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Control for Schrödinger operators on 2-tori: rough potentials

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Abstract. For the Schrödinger equation $(i\partial_t + \Delta)u = 0$ on a torus, an arbitrary non-empty open set Ω provides control and observability of the solution: $\|u|_{t=0}\|_{L^2(\mathbb{T}^2)} \le K_T \|u\|_{L^2((0,T)\times\Omega)}$. We show that the same result remains true for $(i\partial_t + \Delta - V)u = 0$ where $V \in L^2(\mathbb{T}^2)$, and $\mathbb{T}^2$ is a (rational or irrational) torus. That extends the results of [1], and [8] where the observability was proved for $V \in C(\mathbb{T}^2)$ and conjectured for $V \in L^\infty(\mathbb{T}^2)$. The higher dimensional generalization remains open for $V \in L^\infty(\mathbb{T}^n)$.

1. Introduction

The purpose of this paper is to prove a case of the conjecture made by the last two authors in [8]. It concerned control and observability for Schrödinger operators on tori with L^∞ potentials. Here we prove that for two-dimensional tori the desired results are valid for potentials which are merely in L^2.

To state the result consider

$$\mathbb{T}^2 := \mathbb{R}^2/A\mathbb{Z} \times B\mathbb{Z}, \qquad A, B \in \mathbb{R} \setminus \{0\}, \qquad V \in L^2(\mathbb{T}^2),$$

$$(-\Delta + V(z) - \lambda)u(z) = f(z), \qquad z \in \mathbb{T}^2, \tag{1.1}$$

and

$$i\partial_t u(t,z) = (-\Delta + V(z))u(t,z), \qquad z \in \mathbb{T}^2, \tag{1.2}$$

The first theorem concerns solutions of the stationary Schrödinger equation and is applicable to high energy eigenfunctions:

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Theorem 1. *Let $\Omega \subset \mathbb{T}^2$ be a non-empty open set. There exists a constant $K = K(\Omega)$, depending only on Ω, such that for any solution of (1.1) we have*

$$\|u\|_{L^2(\mathbb{T}^2)} \leq K(\|f\|_{L^2(\mathbb{T}^2)} + \|u\|_{L^2(\Omega)}). \tag{1.3}$$

Theorem 1 can be deduced from the following dynamical result:

Theorem 2. *Let $\Omega \subset \mathbb{T}^2$ be a non-empty open set and let $T > 0$. There exists a constant K, depending only on Ω, T and V, such that for any solution of (1.2) we have*

$$\|u(0,\cdot)\|^2_{L^2(\mathbb{T}^2)} \leq K \int_0^T \|u(t,\cdot)\|^2_{L^2(\Omega)}\, dt. \tag{1.4}$$

An estimate of this type is called *an observability* result. Once we have it, the HUM method (see [19]) automatically provides the following *control* result:

Theorem 3. *Let $\Omega \subset \mathbb{T}^2$ be any non-empty open set and let $T > 0$. For any $u_0 \in L^2(\mathbb{T}^2)$, there exists $f \in L^2([0,T] \times \Omega)$ such that the solution of the equation*

$$(i\partial_t + \Delta - V(z))u(t,z) = f \mathbb{1}_{[0,T]\times\Omega}(t,z), \qquad u(0,\cdot) = u_0,$$

satisfies

$$u(T,\cdot) \equiv 0.$$

In the case of $V \equiv 0$ (and rational tori) the estimates (1.3) and (1.4) were proved by Jaffard [13] and Haraux [12] (in dimension 2) and Komornik [16] (in higher dimensions) using Kahane's work [17] on lacunary Fourier series. For $V \in C^\infty(\mathbb{T}^2)$ the results above were proved by the last two authors [8], and for a class potentials including continuous potentials on $\mathbb{T}^n$, by Anantharaman–Macia [1]. The paper [1] resolves other questions concerning semiclassical measures on tori and contains further references; see also [4]. For a presentation of other aspects of control theory for the Schrödinger equation we refer to [18]—see also [6, §3].

The paper is organized as follows. In §2 we present dispersive estimates which allow approximation of rough potentials by smooth potentials. In §3 we refine some of the one-dimensional observability estimates and show that they hold for potentials $W \in L^p(\mathbb{T}^1)$, $p > 1$. The next §4 is devoted to semiclassical observability estimates for a family of smooth potentials compact in $L^2(\mathbb{T}^2)$. In the following section an observability result is proved for general tori with constants uniform in a compact set in L^2 (Proposition 5.1(i)). Combined with the results from §2, that gives the proof of the theorem.

2. A priori estimates for solutions to Schrödinger equations

The proof of observability for rough potentials will follow from observability for smooth potentials with estimates controlled by constants depending only on L^2 norms of the potential. The approximation argument uses dispersion estimates for the Schrödinger group on the torus and we first show that these estimates hold in the presence of a potential.

2.1. *The case of* $\mathbb{T}^1$

We start with the simpler case of one-dimensional equations. It will be needed in §3 but it also introduces the idea of the proof in an elementary setting.

We first make some general comments. The operator $-\partial_x^2 + W$, $W \in L^1(\mathbb{T}^1)$, is defined by Friedrichs' extension (see for instance [10, Theorem 4.10]) using the quadratic form

$$q(v, v) = \int_{\mathbb{T}^1} \left(|\partial_x v(x)|^2 + W(x)|v(x)|^2\right) dx, \qquad v \in H^1(\mathbb{T}^1),$$

which is bounded from below since

$$\begin{aligned}\left|\int_{\mathbb{T}^1} W(x)|v(x)|^2 \, dx\right| &\leq C\|W\|_{L^1}\|u\|_{L^\infty}^2 \leq C\|W\|_{L^1}\|\partial_x v\|_{L^2}\|v\|_{L^2} \\ &\leq -C\epsilon\|W\|_{L^1}\|\partial_x v\|_{L^2}^2 - \frac{C}{\epsilon}\|W\|_{L^1}\|v\|_{L^2}^2.\end{aligned}$$

Hence $P = -\partial_x^2 + W$ defined on $C^\infty(\mathbb{T}^1)$ has a unique self-adjoint extension with the domain containing $H^1(\mathbb{T}^1)$. When $W \in L^2(\mathbb{T}^1)$ the operator is self-adjoint with the domain $H^2(\mathbb{T}^1)$. The resolvent, $(-\partial_x^2 + W - z)^{-1}$, $z \notin \mathbb{R}$, is compact and the spectrum is discrete with eigenvalues $\lambda_j \to \infty$.

The following estimate applies to solutions of the Schrödinger equation satisfying the Floquet periodicity conditions

$$v(x + 2\pi) = e^{2\pi ik} v(x), \tag{2.1}$$

or equivalently to solutions of the Schrödinger equation with ∂_x replaced by $\partial_x + ik$. (We note that $u(x) := e^{-ikx} v(x)$ is periodic and $\partial_x v(x) = e^{ikx}(\partial_x + ik)u(x)$.)

Proposition 2.1. *For any $W \in L^2(\mathbb{T}^1)$, there exists $C > 0$ such that for any $k \in [0, 1)$ and $u_0 \in L^2(\mathbb{T}^1)$, the solution to the Schrödinger equation*

$$(i\partial_t + (\partial_x + ik)^2 - W)u = 0, \qquad v|_{t=0} = u_0, \tag{2.2}$$

satisfies

$$\|u\|_{L^\infty(\mathbb{T}_x^1; L^2(0,T))} \leq C(1 + \sqrt{T})(1 + \|W\|_{L^2(\mathbb{T}^1)})\|u_0\|_{L^2(\mathbb{T}^1)}. \tag{2.3}$$

Proof. For $W \equiv 0$ we put $T = 2\pi$ so that, with $c_n = \hat{u}_0(n)$, we have

$$\begin{aligned}\|e^{it\partial_x^2} u_0\|_{L_x^\infty L_t^2}^2 &= \sup_x \int_0^{2\pi} \Big|\sum_{n\in\mathbb{Z}} c_n e^{-it|n+k|^2 + inx}\Big|^2 dt \\ &= \sup_x \sum_{n,m\in\mathbb{Z}} \int_0^{2\pi} e^{i(|n+k|^2 - |m+k|^2)t} e^{i(n-m)x} c_n \bar{c}_m \, dt \\ &= \sup_x \sum_{n\in\mathbb{Z}} \Big| \sum_{\substack{m\in\mathbb{Z} \\ \pm(m+k)=n+k}} c_m e^{imx} \Big|^2 \leq 4\sum_{n\in\mathbb{Z}} |c_n|^2 \leq C\|u_0\|_{L^2(\mathbb{T}^1)}^2.\end{aligned} \tag{2.4}$$

(We note that $\pm(m+k) = n+k$ has one solution only when $k \neq 0$, $1/2$ and two solutions $m = \pm n$ for $k = 0$ and $m = n$, $-n-1$ for $k = 1/2$.) For a non-zero potential $W \in L^2(\mathbb{T}^1)$ we use Duhamel's formula to write

$$u(t) = e^{it\partial_x^2} u_0 + \frac{1}{i}\int_0^t e^{i(t-s)\partial_x^2}(Wu(s))\,ds.$$

Applying (2.4) (now with a small $T > 0$) and the Minkowski inequality we obtain

$$\begin{aligned}\|u\|_{L_x^\infty L_t^2(0,T)} &\leq C\|u_0\|_{L_x^2} + \int_0^T \|\mathbb{1}_{s<t} e^{i(t-s)\Delta}(Wu(s))\|_{L_x^\infty L_s^2(0,T)}\,ds \\ &\leq C\|u_0\|_{L_x^2} + \int_0^T \|e^{i(t-s)\Delta}(Wu(s))\|_{L_x^\infty L_s^2(0,T)}\,ds \\ &\leq C\|u_0\|_{L_x^2} + C\int_0^T \|Wu(s)\|_{L_x^2}\,ds \\ &\leq C\|u_0\|_{L_x^2} + C\sqrt{T}\,\|W\|_{L^2}\|u\|_{L_x^\infty L_t^2(0,T)}. \end{aligned} \tag{2.5}$$

