 2} \int_\Omega a_{\alpha\beta} \partial^\alpha v \partial^\beta \overline{v}\, dX$$

with $a_{\alpha\beta} = a_{\beta\alpha} \in \mathbb{R}$. The foregoing arguments therefore show that

- *An elliptic operator with real coefficients has a coercive quadratic form in the sense of Aronszajn's definition if and only if it has a coercive quadratic form with real symmetric coefficients satisfying inequality* (1.19).

By using (1.22) and the contrapositive lemma again it follows that

- *If an elliptic operator with real coefficients has no coercive quadratic form over* $W^{2,2}(\Omega)$*, then it has no coercive quadratic form over the corresponding real Sobolev space either.*

Thus the second statement of the main theorem at the beginning of this article follows from the first.

When Ω is a bounded Lipschitz domain, the Gagliardo–Nirenberg inequalities for $v \in W^{m,2}(\Omega)$,

$$|v|_j^2 \leq \epsilon |v|_m^2 + C_{\epsilon,j} |v|_0^2,$$

hold for any $\epsilon > 0$ and $0 \leq j \leq m-1$ with $C_{\epsilon,j}$ also depending on Ω but independent of v (see [Agm65, p. 25], for example). Therefore, by the coerciveness condition (1.7) the quadratic form (1.3) will be coercive in a bounded Lipschitz domain if and only if its *principal part* $\sum_{|\alpha|=|\beta|=2} \int_\Omega a_{\alpha\beta} \partial^\alpha v \partial^\beta \overline{v}\, dX$ is coercive. By the homogeneity of L_γ, the principal part of its associated constant coefficient form is also associated to L_γ. If a bilinear or quadratic form only consists of its principal part, it will be called *homogeneous*.

In sum, to prove the main theorem it suffices to prove the following.

Theorem 1.5. *Given any homogeneous fourth order linear elliptic real constant coefficient operator* L_γ (1.2)*, there are bounded convex domains* Ω *of* $\mathbb{R}^6$ *in which all homogeneous real symmetric constant coefficient quadratic forms* $A[v] = \sum_{|\alpha|=|\beta|=2} \int_\Omega a_{\alpha\beta} \partial^\alpha v \partial^\beta \overline{v}\, dX$ *associated to* L_γ *are noncoercive over* $W^{2,2}(\Omega)$*, i.e. the coerciveness estimate*

$$A[v] + c_0 \int_\Omega |v|^2\, dX \geq c|v|_2^2 \quad \text{for all } v \in W^{2,2}(\Omega) \tag{1.23}$$

must fail for all constants c_0*, all constants* $c > 0$ *and all such* $A[v]$ *associated to* L_γ*.*

1.3. Contents and guide to the proof

Contents

Theorem 1.5 is of interest in itself if, for example, lack of coercive Rellich identities is of concern. The above reduction to real forms and the supporting arguments of Section 2 can be omitted in order to more easily get to the construction demonstrating the absence of coerciveness estimates. With the notation of Section 5, the convex domains are defined in Section 8. All possible constant real coefficient associated forms make up a (large) finite-dimensional affine space that can be viewed as a vector space of null quadratic forms (Section 4) translated by a noncoercive formally positive form (3.3). This formally positive form vanishes on an infinite-dimensional subspace $\mathbf{X} \subset W^{2,2}$ of separated solutions (Section 6). By standard use of Rellich's compactness theorem (Lemma 4.1), proving the absence of coerciveness estimates for all forms reduces to showing that each null form takes only nonpositive values on some infinite dimensional subspace of $\mathbf{X}$. Separated solutions $E(z^n, Q)$ (8.3) are singled out for the purpose of constructing these subspaces. The polynomials Q in these solutions are any linear combinations of derivatives of the polynomial (6.3). A lengthy argument (Sections 9, 10) reduces the space of null forms to a 4-dimensional affine space (14.1) in the following sense (Lemma 10.1): If some null form is coercive over $\mathbf{X}$, then there is a null form from the 4-dimensional affine space that is coercive over each of four subspaces of $\mathbf{X}$ defined in Subsection 10.1. Orthogonality properties, of separated solutions when acted upon by forms from the 4-dimensional affine space, show that nonpositivity over a subspace is equivalent to nonpositivity over a sequence of the separated solutions (Lemma 12.1). This setup for the final estimates of Sections 15–18 is stated in 3 of Section 14. The basis elements for the affine space of forms (14.1) are given in (5.2) and Section 9. The formulas for the forms of Section 9 are valid on $\mathbf{X}$ only (see Lemma 9.1). By constructing various sequences of separated solutions, subspaces of $\mathbf{X}$ are defined that impose necessary conditions on the coefficients of these basis elements in order for a linear combination to be coercive. The first construction, based on letting Q be the constant polynomial, is made in Subsection 4.2 and shows that the null form N_0 must appear as it does in (14.1). Five more sequences and subspaces are constructed in Sections 15–17. The inequalities they force on the coefficients of the remaining four null forms of (14.1) are shown to be incompatible in Section 18, finishing the proof.

The construction of the sequences generally begins by choosing Q to be an undetermined linear combination of polynomials from one of the four polynomial subspaces $\widehat{\mathbf{T}}, \widehat{\mathbf{S}}_1, \ldots$ from which the four subspaces of separated solutions are defined (Subsection 10.1). The null forms (14.1) are applied and each of the terms of the resulting sequence is minimized by choosing values for the undetermined coefficients of the separated solution. A condition on the coefficients of the null forms is obtained when the sequence can be made negative for all n large enough. The procedure is illustrated in Section 12. In addition, Remark 12.4 illustrates the role played by one of two Lipschitz constants, M, in the definition of the convex domains. In the end, the subspaces found in this article by the methods described also impose the requirements on the convex domains that M be confined to a bounded interval away from *zero* and that the second Lipschitz constant be large enough depending on the parameter $0 < \gamma < 1/3$, thereby identifying a collection of convex domains for each L_γ in which no coercive form is available.

In general it is difficult to determine if a given quadratic form is coercive. A notable exception seems to be for those that are formally positive. The Aronszajn–Smith theorem gives an algebraic characterization (see also [Agm65]).

Theorem 1.6 ([Aro54], [Smi70]). *Let $p_1, p_2, \ldots \in \mathbb{C}[X_1, \ldots, X_n]$ denote a finite number of complex polynomials all homogeneous of degree m, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then a necessary and sufficient condition for the coercive estimate over $W^{m,2}(\Omega)$,*

$$\sum_j \int_\Omega |p_j(\partial)v|^2\, dX + c_0 \int_\Omega |v|^2\, dX \geq c|v|_m^2, \qquad c > 0,\ c_0 \in \mathbb{R}, \tag{1.24}$$

is that there be no solution $z \in \mathbb{C}^n \setminus \{0\}$ to the system of algebraic equations

$$p_1(z) = p_2(z) = \cdots = 0.$$

Often an elliptic operator with an associated formally positive form that is seen to be not coercive has other formally positive forms that are coercive by this characterization. To prove that this is not always the case the author constructed [Ver10] the positive definite polynomial (3.1) from which the operators L_γ are derived. The L_γ have no coercive formally positive forms but they do have coercive indefinite forms in C^2 domains, as shown in [Ver12] (Subsection 3.2 below). Consequently, the question arises as to the existence of a coercive indefinite form, in Smith's Lipschitz domains, when there are no formally positive coercive forms for an operator. The answer here is that convex domains lack sufficient regularity for such an outcome, at least in the case of constant coefficient operators and forms.

The algebraic structure of the operators, the definition of the domains and the ability to compute accurately in the last sections are all closely related. The space **X** of separated solutions, described in Sections 6.1–6.3, is the solution space of the over-determined system (6.1) that is derived from the sum of squares representation of each L_γ. While the algebraic structure of the separated solutions is independent of the domains, integrability is not (Remark 6.2 and Example 6.3). The null forms exist independently of the operators and domains. Symmetries in the complex variable z of the domains and the separated solutions effect the first reduction in their number (Section 9), while symmetries in the variable $s \in \mathbb{R}^3$ are used in Section 10 (see Subsection 8.2). Section 13 gives an algorithm for minimizing the terms of the sequences that result from applying the null forms to the chosen subspaces of separated solutions. An appendix details the calculations required by the algorithm in the case of the main estimate of Section 16.

2. Supporting introductory arguments

1. Let the bilinear form $B[u, v]$ be the inner product $(u, v)_m$ on $W^{m,2}(\Omega)$, let $S : W^{m,2}(\Omega) \to W^{m,2}(\Omega)$ be bounded linear, and let $K[v] = \int_\Omega |Sv|^2\, dX$ where Ω is any domain in which the Rellich compactness theorem holds. That K *is completely continuous with respect to the norm* $\|\ \|_m$ may be seen as a consequence of the classical Hilbert space argument (see [Fol95, p. 251], for example) that the strong coerciveness of B provides a bounded self-adjoint linear injection $T : L^2(\Omega) \to W^{m,2}(\Omega) \subset L^2(\Omega)$ that is compact as a map $L^2 \to L^2$ (Rellich) and satisfies

$$\int_\Omega f\,\overline{v}\,dX = B[Tf, v], \qquad v \in W^{m,2}(\Omega). \tag{2.1}$$

By the spectral theorem, the normalized eigenfunctions of T form an orthonormal basis $\{u_j\}$ for L^2 with $Tu_j = \alpha_j u_j$ and $0 < \alpha_j \to 0$ as $j \to \infty$. By (2.1), $B[u_j, u_k] = 0$ when $j \neq k$. Therefore for any $v = \sum_{j=N}^{\infty} c_j u_j$ it follows that $\int_\Omega |v|^2\,dX = \sum_{j=N}^{\infty} \alpha_j B[c_j u_j, c_j u_j] \leq \max_{j\geq N} a_j B[v, v]$. And therefore given $\epsilon > 0$ and N large enough, (1.5) follows by defining $K_\epsilon[v] := \int_\Omega |SP_N v|^2 \leq \|S\|^2 \int_\Omega |P_N v|^2$ with P_N the orthogonal projection in $W^{m,2}$ onto $V_\epsilon = \operatorname{span}\{u_j\}_{j=N}^{\infty}$. Or one can apply Rellich's theorem directly to the task.

2. In general, Hermitian quadratic forms $B[v]$ may be defined directly as functionals satisfying, for all complex numbers α, β and vectors u, v,

$$B[\alpha u + \beta v] = |\alpha|^2 a(u, v) + \alpha\overline{\beta} b(u, v) + \overline{\alpha}\beta c(u, v) + |\beta|^2 d(u, v)$$

where a, b, c and d are functionals independent of α and β. It may be shown that if $B[v]$ is Hermitian quadratic, then so is $\operatorname{Re} B[v]$. Hermitian quadratic forms uniquely determine Hermitian bilinear forms $B[u, v]$ by the requirement $B[v, v] = B[v]$. The one-to-one correspondence is given by $B[u, v] = \frac{1}{4}\sum_{k=0}^{3} i^k B[u + i^k v]$ (see [Aro61, p. 31]). In the case of the bounded integro-differential quadratic forms (1.4), it follows that (1.8) is the unique bilinear form and inherits the bound $|A[u, v]| \leq C_A \|u\|_m \|v\|_m$ from (1.4).

As a consequence of this, the triangle inequality, and Young's inequality, there is a constant $C_{A,\epsilon}$ for each $\epsilon > 0$ such that

$$|A[v]| \leq |A[u + v]| + \epsilon\|v\|_m^2 + C_{A,\epsilon}\|u\|_m^2 \tag{2.2}$$

for any u and v.

3. Consider the coerciveness condition

$$|A[v]| + |K[v]| \geq c\|v\|_m^2 \tag{2.3}$$

in place of (1.6). Again $|A[v]| \geq \frac{1}{2}c\|v\|_m^2$ follows for all $v \in V$, a subspace of finite codimension. Let U be the finite-dimensional *orthogonal complement of* V *in* $W^{m,2}$ and take $u \in U$, $v \in V$. By choosing ϵ in (2.2) and by orthogonality in $W^{m,2}$ it follows that

$$\tfrac{1}{4}c\|u + v\|_m^2 \leq |A[u + v]| + C\|u\|_m^2 \tag{2.4}$$

with C depending only on c and C_A. But the real valued functional $w = u + v \mapsto C\|u\|_m^2$ is a well defined quadratic form on all $w \in W^{m,2}$ and has finite rank. Therefore, (2.3) *is not more general than* (1.6).

4. Moreover, for a smaller $c > 0$, the last term in (2.4) may be replaced by a constant times the squared L^2 norm $\|u + v\|_0^2$ for all $u \in U$, $v \in V$. This follows by orthogonality in $W^{m,2}$ when $0 \le C\|u\|_m^2 \le \frac{1}{8}c\|v\|_m^2$. Otherwise it may be assumed that

$$\|u\|_m^2 = 1 \quad \text{and} \quad \|v\|_m^2 \le 8C/c. \tag{2.5}$$

Since U is finite-dimensional there is equivalence of norms $\|u\|_0^2 \le \|u\|_m^2 \le C_U\|u\|_0^2$. Hence, for any sequence of functions $\{u + v\}$ there will be a subsequence such that $\{v_j\}$ converges weakly in $W^{m,2}$ to v while $\{u_j\}$ converges strongly to a *nonzero* $u_0 \in U$. By the weak convergence, $0 = (v_j, u)_m \to (v, u)_m$ for all $u \in U$ so that $v \in V$. Now if there is no constant $c' > 0$ satisfying $\|u+v\|_0^2 \ge c'$ for all functions $u+v$ satisfying (2.5), then there are sequences as described with weak and strong limits $v \in V$ and *nonzero* $u_0 \in U$ respectively and $v = -u_0$ a.e. in L^2. But also $v = -u_0$ in $W^{m,2}$ by weak convergence since integration of v against the derivatives $\partial^\alpha \phi$ $(0 \le |\alpha| \le m)$ of test functions is a bounded linear functional on $W^{m,2}$. Consequently, there exists $c' > 0$ such that the inequality $\frac{1}{4}c\|u + v\|_m^2 \le |A[u+v]| + (CC_U/c')\|u+v\|_0^2$ follows from (2.4) in the case of (2.5). By both cases then,

- *The coerciveness condition* (1.7) *is equivalent to Aronszajn's condition* (1.6).

5. Let $B[x, y]$ be a bounded Hermitian bilinear form on a Hilbert space H with norm $\|\ \|$. Let $V \subset H$ be a closed subspace with orthogonal complement $V^\perp$. Suppose $|B[x, x]| \ge c\|x\|^2$, with some $c > 0$, for all $x \in V$. Each $y \in V^\perp$ yields a bounded linear functional $B[x, y]$ on $x \in V$. Applying Lax–Milgram to the Hilbert space V there is a unique $v = v_y \in V$ such that $B[x, y] = B[x, v]$ for all $x \in V$. Put $w = w_y = y - v_y$; the set W of all such w from all $y \in V^\perp$ is, by the boundedness of the linear projection operators, a closed subspace of H with the additional properties that $H = V \oplus W$ is the direct sum, and $B[x, w] = 0$ for each $w \in W$ and for all $x \in V$.