Hence

$$\|u\|_{L_x^\infty L_t^2(0,T)} \leq 2C\|u\|_{L_x^2} \quad \text{if } \sqrt{T}\,\|W\|_{L^2} \leq 1/4. \tag{2.6}$$

To obtain the estimate for multiples of T satisfying (2.6) we note that, by the invariance of the L_x^2 norm of $u(t)$, $\int_{(k-1)T}^{kT} \|u(t)\|_{L_x^\infty}^2\,dt \leq 2C\|u((k-1)t)\|_{L_x^2} = 2C\|u_0\|_{L_x^2}$. Iterating this inequality gives (2.3). □

2.2. *The case of two-dimensional tori*

We now assume that $A = 2\pi$, $B = 2\pi\gamma^{-1} > 0$ in the definition of $\mathbb{T}^2$. The case of general A, B follows by rescaling. For $n = (n_1, n_2) \in \mathbb{Z}^2$, we set

$$|n| = \sqrt{n_1^2 + \gamma n_2^2}, \qquad n \cdot x = n_1 x_1 + \gamma n_2 x_2. \tag{2.7}$$

We start with some general observations. If $V \in L^2(\mathbb{T}^2; \mathbb{R})$ then $-\Delta + V$ on $C^\infty(\mathbb{T}^2)$ is a symmetric operator. Also, by Sobolev inequalities,

$$(-\Delta + i)^{-1} : L^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2) \hookrightarrow C^{1-\varepsilon}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2),$$

is a compact operator. Hence, as multiplication by $V \in L^2$ is bounded $L^\infty \to L^2$, $V(-\Delta + i)^{-1}$ is a compact operator on L^2. It follows that the operator $-\Delta + V$ is essentially self-adjoint and has a discrete spectrum (see for instance [10, Theorem 4.19]). Since for $u \in H^2(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$, we have $Vu \in L^2$, the domain is equal to $H^2(\mathbb{T}^2)$. In particular,

$$u(t) := e^{it(\Delta - V)} u_0 \in C^0(\mathbb{R}_t; H^2(\mathbb{T}^2)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{T}^2)),$$

and

$$u(t) = e^{it\Delta} u_0 + \frac{1}{i}\int_0^t e^{i(t-s)\Delta}(Vu(s))\,ds. \tag{2.8}$$

Proposition 2.2. *Let $T > 0$. For any compact subset $\mathcal{V} \subset L^2(\mathbb{T}^2)$, there exist $C(\mathcal{V})$ and $\epsilon > 0$ such that for any $V \in \mathcal{V} + B(0,\epsilon) \subset L^2(\mathbb{T}^2)$ and any*

$$v_0 \in L^2(\mathbb{T}^2), \quad f \in L^1((0,T); L^2(\mathbb{T}^2)) + L^{4/3}(\mathbb{T}^2; L^2(0,T)),$$

the solution to

$$(i\partial_t + (\Delta - V))u = f, \quad u|_{t=0} = v_0, \tag{2.9}$$

satisfies

$$\|u\|_{L^\infty((0,T);L^2(\mathbb{T}^2))\cap L^4(\mathbb{T}^2_x;L^2(0,T))} \leq C(\mathcal{V})\big(\|v_0\|_{L^2(\mathbb{T}^2)} + \|f\|_{L^1((0,T);L^2(\mathbb{T}^2))+L^{4/3}(\mathbb{T}^2;L^2(0,T))}\big). \tag{2.10}$$

Before proving this result, let us show how it implies that Jaffard's result (Theorem 2 with $V = 0$) is stable under perturbation with potentials small in $L^2(\mathbb{T}^2)$:

Corollary 2.3. *For any non-empty open set Ω and $T > 0$, there exist constants $\kappa, K > 0$ such that for $V \in L^2(\mathbb{T}^2)$,*

$$\|V\|_{L^2(\mathbb{T}^2)} \leq \kappa \ \Rightarrow \ \|u_0\|^2_{L^2(\mathbb{T}^2)} \leq K \int_0^T \|e^{-it(-\Delta+V)}u_0\|^2_{L^2(\Omega)}\,dt,$$

for any $u_0 \in L^2(\mathbb{T}^2)$.

Proof. The Duhamel formula gives

$$u = e^{-it(-\Delta+V)}u_0 = e^{it\Delta}u_0 + \frac{1}{i}\int_0^t e^{i(t-s)\Delta}(Vu(s))\,ds,$$

and Jaffard's result (estimate (1.4) for $V = 0$) applies to the first term. Hence, for a constant K_0 depending on Ω and T,

$$\begin{aligned}
\|u_0\|_{L^2(\mathbb{T}^2)} &\leq K_0 \int_0^T \|e^{it\Delta}u_0\|^2_{L^2(\Omega)}\,dt \\
&= K_0 \int_0^T \left\| e^{it(\Delta-V)}u_0 - \frac{1}{i}\int_0^t e^{i(t-s)\Delta}(Vu(s))\,ds \right\|^2_{L^2(\Omega)} dt \\
&\leq 2K_0 \int_0^T \|e^{it(\Delta-V)}u_0\|^2_{L^2(\Omega)}\,dt \\
&\quad + 2K_0T \left\| \int_0^t e^{i(t-s)\Delta}(Vu(s))\,ds \right\|^2_{L^\infty((0,T);L^2(\mathbb{T}^2))}.
\end{aligned} \tag{2.11}$$

We now use Proposition 2.2 with $\mathcal{V} = \{V\}$, $v_0 = 0$ and $f = Vu$ to obtain

$$\begin{aligned}
\left\| \int_0^t e^{i(t-s)\Delta}(Vu(s))\,ds \right\|_{L^\infty((0,T);L^2(\mathbb{T}^2))} &\leq C\|Vu\|_{L^{4/3}(\mathbb{T}^2;L^2(0,T))} \\
&\leq C\|V\|_{L^2(\mathbb{T}^2)}\|u\|_{L^4(\mathbb{T}^2;L^2(0,T))}.
\end{aligned}$$

Applying Proposition 2.2 to the right-hand side, now with $v_0 = u_0$, $f = 0$, gives

$$\left\| \int_0^t e^{i(t-s)\Delta}(Vu(s))\, ds \right\|_{L^\infty((0,T);L^2(\mathbb{T}^2))} \le C\|V\|_{L^2(\mathbb{T}^2)}\|u_0\|_{L^2(\mathbb{T}^2)},$$

so that (2.11) becomes

$$\|u\|^2_{L^2(\mathbb{T}^2)} \le 2K_0 \int_0^T \|e^{it(\Delta-V)}u_0\|^2_{L^2(\Omega)}\, dt + 2CK_0T\|V\|^2_{L^2(\mathbb{T}^2)}\|u_0\|^2_{L^2(\mathbb{T}^2))}.$$

To conclude, it suffices to take $2CK_0T\kappa^2 \le 1/2$. (We note that since K_0 depends on Ω and T while C depends on T, we have no other choice than taking $\kappa > 0$ small.) □

Remark. In §5 we will eliminate the smallness assumption on $\|V\|_{L^2}$ and that will prove Theorem 2.

The proof of Proposition 2.2 proceeds in several steps. We start by proving the estimate for $V = 0$, then we prove the general case by a perturbation argument.

The next proposition is a "fuzzy" version of the classical estimate of Zygmund:

$$\exists C > 0\ \forall \tau \in \mathbb{N}, \quad \Big\| \sum_{n\in\mathbb{Z}^2,\, |n|^2=\tau} c_n e^{in\cdot x} \Big\|^2_{L^4(\mathbb{T}^2)} \le C \sum_{n\in\mathbb{Z}^2,\, |n|^2=\tau} |c_n|^2, \tag{2.12}$$

and it is motivated by the Córdoba square function estimate [9]:

Proposition 2.4. *There exists $C > 0$ such that for any $\kappa \ge 0$ and $0 < h < 1$, and any $u \in L^2(\mathbb{T}^2)$ satisfying*

$$\hat{u}(n) = 0 \quad \text{for } n \notin \mathcal{B}(\kappa, h) := \left\{n \in \mathbb{Z}^2;\ \left|h^2|n|^2 - 1\right| \le \kappa^2 h^2\right\},$$

we have

$$\|u\|_{L^4(\mathbb{T}^2)} \le \begin{cases} C(1+\kappa)^{1/4}(1+\kappa^2 h)^{1/4}\|u\|_{L^2(\mathbb{T}^2)} & \text{if } \kappa \le h^{-1}, \\ C(1+\kappa)^{1/2}\|u\|_{L^2(\mathbb{T}^2)} & \text{if } \kappa \ge h^{-1}. \end{cases} \tag{2.13}$$

We note that the case of $\kappa = 0$ in (2.13) is (2.12), while $\kappa = h^{-1}$ is simply Sobolev embeddings and $\kappa = h^{-1/2}$ is Sogge's estimate for spectral projectors [22], [23, Theorem 10.11] (for which we give an arithmetic proof below).

Proof. We first note that we can assume that $\kappa \ge 1$ as the sets $\mathcal{B}(\kappa, h)$ increase with increasing κ.

For a constant $\delta > 0$, to be fixed later, we distinguish two regimes: $\kappa h \ge \delta$ and $\kappa h \le \delta$. In the first regime, the estimate follows from the Sobolev embedding $H^{1/2}(\mathbb{T}^2) \to L^4(\mathbb{T}^2)$: $\hat{u}(n) = 0$ unless $|n|^2 \le h^{-2} + \kappa^2 \le (1/\delta + 1)\kappa^2$, and this implies

$$\|u\|_{H^{1/2}(\mathbb{T}^2)} \le C_\delta \kappa^{1/2}\|u\|_{L^2}.$$

From now on we assume that $h\kappa \le \delta$. In this regime, we can change the set $\mathcal{B}(\kappa, h)$ to

$$\mathcal{A}(\kappa, h) := \left\{n \in \mathbb{Z}^2;\ \left|h|n| - 1\right| \le \kappa^2 h^2\right\}.$$

The idea is to prove an *arithmetic version of the Córdoba square function estimate* [9]. Indeed, the usual version allows one only to work with $\kappa \geq h^{-1/2}$ (the uncertainty principle). Our version below allows us to get estimates all the way down to $\kappa \sim 1$ (that is, much beyond the uncertainty principle). We first notice that we can also assume that the spectrum of u is also contained in the upper quadrant $\{z \in \mathbb{C};\ \operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0\}$ of the plane (here and in what follows we identify $\mathbb{R}^2$ with $\mathbb{C}$). Indeed, if the result is true for the upper quadrant, by symmetry, it is true for any quadrant, and with a different constant in the general case. Then we decompose the intersection of the annulus with this quadrant into a disjoint union of angular sectors of angles $h\kappa$:

$$\mathcal{A}(\kappa, h) \cap \{\operatorname{Im} z \geq 0, \operatorname{Re} z \geq 0\} = \bigcup_{\alpha=0}^{N_{\kappa,h}} \mathcal{A}_\alpha(\kappa, h), \qquad N_{\kappa,h} := \left[\frac{\pi}{2h\kappa}\right],$$

where

$$\mathcal{A}_\alpha(\kappa, h) := \left\{z;\ \operatorname{Re} z \geq 0,\ \operatorname{Im} z \geq 0,\ \left|h|z| - 1\right| \leq \kappa^2 h^2,\ \arg(z) \in [\alpha h\kappa, (\alpha+1)h\kappa)\right\}.$$

The proof relies on the following geometric lemma which will be proved in Appendix B:

Lemma 2.5. *Fix $\delta > 0$ small enough. Then there exists $Q \in \mathbb{N}$ such that for any $0 < h < 1$ and any $1 \leq \kappa \leq \delta/h$, we have*

$$\begin{aligned} &\forall \alpha, \beta, \alpha', \beta' \in \{0, 1, \ldots, N_{\kappa,h}\},\\ &\qquad (\mathcal{A}_\alpha(\kappa, h) + \mathcal{A}_\beta(\kappa, h)) \cap (\mathcal{A}_{\alpha'}(\kappa, h) + \mathcal{A}_{\beta'}(\kappa, h)) \neq \emptyset\\ &\qquad\quad \Rightarrow\ |\alpha - \alpha'| + |\beta - \beta'| \leq Q \text{ or } |\alpha - \beta'| + |\beta - \alpha'| \leq Q. \end{aligned} \tag{2.14}$$

We apply the lemma as folllows. We have

$$u = \sum_{\alpha=0}^{N_{\kappa,h}} U_\alpha, \qquad u^2 = \sum_{\alpha,\beta=0}^{N_{\kappa,h}} U_\alpha U_\beta, \qquad U_\alpha := \sum_{n \in \mathbb{Z}^2 \cap \mathcal{A}_\alpha(\kappa,h)} u_n e^{in\cdot x},$$

and hence

$$\|u\|^4_{L^4(\mathbb{T}^2)} = \sum_{\alpha,\beta,\alpha',\beta'=0}^{N_{\kappa,h}} \int_{\mathbb{T}^2} U_\alpha U_\beta \overline{U}_{\alpha'} \overline{U}_{\beta'}(x)\, dx. \tag{2.15}$$

The integral vanishes unless

$$(\mathcal{A}_\alpha(\kappa, h) + \mathcal{A}_\beta(\kappa, h)) \cap (\mathcal{A}_{\alpha'}(\kappa, h) + \mathcal{A}_{\beta'}(\kappa, h)) \neq \emptyset$$

as otherwise

$$n \in \mathbb{Z}^2 \cap \mathcal{A}_\alpha,\ m \in \mathbb{Z}^2 \cap \mathcal{A}_\beta,\ p \in \mathbb{Z}^2 \cap \mathcal{A}_{\alpha'},\ q \in \mathbb{Z}^2 \cap \mathcal{A}_{\beta'}\ \Rightarrow\ n + m - (p + q) \neq 0,$$

and, using the inner product (2.7), $\int_{\mathbb{T}^2} e^{ix\cdot(n+m-p-q)}\, dx = 0$. Lemma 2.5 then shows that we can restrict the sum in (2.15) to the subset of indices $(\alpha, \beta, \alpha', \beta')$ satisfying

$$|\alpha - \alpha'| + |\beta - \beta'| \leq Q \quad \text{or} \quad |\alpha - \beta'| + |\beta - \alpha'| \leq Q.$$

This and an application of Hölder's inequality,

$$\left|\int_{\mathbb{T}^2} U_\alpha U_\beta \overline{U}_{\alpha'} \overline{U}_{\beta'}(x)\,dx\right| \le \|U_\alpha\|_{L^4(\mathbb{T}^2)} \|U_\beta\|_{L^4(\mathbb{T}^2)} \|U_{\alpha'}\|_{L^4(\mathbb{T}^2)} \|U_{\beta'}\|_{L^4(\mathbb{T}^2)}$$
$$\le \begin{cases} (\|U_\alpha\|^2_{L^4(\mathbb{T}^2)} + \|U_{\alpha'}\|^2_{L^4(\mathbb{T}^2)})(\|U_\beta\|^2_{L^4(\mathbb{T}^2)} + \|U_{\beta'}\|^2_{L^4(\mathbb{T}^2)}) \\ (\|U_\alpha\|^2_{L^4(\mathbb{T}^2)} + \|U_{\beta'}\|^2_{L^4(\mathbb{T}^2)})(\|U_\beta\|^2_{L^4(\mathbb{T}^2)} + \|U_{\alpha'}\|^2_{L^4(\mathbb{T}^2)}) \end{cases}$$

give

$$\|u\|^4_{L^4(\mathbb{T}^2)} \le C Q^2 \Big(\sum_{\alpha=0}^{N_{\kappa,h}} \|U_\alpha\|^2_{L^4(\mathbb{T}^2)}\Big)^2. \tag{2.16}$$

To estimate the norms of U_α we write

$$\begin{aligned} \|U_\alpha\|_{L^4(\mathbb{T}^2)} &\le C \|U_\alpha\|^{1/2}_{L^\infty(\mathbb{T}^2)} \|U_\alpha\|^{1/2}_{L^2(\mathbb{T}^2)} \\ &\le \Big(\sum_{n\in\mathbb{Z}^2\cap\mathcal{A}_\alpha(\kappa,h)} |u_n|\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}^2\cap\mathcal{A}_\alpha(\kappa,h)} |u_n|^2\Big)^{1/4} \\ &\le C|\mathbb{Z}^2 \cap \mathcal{A}_\alpha(\kappa,h)|^{1/4} \|U_\alpha\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{2.17}$$

To estimate the number of integral points in $\mathcal{A}_\alpha(\kappa,h)$, we first notice that $\mathcal{A}_\alpha(\kappa,h)$ is included in a rectangle of height $1+\kappa$ and width $1+3\kappa^2 h$.

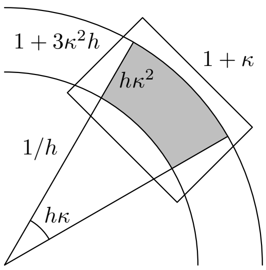

Fig. 1. The angular region $\mathcal{A}_\alpha(\kappa,h)$ fitted inside a rectangle.

Now, the number of integral points in any rectangle of height H and width W is bounded by $C\max(H,1)\max(W,1)$. (To see this, notice that open discs of radius $1/2$ centered at the integer points are pairwise disjoint and are all included in a rectangle of height $H+1$ and width $W+1$.) Hence, recalling that $\kappa h \le \delta$, we have

$$|\mathbb{Z}^2 \cap \mathcal{A}_\alpha(\kappa,h)| \le C(1+\kappa)(1+3\kappa^2 h) \le C(1+\kappa)^2.$$

Combining this with (2.17) and (2.16) gives

$$\|u\|^4_{L^4(\mathbb{T}^2)} \le C(1+\kappa)(1+\kappa^2 h)\|u\|^4_{L^2(\mathbb{T}^2)},$$

concluding the proof. □

The next step in the proof of Proposition 2.2 is an optimal (at least in terms of the spectral region where it holds) resolvent estimate—see Kenig–Dos Santos–Salo [11, Remark 1.2] and Bourgain–Shao–Sogge–Yao [3] for related results.