6. Let ς denote a smooth vector field of $\mathbb{R}^n$ that is transverse to the boundary of a bounded Lipschitz domain Ω with outer unit normal vector ν, i.e. $\varsigma \cdot \nu \ge c_\Omega > 0$ a.e.(ds). The existence of such vector fields is equivalent to the existence of smooth approximating domains [Neč62]. By compactness of $\partial\Omega$ and by classical ODE theory there is then a unique smooth flow $Y(X; t) \in \mathbb{R}^n$, defined for all X in a neighborhood of $\partial\Omega$ and all t in an open interval containing the origin, that solves the system $\frac{d}{dt}Y(X; t) = \varsigma(Y(X; t))$, $Y(X; 0) = X$. By uniqueness for the initial value problem the map $X \mapsto Y(X; t)$ is a diffeomorphism for each t. Define $\partial\Omega_t = \{Y(X; t) : X \in \partial\Omega\}$. For all $|t|$ small enough these will be the boundaries of Lipschitz domains Ω_t with equivalent Lipschitz characters, $\varsigma \cdot \nu \ge c_\Omega/2$. Fix one such domain $\Omega_{t_1} \subset \Omega$, $t_1 < 0$, and consider the annular region $\mathcal{A} = \Omega \setminus \overline{\Omega_{t_1}}$. Local patches of $\mathcal{A}$ can be defined by $\mathcal{A}_{\omega,\psi} = \{Y(X', 0; t) : t_1 + \psi(X') < t < \psi(X')$ and $Y(X', 0; \psi(X')) \in \partial\Omega\}$ where $X = (X', X_n)$ is a rotation and translation of the rectangular coordinate system of $\mathbb{R}^n$ and where $\omega \subset \mathbb{R}^{n-1}$ is open. The map $(X', t) \mapsto Y(X', 0; t)$ onto $\mathcal{A}_{\omega,\psi}$ is also a diffeomorphism. The Jacobian of this transformation, $\det[\nabla' Y\ \partial_t Y]$, when evaluated at points

$(X', t+\psi(X'))$, $t_1 < t < 0$, can also be written

$$\det\left([\nabla' Y\ \partial_t Y]\begin{bmatrix} I & 0 \\ \nabla'\psi & 1 \end{bmatrix}\right) = \det[\nabla_{X'} Y\ \partial_t Y]$$

by the chain rule. Thus the area of a patch is calculated to be $\int_{\mathcal{A}_{\omega,\psi}} dY = \int_{t_1}^0 dt \int_\omega |\det[\nabla_{X'} Y\ \ \varsigma]|\, dX'$. For each t the boundary $\partial\Omega_t$ is locally parameterized by $X' \mapsto Y(X', 0; t+\psi(X'))$. Consequently, the surface area Jacobian for $X' \to \partial\Omega_t$ is a.e.(ds) equal to $|\det[\nabla_{X'} Y\ \nu]|$ where ν is the unit normal vector to $\partial\Omega_t$. Because ς and $\varsigma \cdot \nu\ \nu$ differ by a vector tangent to $\partial\Omega_t$,

$$\int_{\mathcal{A}_{\omega,\psi}} dY = \int_{t_1}^0 dt \int_\omega \varsigma\cdot\nu\ |\det[\nabla_{X'} Y\ \nu]|\, dX' = \int_{t_1}^0 dt \int_{\partial\Omega_t \cap \mathcal{A}_{\omega,\psi}} \varsigma\cdot\nu\, ds_t.$$

7. Most of the proof [Aro61, pp. 37–39] of Aronszajn's theorem, as stated for Sobolev spaces in Theorem 1.4 above, will be given. The proof is based on Hausdorff's convexity of range theorem for general quadratic forms B. A *quadratic norm* on a vector space is the square root of a quadratic form that takes only positive (> 0) values for all nonzero vectors.

Theorem 2.1 (Hausdorff [Hau19]). *Let B be a quadratic form on a vector space V with quadratic norm $\|\ \|$. Then the range of $B[v]/\|v\|^2$ over $V \setminus \{0\}$ is a convex subset of the complex plane.*

The proof reduces immediately to the case of a two-dimensional vector space V and can be found on pp. 37–38 of [Aro61].

To prove Theorem 1.4, if $A[v]$ is coercive over $W^{m,2}(\Omega)$, then there are a constant $c > 0$ and a subspace V of finite codimension such that $|A[v]|/\|v\|^2 \geq c$ for all $v \in V \setminus \{0\}$. The convex range of $A[v]/\|v\|^2$ is disjoint from the interior of the disc of radius c centered at the origin. Therefore there is a separating line which can be taken as a tangent line to the disc. There is then an angle so that $\mathrm{Re}(e^{i\theta}A[v]) \geq c\|v\|_m^2$ for all $v \in V$. Since $A[v]$ is also bounded, there is an interval of angles. Again letting U be the orthogonal complement of V and uniquely writing each element of $W^{m,2}$ as $u+v$, define the quadratic form $K[u+v] = -\mathrm{Re}(e^{i\theta}A[u,v]) - \mathrm{Re}(e^{i\theta}A[v,u]) - \mathrm{Re}(e^{i\theta}A[u]) + c\|u\|_m^2$. (See Subsection 2 above.) By orthogonality $c\|u+v\|_m^2 \leq \mathrm{Re}(e^{i\theta}A[v]) + c\|u\|_m^2 = \mathrm{Re}(e^{i\theta}A[u+v]) + K[u+v]$. Since K vanishes identically on V, Aronszajn's Theorem 1.4 is proved.

8. The *contrapositive lemma* follows:

Lemma 2.2. *Let V be a closed subspace of $W^{m,2}(\Omega)$ and A a Hermitian symmetric quadratic form. The coerciveness estimate*

$$A[v] + c_0 \int_\Omega |v|^2\, dX \geq c\|v\|_m^2 \quad \textit{for all } v \in V \tag{2.6}$$

does not *hold for any c_0 and $c > 0$ if and only if there exists a sequence $\{v_j\}_{j=1}^\infty$ of nonzero* *elements of V such that*

$$A[v_j] + j \int_\Omega |v_j|^2 \, dX \le \frac{1}{j} \|v_j\|_m^2, \qquad j = 1, 2, \dots. \tag{2.7}$$

Proof. Necessity follows directly.

For sufficiency, if both inequalities (2.6) and (2.7) hold, then for all $1/j < c$,

$$A[v_j] + j \int_\Omega |v_j|^2 \, dX \le A[v_j] + c_0 \int_\Omega |v_j|^2 \, dX,$$

so that $j \le c_0$, a contradiction. □

For sufficiency it does not work to replace (2.7) with $A[v_j] + c_1 \int_\Omega |v_j|^2 \, dX \le j^{-1} \|v_j\|_m^2$ for some constant c_1. This is because it is possible, for a given c_1, that there is a nonzero v such that $A[v] + c_1 \int_\Omega |v|^2 \, dX \le 0$.

3. Operator symbols and formally positive forms for L_γ

3.1. Algebraic properties of the operator symbols

The Fourier symbols for the operators L_γ, after a slight change of variables (the inverse of (3.2) below), are certain real homogeneous polynomials

$$\begin{aligned} &p_\gamma(u, v, w, x, y, z) \\ &= (u^2 + v^2 + vw)^2 + (w^2 - \gamma(x^2 + y^2 + z^2))^2 + (wx - yz)^2 + (wy - zx)^2 + (wz - xy)^2, \end{aligned} \tag{3.1}$$

$0 < \gamma < 1/3$, constructed in [Ver10, p. 244]. Write any of these as $p = \sum_{j=1}^5 q_j^2$.

As a sum of squares of the real polynomials q_j, the positive definiteness of a p can be deduced by showing that there are no roots in $\mathbb{R}^6 \setminus \{0\}$ for the system of equations

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 0, \quad q_4 = 0, \quad q_5 = 0.$$

The only roots for this system are the scalar multiples of the roots $(1, \pm i, 0, 0, 0, 0)$. Further, the existence of these nontrivial complex roots to the system is an integral part of the construction (3.1) and acquires significance because of another objective of the construction. The representation of the polynomials (3.1) as sums of squares is essentially unique. The idea of unique representation is taken from [CLR95] and is also explained in [Ver10], [Ver12]. Uniqueness of representation is proved for the polynomials (3.1) in [Ver10, Theorem 4.1]. It can be accurately expressed here by saying that if $p = \sum r_k^2$ is any other sum of squares representation for p, then the r_k *are necessarily linear combinations of the* q_j. Thus the system $r_1 = 0, \ r_2 = 0, \dots$ must also have the roots $(1, \pm i, 0, 0, 0, 0)$.

These algebraic properties of the polynomials (3.1) also hold for any polynomial of the form $q = \sum_{j=1}^5 c_j q_j^2$ where $c_j > 0$ $(j = 1, \dots, 5)$. For if $q = \sum r_k^2$ with some r_k *not* *a linear combination of the* q_j, then by defining $c^* = \max\{c_j\}$, any polynomial of (3.1),

$$p = \sum_{j=1}^{5}\left(1 - \frac{c_j}{c^*}\right)q_j^2 + \frac{1}{c^*}\sum r_k^2,$$

would also have such a sum of squares representation. This contradiction together with the change of variables in (3.1)

$$u \to \tfrac{1}{4}u, \qquad v \to \tfrac{1}{4}v - \tfrac{1}{2}w, \qquad w \to w \tag{3.2}$$

proves that the symbols for the operators L_γ have the same algebraic properties as just described for the polynomials (3.1) (see [Ver10, Corollary 1.5 and Theorem 3.1]).

3.2. Noncoerciveness of the formally positive forms

The sum of squares representation (1.2) for an operator L_γ directly yields the associated *formally positive quadratic form*

$$\begin{aligned} G[v] &= \sum_{|\alpha|=|\beta|=2} \int_\Omega g_{\alpha\beta}\partial^\alpha v \partial^\beta \overline{v}\, dX \\ &= \int_\Omega \left(\left| \left(\frac{\partial_1^2 + \partial_2^2}{4} - \partial_3^2 \right) v \right|^2 + |\partial_3^2 v - \gamma(\partial_4^2 + \partial_5^2 + \partial_6^2)v|^2 \right. \\ &\quad \left. + |(\partial_3\partial_4 - \partial_5\partial_6)v|^2 + |(\partial_3\partial_5 - \partial_6\partial_4)v|^2 + |(\partial_3\partial_6 - \partial_4\partial_5)v|^2 \right) dX, \end{aligned} \tag{3.3}$$

where each coefficient $g_{\alpha\beta}$ is uniquely determined as the sum of the coefficients of the corresponding terms $\partial^\alpha v \partial^\beta \overline{v}$ that result from expanding each of the five squares. The form (3.3) is a real symmetric quadratic form and the $g_{\alpha\beta}$ are the entries of a symmetric 21×21 positive semidefinite matrix. Conversely, any real symmetric quadratic form with coefficients that make a positive semidefinite matrix is a formally positive quadratic form. Positive semidefinite matrices that represent the sums of squares of real polynomials are known as *Gram matrices* [CLR95].

The assertion that L_γ has a unique sum of squares representation (uniqueness of the *sos* representation (3.1), (1.2)) has a well defined meaning in terms of associated real symmetric quadratic forms. It means that the form $G[v]$ of (3.3) is the only associated real symmetric quadratic form for L_γ that has coefficients that make a positive semidefinite matrix. All other associated real symmetric quadratic forms are *algebraically indefinite* and not formally positive (see [Ver12]).

Now the algebraic system of equations $q_1 = 0, \ldots, q_5 = 0$ that corresponds to each L_γ, from its sum of squares representation (1.2), has roots in $\mathbb{C}^6 \setminus \{0\}$. It follows by Aronszajn's algebraic characterization of coercive formally positive forms (Theorem 1.6) that the only formally positive real symmetric form for an L_γ, viz. $G[v]$, must be *noncoercive* in any bounded open set. By the reduction to real forms in the introduction or by arguments in [Ver12],

- *There are no coercive formally positive Hermitian quadratic forms* (1.3) *for the operators* L_γ *in any domain.*

It is possible, however, for an algebraically indefinite form to be coercive over $W^{m,2}(\Omega)$. For example, Agmon [Agm61] shows this to be true for certain forms associated to the bi-Laplacian. Using Agmon's characterization of general coercive forms [Agm58], it is shown in [Ver12] that certain algebraically indefinite forms associated to the L_γ are coercive over $W^{2,2}$, as long as the domains Ω have C^2 boundaries.

Remark 3.1. The polynomial $w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4wxyz$ was discovered by Choi and Lam [CL77], and is an example of a Motzkin type positive semidefinite polynomial that cannot be written as a sum of squares. The polynomials p_γ were constructed in [Ver10] by continuously perturbing Choi and Lam's polynomial into the cone of positive definite polynomials until a sum of squares was produced. This yielded the elliptic operators $\widehat{L}_\gamma$ in (6.2) below. By adding the first square in (3.1), both the uniqueness and the noncoerciveness of the formally positive form G was achieved.

4. Null forms and a basic argument

4.1. Null forms

Given an L_γ and its associated formally positive form $G[v]$, any other homogeneous real symmetric constant coefficient quadratic form $A[v] = \sum_{|\alpha|=|\beta|=2} \int_\Omega a_{\alpha\beta}\partial^\alpha v \partial^\beta \overline{v}\, dX$ associated to L_γ must differ from G by a *null quadratic form*

$$N[v] = \sum_{|\alpha|=|\beta|=2} \int_\Omega n_{\alpha\beta}\partial^\alpha v \partial^\beta \overline{v}\, dX, \tag{4.1}$$

also with real symmetric coefficients and satisfying $\sum_{|\alpha|=|\beta|=2} n_{\alpha\beta}\partial^{\alpha+\beta} = 0$.

Define

$$\mathrm{Re} \int_\Omega (\partial^\alpha v \partial^\beta \overline{v} - \partial^{\alpha'} v \partial^{\beta'} \overline{v})\, dX \tag{4.2}$$

to be an *elementary null form* whenever $\alpha + \beta = \alpha' + \beta'$ and $|\alpha| = |\beta| = |\alpha'| = |\beta'|$. In $\mathbb{R}^6$ for fourth order homogeneous operators, a basis for all null quadratic forms can be formed by a collection of 105 elementary null forms [Ver10, p. 240].