Proposition 2.6. *For any compact subset $\mathcal{V} \subset L^2(\mathbb{T}^2)$, there exist $C(\mathcal{V})$ and $\epsilon > 0$ such that for any $V \in \mathcal{V} + B(0, \epsilon)$, any $f \in C^\infty(\mathbb{T}^2)$ and any $\tau \in \mathbb{C}$ with $|\mathrm{Im}\,\tau| \geq 1$,*

$$\|(-\Delta + V - \tau)^{-1} f\|_{L^4(\mathbb{T}^2)} \leq C\|f\|_{L^{4/3}(\mathbb{T}^2)}. \tag{2.18}$$

We deduce it from Proposition 2.4 and the following elementary result:

Lemma 2.7. *Assume that $\mathcal{V}$ is a compact subset of $L^2(\mathbb{T}^2)$. Then for any $\delta > 0$ there exists $C_\delta > 0$ and for any $V \in \mathcal{V}$ there exists $V_\delta \in L^\infty(\mathbb{T}^2)$ such that*

$$\|V_\delta - V\|_{L^2(\mathbb{T}^2)} \leq \delta, \qquad \|V_\delta\|_{L^\infty(\mathbb{T}^2)} \leq C_\delta.$$

Proof. This is obvious for $\mathcal{V} = \{V_0\}$ since $L^\infty \subset L^2$ is dense. Applying it with δ replaced by $\delta/2$ the statement remains true for V with $\|V - V_0\|_{L^2} \leq \delta/2$. A covering argument provides the result for a general compact set in L^2. □

Proof of Proposition 2.6. For $\mathrm{Re}\,\tau \leq C$ for any fixed C, we get (2.18) directly. Indeed, from $(-\Delta - \tau + V)u = f$, multiplying by $\overline{u}$, integrating by parts and taking real and imaginary parts, we get

$$\|\nabla u\|^2_{L^2(\mathbb{T}^2)} - \mathrm{Re}\,\tau\|u\|^2_{L^2(\mathbb{T}^2)} \leq \left\|V|u|^2\right\|_{L^1(\mathbb{T}^2)} + \|u\|_{L^4(\mathbb{T}^2)}\|f\|_{L^{4/3}(\mathbb{T}^2)},$$
$$|\mathrm{Im}\,\tau|\,\|u\|^2_{L^2(\mathbb{T}^2)} \leq \|u\|_{L^4(\mathbb{T}^2)}\|f\|_{L^{4/3}(\mathbb{T}^2)}.$$

Since $|\mathrm{Im}\,\tau| \geq 1$, the Sobolev embedding and Lemma 2.7 imply

$$\begin{aligned}\|u\|^2_{L^4(\mathbb{T}^2)} &\leq C\|u\|^2_{H^1(\mathbb{T}^2)}\\ &\leq C\big(\|V_\delta - V\|_{L^2(\mathbb{T}^2)}\|u\|^2_{L^4(\mathbb{T}^2)} + \|V_\delta\|_{L^\infty(\mathbb{T}^2)}\|u\|^2_{L^2(\mathbb{T}^2)} + \|u\|_{L^4(\mathbb{T}^2)}\|f\|_{L^{4/3}(\mathbb{T}^2)}\big)\\ &\leq C(\delta + \epsilon)\|u\|^2_{L^4(\mathbb{T}^2)} + C(\|V_\delta\|_{L^\infty(\mathbb{T}^2)} + 1)\|u\|_{L^4(\mathbb{T}^2)}\|f\|_{L^{4/3}(\mathbb{T}^2)}\end{aligned}$$

and choosing $\epsilon < \delta = \frac{1}{4}C$ gives the result.

For $\mathrm{Re}\,\tau > C$ we start with the case of $V = 0$ and notice that

$$(-\Delta - \tau)^{-1} = (-\Delta - \tau)^{-1/2}((-\Delta - \bar{\tau})^{-1/2})^* : L^{4/3} \to L^4$$

follows from $(-\Delta - \tau)^{-1/2} : L^2 \to L^4 = (L^{4/3})^*$. Here the square root is defined using the spectral theorem and the branches chosen for $\pm\,\mathrm{Im}\,\tau > 1$ so that

$$(\lambda - \tau)^{1/2}\,\overline{(\lambda - \bar{\tau})^{1/2}} = \lambda - \tau, \qquad \lambda \geq 0.$$

Hence we need to prove that

$$\|u\|_{L^4(\mathbb{T}^2)} \leq C\|f\|_{L^2(\mathbb{T}^2)}, \qquad u := (-\Delta - \tau)^{-1/2} f.$$

To use Proposition 2.4 we write the resolvent applied to f using the Fourier series

$$u=\sum_n \frac{f_n}{(|n|^2-\tau)^{1/2}}e^{in\cdot x}=u_0+\sum_{j=1}^\infty u_j,\qquad u_j:=\sum_{2^{j-1}\le ||n|^2-\operatorname{Re}\tau|<2^j}\frac{f_n}{(|n|^2-\tau)^{1/2}}e^{in\cdot x}.$$

We note that $u_0=\sum_{||n|^2-\operatorname{Re}\tau|<1} f_n(|n|^2-\tau)^{-1/2}e^{in\cdot x}$ and hence Proposition 2.4 gives

$$\|u_0\|_{L^4(\mathbb{T}^2)}\le C\|f\|_{L^2(\mathbb{T}^2)}.$$

Applying (2.13) to u_j's with $h=(\operatorname{Re}\tau)^{-1/2}$ and $\kappa=2^{j/2}$ gives

$$\begin{aligned}\|u-u_0\|_{L^4(\mathbb{T}^2)}&\le C\sum_j 2^{j/4}\|u_j\|_{L^2(\mathbb{T}^2)}\\&\le\Big(\sum_{j=1}^\infty 2^{-j/2}\Big)^{1/2}\Big(\sum_{j=1}^\infty 2^j\sum_{2^{j-1}\le ||n|^2-\operatorname{Re}\tau|<2^j}\frac{|f_n|^2}{\big||n|^2-\tau\big|}\Big)^{1/2}\\&\le C\|f\|_{L^2(\mathbb{T}^2)},\end{aligned}$$

which concludes the proof of Proposition 2.6 for $V=0$.

The general case $V\neq 0$ follows from the same perturbation argument as in the case $\operatorname{Re}\tau\le C$. Indeed, from $(-\Delta-\tau)u=-Vu+f$, we deduce

$$|\operatorname{Im}\tau|\,\|u\|^2_{L^2(\mathbb{T}^2)}\le\|u\|_{L^4(\mathbb{T}^2)}\|f\|_{L^{4/3}(\mathbb{T}^2)},$$

and from the resolvent estimate for $V=0$,

$$\begin{aligned}\|u\|_{L^4(\mathbb{T}^2)}&\le C\|Vu\|_{L^{4/3}(\mathbb{T}^2)}+\|f\|_{L^{4/3}(\mathbb{T}^2)}\\&\le C\big(\|V_\delta-V\|_{L^2(\mathbb{T}^2)}\|u\|_{L^4(\mathbb{T}^2)}+\|V_\delta\|_{L^\infty(\mathbb{T}^2)}\|u\|_{L^2(\mathbb{T}^2)}+\|f\|_{L^{4/3}(\mathbb{T}^2)}\big)\\&\le C\delta\|u\|_{L^4(\mathbb{T}^2)}+C\big(\|V_\delta\|_{L^\infty(\mathbb{T}^2)}\|u\|^{1/2}_{L^4(\mathbb{T}^2)}\|f\|^{1/2}_{L^{4/3}(\mathbb{T}^2)}+\|f\|_{L^{4/3}(\mathbb{T}^2)}\big).\end{aligned}$$

Choosing δ small enough gives the desired estimate. □

Proof of Proposition 2.2. Let us first study the contribution of v_0. Putting $Tu_0=e^{it(\Delta-V)}u_0$ we have

$$TT^*f=\int_0^T e^{i(t-s)(\Delta-V)}f(s)\,ds=\int_0^t e^{i(t-s)(\Delta-V)}f(s)\,ds+\int_t^T e^{i(t-s)(\Delta-V)}f(s)\,ds.$$

To prove that $T:L^2(\mathbb{T}^2)\to L^4(\mathbb{T}^2_x;L^2(0,T))$ it suffices to prove that

$$TT^*:L^{4/3}(\mathbb{T}^2_x;L^2(0,T))\to L^4(\mathbb{T}^2_x;L^2(0,T)),$$

and we will show it for the two operators on the right-hand side, say the first one. That means showing that for solutions to $(i\partial_t+\Delta-V)v=f$, $v|_{t=0}=0$, we have

$$\|v\|_{L^4(\mathbb{T}^2;L^2(0,T))}\le C\|f\|_{L^{4/3}(\mathbb{T}^2;L^2(0,T))}. \tag{2.19}$$

Let $U = ve^{-t}\mathbb{1}_{t>0}$ and $F = fe^{-t}\mathbb{1}_{0<t<T}$. We have $(i\partial_t + \Delta - V + i)U = F$ and hence, by taking the Fourier transform in t,

$$(\Delta - V + i - \tau)\widehat{U} = \widehat{F}.$$

Proposition 2.6 now shows that for any $\tau \in \mathbb{R}$,

$$\|\widehat{U}(\tau)\|_{L^4(\mathbb{T}^2)} \leq C\|\widehat{F}(\tau)\|_{L^{4/3}(\mathbb{T}^2)},$$

which implies

$$\begin{aligned}\|u\|_{L^4(\mathbb{T}^2_x;L^2(0,T))} &\leq C\|U\|_{L^4(\mathbb{T}^2_x;L^2(\mathbb{R}_t))} = C\|\widehat{U}\|_{L^4(\mathbb{T}^2_x;L^2(\mathbb{R}_\tau))}\\ &\leq C\|\widehat{U}\|_{L^2(\mathbb{R}_\tau;L^4(\mathbb{T}^2_x))} \leq C'\|\widehat{F}\|_{L^2(\mathbb{R}_\tau;L^{4/3}(\mathbb{T}^2_x))}\\ &\leq C'\|\widehat{F}\|_{L^{4/3}(\mathbb{T}^2_x;L^2(\mathbb{R}_\tau))} = C'\|F\|_{L^{4/3}(\mathbb{T}^2_x;L^2(0,T))},\end{aligned} \tag{2.20}$$

concluding the proof of (2.19).