4.2. A basic argument

The noncoerciveness of $G[v]$ established above can be seen directly, without Aronszajn's characterization, by observing that $G[v]$ vanishes whenever $v(X) = h(X_1, X_2)$ for any harmonic function of two variables that defines v in $W^{2,2}(\Omega)$. Since these functions form an infinite-dimensional subspace, coerciveness of G over $W^{2,2}(\Omega)$,

$$G[v] + c_0 \int_\Omega |v|^2\, dX \geq c|v|_2^2, \qquad c > 0,$$

contradicts the Rellich compactness theorem by the following lemma (see, for example, [Agm65, Lemma 11.6]).

Lemma 4.1. *Let Ω be a bounded domain in which Rellich's compactness theorem holds. Suppose $A[v]$ is coercive over $W^{m,2}(\Omega)$ while $A[v] \leq 0$ for all v in a subspace $V \subset W^{m,2}(\Omega)$. Then V is finite-dimensional.*

Proof. Suppose V is not finite-dimensional. Being a subspace of $L^2(\Omega)$, V contains an infinite sequence of orthonormal functions. By the coerciveness estimate (1.23), the inequality $A[v] \le 0$ and Rellich compactness there is then a subsequence $\{v_j\}$ that converges in L^2. Now $\|v_{j+1} - v_j\|_0 \to 0$ contradicts orthonormality. □

Consequently, any null quadratic form N for which $A[v] = G[v] + N[v]$ might be coercive, is necessarily a *positive* multiple of the elementary null form

$$N_0[v] = \frac{1}{2} \operatorname{Re} \int_\Omega (|\partial_1 \partial_2 v|^2 - \partial_1^2 v \partial_2^2 \overline{v})\, dX \tag{4.3}$$

plus some linear combination of the remaining 104 basis null forms. This follows because N_0 is positive on all but a finite-dimensional subspace of the space of harmonic functions $h(X_1, X_2)$. But all other elementary null forms vanish on every harmonic function $h(X_1, X_2)$. Therefore we have the following approach to proving Theorem 1.5.

- *To prove the noncoerciveness for the forms A of Theorem 1.5, it suffices to show that whenever a choice of $b > 0$ and a choice of null form N is given, there is an infinite-dimensional subspace on which $G[v] + bN_0[v] + N[v] \le 0$.*

5. Change of notation

In the remainder of this article, $\mathbb{R}^6$ will be thought of as $\mathbb{C} \times \mathbb{R} \times \mathbb{R}^3$. The operators L_γ are written

$$L_\gamma = (\partial\overline{\partial} - \partial_t^2)^2 + (\partial_t^2 - \gamma\triangle)^2 + (\partial_t\partial_1 - \partial_2\partial_3)^2 + (\partial_t\partial_2 - \partial_3\partial_1)^2 + (\partial_t\partial_3 - \partial_1\partial_2)^2, \tag{5.1}$$

$0 < \gamma < 1/3$. We write $z \in \mathbb{C}$, $z = x + iy$ for $(x, y) \in \mathbb{R}^2$, $\partial = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $t \in \mathbb{R}$, $\partial_t = \partial/\partial t$, $s = (s_1, s_2, s_3) \in \mathbb{R}^3$, $\partial_j = \partial/\partial s_j$ $(j = 1, 2, 3)$, and $\triangle = \partial_1^2 + \partial_2^2 + \partial_3^2$.

In this notation the elementary null form N_0 (4.3) is

$$N_0[v] = \int_\Omega (|\partial^2 v|^2 + |\overline{\partial}^2 v|^2 - 2|\partial\overline{\partial} v|^2). \tag{5.2}$$

For economy of notation we will also write $\partial_t v = v_t$, $\partial_k v = v_k$, $\partial_t\partial_j v = v_{tj}$, etc. By ∇v is meant the gradient in s, (v_1, v_2, v_3), so that $\partial\nabla v = (\partial v_1, \partial v_2, \partial v_3)$ and $|\partial\nabla v|^2 = |\partial v_1|^2 + |\partial v_2|^2 + |\partial v_3|^2$, etc. Polar coordinates $z = re^{i\theta}$ in $\mathbb{C}$, and $s = \rho\sigma$, $\rho = |s|$, $\sigma \in \mathbb{S}^2$ in $\mathbb{R}^3$ will also be used.

6. Two over-determined systems and their solution spaces

6.1. The elliptic noncoercive over-determined system and its solution space **X**

The algebraic system of equations $q_1 = 0, \ldots, q_5 = 0$ that corresponds to each L_γ from Section 3 are symbol equations for an elliptic second order over-determined system (*o-ds*)

$$\partial\overline{\partial} v - v_{tt} = v_{tt} - \gamma\triangle v = v_{t1} - v_{23} = v_{t2} - v_{31} = v_{t3} - v_{12} = 0. \tag{6.1}$$

Ellipticity here is the same as the ellipticity of the sum of squares operators L_γ (see [Agm65, Definition 6.3]).

Define $\mathbf{X} \subset W^{2,2}(\Omega)$ to be the linear space of weak solutions to the *o-ds* (6.1) in Ω. The solution space $\mathbf{X}$ is the subspace on which the formally positive quadratic form $G[v]$ vanishes identically. As mentioned in Section 4, $\mathbf{X}$ is infinite-dimensional. It is also *self-adjoint*, i.e. $v \in \mathbf{X}$ if and only if $\overline{v} \in \mathbf{X}$.

Because the elliptic *o-ds* is homogeneous and constant coefficient, interior regularity theory shows that each weak solution $v \in \mathbf{X}$ is *infinitely differentiable* when redefined on a set of measure zero (see [Agm65, Theorem 6.6]).

6.2. *The reduced over-determined system*

By omitting the first equation from the elliptic *o-ds* (6.1) one obtains a *reduced* elliptic *o-ds* in $\mathbb{R}^4$. It corresponds to the elliptic fourth order operator

$$\widehat{L}_\gamma = (\partial_t^2 - \gamma\triangle)^2 + (\partial_t\partial_1 - \partial_2\partial_3)^2 + (\partial_t\partial_2 - \partial_3\partial_1)^2 + (\partial_t\partial_3 - \partial_1\partial_2)^2, \tag{6.2}$$

which like L_γ has a unique sum of squares representation and therefore a unique formally positive quadratic form [Ver10, Theorem 3.1]. This form is coercive and the reduced *o-ds* can therefore be thought of as coercive.

The following proposition is proved by applying Hilbert's *Nullstellensatz* for a different end but in a way almost identical to part of the proof of Smith's theorem extending Aronszajn's algebraic characterization of coercive formally positive forms to bounded Lipschitz domains (see [Agm65, p. 161]).

Proposition 6.1. *There is a positive integer d such that for every infinitely differentiable function $v(t,s)$ that solves the reduced o-ds $v_{tt} - \gamma\triangle v = v_{t1} - v_{23} = v_{t2} - v_{31} = v_{t3} - v_{12} = 0$ in any open set of $\mathbb{R}^4$ it follows that v is a polynomial of degree no more than d.*

Proof. Because the operator symbols q_2, q_3, q_4, q_5 from (3.1) share no common nonzero complex root, Hilbert's *Nullstellensatz* implies there are polynomials $Q_{\alpha,j}$ such that $\partial^\alpha = \sum_{j=2}^5 Q_{\alpha,j}(\partial)q_j(\partial)$ for every multi-index α with order greater than some number depending only on the q_j. An application of Taylor's theorem yields the result. ◻

In fact $d = 4$ and the solution space $\widehat{\mathbf{X}}$ of the reduced system has a basis of 16 real homogeneous polynomials. The basis elements can be taken to be derivatives of the *unique homogeneous fourth degree element*

$$P(t,s) = \frac{3\gamma}{4!}t^4 + \frac{1}{2^2}|s|^2t^2 + s_1s_2s_3t + \frac{1}{2^2}(s_1^2s_2^2 + s_2^2s_3^2 + s_3^2s_1^2) + \frac{\gamma^{-1}-2}{4!}(s_1^4 + s_2^4 + s_3^4). \tag{6.3}$$

These assertions can be verified as follows.

By inspection the set

$$\mathcal{B} = \{P, P_t, P_1, P_2, P_3, P_{t1}, P_{t2}, P_{t3}, P_{11}, P_{22}, P_{33}, t, s_1, s_2, s_3, 1\} \tag{6.4}$$

is *linearly independent* (the four remaining second order derivatives are in the span of $\mathcal{B}$

since P is a solution to the *o-ds*). That $\mathcal{B}$ spans $\widehat{\mathbf{X}}$ is a consequence of the uniqueness of P, which will be deduced in three steps.

Re-index the polynomials from (6.2) and (3.1),

$$q_0(t,s) = t^2 - \gamma|s|^2, \quad q_1(t,s) = ts_1 - s_2s_3, \quad q_2(t,s) = ts_2 - s_3s_1, \quad q_3(t,s) = ts_3 - s_1s_2.$$

Let $\mathbb{C}[x]$ denote the polynomial ring over $\mathbb{C}$ in $x = (x_1, x_2, \dots)$ and let $\mathbb{C}[x]_d$ denote the vector space over $\mathbb{C}$ of polynomials homogeneous of degree d in x.

First, the set $\mathcal{C}_3 = \{tq_0, s_jq_0, tq_j, s_jq_k : 1 \le j,\ k \le 3\}$ of 16 polynomials is linearly independent in $\mathbb{C}[t,s]_3$. This is because no dependence relation can nontrivially contain tq_0 as it is the only polynomial with t^3; likewise s_j^3 is exclusive to s_jq_0 which thus being eliminated from any dependence relation leaves t^2s_j exclusively to tq_j which are therefore also eliminated. For the remaining nine elements of $\mathcal{C}_3$, s_j^2 is exclusive to s_jq_j, which are therefore eliminated. The remaining six elements of $\mathcal{C}_3$ each contain and are in one-to-one correspondence with $s_j^2s_k$ $(j \ne k)$, and are eliminated.

Second, there is the *dual relation*, $\overline{q(\partial_t,\partial)}q(t,s) > 0$, for every nonzero homogeneous polynomial $q \in \mathbb{C}[t,s]$ (see [SW71, p. 139] for example). Let q be any linear combination of the polynomials from $\mathcal{C}_3$ and let P' be any linear combination of the third degree polynomials from $\mathcal{B}$. Since $\overline{q(\partial_t,\partial)}P'(t,s) = 0$, it follows that $q + P' = 0$ will imply $0 = (\overline{q(\partial_t,\partial)} + \overline{P'(\partial_t,\partial)})P'(t,s) = \overline{P'(\partial_t,\partial)}P'(t,s)$, whence $P' = 0$. Because $\mathbb{C}[t,s]_3$ has dimension 20 it then follows that $\mathcal{C}_3 \cup \{P_t, P_1, P_2, P_3\}$ is a basis for $\mathbb{C}[t,s]_3$. It also follows from the dual relation that no nontrivial q in the span of $\mathcal{C}_3$ can be in the solution space $\widehat{\mathbf{X}}$. Consequently, it has been shown that all homogeneous solutions of degree 3 to the reduced *o-ds* are in $\operatorname{span}\{P_t, P_1, P_2, P_3\}$, as asserted.

Third, suppose there is a homogeneous fourth degree solution that is not a multiple of P. Then there is a linear combination of the two solutions, Q, such that $Q_t = \sum_{j=1}^3 a_jP_j$, whence $Q = \sum_{j=1}^3 a_j \int P_j dt + r(s)$ where r is homogeneous of degree 4. Applying the operators $\partial_k q_k(\partial_t,\partial)$, $k = 1,2,3$, and noting that $\partial_1\partial_2\partial_3 P_j = 0$ it follows that

$$\begin{aligned} 0 = a_1(\gamma^{-1} - 2)s_1 + a_2s_2 + a_3s_3 - r_{123} &= a_1s_1 + a_2(\gamma^{-1} - 2)s_2 + a_3s_3 - r_{123} \\ &= a_1s_1 + a_2s_2 + a_3(\gamma^{-1} - 2)s_3 - r_{123}. \end{aligned}$$

Since $\gamma \ne 1/3$ it follows that $a_1 = a_2 = a_3 = 0$ and Q is a polynomial in s only. One observes that the ideal $\langle \triangle, \partial_2\partial_3, \partial_3\partial_1, \partial_1\partial_2 \rangle \subset \mathbb{C}[\partial]$ contains $\mathbb{C}[\partial]_3$. Consequently, the vanishing of $Q(s)$ under each of $\partial_j q_0(\partial_t,\partial)$, $\partial_j q_k(\partial_t,\partial)$, $1 \le j,k \le 3$, implies $Q = 0$. Thus P is unique.

Finally, any homogeneous fifth degree solution R must then have each of its first order derivatives equal to a multiple of P. If R is not identically *zero*, this can only occur if R is a *nonzero* multiple of the monomial x^5 with x evaluated at a linear combination $b_0t + b_1s_1 + b_2s_2 + b_3s_3$ so that P would similarly be a multiple of x^4, which is not true.

6.3. Algebraic structure of the solutions in $\mathbf{X}$

Let $\Omega \subset \mathbb{R}^6$ be a bounded convex domain and let $v \in \mathbf{X} \subset W^{2,2}(\Omega)$. Fix $z = x + iy$ so that the set $\{(z,t,s) : (t,s) \in \mathbb{R}^4\}$ has nonempty intersection with Ω. Then for z fixed,

$v(z,t,s)$ is the restriction of a polynomial from $\widehat{\mathbf{X}}$ with complex coefficients. Therefore

$$v(z,t,s) = \sum_j f_j(z) p_j(t,s) \tag{6.5}$$

where, by the previous section, the p_j can be taken to form a basis of real homogeneous polynomials for $\widehat{\mathbf{X}}$, and the coefficients $f_j(z)$ are in $\mathbb{C}$. Since $v \in C^\infty(\Omega)$, the f_j are also infinitely differentiable in z. Apply the operator $\partial\overline{\partial} - \partial_t^2$ to (6.5), obtaining

$$0 = \sum_j \partial\overline{\partial} f_j p_j - \sum_j f_j \partial_t^2 p_j. \tag{6.6}$$

The second sum contains only polynomials from $\widehat{\mathbf{X}}$ of at most degree 2. By linear independence,

- *The coefficient in v of each p_j that is of degree 4 or 3 must satisfy $\partial\overline{\partial} f_j = 0$.*

Using this and applying $\partial\overline{\partial}$ to (6.6) yields the second sum now over only the constant polynomial. Therefore by the first sum,

- $(\partial\overline{\partial})^2 f_j = 0$ *for each p_j of degree 2 or 1.*

Finally,

- $(\partial\overline{\partial})^3 f_j = 0$ *if f_j is the coefficient of the constant polynomial in v.*

6.4. *Elementary solutions of* $\mathbf{X}$

These are based on a *general polynomial* (not necessarily homogeneous) $Q \in \widehat{\mathbf{X}}$ and a given complex valued harmonic function $h(z)$. They are in $W^{2,2}(\Omega)$ by definition, and take the form (6.5) as

$$v = E(h, Q) := hQ + \mathcal{G}(h) Q_{tt} + \mathcal{G}(\mathcal{G}(h)) Q_{tttt} \tag{6.7}$$

where $\mathcal{G}(f)$ denotes any function of z satisfying $\partial\overline{\partial}\mathcal{G}(f) = f$ (as long as $v \in W^{2,2}(\Omega)$).

Remark 6.2. *Any $v \in \mathbf{X}$ can be written as a linear combination of functions* (6.7)*, $E(h_j, p_j)$, $1 \leq j \leq 16$, where the p_j are the homogeneous basis polynomials for $\widehat{\mathbf{X}}$ from* (6.5). To see this let v be as in (6.5) and let d be the highest degree of any p_j with *nonzero* coefficient $f_j(z)$. Then by the same arguments as those following (6.5) the coefficients of basis elements of degrees d and $d-1$ are harmonic, while those of basis elements of degrees $d-2$ and $d-3$ are biharmonic, etc. For each $Q = p_j$ of degree d in (6.5) subtract from v the corresponding solution (6.7) with $h = h_j = f_j$. The result is still a solution but with new f_j's and with highest degree now at most $d-1$ so that the coefficients of basis elements of degrees $d-1$ and $d-2$ are now harmonic, etc. With a few iterations this decomposes v into a linear combination of solutions $E(h_j, p_j)$, as asserted.

However, even though $v \in W^{2,2}(\Omega)$ and even though each $E(h_j, p_j)$ satisfies the *o-ds* (6.1), *it does not follow that the $E(h_j, p_j)$ are in* $W^{2,2}(\Omega)$. This loss of integrability can be seen in the following example.

Example 6.3. Take $p_1 = s_1 + t$ and $p_2 = s_1 - t$ for basis elements. Now consider the convex domain in $\mathbb{R}^6$ defined by

$$\Omega = \{(z, t, s) : |z| < t < 1 - |s|\}$$

and let $v = 2f(z)s_1 = E(f, p_1) + E(f, p_2)$ for a holomorphic function f in the disc $\mathbb{D} = \{z : |z| < 1\}$. Denote Lebesgue measure in the disc by $dm = dm(z)$. Then

$$\int_\Omega |\partial^2 v|^2 = 4 \int_{\mathbb{D}} |f''|^2 \, dm \int_{|s|<1-|z|} s_1^2 \, ds \int_{|z|}^{1-|s|} dt = \frac{8\pi}{45} \int_{\mathbb{D}} |f''|^2 (1 - |z|)^6 \, dm,$$

while

$$\int_\Omega |\partial^2 E(f, p_j)|^2 = \int_\Omega |f''|^2 (s_1^2 + t^2)$$
$$\geq \int_{\mathbb{D}} |f''|^2 \, dm \int_{|z|}^1 t^2 \, dt \int_{|s|<1-t} ds = \frac{\pi}{45} \int_{\mathbb{D}} |f''|^2 (1 - |z|)^4 (1 + 4|z| + 10|z|^2) \, dm,$$

$j = 1, 2$. Then for example, if $f = 1/(1 - z)$ the first integration converges while the second does not; computing the norms of the remaining derivatives shows that $2s_1/(1 - z) \in W^{2,2}(\Omega)$.

7. The basic argument

For a bounded convex domain Ω the unique formally positive form $G[v]$ (3.3) vanishes identically on the *o-ds* solution space $\mathbf{X}$. Therefore by the basic argument of Section 4 and normalizing $b = 1$,

- *To prove Theorem 1.5, it suffices to show that whenever a choice of null form N is given, there is an infinite-dimensional subspace of* $\mathbf{X}$ *on which* $N_0[v] + N[v] \leq 0$.

Here N_0 is the null form of (5.2) and (4.3) that has been shown to be necessary for coerciveness because it and only it is positive on the harmonic solutions.

8. The convex domains $\Omega_{M,T}$ and solution spaces $\mathbf{X}_{M,T}$

8.1. The domains

Parameters $M > 0$ and $T > 0$ will be called *Lipschitz constants*. Define

$$\Omega = \Omega_{M,T} = \{(z, t, s) : T|s| < t < (1 - |z|)/M\}. \tag{8.1}$$

In the polar coordinates for $\mathbb{C}$ and $\mathbb{R}^3$,

$$\Omega = \{(re^{i\theta}, t, \rho\sigma) : T\rho < t < (1 - r)/M\}.$$

Let $0 < \phi < 1$, $z = (1-\phi)z_1 + \phi z_2$, $t = (1-\phi)t_1 + \phi t_2$ and $s = (1-\phi)s_1 + \phi s_2$. Then

$$T|s| \le T(1-\phi)|s_1| + T\phi|s_2| < t = (1-\phi)t_1 + \phi t_2 < (1-\phi)\frac{1-|z_1|}{M} + \phi\frac{1-|z_2|}{M}$$
$$= \frac{1-(1-\phi)|z_1| - \phi|z_2|}{M} \le \frac{1-|z|}{M}.$$

Therefore Ω is bounded and convex.

The outer unit normal vector does not vary continuously on the boundary of Ω, hence $\partial\Omega \notin C^1$.

8.2. *Invariance of the solution spaces* $\mathbf{X} = \mathbf{X}_{M,T}$ *under transformations*

Let $\mathbf{X}_{M,T}$ denote the solution space $\mathbf{X}$ of the *o-ds* (6.1) in the domain $\Omega_{M,T}$.

By the structure of the solutions (6.5), reflection and rotation invariance of the Laplacian $\partial\overline{\partial}$ and the property of $\Omega_{M,T}$ that each nonempty $D_{t,s} = \{z : (z,t,s) \in \Omega_{M,T}\}$ is a disc (of radius $1 - Mt$) centered at the *origin*, it follows that

- $\mathbf{X}_{M,T}$ *is invariant under unitary transformations of* $\mathbb{C}$.

That is, $v(z,t,s) \in \mathbf{X}_{M,T}$ if and only if $v(\overline{z},t,s)$ and $v(e^{i\phi}z,t,s)$, for each $0 \le \phi \le 2\pi$, are also solutions.

Though the reduced solution space $\widehat{\mathbf{X}}$ is not invariant under unitary transformations of $\mathbb{R}^3$,

- $\widehat{\mathbf{X}}$ *is invariant under permutations of* $\{s_1, s_2, s_3\}$, *as also is* $\mathbf{X}_{M,T}$.

This can be seen by inspecting the polynomial basis $\mathcal{B}$ (6.4).

8.3. *Elementary solutions in* $\mathbf{X}_{M,T}$

Let $f(z)$ be a holomorphic function in the unit disc $\mathbb{D}$. Denote by f_{-1}, f_{-2} holomorphic primitives of f in $\mathbb{D}$, i.e. $f''_{-2} = f'_{-1} = f$. For $Q \in \widehat{\mathbf{X}}$, $\Omega = \Omega_{M,T}$ and $fQ \in W^{2,2}(\Omega)$ define elementary solutions (6.7) by

$$E(f,Q) = fQ + \overline{z} f_{-1} Q_{tt} + \tfrac{1}{2}\overline{z}^2 f_{-2} Q_{tttt} \tag{8.2}$$

together with their complex conjugates.

Below we will make use of sequences of elementary solutions (8.2) defined by

$$v_n = E(z^n, Q) = z^n\big(Q + (n+1)^{-1}|z|^2 Q_{tt} + \tfrac{1}{2}(n+1)^{-1}(n+2)^{-1}|z|^4 Q_{tttt}\big),$$

$n = 1, 2, \ldots$, where the $Q \in \widehat{\mathbf{X}}$ are

$$\text{(a) nonzero,} \quad \text{(b) allowed to depend on } n. \tag{8.3}$$

We observe that

$$\partial E(z^n, Q) = nE(z^{n-1}, Q) \quad \text{and} \quad \overline{\partial} E(z^n, Q) = (n+1)^{-1} E(z^{n+1}, Q_{tt}).$$

9. Narrowing the space of null forms I

In this section the following argument will be successively applied. Given a null form N (4.1) that is assumed to be coercive over $\mathbf{X} = \mathbf{X}_{M,T}$ there is a null form N', also coercive over $\mathbf{X}$, that has a simpler structure than N, e.g. is a linear combination of fewer elementary null forms (4.2), or exhibits more symmetry. This argument results in a reduced space of null forms that must contain a coercive null form whenever some null form has been hypothesized to be coercive. In subsequent sections it will be shown that no null form in this reduced space can be coercive over $\mathbf{X}$.

We denote $\partial_t = \partial_0$ and again use the derivatives ∂_x and ∂_y. Subscripts j, k, l will denote derivatives in t or s. An integrand

$$\mathrm{Re}(\partial^\alpha v \partial^\beta \overline{v} - \partial^{\alpha'} v \partial^{\beta'} \overline{v})$$

of an elementary null form will also be called a null form.

9.1. Coerciveness preserving transformations of null forms

Both $\mathbf{X}$ and the Sobolev norm $\|v\|_2$ are invariant under unitary transformations in x and y. If it is assumed that $N[v] + c_0 \int_\Omega |v|^2\, dX \geq c|v|_2^2$ for some $v \in \mathbf{X}$, then the same inequality holds for all $v^\phi \in \mathbf{X}$ where $v^\phi(z, t, s) = v(e^{i\phi} z, t, s)$. Averaging over $0 \leq \phi \leq 2\pi$ yields

$$\frac{1}{2\pi} \int_0^{2\pi} N[v^\phi]\, d\phi + c_0 \int_\Omega |v|^2\, dX \geq c|v|_2^2$$

where the leftmost term is again a quadratic form in v that is coercive over $\mathbf{X}$. After changing variables $e^{i\phi} z \to z$ and applying Fubini's theorem this quadratic form is also seen to be a null form.

Using the unitary transformation of reflection and averaging also preserves coerciveness on $\mathbf{X}$:

$$\tfrac{1}{2}(N[v] + N[\widetilde{v}]) + c_0 \int_\Omega |v|^2\, dX \geq c|v|_2^2$$

where $\widetilde{v}(z, t, s) = v(\overline{z}, t, s)$, and results in a null form.

For example, the elementary null form (5.2) is preserved by each of the above transformations and averagings, $(2\pi)^{-1} \int_0^{2\pi} N_0[v^\phi]\, d\phi = \frac{1}{2}(N_0[v] + N_0[\widetilde{v}]) = N_0[v]$. In general, however, null forms are transformed into new null forms by these averagings. We will examine various cases of this. The cases are described by the types of derivatives occurring in the multi-index $\alpha + \beta$ for an elementary null form (4.2).

For a function F of the complex variable z the following conventions will be used:

$$\partial F = F_z, \quad \overline{\partial} F = F_{\overline{z}}, \quad F^\phi(z) = F(e^{i\phi} z),$$

so that the chain rule is written

$$\partial F(e^{i\phi} z) = e^{i\phi} F_z(e^{i\phi} z), \quad \text{i.e.} \quad \partial F^\phi = e^{i\phi} F_z^\phi,$$

and similarly

$$\overline{\partial} F^{\phi} = e^{-i\phi} F^{\phi}_{\overline{z}} \,.$$

The chain rules for reflection are written

$$\partial F(\overline{z}) = F_{\overline{z}}(\overline{z}) \quad \text{and} \quad \overline{\partial} F(\overline{z}) = F_z(\overline{z}).$$

9.2. *Reducing null forms containing xy-derivatives*

9.2.1. Elementary null forms containing only xy-derivatives. The only nonzero elementary null form containing only xy-derivatives is $N_0[v]$ of (4.3) and (5.2).

9.2.2. Exactly one or three xy-derivatives. Applying the null form $\mathrm{Re}(v_{xj}\overline{v}_{kl} - v_{kj}\overline{v}_{xl})$ to v^{ϕ}, integrating and then changing variables yields

$$\begin{aligned}\mathrm{Re}\int_{\Omega}&\big((\partial+\overline{\partial})v^{\phi}_j\overline{v^{\phi}_{kl}} - v^{\phi}_{kj}(\partial+\overline{\partial})\overline{v^{\phi}_l}\big)\\ &= \mathrm{Re}\int_{\Omega}\big((e^{i\phi}v^{\phi}_{jz} + e^{-i\phi}v^{\phi}_{j\overline{z}})\overline{v^{\phi}_{kl}} - v^{\phi}_{kj}(e^{i\phi}\overline{v^{\phi}_{l\overline{z}}} + e^{-i\phi}\overline{v^{\phi}_{lz}})\big)\\ &= \mathrm{Re}\int_{\Omega}\big(e^{i\phi}(\partial v_j\overline{v}_{kl} - v_{kj}\partial\overline{v}_l) + e^{-i\phi}(\overline{\partial}v_j\overline{v}_{kl} - v_{kj}\overline{\partial}\overline{v}_l)\big).\end{aligned}$$

Averaging over $0 \le \phi \le 2\pi$ transforms this type of form to the zero form.

The same steps produce the same conclusion for any elementary null form with three xy-derivatives.

9.2.3. Exactly two xy-derivatives. There are three nonvanishing types of these represented by

(i) $\mathrm{Re}(v_{xx}\overline{v}_{jk} - v_{xk}\overline{v}_{jx})$,
(ii) $\mathrm{Re}(v_{xy}\overline{v}_{jk} - v_{xk}\overline{v}_{jy})$,
(iii) $\mathrm{Re}(v_{xk}\overline{v}_{jy} - v_{xj}\overline{v}_{ky})$, $j \ne k$.

Applying the first to v^{ϕ} as above yields

$$\mathrm{Re}\int_{\Omega}\big((e^{2i\phi}\partial^2 v + 2\partial\overline{\partial}v + e^{-2i\phi}\overline{\partial}^2 v)\overline{v}_{jk} - (e^{i\phi}\partial v_k + e^{-i\phi}\overline{\partial}v_k)(e^{i\phi}\partial\overline{v}_j + e^{-i\phi}\overline{\partial}\overline{v}_j)\big).$$

Averaging in ϕ transforms type (i) to

$$\mathrm{Re}\int_{\Omega}(2\partial\overline{\partial}v\overline{v}_{jk} - \partial v_k\overline{\partial}\overline{v}_j - \overline{\partial}v_k\partial\overline{v}_j). \tag{9.1}$$

Similarly (ii) transforms to

$$\mathrm{Re}\int_{\Omega}(-i\overline{\partial}v_k\partial\overline{v}_j + i\partial v_k\overline{\partial}\overline{v}_j), \tag{9.2}$$

which is nonzero only when $k \ne j$.