Part of the nonhomogeneous estimate in (2.10),

$$\|v\|_{L^\infty((0,T);L^2(\mathbb{T}^2))\cap L^4(\mathbb{T}^2_x;L^2(0,T))} \leq C\|f\|_{L^1((0,T);L^2(\mathbb{T}^2))},$$

follows from the boundedness of the operator T from L^2 to $L^4(\mathbb{T}^2; L^2(0,T))$ and the Minkowski inequality. Finally, since the dual of the operator $f \mapsto \int_0^t e^{i(t-s)(\Delta-V)}f(s)\,ds$ is $g \mapsto \int_t^T e^{i(t-s)(\Delta-V)}g(s)\,ds$, we also get

$$\|u\|_{L^\infty((0,T);L^2(\mathbb{T}^2))} \leq C\|f\|_{L^1((0,T);L^2(\mathbb{T}^2))+L^{4/3}(\mathbb{T}^2;L^2(0,T))},$$

which concludes the proof of Proposition 2.2. □

We conclude this section with a continuity result which will be useful later:

Proposition 2.8. *Consider a sequence $\{V_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{T}^2)$ converging to $V \in L^2(\mathbb{T}^2)$. Then there exists $C > 0$ such that for any $v_0 \in L^2(\mathbb{T}^2)$,*

$$\|e^{-it(-\Delta+V)}v_0 - e^{-it(-\Delta+V_n)}v_0\|_{L^\infty((0,T);L^2(\mathbb{T}^2))} \leq C\|V-V_n\|_{L^2(\mathbb{T}^2)}\|u_0\|_{L^2(\mathbb{T}^2)}. \tag{2.21}$$

Remark. The result in Proposition 2.8 can be stated more generally, for a compact subset $\mathcal{V} \subset L^2(\mathbb{T}^2)$, and is equivalent to the Lipschitz continuity of the map

$$V \in \mathcal{V} \subset L^2(\mathbb{T}^2) \mapsto e^{-it(-\Delta+V)} \in L^\infty((0,T); \mathcal{L}(L^2(\mathbb{T}^2))).$$

A slight modification of the proof presented here shows that it is in fact also Lipschitz on bounded subsets of L^p, $p > 2$. It would be interesting to investigate such properties on other manifolds, as they seem to depend strongly on the geometry. Indeed, the analysis in [5, Theorem 2] is likely to give that on spheres, there exists a sequence of potentials $\{V_n\}_{n\in\mathbb{N}}$ such that for any $T > 0$ and $p < \infty$,

$$\lim_{n\to\infty}\|V_n\|_{L^p(\mathbb{S}^2)} = 0, \quad \text{but} \quad \lim_{n\to\infty}\|e^{it\Delta} - e^{it(\Delta-V_n)}\|_{L^\infty((0,T);\mathcal{L}(L^2(\mathbb{S}^2)))} > 0.$$

Proof of Proposition 2.8. Let $u = e^{it(\Delta - V)}v_0$ and $u_n = e^{it(\Delta - V_n)}v_0$, so that the Duhamel formula gives

$$u - u_n = \frac{1}{i}\int_0^t e^{i(t-s)(\Delta - V)}((V_n - V)u_n(s))\,ds.$$

Proposition 2.2 applied with $\mathcal{V} = \{V\}$, $v_0 = 0$ and $f = (V_n - V)u_n$, and Hölder's inequality give

$$\begin{aligned}\|u_V - u_n\|_{L^\infty((0,T);L^2(\mathbb{T}^2_x))} &\le C\|(V - V_n)u_n\|_{L^{4/3}(\mathbb{T}^2;L^2(0,T))}\\ &\le C\|V - V_n\|_{L^2(\mathbb{T}^2)}\|u_n\|_{L^4(\mathbb{T}^2;L^2(0,T))}.\end{aligned}$$

Applying Proposition 2.2 again, now with $\mathcal{V} = \{V_n;\ n \in \mathbb{N}\} \cup \{V\}$, and $f = 0$, we estimate the right-hand side to obtain the desired estimate:

$$\|u_V - u_n\|_{L^\infty((0,T);L^2(\mathbb{T}^2_x))} \le C\|V - V_n\|_{L^2(\mathbb{T}^2)}\|v_0\|_{L^2(\mathbb{T}^2_x)}. \qquad \square$$

3. One-dimensional observability estimates

In this section we consider the one-dimensional analog of our result which we prove for L^p potentials, $p > 1$. In applications to control and observability on 2-tori we will use it only for $p = 2$ but the finer estimate may be of independent interest.

Let us recall (see Section 2.1) that the operator $-\partial_x^2 + W$, $W \in L^1(\mathbb{T}^1)$, which is defined on $C^\infty(\mathbb{T}^1)$ has a unique self-adjoint extension with the domain containing $H^1(\mathbb{T}^1)$ (if $W \in L^2(\mathbb{T}^1)$ the domain is $H^2(\mathbb{T}^1)$). The resolvent $(-\partial_x^2 + W - z)^{-1}$, $z \notin \mathbb{R}$, is compact and the spectrum is discrete with eigenvalues $\lambda_j \to \infty$.

We have the following one-dimensional observability result which holds for functions satisfying the Floquet boundary conditions:

Proposition 3.1. *Assume that $W \in L^p(\mathbb{T}^1)$, $p > 1$, and $\omega \subset \mathbb{T}^1$ is a non-empty open set. Then for any $T > 0$ there exists $K_0 > 0$ such that for any $k \in \mathbb{R}$ and $v \in L^2(\mathbb{T}^1)$,*

$$\|v\|^2_{L^2(\mathbb{T}^1)} \le K_0 \int_0^T \|e^{it((\partial_x + ik)^2 - W)}v\|^2_{L^2(\omega)}\,dt. \tag{3.1}$$

Let us first notice that conjugating with $e^{ix[k]}$, we can replace k by $k - [k]$ and hence assume that $k \in [0, 1]$. We first prove the stationary version following the elementary approach of [7]:

Proposition 3.2. *Under the assumptions of Proposition 3.1 there exists $C_1 = C_1(\omega, \|W\|_{L^p(\mathbb{T}^1)})$ such that for any $\tau \in \mathbb{R}$, any solution to*

$$(-(\partial_x + ik)^2 + W - \tau)u = g$$

satisfies

$$\|u\|_{L^2(\mathbb{T}^1)} \le C_1(\langle\tau\rangle^{-1/2}\|g\|_{L^2(\mathbb{T}^1)} + \|u\|_{L^2(\omega)}). \tag{3.2}$$

This follows from the following result which holds for $W = 0$.

Lemma 3.3. *Let $\omega \subset \mathbb{T}^1$ be an open set. Then there exists a constant $C_0 = C_0(\omega)$ such that for any $k \in \mathbb{R}$ and any $u \in H^1(\mathbb{T}^1)$ satisfying*

$$(-(\partial_x + ik)^2 - \tau)u = f + g, \tag{3.3}$$

we have

$$\|u\|_{L^2(\mathbb{T}^1)} + \langle\tau\rangle^{-1/2}\|\partial_x u\|_{L^2(\mathbb{T}^1)} \leq C_0\big(\|f\|_{H^{-1}(\mathbb{T}^1)} + \langle\tau\rangle^{-1/2}\|g\|_{L^2(\mathbb{T}^1)} + \|u\|_{L^2(\omega)}\big). \tag{3.4}$$

Proof. We start by showing that there exists a constant C such that for any $k \in \mathbb{R}$ and any $u \in H^1(\mathbb{T}^1)$ satisfying

$$(-(\partial_x + ik)^2 - \tau)u = (\partial_x + ik)f + g, \tag{3.5}$$

we have

$$\|u\|_{L^2(\mathbb{T}^1)} \leq C\big(\|f\|_{L^2(\mathbb{T}^1)} + \langle\tau\rangle^{-1/2}\|g\|_{L^2(\mathbb{T}^1)} + \|u\|_{L^2(\omega)}\big). \tag{3.6}$$

The elementary proof given in [7] shows that it is true for $k = 0$. For any solution U to (3.5), let $v = e^{-ikx}u$, which is no longer periodic but satisfies the Floquet conditions (2.1) and

$$(-\partial_x^2 - \tau)v = \partial_x F + G, \quad F = e^{-ikx}f, \quad G = e^{-ikx}g.$$

Choosing a parametrization on $\mathbb{T}^1$ so that $2\pi \in \omega$ we take $\chi \in C^\infty(\mathbb{T}^1)$ equal to one in a neighbourhood of $\mathbb{T}^1 \setminus \omega$, and vanishing in a neighbourhood of 2π. Hence, $\operatorname{supp}\chi v \subset (\epsilon, 2\pi - \epsilon)$ and $u = \chi v$ defines a function on $\mathbb{T}^1$ such that

$$(-\partial_x^2 - \tau)u = \partial_x(\chi F + 2\chi' v) + \chi G - \chi' F - \chi'' v.$$

Applying (3.6) for $k = 0$, we obtain, using the properties of χ,

$$\begin{aligned}\|\chi v\|_{L^2(\mathbb{T}^1)} &\leq C\big(\|\chi F + 2\chi' v\|_{L^2(\mathbb{T}^1)} + \langle\tau\rangle^{-1/2}\|\chi G - \chi' F - \chi'' v\|_{L^2(\mathbb{T}^1)}\big) \\ &\leq C'\big(\|F\|_{L^2(\mathbb{T}^1)} + \langle\tau\rangle^{-1/2}\|G\|_{L^2(\mathbb{T}^1)} + \|v\|_{L^2(\omega)}\big),\end{aligned}$$

which, coming back to u, implies that (3.6) holds for any k.

Since for $k \in [0, 1]$,

$$\begin{aligned}\|f\|_{H^{-1}} &= \inf\{\|F\|_{L^2} + \|ikF + H\|_{L^2};\ f = (\partial_x + ik)F + H\} \\ &\geq \tfrac{1}{2}\inf\{\|F\|_{L^2} + \|H\|_{L^2};\ f = (\partial_x + ik)F + H\},\end{aligned}$$

the estimate on $\|u\|_{L^2(\mathbb{T}^1)}$, $u(x) = e^{ikx}v(x)$, in (3.4) follows from (3.6).