Form (iii) transforms to

$$\mathrm{Re}\int_\Omega \big(i\overline{\partial} v_k \partial \overline{v}_j - i\partial v_k \overline{\partial}\overline{v}_j - i\overline{\partial} v_j \partial \overline{v}_k + i\partial v_j \overline{\partial}\overline{v}_k\big) = 2Re\int_\Omega (i\overline{\partial} v_k \partial \overline{v}_j - i\partial v_k \overline{\partial}\overline{v}_j),$$

i.e. (9.2) again.

Now applying form (9.2) to reflected solutions $\widetilde{v} = v(\overline{z}, t, s)$ and then changing variables yields

$$\mathrm{Re}\int_\Omega (-i\widetilde{v}_{kz}\overline{\widetilde{v}_{jz}} + i\widetilde{v}_{k\overline{z}}\overline{\widetilde{v}_{j\overline{z}}}) = \mathrm{Re}\int_\Omega (-i\partial v_k \overline{\partial}\overline{v}_j + i\overline{\partial} v_k \partial \overline{v}_j).$$

Averaging with (9.2) it follows that all forms (ii) and (iii) transform to zero.

9.2.4. Two xy-derivatives with t-derivatives. From (9.1) follow two types, viz.

$$\mathrm{Re}(2\partial\overline{\partial} v \overline{v}_{tk} - \partial v_k \overline{\partial}\overline{v}_t - \overline{\partial} v_k \partial \overline{v}_t), \qquad k = 1, 2, 3, \tag{9.3}$$

and $2\partial\overline{\partial} v \overline{v}_{tt} - |\partial v_t|^2 - |\overline{\partial} v_t|^2$ which on $\mathbf{X}$ is, by (6.1),

$$N_t[v] = \int_\Omega (2|v_{tt}|^2 - |\partial v_t|^2 - |\overline{\partial} v_t|^2). \tag{9.4}$$

9.2.5. Two xy-derivatives with no t-derivatives. Again two types follow from (9.1): first,

$$\mathrm{Re}(\partial\overline{\partial} v \overline{v}_{jk} - \partial v_k \overline{\partial}\overline{v}_j - \overline{\partial} v_k \partial \overline{v}_j), \qquad 1 \le j < k \le 3; \tag{9.5}$$

second, $\mathrm{Re}(2\partial\overline{\partial} v \overline{v}_{jj} - |\partial v_j|^2 - |\overline{\partial} v_j|^2)$, $j = 1, 2, 3$. Since both $\mathbf{X}_{M,T}$ and $\Omega_{M,T}$ are invariant under permutations in s, this second form can be transformed by s permutations in a manner that is similar to the way above that (9.2) was transformed to zero by transposing x and y and averaging. Here, when $j = 1$, averaging with the two transpositions $(1, 2)$ and $(1, 3)$ yields $2\partial\overline{\partial} v \triangle \overline{v} - |\partial\nabla v|^2 - |\overline{\partial}\nabla v|^2$, which by the *o-ds* (6.1) becomes $(2/\gamma)|v_{tt}|^2 - |\partial\nabla v|^2 - |\overline{\partial}\nabla v|^2$. We use this to define

$$N_s[v] = \int_\Omega (2|v_{tt}|^2 - \gamma|\partial\nabla v|^2 - \gamma|\overline{\partial}\nabla v|^2). \tag{9.6}$$

The same transformation is obtained for $j = 2$ and 3.

9.3. Reducing null forms with t-derivatives and no xy-derivatives

9.3.1. Two t-derivatives. We have

$$\mathrm{Re}(v_{tt}\overline{v}_{jk} - v_{tk}\overline{v}_{jt}), \qquad 1 \le j < k \le 3. \tag{9.7}$$

The $j = k$ cases transform by way of transpositions in s to $\gamma^{-1}|v_{tt}|^2 - |\nabla v_t|^2$ as with (9.6). Define

$$N_{ts}[v] = \int_\Omega (|v_{tt}|^2 - \gamma|\nabla v_t|^2). \tag{9.8}$$

9.3.2. One t-derivative. For $1 \leq j, k \leq 3$ we have

$$\mathrm{Re}(v_{tj}\overline{v}_{kk} - v_{tk}\overline{v}_{kj}), \qquad j \neq k. \tag{9.9}$$

For forms like $\mathrm{Re}(v_{t1}\overline{v}_{23} - v_{t3}\overline{v}_{21})$, transformation by transposing (1 and 3 here) as above yields the zero null forms.

9.4. Reducing null forms containing only s-derivatives

For j, k, l distinct we have

$$\mathrm{Re}(v_{jj}\overline{v}_{kl} - v_{jl}\overline{v}_{kj}). \tag{9.10}$$

The form $\mathrm{Re}(v_{11}\overline{v}_{22} - |v_{12}|^2)$ can be transformed using transpositions $(1, 3)$ to $\mathrm{Re}((v_{11} + v_{33})\overline{v}_{22} - |v_{12}|^2 - |v_{32}|^2) = \mathrm{Re}(\triangle v\overline{v}_{22} - |\nabla v_2|^2)$, which in turn transforms by transpositions to $|\triangle v|^2 - |\nabla\nabla v|^2$ where $\nabla\nabla v$ denotes the Hessian matrix for s-differentiations. By (6.1) then, define

$$N_{ss}[v] = \int_\Omega (|v_{tt}|^2 - \gamma^2|\nabla\nabla v|^2). \tag{9.11}$$

9.5. A narrowed space of null forms

The computations of this section and the basic argument of Section 7 yield

Lemma 9.1. *If a null form N is coercive over $\mathbf{X} = \mathbf{X}_{M,T}$, then there exists a null form $N_0 + N'$, also coercive over $\mathbf{X}$, where N', when restricted to $\mathbf{X}$, is a linear combination of null forms N_t, N_s, N_{ts} and N_{ss} together with elementary null forms that have integrands (9.3), (9.5), (9.7), (9.9) and (9.10).*

10. Narrowing the space of null forms II

10.1. Some orthogonal subspaces of $\mathbf{X}_{M,T}$

Recall the fourth degree polynomial P (6.3) and the basis $\mathcal{B}$ (6.4). Partition $\mathcal{B}$ as

$$\mathcal{T} = \{P, P_t, P_{11}, P_{22}, P_{33}, t, 1\},$$
$$\mathcal{S}_1 = \{P_1, P_{t1}, s_1\}, \qquad \mathcal{S}_2 = \{P_2, P_{t2}, s_2\}, \qquad \mathcal{S}_3 = \{P_3, P_{t3}, s_3\}.$$

Define the corresponding spans $\widehat{\mathbf{T}}$, $\widehat{\mathbf{S}}_1$, $\widehat{\mathbf{S}}_2$ and $\widehat{\mathbf{S}}_3$ in $\widehat{\mathbf{X}}$.

By inspection and using the *o-ds* (6.1), first derivatives map the spans as follows:

$$\begin{aligned}
\partial_t &: \widehat{\mathbf{T}} \to \widehat{\mathbf{T}} \quad \text{and} \quad \widehat{\mathbf{S}}_j \to \widehat{\mathbf{S}}_j, \qquad j = 1, 2, 3,\\
\partial_1 &: \widehat{\mathbf{T}} \to \widehat{\mathbf{S}}_1 \to \widehat{\mathbf{T}} \quad \text{and} \quad \widehat{\mathbf{S}}_2 \to \widehat{\mathbf{S}}_3 \to \widehat{\mathbf{S}}_2,\\
\partial_2 &: \widehat{\mathbf{T}} \to \widehat{\mathbf{S}}_2 \to \widehat{\mathbf{T}} \quad \text{and} \quad \widehat{\mathbf{S}}_3 \to \widehat{\mathbf{S}}_1 \to \widehat{\mathbf{S}}_3,\\
\partial_3 &: \widehat{\mathbf{T}} \to \widehat{\mathbf{S}}_3 \to \widehat{\mathbf{T}} \quad \text{and} \quad \widehat{\mathbf{S}}_1 \to \widehat{\mathbf{S}}_2 \to \widehat{\mathbf{S}}_1.
\end{aligned}$$

In particular, two polynomials from different spans are mapped by the same derivative, of any order, to different spans.

Also by inspection each basis element of $\mathcal{T}$ is orthogonal in $L^2(\mathbb{S}^2, d\sigma)$ to each element of the $\mathcal{S}_j$, as is each element of $\mathcal{S}_j$ to each of $\mathcal{S}_k$, $j \neq k$.

Now define subspaces $\mathbf{T}$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ of $\mathbf{X}_{M,T}$ to be those solutions v in which only polynomials from $\widehat{\mathbf{T}}, \widehat{\mathbf{S}}_1, \ldots$ respectively appear in the representation (6.5). If v is taken from one of these subspaces, derivatives, up to order two, in ∂ and $\overline{\partial}$ of v remain, as L^2 functions, in that subspace. Also in this sense derivatives up to order two in t and s map between the subspaces according to the way they mapped between the corresponding spans above. Further, the $L^2(\mathbb{S}^2)$ orthogonality between spans induces an $L^2(\Omega_{M,T})$ orthogonality between subspaces since nonempty sets $B_{t,z} = \{s : (z, t, s) \in \Omega_{M,T}\}$ are balls (of radius t/T) centered at the origin of $\mathbb{R}^3$.

By these observations, if v and w are taken from distinct subspaces, then the $W^{2,2}$ inner product $(v, w)_2$ is zero. Indeed, each term of the inner product vanishes. Thus the subspaces defined above are mutually orthogonal in $\mathbf{X}_{M,T}$.

10.2. Coercive null forms over $\mathbf{X}$ *restricted to* $\mathbf{T}$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$

Consider the null forms (9.3), (9.5), (9.7), (9.9) and (9.10) restricted to any one of the subspaces defined here. By applying the mapping properties of derivatives between subspaces and the L^2 orthogonality between them it follows by inspection that each term in each of the null forms vanishes upon integration. By Lemma 9.1 this establishes

Lemma 10.1. *If any null form is coercive over* $\mathbf{X} = \mathbf{X}_{M,T}$*, then there exists a null form which, when restricted to* $\mathbf{X}$*, is a linear combination of the null forms* N_0, N_t, N_s, N_{ts} *and* N_{ss} *only, and which is coercive over each of the subspaces* $\mathbf{T}$, $\mathbf{S}_1$, $\mathbf{S}_2$ *and* $\mathbf{S}_3$.

11. The basic argument

Because of Lemma 10.1 the basic argument can now be stated as follows:

- *To prove Theorem 1.5, it suffices to show that for each null form on* $\mathbf{X}$,

$$N = N_0 + \epsilon N_t + \delta_s N_s + \delta_{ts} N_{ts} + \delta_{ss} N_{ss}, \tag{11.1}$$

where ϵ, δ_s, δ_{ts} *and* δ_{ss} *are real numbers, and there is an infinite-dimensional subspace* $\mathbf{X}_N$ *contained in one of* $\mathbf{T}$, $\mathbf{S}_1$, $\mathbf{S}_2$ *or* $\mathbf{S}_3$ *such that* $N[v] \le 0$ *for all* $v \in \mathbf{X}_N$.

12. $N = N_0$ does not suffice for coerciveness

Let $Q(t, s) = \frac{1}{2}t^2 + \frac{1}{6\gamma}|s|^2 + X_1 t + X_0 \in \widehat{\mathbf{T}}$ and consider elementary solutions (8.2), $v = E(f, Q) \in \mathbf{T} \subset \mathbf{X} = \mathbf{X}_{M,T}$, where f is holomorphic and X_0, X_1 are real numbers that will be chosen depending on f, as stated in (8.3). Then

$$\partial^2 v = f'' Q + \overline{z} f', \qquad \overline{\partial}^2 v = 0, \qquad \partial\overline{\partial} v = f$$

and therefore we want to show that

$$N_0[v] = \int_\Omega (|f'' Q + \bar{z} f'|^2 - 2|f|^2) \le 0 \tag{12.1}$$

over an infinite-dimensional subspace of such solutions. When $f = z^n$ and $v_n = E(z^n, Q)$ as in (8.3), one obtains

$$\partial^2 v_n = z^{n-2}(n(n-1)Q + n|z|^2), \qquad n = 2, 3, \ldots,$$

so that by orthogonality over $\mathbb{D}$,

$$N_0[c_1 v_n + c_2 v_m] = |c_1|^2 N_0[v_n] + |c_2|^2 N_0[v_m], \qquad n \ne m. \tag{12.2}$$

Consequently, showing $N_0[v_n] \le 0$ for an increasing infinite sequence of n yields an infinite-dimensional subspace and establishes the noncoerciveness of N_0 over $\mathbf{X}$.

This orthogonality works more generally by definition of the domains $\Omega_{M,T}$, definition of the null forms (11.1) and the differentiability properties of the elementary solutions (8.3), as can be seen by inspection. We state it as a lemma.

Lemma 12.1. *Let N be any null form (11.1), and let Ω be any of the convex domains (8.1). Suppose $N[v_n] \le 0$ on Ω for an increasing subsequence of elementary solutions $v_n = E(z^n, Q)$ of (8.3). Then $N[v] \le 0$ on an infinite-dimensional subspace of* $\mathbf{X}$.

One then wants to minimize each $N_0[v_n]$ over the real coefficients X_0, X_1 in hope of obtaining a sequence of inequalities (12.1). This is the straightforward task of minimizing quadratic polynomials, here in the variables $X = (X_0, X_1)$, with coefficients derived from definite integrations over Ω. That this results in the inequality (12.1) will be proved. Why one might hypothesize this to result in the inequality (12.1) can be partially explained.

Definition 12.2. Let $F(z, t, s)$ be a function defined on $\Omega = \Omega_{M,T}$. Let $C(M, T)$ denote the quantity $3M^4T^3/(2\pi^2)$. Define the integral

$$\begin{aligned}\int_{M,T} F\left(z, \frac{1-|z|}{M}t, \frac{1-|z|}{MT}ts\right) &:= C(M,T)\int_\Omega F(z,t,s)\\ &= C(M,T)\int_{\mathbb{D}} dm(z)\int_0^{(1-|z|)/M} dt \int_{|s|<t/T} F(z,t,s)\, ds.\end{aligned}$$

By scalings $s \to \frac{t}{T}s$, then $t \to \frac{1-|z|}{M}t$ and Fubini,

$$\begin{aligned}&\int_{M,T} F\left(z, \frac{1-|z|}{M}t, \frac{1-|z|}{MT}ts\right)\\ &\quad = \frac{3}{4\pi}\int_{|s|<1} ds\, 4\int_0^1 t^3\, dt\, \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^1 F\left(re^{i\theta}, \frac{1-r}{M}t, \frac{1-r}{MT}ts\right)(1-r)^4 r\, dr.\end{aligned} \tag{12.3}$$

Integration over Ω is seen as integration over $\mathbb{D}$ of a scaled integrand against four powers of the *distance to the boundary function* $1-|z| = 1-r$, with the angular integration averaged over $0 \le \theta \le 2\pi$, and then averaged over the unit interval, with respect to the measure $t^3 dt$, and over the unit ball in $\mathbb{R}^3$.

Formulating the desired inequality (12.1) this way gives

$$\int_{M,T} \left| f'' \frac{(1-|z|)^2}{M^2} \frac{1}{2} t^2 \left(1 + \frac{|s|^2}{3\gamma T^2}\right) + X_1 f'' \frac{1-|z|}{M} t + X_0 f'' + \bar{z} f' \right|^2 \\ - \frac{1}{\pi} \int_{\mathbb{D}} |f|^2 (1-|z|)^4 \, dm(z) \le 0. \qquad (12.4)$$

When $f = z^n$ the negative term is precisely

$$-2 \int_0^1 r^{2n+1} (1-r)^4 \, dr = -2B(2n+2, 5)$$

where B is the *beta function* [Rud76, p. 193]. The first integral yields a linear combination of beta functions. In general $B(2n+k, h) = O(n^{-h})$ as $n \to \infty$ so the first integral may not be allowed to decay slower than $O(n^{-5})$ if the inequality (12.1) and (12.4) is to hold for a sequence of n.

Classical interior estimates for holomorphic or harmonic functions yield the principle that each power of the distance to the boundary function removes a derivative from f. Consequently, in the first integral the first term of the sum being squared can be thought of as fM^{-2} which when squared integrates just as the negative integral, i.e. as $O(n^{-5})$ when $f = z^n$, but with a small coefficient for M sufficiently large. It can be discounted if the remaining terms also integrate as $O(n^{-5})$ with small enough coefficients. The integral of the square of the fourth term is precisely $(2\pi)^{-1} \int_{\mathbb{D}} |f'|^2 |z|^2 (1-|z|)^4 dm = n^2 B(2n+2, 5) = O(n^{-3})$ when $f = z^n$ and is therefore an obstacle to the desired inequality. In this case the third and fourth terms together can be written $X_0 f'' + \bar{z} f' = n(|z|^2 - 1) z^{n-2}$ by choosing $X_0 = \frac{-1}{n-1}$. In principle this is again like f, the distance to the boundary effectively canceling the n. Its square integral is $n^2(B(2n-2, 7) + 2B(2n-1, 7) + B(2n, 7)) = O(n^{-5})$, with a larger coefficient than the $2B(2n+2, 5)$. Hence, consider also $X_1 = X_{11} \frac{M}{n-1}$ with $X_{11} = O(1)$ to be determined. Now the second term also behaves like z^n. A quadratic minimization yields

$$\min_{X_{11}} \int_{M,T} \left| X_{11} n z^{n-2} (1-|z|) t + n z^{n-2} (|z|^2 - 1) \right|^2 - \frac{1}{\pi} \int_{\mathbb{D}} |z|^{2n} (1-|z|)^4 \, dm \\ = -\frac{3}{5n^5} + O(n^{-6}). \qquad (12.5)$$

The minimum is achieved at $X_{11} = \frac{6}{5} \frac{4n+3}{2n+5}$. Setting $X_{11} = 12/5$ yields the same asymptotic expression.

With $f = z^n$ and with these choices for X, the general inequality $(a+b)^2 \le (1+c^2)a^2 + (1+c^{-2})b^2$ proves that (12.4) is true for all M and n large enough. A better result than (12.5), $-\frac{3}{4n^5} + O(n^{-6})$, is obtained by setting $X_0 = \frac{-1}{n-1} + X_{02} \frac{1}{n(n-1)}$ and minimizing over X_{02} and X_{11}. The values $X_{02} = 1$ and $X_{11} = 2$ in place of the minimizers also suffice. The best result is by minimizing (12.4) directly. A method for computing these asymptotic results is discussed in the following section.

Remark 12.3. A more precise computation (16.6) will show that N_0 does not suffice for coerciveness whenever $8M^4 > 5$.

Remark 12.4. The roles of the Lipschitz constant M and therefore of the distance to the boundary function can be seen when they are absent, by considering the domains obtained by letting $M \to 0$ and cutting off at height $t = H$,

$$\Omega^H_{0,T} = \{(z, t, s) : z \in \mathbb{D} \text{ and } T|s| < t < H\}.$$

Scaling $s \to ts$ and forming the average in the ts-integration allows one to write inequality (12.1) for $\Omega^H_{0,T}$ as

$$\frac{3T^3}{\pi H^4} \int_{|s|<1/T} \int_0^H t^3 \, dt \, ds \int_{\mathbb{D}} (|f''Q + \bar{z}f'|^2 - 2|f|^2) \, dm \le 0 \tag{12.6}$$

where Q has been scaled. Denote the average of a function $F(t, s)$, with respect to $t^3 dtds$ here, as $[F]$. Then by the arithmetic-geometric mean inequality,

$$\left[\int_{\mathbb{D}} |f''Q + \bar{z}f'|^2 \, dm\right] = \int_{\mathbb{D}} (|f''|^2[Q^2] + 2Ref''\bar{z}\overline{f'}[Q] + |z|^2|f'|^2) \, dm$$
$$\ge \int_{\mathbb{D}} (|f''|^2[Q^2] - |f''|^2[Q]^2 - |z|^2|f'|^2 + |z|^2|f'|^2) \, dm = ([Q^2] - [Q]^2) \int_{\mathbb{D}} |f''|^2 \, dm.$$

Since Q is not constant, this and (12.6) imply

$$\int_{\mathbb{D}} |f''|^2 \, dm \le C_Q \int_{\mathbb{D}} |f|^2 \, dm \tag{12.7}$$

where C_Q is a constant depending only on the integral averages of Q. By Gagliardo–Nirenberg inequalities and Rellich compactness, (12.7) cannot hold for an infinite-dimensional subspace of $W^{2,2}(\mathbb{D})$. Similar uses of the arithmetic-geometric mean inequality show that $v = fQ + \bar{z}f_{-1} \in W^{2,2}(\Omega^H_{0,T})$ if and only if $f \in W^{2,2}(\mathbb{D})$. Thus the above proof that N_0 is noncoercive in the domains $\Omega_{M,T}$ fails in these domains.

It is also to be remarked that the failure exhibited here remains when $\mathbb{D}$ is replaced by any bounded domain of $\mathbb{C}$ satisfying, for example, the segment property.

13. Computing asymptotics

Let A be a real symmetric positive definite $m \times m$ matrix with entries a^j_k, $0 \le j, k \le m-1$. Let $X = (X_0, \dots, X_{m-1}) \in \mathbb{R}^m$ and define $a_m = (a^0_m, \dots, a^{m-1}_m) \in \mathbb{R}^m$. Define $a^m_m \in \mathbb{R}$. Let a_k denote the column vectors of A. Denote the transpose of a_k by a^T_k.

Then the quadratic in the variables X,

$$F(X) = X \cdot AX + 2a_m \cdot X + a_m^m,$$

reaches its minimum when $AX = -a_m$. The minimum value is $-a_m \cdot A^{-1}a_m + a_m^m$. If A_k denotes the matrix A with its kth column replaced by $-a_m$, then the minimizing X_j, $0 \le j \le m-1$, can be realized by Cramer's rule as the ratio of determinants $X_j = \det A_j / \det A$. From this and cofactor expansion (e.g. along the last row of $\check{A}$) it can be seen that the minimum value of F is $\det \check{A} / \det A$ where $\check{A}$ is the $(m+1)\times(m+1)$ matrix $\left[\begin{smallmatrix} A & a_m \\ a_m^T & a_m^m \end{smallmatrix}\right]$.

Put $Q = X_4 q_4 + X_3 q_3 + X_2 q_2 + X_1 q_1 + X_0$ where each q_j is a polynomial of $\widehat{\mathbf{X}}$ of homogeneity j. Let v_n be the elementary solutions $E(z^n, Q)$ of (8.3) and let N be any null form (11.1). For each n, matrix entries a_k^j are defined by $a_k^j = \frac{1}{2}\frac{\partial}{\partial X_j}\frac{\partial}{\partial X_k} N[v_n]$, $0 \le j, k \le 4$. By what was said immediately after (4.3), $a_0^0 > 0$. In order to find the X_j such that $N[v_n] < 0$ one successively computes, for $h = 1, 2, 3, 4$, the determinants of the $(h+1)\times(h+1)$ matrices with entries a_k^j, $0 \le j, k \le h$, until a negative result is obtained. By the above discussion the negative result is the determinant of $\check{A}$, thus identifying $\check{A}$ and then giving the matrix A, the minimizing $X_0, \ldots, X_{h-1}$ with $X_h = 1$ and $X_{h+1} = \cdots = X_4 = 0$ together with the negative minimum value of $N[v_n]$.

Unless, of course, there is no negative determinant.

As seen in the last section, each matrix element a_k^j will be a rational function in the variables n, M and T. Then so will the determinants, the minimizing X_j and the minimum values of the $N[v_n]$.

14. Change of notation and the basic argument

1. On $\Omega = \Omega_{M,T}$ define the integration

$$\int F(z,t,s) := C(M,T)\int_\Omega F(z,t,s)\,dm(z)\,dt\,ds = \int_{M,T} F\left(z, \frac{1-|z|}{M}t, \frac{1-|z|}{MT}ts\right)$$

where the quantity $C(M,T)$ and the last integral are from Definition 12.2.

2. Redefine the null forms $N_0, N_t, N_s, N_{ts}, N_{ss}$ of (5.2), (9.4), (9.6), (9.8), (9.11) by replacing $\int_\Omega$ with $\int$.

3. By Lemma 12.1 and the basic argument of Section 11,

- *To prove Theorem 1.5, it suffices to show that for each null form on* $\mathbf{X}$,

$$N = N_0 + \epsilon N_t + \delta_s N_s + \delta_{ts} N_{ts} + \delta_{ss} N_{ss}, \tag{14.1}$$

where ϵ, δ_s, δ_{ts} and δ_{ss} are real numbers, there is a sequence of nonzero polynomials Q in one of $\widehat{\mathbf{T}}$, $\widehat{\mathbf{S}}_1$, $\widehat{\mathbf{S}}_2$ or $\widehat{\mathbf{S}}_3$ and a corresponding subsequence of elementary solutions $v_n = E(z^n, Q)$ such that $N[v_n] \le 0$ for all members of the subsequence.

15. A bound on ϵ

Lemma 15.1. *For all $M, T > 0$, a necessary condition for the coerciveness of N (11.1) over $\mathbf{X}_{M,T}$ is that $\epsilon < M^{-2}$.*

Proof. Choose $Q = t + X_0$ and consider elementary solutions (8.2), $v = E(f, Q) = fQ \in \mathbf{T}$. Then $v_{tt} = 0$ and

$$N_s[v] = N_{ts}[v] = N_{ss}[v] = 0.$$

In addition $\overline{\partial}^2 v = \overline{\partial} v_t = 0$ so that

$$N[v] = \int (|f''|^2(t + X_0)^2 - \epsilon|f'|^2). \tag{15.1}$$

Choosing $f = z^n$ and defining v_n (8.3) we use the basic argument of Section 14. For each n choose $X_0 = \frac{-4}{M(2n+3)}$. Integrating (15.1) according to Definition 12.2 yields

$$\int_{M,T} \left(n^2(n-1)^2|z|^{2n-4}\left(\frac{1-|z|}{M}t + X_0\right)^2 - \epsilon n^2|z|^{2n-2}\right) = \frac{-6n^2(\epsilon M^2 - 1) - 3n(3\epsilon M^2 + 2)}{M^2(2n+3)^2(2n+1)(n+2)(n+1)}, \tag{15.2}$$

which is negative whenever $\epsilon \geq M^{-2}$. For each N, with ϵ in this range, the solutions $v_n = z^n(t - 4M^{-1}(2n+3)^{-1})$ generate the infinite-dimensional subspace $\mathbf{X}_N \subset \mathbf{T} \subset \mathbf{X}_{M,T}$ required by the basic argument. □

Remark 15.2. Computing the minimum over X_0 of (15.1) (15.2) directly by the ratio $\det \check{A}/\det A = \det \check{A}/a_0^0$ of Section 13 yields the sum of rational functions $-\frac{3}{4}\frac{\epsilon M^2-1}{M^2 n^3} + O(n^{-4})$. This is positive for all $\epsilon < M^{-2}$ and n large enough so that the bound of this section is sharp.

We record that $a_0^0 = n^2(n-1)^2 B(2n-2, 5) = \frac{4!}{2^5 n} + O(n^{-2}) = \frac{3}{4n} + O(n^{-2}) > 0$ and that $\det \check{A} = -\frac{9}{16}\frac{\epsilon M^2-1}{M^2 n^4} + O(n^{-5})$, positive ($n$ large enough) if and only if $\epsilon < M^{-2}$.

16. Main estimate and lemma

Let $X = (X_0, X_1) \in \mathbb{R}^2$. Let $Q(t, s) = \frac{1}{2}t^2 + \frac{1}{6\gamma}|s|^2 + X_1 t + X_0 \in \widehat{\mathbf{T}}$, and let $v = fQ + \overline{z} f_{-1}$ be elementary solutions (8.2) in $\Omega_{M,T}$. Then

$$N_0[v] + \epsilon N_t[v] = \int (|f''Q + \overline{z} f'|^2 - 2(1-\epsilon)|f|^2 - \epsilon|f'Q_t|^2).$$

16.1. The estimate

We have the following *negative* bound from above, uniformly over $\epsilon < M^{-2}$, when $f = z^n$.

Lemma 16.1. *There are numbers $\mu_0, \mu_1 > 0$ independent of n, M, T, and γ; a constant T_γ depending only on the parameter $0 < \gamma < 1/3$; and an open interval I containing the closed interval $[1, 2]$, such that for all $T > T_\gamma$, all $M^2 \in I$ and all $-\infty < \epsilon < M^{-2}$,*

$$\min_{X\in\mathbb{R}^2}\Bigg[\int_{M,T}|z|^{2n-4}\bigg(n(n-1)\bigg(\frac{1}{2}\frac{(1-|z|)^2}{M^2}t^2\bigg(1+\frac{|s|^2}{3\gamma T^2}\bigg)+X_1\frac{1-|z|}{M}t+X_0\bigg)+n|z|^2\bigg)^2$$
$$-\int_{M,T}\bigg(2(1-\epsilon)|z|^{2n}+\epsilon|z|^{2n-2}\bigg(n\frac{1-|z|}{M}t+X_1 n\bigg)^2\bigg)\Bigg]$$
$$\le -\frac{\mu_0+\mu_1(M^{-2}-\epsilon)}{n^5}+O(n^{-6}), \qquad (16.1)$$

where the leading coefficient of $O(n^{-6})$ is allowed to depend on M, T and ϵ.