To estimate $(\partial_x + ik)u$ we write

$$\begin{aligned}\|(\partial_x + ik)u\|^2_{L^2(\mathbb{T}^1)} &= \langle(-(\partial_x + ik)^2 - \tau)u, u\rangle_{L^2(\mathbb{T}^1)} + \tau\|u\|^2_{L^2(\mathbb{T}^1)} \\ &= \langle f + g, u\rangle_{L^2(\mathbb{T}^1)} + \tau\|u\|^2_{L^2(\mathbb{T}^1)} \\ &\leq \|f\|_{H^{-1}(\mathbb{T}^1)}\|u\|_{H^1(\mathbb{T}^1)} + \|g\|_{L^2(\mathbb{T}^1)}\|u\|_{L^2(\mathbb{T}^1)} + \langle\tau\rangle\|u\|^2_{L^2(\mathbb{T}^1)} \\ &\leq \tfrac{1}{2}\|(\partial_x + ik)u\|^2_{L^2(\mathbb{T}^1)} + C\|f\|^2_{H^{-1}(\mathbb{T}^1)} + C\|g\|^2_{L^2(\mathbb{T}^1)} + C\langle\tau\rangle\|u\|^2_{L^2(\mathbb{T}^1)}.\end{aligned}$$

Using the estimate for $\|u\|_{L^2(\mathbb{T}^1)}$ and the fact that $k \in [0, 1]$ we obtain (3.4). □

Proof of Proposition 3.2. With the constant C_1 depending on τ (but not on k) the estimate (3.2) follows from the conjugation $u \mapsto v = e^{-ikx}u$ and the unique continuation property for $-\partial_x^2 + W$, $W \in L^p$, $p > 1$. As pointed out in [14], this result is implicit in the paper of Schechter–Simon [21].

To obtain the dependence of constants for large $\langle \tau \rangle$ we first observe that interpolation between the H^{-1} and L^2 estimates in Lemma 3.3 shows that if $(-(\partial_x + ik)^2 - \tau)u = g + f$, then

$$\|u\|_{L^2} + \langle \tau \rangle^{-1/2} \|\partial_x u\|_{L^2} \le C \langle \tau \rangle^{-1/2} \|g\|_{L^2} + C \langle \tau \rangle^{(s-1)/2} \|f\|_{H^{-s}} + C\|u\|_{L^2(\omega)}$$

for $0 \le s \le 1$. As a consequence, if $(-(\partial_x + ik)^2 - \tau)u = g - Wu$, then

$$\|u\|_{L^2} \le C\langle \tau \rangle^{-1/2} \|g\|_{L^2} + C\langle \tau \rangle^{(s-1)/2} \|Wu\|_{H^{-s}} + C\|u\|_{L^2(\omega)}. \tag{3.7}$$

By Sobolev embeddings, for any $s < 1/2$, there exists $C > 0$ such that for any $u \in H^s(\mathbb{T}^1)$,

$$\|u\|_{L^{2/(1-2s)}(\mathbb{T}^1)} \le C\|u\|_{H^s(\mathbb{T}^1)}.$$

By duality, we deduce $L^{2/(1+2s)}(\mathbb{T}^1) \to H^{-s}(\mathbb{T}^1)$. Choosing $s = 1/(2p) < 1/2$, and applying Hölder's inequality we obtain

$$\begin{aligned} \|Wu\|_{H^{-s}} &\le C\|Wu\|_{L^{2/(1+2s)}} \le C\|W\|_{L^p}\|u\|_{L^{2/(1-2s)}} \\ &\le C\|W\|_{L^p}\|u\|_{H^s} \le C'\|W\|_{L^p}\|u\|_{L^2}^{1-s}(\|u\|_{L^2} + \|\partial_x u\|_{L^2})^s \\ &\le C'\|W\|_{L^p}\big(\langle \tau \rangle^{(1+\delta)s^2/(2(1-s))}\|u\|_{L^2} + \langle \tau \rangle^{-(1+\delta)s/2}\|\partial_x u\|_{L^2}\big). \end{aligned}$$

Combining this with (3.7) yields

$$\begin{aligned} \|u\|_{L^2} + \langle \tau \rangle^{-1/2}\|\partial_x u\|_{L^2} &\le C\langle \tau \rangle^{-1/2}\|g\|_{L^2} + C\|u\|_{L^2(\omega)} \\ &\quad + C_2\langle \tau \rangle^{(s-1)/2}\langle \tau \rangle^{(1+\delta)s^2/(2(1-s))}\|u\|_{L^2} + C_3\langle \tau \rangle^{(s-1)/2}\langle \tau \rangle^{-(1+\delta)s/2}\|u\|_{H^1}. \end{aligned}$$

Since $0 < s < 1$, taking $\langle \tau \rangle$ large enough allows us to absorb the last term on the right-hand side in the left-hand side. The same is true for the third term since

$$\frac{(1+\delta)s^2}{2(1-s)} + \frac{s-1}{2} = \frac{-1 + 2s + \delta s^2}{1-s},$$

which is negative for $0 < s < 1/2$ if we choose δ small enough. □

Proof of Proposition 3.1. Let us now show how to pass from the estimate in Proposition 3.2 to an observability result. This was already achieved in [6] in a more general semiclassical setting. For completeness we present a simple version of it here—see [20].

For $\chi \in C_0^\infty(\mathbb{R})$, put $w = \chi(t)e^{itP}u_0$, which solves

$$(i\partial_t + P)w = i\chi'(t)e^{itP}u_0 = v, \qquad P := -(\partial_x + ik)^2 + W(x).$$

Taking Fourier transforms with respect to time, we get

$$(P - \tau)\widehat{w}(\tau) = \widehat{v}(\tau).$$

Using the estimate in Proposition 3.2, we write

$$\|\widehat{w}(\tau)\|_{L^2(\mathbb{T}^1)} \le \frac{C}{1+\sqrt{|\tau|}}\|\widehat{v}(\tau)\|_{L^2(\mathbb{T}^1)} + C\|\widehat{w}(\tau)\|_{L^2(\omega)}.$$

Now, taking the L^2 norm with respect to the τ variable gives

$$\|\widehat{w}(\tau)\|_{L^2(\mathbb{R}_\tau\times\mathbb{T}^1)} \le \frac{C}{1+\sqrt{N}}\|\widehat{v}(\tau)\|_{L^2(\mathbb{R}_\tau\times\mathbb{T}^1)} + C\|\widehat{w}(\tau)\|_{L^2(\mathbb{R}_\tau\times\omega)} + \left(\int_{|\tau\le N}\|\widehat{v}(\tau)\|^2_{L^2(\mathbb{T}^1)}\,d\tau\right)^{1/2}.$$

From this we notice that

$$\|\widehat{w}(\tau)\|_{L^2(\mathbb{R}_\tau\times\mathbb{T}^1)} = \|u_0\|_{L^2(\mathbb{T}^1)}\|\chi\|_{L^2(\mathbb{R})}, \qquad \|\widehat{v}(\tau)\|_{L^2(\mathbb{R}_\tau\times\mathbb{T}^1)} = \|u_0\|_{L^2(\mathbb{T}^1)}\|\chi'\|_{L^2(\mathbb{R})},$$
$$\|\widehat{w}(\tau)\|_{L^2(\mathbb{R}_\tau\times\omega)} = \|\chi(t)e^{itP}u_0\|_{L^2(\mathbb{R}_t\times\mathbb{T}^1)}.$$

Hence we deduce that if

$$\frac{C\|\chi'\|_{L^2}}{\|\chi\|_{L^2}(1+\sqrt{N})} \le \frac{1}{2},$$

then

$$\|u_0\|_{L^2_x} \le C'\|\chi(t)e^{itP}u_0\|_{L^2(\mathbb{R}_t\times\mathbb{T}^1_x)} + C'\left(\int_{|\tau|\le N}\|\widehat{v}(\tau)\|^2_{L^2_x}\,d\tau\right)^{1/2}. \tag{3.8}$$

To understand the last term on the right-hand side we define Sobolev norms associated to P. Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(\mathbb{T}^1)$ consisting of eigenfuctions of P. We then put

$$\|u\|^2_{H^k_P} := \sum_{j=1}^\infty \langle\lambda_n\rangle^{2k}|u_n|^2, \qquad P\varphi_n = \lambda_n\varphi_n, \qquad u_n := \langle u, \varphi_n\rangle.$$

In this notation $w = \chi(t)\sum_n u_n e^{-it\lambda_n}\varphi_n$, and

$$\widehat{v}(\tau) = \sum_n \widehat{\chi}'(\tau-\lambda_n)u_n\varphi_n.$$

Hence

$$\begin{aligned}\int_0^N \|\widehat{v}(\tau)\|^2_{L^2_x}\,d\tau &= \sum_{n=1}^\infty |u_n|^2\int_0^N |(\tau-\lambda_n)\hat{\chi}(\tau-\lambda_n)|^2\,d\tau\\ &= \sum_{n=1}^\infty |u_n|^2\int_0^N \mathcal{O}(\langle\tau-\lambda_n\rangle^{-\infty})\,d\tau\\ &\le C_{N,M}\sum_{n=1}^\infty\langle\lambda_n\rangle^{-M}|u_n|^2 = C_{N,M}\|u\|^2_{H_P^{-M}}\end{aligned}$$

for any M. Taking $M=2$ and combining this with (3.8) we obtain

$$\|u_0\|_{L^2(\mathbb{T}^1)} \le C\|\chi(t)e^{itP}u_0\|_{L^2(\mathbb{R}_t\times\omega)} + C\|u_0\|_{H_P^{-2}}. \tag{3.9}$$

To complete the proof, it remains to eliminate the last term on the right-hand side of (3.9). For this, we apply the now classical uniqueness-compactness argument of Bardos–Lebeau–Rauch [2] (see also [8, §4]) or the direct argument presented in the Appendix. We note that both approaches rely on the unique continuation property of $-(\partial_x+ik)^2+W(x)$, $W \in L^p(\mathbb{T}^1)$, $p > 1$. Notice also that in this argument, to get the independence of the constant from $k \in [0, 1]$, it is enough to use the compactness of $[0, 1]$. □