Proof. Consider first the *modified minimization problem* that would arise if $Q(t,s)$ were replaced by $Q(t,0)$, i.e. remove the factor $1+\frac{|s|^2}{3\gamma T^2}$ from the first square being integrated.

In the fashion of Section 12 write $X_0 = \frac{-1}{n-1} + X_{02}\frac{1}{n(n-1)}$ and $X_1 = X_{11}\frac{1}{n}$ where X_{02} and X_{11} are to be $O(1)$ as $n \to \infty$. With $X_2 = 1$ the first square is now

$$r^{2n-4}\bigg(X_2\bigg(n(n-1)\frac{1}{2}\frac{(1-r)^2}{M^2}t^2+n(r^2-1)\bigg)+X_{11}(n-1)\frac{1-r}{M}t+X_{02}\bigg)^2. \quad (16.2)$$

By the heuristics of Section 12, after integration the coefficient of each of the quadratic monomials X_{02}^2, $X_{02}X_{11}$, etc. will be $O(n^{-5})$. This is also the case for the second square

$$-2(1-\epsilon)r^{2n}X_2^2 \qquad (16.3)$$

and the third which now reads

$$-\epsilon r^{2n-2}\bigg(X_2 n\frac{1-r}{M}t+X_{11}\bigg)^2. \qquad (16.4)$$

Thus by integrating (16.1) , the matrix entries a_k^j, $0 \le j, k \le 2$, of Section 13 that arise as coefficients of the quadratic monomials X_{02}^2, $X_{02}X_{11}, \ldots$ will all be rational functions of n and will be $O(n^{-5})$ as $n \to \infty$. The entries a_0^0, a_1^0 and a_1^1 which form the symmetric matrix A of Section 13 arise from integrating (16.2) and (16.4) and taking the coefficients of X_{02}^2, $X_{02}X_{11}$ and X_{11}^2. By inspecting (16.2) and (16.4), a_0^0 and a_1^0 must be independent of ϵ, while a_1^1 will be a polynomial in ϵ of degree 1 (for each n). The entries a_2^0, a_2^1, a_2^2, the coefficients of the monomials containing X_2, complete the matrix $\check{A}$. The first entry, arising only from (16.2), is independent of ϵ, while a_2^1 and a_2^2, from (16.2)–(16.4), are polynomials in ϵ of degree 1. Consequently, $\det A$ is $O(n^{-10})$ and of degree 1 in ϵ, while $\det \check{A}$ is $O(n^{-15})$ and of degree 2 in ϵ.

Computing explicitly (see Appendix), we find

$$\det A = \frac{9}{16}\,\frac{M^{-2}-\epsilon}{n^{10}} + O(n^{-11}) \tag{16.5}$$

and

$$\det \check{A} = -\frac{27}{512M^6n^{15}}\big(8M^4(2M^2-1)(\epsilon-M^{-2})^2 + M^2(24M^4-1)(\epsilon-M^{-2}) + 2(4M^2-1)^2\big) + O(n^{-16}). \tag{16.6}$$

When written out, the first determinant (16.5) differs from the determinant of the $\check{A}$ of Remark 15.2 by a factor of exactly $n^{-4}(n-1)^{-2}$, as can be seen by comparing (15.2) with (16.2) and (16.4). It is, like that determinant, *positive* if and only if $\epsilon < M^{-2}$.

Therefore, in order for the minimum of the modified (16.1), viz. $\det \check{A}/\det A$, to be negative for all $\epsilon < M^{-2}$ and n large enough, the parenthetic quadratic, in the indeterminant $\epsilon - M^{-2}$, of (16.6) must remain positive. This can be arranged because its discriminant equals

$$M^4(576M^8 - 2048M^6 + 2000M^4 - 640M^2 + 65).$$

Here the parenthetic quartic in M^2 is *negative* on an open interval containing $1 \le M^2 \le 2$, while the coefficient of $(\epsilon - M^{-2})^2$ in (16.6) is *positive* for these M.

Moreover, for M in this range, $\det \check{A}$ is *negative* at the endpoint $\epsilon = M^{-2}$, while the leading term of $\det A$ decreases to 0 as $\epsilon \uparrow M^{-2}$. Likewise $\det \check{A}$ behaves like $-\epsilon^2$ as $\epsilon \to -\infty$, while $\det A$ behaves like $|\epsilon|$. Thus $\det \check{A}/\det A < 0$ for all $\epsilon < M^{-2}$ and diverges to $-\infty$ at both endpoints of that interval.

Consequently, there are numbers $\mu_0, \mu_1 > 0$ such that the minimum of the modified problem satisfies inequality (16.1) for the stated range of M.

Remark 16.2. The leading coefficient of the rational $O(n^{-6})$ function in (16.1) will contain the factor $\epsilon - M^{-2}$ in its denominator.

With patience one can carry out by hand the calculations leading to (16.5) and (16.6). However, in order to finish the proof of the lemma we will verify by a perturbation argument that the modified case, now completed, implies the original inequality (16.1).

By adding $X_2 n(n-1)\frac{1}{2}\frac{(1-r)^2}{M^2}t^2\frac{|s|^2}{3\gamma T^2}$ to the square (16.2) one returns to the original problem. The only effect is to replace a_2^0, a_2^1 and a_2^2 with $a_2^0 + r_2^0$, $a_2^1 + r_2^1$ and $a_2^2 + r_2^2$ respectively where each r_k^j is independent of ϵ and is $O(n^{-5})$. In addition each r_k^j is $O\big(\frac{1}{\gamma T^2}\big)$ as $T \to \infty$.

Therefore the matrix A is unchanged, and $\check{A}$ is replaced by $\check{A} + R$ where $R = (r_k^j)$ is symmetric with the r_k^j as defined and with also $r_k^j = 0$, $0 \le j, k \le 1$. Using the previous observations about the a_k^j, inspection shows that $\det(\check{A} + R) - \det \check{A}$ is of degree 1 in ϵ.

Also by inspection it is $O(\frac{1}{\gamma T^2})$. With M confined to a compact interval, then, there are *positive* numbers a and b such that

$$|\det(\check{A}+R)-\det\check{A}|\leq\frac{a+b(M^{-2}-\epsilon)}{\gamma T^2n^{15}}+O(n^{-16}).$$

Because it has been arranged that the quadratic in ϵ of (16.6) has no real roots, it follows that there is a T_γ such that for all $T>T_\gamma$ and all $\epsilon<M^{-2}$,

$$\frac{a+b(M^{-2}-\epsilon)}{\gamma T^2n^{15}}\leq\frac{1}{2}|\det\check{A}|$$

once n is large enough for each ϵ. Consequently, given $\frac{\det\check{A}}{\det A}\leq-\frac{\mu_0+\mu_1(M^{-2}-\epsilon)}{n^5}+O(n^{-6})$, the minimum of (16.1) satisfies

$$\frac{\det(\check{A}+R)}{\det A}\leq\frac{\det\check{A}+\frac{1}{2}|\det\check{A}|}{\det A}=\frac{1}{2}\frac{\det\check{A}}{\det A}\leq-\frac{1}{2}\frac{\mu_0+\mu_1(M^{-2}-\epsilon)}{n^5}+O(n^{-6})$$

for all n large enough. □

16.2. The main lemma

In addition to $N_0[v]+\epsilon N_t[v]$ when $f=z^n$, we also have, by (9.6),

$$\begin{aligned}N_s[v]&=\int\left(2|f|^2-\gamma|f'\frac{1}{3\gamma}s|^2\right)=\int_{M,T}\left(2|z|^{2n}-\frac{1}{9\gamma}|z|^{2n-2}n^2\frac{(1-|z|)^2}{M^2}t^2\frac{|s|^2}{T^2}\right)\\&=2B(2n+2,5)-\frac{1}{9M^2\gamma T^2}\frac{2}{5}n^2B(2n,7)=2\frac{(2n+1)!4!}{(2n+6)!}-\frac{C_s}{M^2\gamma T^2}\frac{1}{n^5}+O(n^{-6})\\&=\left(\frac{3}{2}-\frac{C_s}{M^2\gamma T^2}\right)\frac{1}{n^5}+O(n^{-6}),\end{aligned}\tag{16.7}$$

where $C_s>0$ is a constant independent of n, M, T, and γ. Also by (9.8),

$$N_{ts}[v]=\int|f|^2=B(2n+2,5)=\frac{3}{4}\frac{1}{n^5}+O(n^{-6}).$$

And by (9.11),

$$N_{ss}[v]=\int\left(|f|^2-\gamma^2\left|f\frac{1}{6\gamma}\nabla\nabla|s|^2\right|^2\right)=\int\left(|f|^2-\frac{1}{3}|f|^2\right)=\frac{1}{2}\frac{1}{n^5}+O(n^{-6}).$$

Let $v_n=z^n(\frac{1}{2}t^2+\frac{1}{6\gamma}|s|^2+X_1t+X_0+(n+1)^{-1}|z|^2)$ be elementary solutions (8.3) where X, for each n, is the minimizer of (16.1). Let N be any null form (11.1) such that $\epsilon<M^{-2}$. Then using Lemma 16.1 and the above, we obtain

$$N[v_n]\leq\left(-\mu_0-\mu_1(M^{-2}-\epsilon)+\left(\frac{3}{2}-\frac{C_s}{M^2\gamma T^2}\right)\delta_s+\frac{3}{4}\delta_{ts}+\frac{1}{2}\delta_{ss}\right)\frac{1}{n^5}+O(n^{-6}).$$

Consequently, we obtain the following coerciveness condition.

Lemma 16.3. *Suppose $1 \le M^2 \le 2$ and let N be any null form (11.1) satisfying $\epsilon < M^{-2}$. There is a constant T_γ, depending only on the parameter $0 < \gamma < 1/3$, such that for all $T > T_\gamma$ a necessary condition for the coerciveness of N over $\mathbf{X}_{M,T}$ is*

$$\left(\frac{3}{2} - \frac{C_s}{M^2\gamma T^2}\right)\delta_s + \frac{3}{4}\delta_{ts} + \frac{1}{2}\delta_{ss} \ge \mu_0 + \mu_1(M^{-2} - \epsilon)$$

where C_s, μ_0 and μ_1 are positive numbers independent of M, T and γ.

17. Bounding the δ's from above

1. Let $v = fs_1 \in \mathbf{S}_1$. Applying any N (11.1) to v yields $N[v] = N_0[v] + \delta_s N_s[v]$. For $f = z^n$,

$$\begin{aligned} N_0[v] &= \int |f''|^2 s_1^2 = \frac{1}{3}\int |f''|^2|s|^2 = \frac{1}{3}\int_{M,T} |z|^{2n-4} n^2(n-1)^2 \frac{(1-|z|)^2}{M^2} t^2 \frac{|s|^2}{T^2} \\ &= \frac{2}{15M^2T^2} n^2(n-1)^2 B(2n-2, 7) = \frac{3}{4M^2T^2}\frac{1}{n^3} + O(n^{-4}). \end{aligned}$$

And

$$N_s[v] = -\gamma \int |f'|^2 = -\gamma n^2 B(2n, 5) = -\gamma \frac{3}{4}\frac{1}{n^3} + O(n^{-4}).$$

Thus $N[v] = \gamma \frac{3}{4}\big(\frac{1}{M^2\gamma T^2} - \delta_s\big)\frac{1}{n^3} + O(n^{-4})$ and the following bound is established.

Lemma 17.1. *For all $M, T > 0$, a necessary condition for the coerciveness of N (11.1) over $\mathbf{X}_{M,T}$ is that $\delta_s \le \frac{1}{M^2\gamma T^2}$.*

2. By subtracting P_{22} from P_{11} where P is the polynomial (6.3) it follows that $s_1^2 - s_2^2$ is an element of $\widehat{\mathbf{T}}$. Let $v = f\frac{1}{2}(s_1^2 - s_2^2)$. Applying any N (11.1) to v yields $N[v] = N_0[v] + \delta_s N_s[v] + \delta_{ss} N_{ss}[v]$. For $f = z^n$,

$$\begin{aligned} N_0[v] &= \int \left| f'' \frac{1}{2}(s_1^2 - s_2^2)\right|^2 = \frac{1}{4}\int_{M,T} |z|^{2n-4} n^2(n-1)^2 \frac{(1-|z|)^4}{M^4} t^4 \frac{(s_1^2 - s_2^2)^2}{T^4} \\ &= \frac{C_0}{M^4T^4}\frac{1}{n^5} + O(n^{-6}) \end{aligned}$$

where $C_0 > 0$ is a constant independent of n, M, T, and γ. Moreover

$$\begin{aligned} N_s[v] &= -\gamma \int |f'|^2(s_1^2 + s_2^2) = -\gamma \frac{2}{3}\int_{M,T} |z|^{2n-2} n^2 \frac{(1-|z|)^2}{M^2} t^2 \frac{|s|^2}{T^2} \\ &= -\gamma \frac{6C_s}{M^2T^2}\frac{1}{n^5} + O(n^{-6}) \end{aligned} \tag{17.1}$$

where C_s is the constant from (16.7). Finally,

$$N_{ss}[v] = -\gamma^2 2 \int |f|^2 = -\gamma^2 \frac{3}{2}\frac{1}{n^5} + O(n^{-6}), \tag{17.2}$$

as is also seen in (16.7).

Consequently, we obtain

Lemma 17.2. *For all $M > 0$ and $T > 0$ a necessary condition for the coerciveness of N (11.1) over $\mathbf{X}_{M,T}$ is that $\frac{1}{2}\delta_{ss} \leq \frac{C_0}{3M^4\gamma^2T^4} - \delta_s \frac{2C_s}{M^2\gamma T^2}$ where C_0 and C_s, the constant of the main lemma, Lemma 16.3, are positive numbers independent of M, T and γ.*

3. Let $Q = s_1 t + s_2 s_3 + X_0 s_1 \in \widehat{\mathbf{S}}_1$ and let $v = fQ$ be elementary solutions. When $f = z^n$,

$$N_t[v] = -\int |f' s_1|^2 = -\frac{3C_s}{M^2T^2}\frac{1}{n^5} + O(n^{-6}) \tag{17.3}$$

by comparison to (17.1). Moreover,

$$N_{ts}[v] = -\gamma \int |f|^2 = -\gamma \frac{3}{4}\frac{1}{n^5} + O(n^{-6}) \tag{17.4}$$

as in (17.2), and

$$N_{ss}[v] = -\gamma^2 2 \int |f|^2 = -\gamma^2 \frac{3}{2}\frac{1}{n^5} + O(n^{-6}). \tag{17.5}$$

Remark 17.3. When multiplied by $-\gamma^{-1}$ the coefficients of $\frac{1}{n^5}$ for N_{ts} and N_{ss} are almost the coefficients of δ_{ts} and δ_{ss} respectively of the main lemma. Replacing $\gamma 3$ with 1 in the N_{ss} coefficient would make them identical to the main lemma coefficients. The restriction $\gamma < 1/3$ will be used in the next section for just this purpose.