For later use we also record the following approximation result:

Proposition 3.4. *Assume that the sequence of potentials W_j converges to W in $L^p(\mathbb{T}^1)$, $p \geq 2$. Then there exists $K_0 > 0$ such that for any $k \in \mathbb{R}$ and $u \in L^2(\mathbb{T}^1)$, and any $j \in \mathbb{N}$,*

$$\|u\|^2_{L^2(\mathbb{T}^1)} \leq K_0 \int_0^T \|e^{it((\partial_x+ik)^2-W_j)}v\|^2_{L^2(\omega)}\,dt. \tag{3.10}$$

Proof. The conclusion follows from Proposition 3.1 by a simple perturbation argument. Put $P = -(\partial_x + ik)^2 + W$ and $P_j = -(\partial_x^2 + ik)^2 + W_j$. Then, according to the Duhamel formula, we have

$$e^{-itP}v = e^{-itP_j}v + \frac{1}{i}\int_0^t e^{-i(t-s)P_j}(W - W_j)e^{-isP}v\,ds,$$

and consequently, according to (2.3), we obtain

$$\begin{aligned}
\|e^{-itP}v - e^{-itP_j}v\|_{L^\infty((0,T);L^2(\mathbb{T}^1))} &\leq C\|(W - W_j)e^{-isP}v\|_{L^1((0,T);L^2(\mathbb{T}^1))}\\
&\leq C\sqrt{T}\,\|W - W_j\|_{L^2(\mathbb{T}^1)}\|e^{-isP}v\|_{L^\infty(\mathbb{T}^1_x;L^2(0,T))}\\
&\leq C\sqrt{T}\,\|W - W_j\|_{L^2(\mathbb{T}^1)}\|v\|_{L^2(\mathbb{T}^1)}.
\end{aligned}$$

According to (3.1) we have

$$\begin{aligned}
\|v\|^2_{L^2(\mathbb{T}^1)} &\leq K_0\int_0^T \|e^{-itP}v\|^2_{L^2(\omega)}\,dt\\
&\leq 2K_0\int_0^T \|e^{-itP_j}v\|^2_{L^2(\omega)}\,dt + 2C^2T\|W - W_j\|^2_{L^2(\mathbb{T}^1)}\|v\|^2_{L^2(\mathbb{T}^1)},
\end{aligned}$$

which implies (3.10) if $\|W - W_j\|_{L^2(\mathbb{T}^1)}$ is small enough. □

4. Semiclassical observation estimates in dimension 2

We revisit and refine the arguments of [8]. The key point in our analysis will be the following variant of [8, Proposition 3.1]. The key difference is that now the main constant is determined in terms of the geometry of the problem and the potential V.

Proposition 4.1. *Suppose that $V_j \in C^\infty(\mathbb{T}^2;\mathbb{R})$ converges to V in the $L^2(\mathbb{T}^2)$ topology. Let $\chi \in C_0^\infty(-1,1)$ be equal to 1 near 0, and define*

$$\Pi_{h,\rho,j}u_0 := \chi\left(\frac{h^2(-\Delta+V_j)-1}{\rho}\right)u_0, \qquad \rho>0.$$

Then for any non-empty open subset Ω of $\mathbb{T}^2$ and $T>0$, there exists a constant $K>0$ such that for any j there exist $\rho_j, h_{0,j}>0$ such that for any $0<h<h_{0,j}$ and $u_0\in L^2(\mathbb{T}^2)$,

$$\|\Pi_{h,\rho_j,j}u_0\|_{L^2}^2 \le K\int_0^T \|e^{-it(-\Delta+V_j)}\Pi_{h,\rho_j,j}u_0\|_{L^2(\Omega)}^2\,dt. \tag{4.1}$$

In the proof we argue by contradiction. We first observe that if the estimate (4.1) is true for some $\rho>0$, then it is true for all $0<\rho'<\rho$. As a consequence, if (4.1) were false then for any j, there would exist sequences

$$h_{n,j}\to 0, \qquad \rho_{n,j}\to 0, \qquad u_{0,n,j}=\Pi_{h_{n,j},\rho_{n,j},j}v_{0,n,j}\in L^2,$$
$$i\partial_t u_{n,j}(t,z)=(-\Delta+V_j(z))u_{n,j}(t,z), \qquad u_{n,j}(0,z)=u_{0,n,j}(z),$$

such that

$$1=\|u_{0,n,j}\|_{L^2}^2, \qquad \int_0^T \|u_{n,j}(t,\cdot)\|_{L^2(\Omega)}^2\,dt \le \frac{1}{K}.$$

Each sequence $n\mapsto u_{n,j}$ is bounded in $L^2_{\text{loc}}(\mathbb{R}\times\mathbb{T}^2)$ and consequently, after possibly extracting a subsequence, there exists a semiclassical defect measure μ_j on $\mathbb{R}_t\times T^*\mathbb{T}^2_z$ such that for any function $\varphi\in C^0_0(\mathbb{R}_t)$ and any $a\in C_0^\infty(T^*\mathbb{T}^2_z)$, we have

$$\langle \mu_j, \varphi(t)a(z,\zeta)\rangle = \lim_{n\to\infty}\int_{\mathbb{R}_t\times\mathbb{T}^2}\varphi(t)(a(z,h_{n,j}D_z)u_{n,j})(t,z)\overline{u}_{n,j}(t,z)\,dt\,dz. \tag{4.2}$$

Furthermore, standard arguments[‡] show that:

- We have
$$\mu_j((t_0,t_1)\times T^*\mathbb{T}^2_z)=t_1-t_0. \tag{4.3}$$
- The measure μ_j on $\mathbb{R}_t\times T^*(\mathbb{T}^2)$ is supported in the set
$$\Sigma := \{(t,z,\zeta)\in\mathbb{R}_t\times\mathbb{T}^2_z\times\mathbb{R}^2_\zeta;\ |\zeta|=1\}$$
and is invariant under the action of the geodesic flow:
$$\xi\cdot\nabla_x(\mu_j)=0. \tag{4.4}$$
- The mass of the measure on Ω is bounded:
$$\mu_j((0,T)\times T^*\Omega)\le 1/K. \tag{4.5}$$

‡ See [1] for a review of recent results about measures used for the Schrödinger equation.

We are going to show that a proper choice of the constant K above contradicts (4.3). When no confusion is likely to occur we will drop the index j for conciseness.

We start by decomposing Σ into its rational and irrational parts. For that we identify $\mathbb{T}^2 \simeq [0, A)_x \times [0, B)_y$ where $A, B \in \mathbb{R} \setminus \{0\}$, and define

$$\Sigma_{\mathbb{Q}} := \Sigma \cap \left\{ \left(t, z, \frac{(Ap, Bq)}{\sqrt{A^2p^2 + B^2q^2}} \right);\ p, q \in \mathbb{Z},\ \gcd(p, q) = 1 \right\}.$$

The flow on $\Sigma_{\mathbb{Q}}$ is periodic. Its complement is the set of irrational points,

$$\Sigma_{\mathbb{R}\setminus\mathbb{Q}} := \Sigma \setminus \Sigma_{\mathbb{Q}},$$

and it also invariant under the flow.

4.1. *The irrational directions*

For simplicity we assume here that $A = B = 2\pi$, that is, $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$, as the argument is the same as in the general case.

Let us first define $\mu_{\mathbb{R}\setminus\mathbb{Q}}$ to be the restriction of the measure μ to $\Sigma_{\mathbb{R}\setminus\mathbb{Q}}$. Since μ is invariant, for any open set $\Omega \subset \mathbb{T}^2$ and any $s \in \mathbb{R}$,

$$\mu_{\mathbb{R}\setminus\mathbb{Q}}((t_1, t_2) \times \Omega \times \mathbb{R}^2) = \mu_{\mathbb{R}\setminus\mathbb{Q}}((t_1, t_2) \times \Phi_s(\Omega \times \mathbb{R}^2))$$

where the flow Φ_s is defined by $\Phi_s(z, \zeta) = (z + s\zeta, \zeta)$. As a consequence, for any $S > 0$,

$$\begin{aligned} \mu_{\mathbb{R}\setminus\mathbb{Q}}((t_1, t_2) \times \Omega \times \mathbb{R}^2) &= \frac{1}{S} \int_0^S \mu_{\mathbb{R}\setminus\mathbb{Q}}((t_1, t_2) \times \Phi_s(\Omega \times \mathbb{R}^2))\, ds \\ &= \int \mathbb{1}_{t \in (t_1, t_2)} \times \frac{1}{S} \int_0^S \mathbb{1}_{(z,\zeta) \in \Phi_s(\Omega \times \mathbb{R}^2)}\, ds\, d\mu_{\mathbb{R}\setminus\mathbb{Q}}. \end{aligned}.$$

The equidistribution theorem shows that for any (z, ζ) in the support of $\mu_{\mathbb{R}\setminus\mathbb{Q}}$,

$$\lim_{S \to \infty} \frac{1}{S} \int_0^S \mathbb{1}_{(z,\zeta) \in \Phi_s(\Omega \times \mathbb{R}^2)}\, ds = \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\mathbb{T}^2)}.$$

Hence the dominated convergence theorem and (4.3) show that

$$\mu_{\mathbb{R}\setminus\mathbb{Q}}((t_1, t_2) \times \Omega \times \mathbb{R}^2) = \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\mathbb{T}^2)} \mu_{\mathbb{R}\setminus\mathbb{Q}}((t_1, t_2) \times \mathbb{T}^2 \times \mathbb{R}^2). \tag{4.6}$$

4.2. *Dense rational directions*

We now consider the restriction of the measure μ to the set of rational directions, $\Sigma_{\mathbb{Q}}$. We first consider the case of p/q for which $p^2 + q^2$ is large (we again assume that $A = B = 1$ as the general argument is the same). In some sense that corresponds to being close to the irrational case.