What remains is

$$\begin{aligned} N_0[v] + \delta_s N_s[v] &= \int \left(|f''|^2((t+X_0)s_1 + s_2 s_3)^2 - \delta_s \gamma |f'|^2((t+X_0)^2 + s_3^2 + s_2^2)\right) \\ &= \int \left(|f''|^2\left((t+X_0)^2 \frac{1}{3}|s|^2 + s_2^2 s_3^2\right) - \delta_s \gamma |f'|^2 \left((t+X_0)^2 + \frac{2}{3}|s|^2\right)\right) \end{aligned}$$

where $\int_{|s|<1} s_1 s_2 s_3 \, ds = 0$ has also been used.

The main lemma puts only a partial restriction on the negativity of δ_s, δ_{ts} and δ_{ss}. Here and in Lemmas 17.1 and 17.2 unrestricted negative values for the δ's are seen to allow the coerciveness that is to be disproved in this article. Thinking of $\delta_s < 0$ then, it will be seen below that here one wants to choose X_0 for each n in order to minimize the term $\int |f'|^2(t+X_0)^2$. The minimizer (compare (15.2)) is $X_0 = \frac{-4}{M(2n+5)}$. Hence with $v_n = z^n\left(s_1 t + s_2 s_3 - \frac{4s_1}{M(2n+5)}\right)$,

$N_0[v_n] + \delta_s N_s[v_n] =$

$$\int_{M,T} n^2(n-1)^2|z|^{2n-4}\left(\left(\frac{1-|z|}{M}t - \frac{4}{M(2n+5)}\right)^2 \frac{1}{3}\frac{(1-|z|)^2}{M^2}t^2\frac{|s|^2}{T^2} + \frac{(1-|z|)^4}{M^4}t^4\frac{s_2^2 s_3^2}{T^4}\right)$$
$$- \delta_s\gamma \int_{M,T} n^2|z|^{2n-2}\left(\left(\frac{1-|z|}{M}t - \frac{4}{M(2n+5)}\right)^2 + \frac{2}{3}\frac{(1-|z|)^2}{M^2}t^2\frac{|s|^2}{T^2}\right)$$
$$= \left(\frac{C_1}{M^4T^2} - \delta_s\gamma\left(\frac{3}{4M^2} + \frac{6C_s}{M^2T^2}\right)\right)\frac{1}{n^5} + O(n^{-6}) \qquad (17.6)$$

where $C_1 > 0$ is independent of n, M, T, and γ; evaluation of the second integral is as in (15.2) and (17.1).

By (17.3)–(17.6) and with $N = N_0 + \epsilon N_t + \delta_s N_s + \delta_{ts} N_{ts} + \delta_{ss} N_{ss}$,

$$N[v_n] = \gamma\left(\frac{C_1}{M^4\gamma T^2} - \epsilon\frac{3C_s}{M^2\gamma T^2} - \delta_s\left(\frac{3}{4M^2} + \frac{6C_s}{M^2T^2}\right) - \frac{3}{4}\delta_{ts} - \gamma\frac{3}{2}\delta_{ss}\right)\frac{1}{n^5} + O(n^{-6}).$$

Consequently, we obtain

Lemma 17.4. *For all $M, T > 0$ and real ϵ, a necessary condition for the coerciveness of N (11.1) over $\mathbf{X}_{M,T}$ is that*

$$\left(\frac{3}{4M^2} + \frac{6C_s\gamma}{M^2\gamma T^2}\right)\delta_s + \frac{3}{4}\delta_{ts} + \gamma\frac{3}{2}\delta_{ss} \le \frac{C_1}{M^4\gamma T^2} - \frac{3C_s}{M^2\gamma T^2}\epsilon$$

where C_1 and C_s are positive numbers independent of M, T and γ.

Remark 17.5. Choosing $Q = s_1 t + s_2 s_3$, i.e. $X_0 = 0$ in $\int |f'|^2(t + X_0)^2$, and forgoing minimization produces $\int |f'|^2 t^2 = \frac{2}{3M^2}n^2 B(2n, 7) + \frac{15}{4M^2}\frac{1}{n^5} + O(n^{-6})$. Thus $\frac{15}{4M^2}$ replaces $\frac{3}{4M^2}$ in the coefficient of δ_s. In the main lemma the quantity $\frac{3}{2}$ is the principal part of the coefficient of δ_s. The inequality $\frac{3}{2} > \frac{3}{4M^2}$ is needed in the argument of the next section.

18. Final inequality

As in the main lemma, Lemma 16.3, fix $1 \le M^2 \le 2$. By Lemma 15.1 a necessary condition for coerciveness of any N (11.1) is $M^{-2} - \epsilon > 0$. With these conditions, Lemma 16.3, Lemmas 17.1, 17.2 and 17.4, together with further restrictions on T, we obtain

$$\mu_0 + \mu_1(M^{-2} - \epsilon) \le \left(\frac{3}{2} - \frac{C_s}{M^2\gamma T^2}\right)\delta_s + \frac{3}{4}\delta_{ts} + \frac{1}{2}\delta_{ss}$$
$$\le \frac{C_1}{M^4\gamma T^2} - \frac{3C_s}{M^2\gamma T^2}\epsilon + \left(\frac{3}{2} - \frac{3}{4M^2} - \frac{(1+6\gamma)C_s}{M^2\gamma T^2}\right)\delta_s + \frac{1}{2}(1-3\gamma)\delta_{ss}$$
$$\le \frac{(1-3\gamma)C_0}{3M^4\gamma^2T^4} + \frac{C_1}{M^4\gamma T^2} - \frac{3C_s}{M^2\gamma T^2}\epsilon + \left(\frac{3}{2} - \frac{3}{4M^2} - \frac{3C_s}{M^2\gamma T^2}\right)\delta_s$$
$$\le \frac{(1-3\gamma)C_0}{3M^4\gamma^2T^4} + \frac{C_1}{M^4\gamma T^2} + \left(\frac{3}{2} - \frac{3}{4M^2} - \frac{3C_s}{M^2\gamma T^2}\right)\frac{1}{M^2\gamma T^2} - \frac{3C_s}{M^2\gamma T^2}\epsilon$$

$$= \frac{1}{\gamma T^2}\left(\frac{(1-3\gamma)C_0}{3M^4\gamma T^2} + \frac{C_1 - 3C_s}{M^4} + \left(\frac{3}{2} - \frac{3}{4M^2} - \frac{3C_s}{M^2\gamma T^2}\right)\frac{1}{M^2}\right) + \frac{1}{\gamma T^2}\frac{3C_s}{M^2}(M^{-2} - \epsilon).$$

The first inequality is the conclusion of Lemma 16.3 for $T > T_\gamma$. The second is Lemma 17.4. The third uses $1 - 3\gamma > 0$ in order to apply Lemma 17.2. The fourth follows from Lemma 17.1 whenever T is large enough so that $0 < \frac{3C_s}{M^2\gamma T^2} \le \frac{3}{2} - \frac{3}{4M^2}$. By choosing T larger still, the positivity of μ_0 and μ_1 of the main lemma is contradicted. The necessary conditions on the coefficients $\epsilon, \delta_s, \delta_{ts}$ and δ_{ss} for coerciveness of the null forms (11.1) have been shown to be inconsistent.

Remark 18.1. The real partial differential operators L_γ (1.2) are not elliptic when $\gamma = 0$ or $\gamma = 1/3$. They are elliptic otherwise. However, in each of the ranges $\gamma < 0$ and $\gamma > 1/3$ the formally positive quadratic form (3.3) is coercive by the theorem of Aronszajn–Smith. The use made of both $\gamma > 0$ and $\gamma < 1/3$ in the final inequality cannot be an artifact of the various $\mathbf{X}_N$ chosen to exhibit noncoerciveness in the lemmas of Sections 15–17.

The proof of the following restatement of Theorem 1.5 and the Theorem stated at the beginning of this article is now complete.

Theorem. *For each elliptic constant coefficient fourth order operator* (1.2) L_γ, $0 < \gamma < 1/3$, *and each Lipschitz constant* M, $1 \le M \le \sqrt{2}$, *there is a real number* $T(\gamma, M)$ *such that for all Lipschitz constants* $T > T(\gamma, M)$ *there is no constant coefficient Hermitian quadratic form* (1.3) *associated to* L_γ *that is coercive over the Sobolev spaces of functions with square integrable derivatives up to order* 2 *in the bounded convex domains* $\Omega_{M,T}$ *of* (8.1).

19. Open problems

1. Construct C^1 domains in which the L_γ or related operators have no constant coefficient coercive forms.

2. Can nonconstant coefficient coercive forms be associated to the L_γ in convex domains and, more generally, in Lipschitz domains?

3. Solve the Neumann problem for constant coefficient higher order elliptic operators and second order Legendre–Hadamard systems in convex domains and, more generally, in Lipschitz domains.

20. Appendix

Using (12.3), the symmetric matrix entries a_k^j, $0 \le j, k \le 2$, in the proof of the main estimate will be computed. The determinants (16.5), (16.6) can then be computed. As explained, the integrands are from (16.2)–(16.4) and the matrix entries are obtained as the coefficients of the quadratic monomials in X_{02}, X_{11} and X_2 in the manner of Section 13. We have

$$a_0^0 = \int_{M,T} r^{2n-4} = \int_0^1 r^{2n-4}(1-r)^4 r\,dr = B(2n-2,5),$$

$$a_1^0 = \int_{M,T} \frac{n-1}{M} t r^{2n-4}(1-r) = \frac{n-1}{M} 4\int_0^1 t^4\,dt \int_0^1 r^{2n-3}(1-r)^5\,dr$$
$$= \frac{n-1}{M}\,\frac{4}{5} B(2n-2,6),$$

$$a_1^1 = \int_{M,T}\left(\frac{(n-1)^2}{M^2} t^2 r^{2n-4}(1-r)^2 - \epsilon r^{2n-2}\right)$$
$$= \frac{(n-1)^2}{M^2} 4\int_0^1 t^5\,dt \int_0^1 r^{2n-3}(1-r)^6\,dr - \epsilon \int_0^1 r^{2n-1}(1-r)^4\,dr$$
$$= \frac{(n-1)^2}{M^2}\,\frac{2}{3} B(2n-2,7) - \epsilon B(2n,5).$$

We have $B(2n+k,h) = \frac{(h-1)!}{2^h n^h} + O(n^{-h-1})$. Write $x \asymp y$ to mean $x = yn^{-5} + O(n^{-6})$. Then

$$a_0^0 \asymp \frac{4!}{2^5} = \frac{3}{4}, \quad a_1^0 \asymp \frac{4}{5M}\,\frac{5!}{2^6} = \frac{3}{2M}, \quad a_1^1 \asymp \frac{2}{3M^2}\,\frac{6!}{2^7} - \epsilon\frac{4!}{2^5} = \frac{3}{4}(5M^{-2}-\epsilon).$$

Computing $a_0^0 a_1^1 - (a_1^0)^2$ yields (16.5).

Moreover,

$$a_2^0 = \int_{M,T}\left(\frac{n(n-1)}{2M^2} t^2 r^{2n-4}(1-r)^2 + n r^{2n-4}(r+1)(r-1)\right)$$
$$= \frac{n(n-1)}{2M^2}\,\frac{2}{3} B(2n-2,7) - nB(2n-1,6) - nB(2n-2,6),$$

$$a_2^1 = \int_{M,T}\left(\frac{n(n-1)^2}{2M^3} t^3 r^{2n-4}(1-r)^3 - \frac{n(n-1)}{M} t r^{2n-4}(r+1)(1-r)^2 - \epsilon\frac{n}{M} t r^{2n-2}(1-r)\right)$$
$$= \frac{n(n-1)^2}{2M^3}\,\frac{4}{7} B(2n-2,8) - \frac{n(n-1)}{M}\,\frac{4}{5}(B(2n-1,7)+B(2n-2,7)) - \epsilon\frac{n}{M}\,\frac{4}{5} B(2n,6),$$

$$a_2^2 = \int_{M,T}\left(\frac{n^2(n-1)^2}{4M^4} t^4 r^{2n-4}(1-r)^4 - \frac{n^2(n-1)}{M^2} t^2 r^{2n-4}(r+1)(1-r)^3\right)$$
$$+ \int_{M,T}\left(n^2 r^{2n-4}(r^2+2r+1)(1-r)^2 - 2(1-\epsilon) r^{2n} - \epsilon\frac{n^2}{M^2} t^2 r^{2n-2}(1-r)^2\right)$$
$$= \frac{n^2(n-1)^2}{4M^4}\,\frac{1}{2} B(2n-2,9) - \frac{n^2(n-1)}{M^2}\,\frac{2}{3}(B(2n-1,8)+B(2n-2,8))$$
$$+ n^2(B(2n,7) + 2B(2n-1,7) + B(2n-2,7)) - 2(1-\epsilon)B(2n+2,5)$$
$$- \epsilon\frac{n^2}{M^2}\,\frac{2}{3} B(2n,7).$$

These satisfy

$$
\begin{aligned}
a_2^0 &\asymp \frac{1}{3M^2}\frac{6!}{2^7} - \frac{5!}{2^6} - \frac{5!}{2^6} = \frac{15}{8}(M^{-2}-2),\\
a_2^1 &\asymp \frac{2}{7M^3}\frac{7!}{2^8} - \frac{8}{5M}\frac{6!}{2^7} - \epsilon\frac{4}{5M}\frac{5!}{2^6} = \frac{3}{2M}\left(\frac{15}{4}M^{-2} - \epsilon - 6\right),\\
a_2^2 &\asymp \frac{1}{8M^4}\frac{8!}{2^9} - \frac{4}{3M^2}\frac{7!}{2^8} + 4\frac{6!}{2^7} - 2(1-\epsilon)\frac{4!}{2^5} - \epsilon\frac{2}{3M^2}\frac{6!}{2^7}\\
&= \frac{15}{4M^2}\left(\frac{21}{8}M^{-2} - \epsilon\right) - \frac{3}{2}\left(\frac{35}{2}M^{-2} - \epsilon\right) + 21.
\end{aligned}
$$

With a little more work $\det \check{A}$ can now be computed and (16.6) obtained. The author found Maple 14 to be a useful tool, especially before it was clear that these kinds of computations would lead to the noncoercivity conclusion.

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