Lemma 4.2. *For any open set* Ω*, there exist* $N \in \mathbb{N}$ *and* $\delta > 0$ *such that for any* $(p,q) \in \mathbb{Z}^2$ *with* $\gcd(p,q) = 1$ *and* $\sqrt{p^2+q^2} \geq N$*,*

$$\liminf_{S\to\infty} \frac{1}{S}\int_0^S \mathbb{1}_{(z,\zeta)\in\Phi_s(\Omega\times\mathbb{R}^2)}\, ds \geq \delta, \qquad \zeta = \frac{(p,q)}{\sqrt{p^2+q^2}}.$$

Proof. For any $z_0 = (x_0, y_0) \in \Omega$ choose $N > 4\pi/\epsilon$ where $B(z_0, 2\epsilon) \subset \Omega$. Assume that $p \geq N/2 > 2\pi/\epsilon$ and that $p \geq q$ (the case of $q \leq p$ is similar). Put

$$s_k := \frac{\sqrt{p^2+q^2}}{p}(2k\pi - x_0), \qquad k = 0, \ldots, p-1.$$

Since p and q are coprime, q is a generator of the group $\mathbb{Z}/p\mathbb{Z}$. Consequently, the points

$$Y_k = \frac{s_k}{\sqrt{p^2+q^2}}q - y_0 \in \mathbb{T}^1$$

are at distance exactly $2\pi/p$ from each other. (Here and below, addition on $\mathbb{T}^1$ is meant mod $2\pi\mathbb{Z}$.) We conclude that for any $z \in \mathbb{T}^1$ there exists

$$J_z \subset \{0, \ldots, p-1\}, \qquad |J_z| = [\epsilon p/\pi], \qquad \text{such that} \quad |y + Y_k - y_0| \leq \epsilon \text{ for } k \in J_z.$$

Since the flow is given by

$$\Phi_{-s}\left((x,y), \frac{(p,q)}{\sqrt{p^2+q^2}}\right) = \left((x,y) - \frac{s}{\sqrt{p^2+q^2}}(p,q), \frac{(p,q)}{\sqrt{p^2+q^2}}\right),$$

for any $k \in J$, we have $\Phi_{-s_k}\big(z, (p,q)/\sqrt{p^2+q^2}\big) \in B(z_0, \epsilon) \times \mathbb{R}^2$. Since $2\pi/p < \epsilon$, we also find that for $|s - s_k| < \epsilon$,

$$\Phi_{-s}\left(z, \frac{(p,q)}{\sqrt{p^2+q^2}}\right) \in B(z_0, 2\epsilon) \times \mathbb{R}^2 \subset \Omega \times \mathbb{R}^2.$$

Hence, using the assumption that $q \leq p$,

$$\int_0^{2\pi\sqrt{p^2+q^2}} \mathbb{1}_{\Phi_{-s}(z,\zeta)\in\Omega\times\mathbb{R}^2}\, ds \geq [\epsilon p/\pi]\epsilon > 2\pi\sqrt{p^2+q^2}\,\delta, \qquad \zeta = (p,q)/\sqrt{p^2+q^2},$$

for some $\delta > 0$. Since the evolution of (z, ζ) is periodic with period $2\pi\sqrt{p^2+q^2}$, the lemma follows. □

Let us now fix N as in Lemma 4.2 and let $\mu_{\mathbb{Q},N}$ be the restriction of $\mu_{\mathbb{Q}}$ to rational directions satisfying $\sqrt{p^2+q^2} \geq N$. As in the study of the irrational directions, Lemma 4.2 and Fatou's Lemma imply

$$\mu_{\mathbb{Q},N}((t_1,t_2) \times \Omega \times \mathbb{R}^2) \geq \delta\mu_{\mathbb{Q},N}((t_1,t_2) \times \mathbb{T}^2 \times \mathbb{R}^2). \tag{4.7}$$

4.3. Isolated rational directions

This section is closest to the arguments of [8, §3]. We allow here existence of points in $\Sigma_{\mathbb{Q}}$ whose evolution misses Ω altogether. The contradiction is derived from that assumption. It is now important to keep A and B arbitrary, $\mathbb{T}^2 = \mathbb{R}^2/A\mathbb{Z} \times B\mathbb{Z}$. The constraints on the constant K will not be only geometric as in §§4.1, 4.2, but will also involve the limit potential V. Hence we return to the notation of (4.2) and keep the index j.

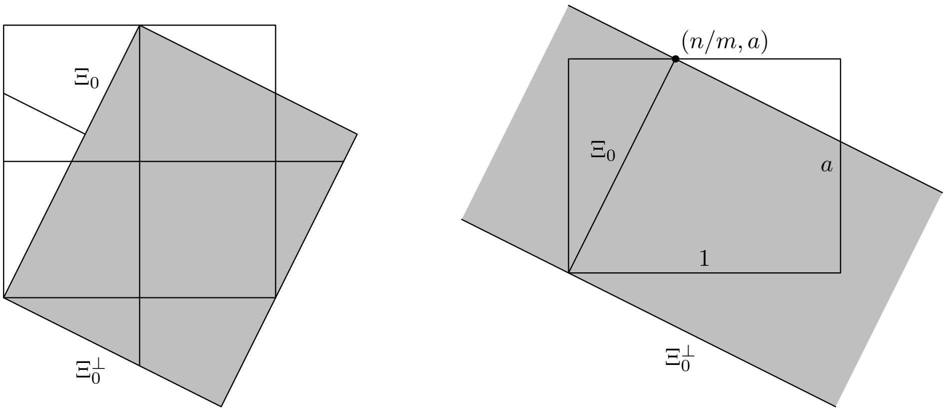

Fig. 2. Left: a rectangle, R, covering a rational torus $\mathbb{T}^2$. In that case we obtain a periodic solution on R. Right: the irrational case; the strip with sides $m\Xi_0 \times \mathbb{R}\Xi_0^\perp$, $\Xi_0 = (n/m, a)$ (not normalized to have norm one) also covers the torus $[0, 1] \times [0, a]$. Periodic functions are pulled back to functions satisfying (4.10). This figure is borrowed from [8].

We consider the restriction of the measure μ to any of the finitely many isolated rational directions:

$$\Xi_0 = \frac{(Ap, Bq)}{\sqrt{A^2p^2 + B^2q^2}}, \qquad \sqrt{p^2 + q^2} \le N. \tag{4.8}$$

We first recall the following simple result [8, Lemma 2.7] (see Fig. 2 for an illustration).

Lemma 4.3. *Suppose that Ξ_0 is given by (4.8) and*

$$F : (x, y) \mapsto z = F(x, y) = x\Xi_0^\perp + y\Xi_0, \qquad \Xi_0^\perp = \frac{1}{\sqrt{A^2p^2 + B^2q^2}}(Bq, -Ap). \tag{4.9}$$

If $u = u(z)$ is periodic with respect to $A\mathbb{Z} \times B\mathbb{Z}$ then

$$F^*u(x + ka, y + \ell b) = F^*u(x, y - k\gamma), \qquad k, \ell \in \mathbb{Z},\ (x, y) \in \mathbb{R}^2, \tag{4.10}$$

where, for any fixed $p, q \in \mathbb{Z}$,

$$a = \frac{-(q^2 + p^2)AB}{\sqrt{A^2p^2 + B^2q^2}}, \qquad b = \sqrt{A^2p^2 + B^2q^2}, \qquad \gamma = -\frac{pq(B^2 - A^2)}{\sqrt{A^2p^2 + B^2q^2}}.$$

When $B/A = r/s \in \mathbb{Q}$ then

$$F^* u(x + k\widetilde{a}, y + \ell b) = F^* u(x, y), \qquad k, \ell \in \mathbb{Z},\ (x, y) \in \mathbb{R}^2,$$

for $\widetilde{a} = (p^2 s^2 + q^2 r^2)a$.

Indeed, with

$$\mathcal{A} = \frac{1}{\sqrt{A^2 p^2 + B^2 q^2}} \begin{pmatrix} qB & pA \\ -pA & qB \end{pmatrix},$$

we have

$$\mathcal{A} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} -pA \\ qB \end{pmatrix}, \qquad \mathcal{A} \begin{pmatrix} a \\ \gamma \end{pmatrix} = \begin{pmatrix} -qA \\ pB \end{pmatrix},$$

which implies

$$u\left(\mathcal{A}\begin{pmatrix} x + ka \\ y + k\gamma + lb \end{pmatrix}\right) = u\left(\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

We now identify $u_{n,j}$ with $F^* u_{n,j}$, and consider the Schrödinger equation on the strip $R = \mathbb{R}_x \times [0, b]_y$ (or the rectangle $R = [0, a]_x \times [0, b]_y$ in the case when $A/B \in \mathbb{Q}$). In this coordinate system, $\Xi_0 = (0, 1)$.

Choosing a function $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 near $(0, 0)$ we define, for $\epsilon > 0$,

$$\chi_\epsilon := \chi\big(((\eta, \zeta) - (0, 1))/\epsilon\big), \qquad \eta, \zeta \in \mathbb{R},$$

and

$$u_{n,j,\epsilon}(x, y) = \chi_\epsilon(h_{n,j} D_x) u_{n,j}.$$

We denote by $\mu_{j,\epsilon}$ the semiclassical measure of the sequence $(u_{n,j,\epsilon})_{n\in\mathbb{N}}$ (j, ϵ are parameters). Since $\mu_{j,\epsilon} = (\chi_\epsilon(\zeta))^2 \mu_j$ (where we skipped the pull-back by F), we have

$$\lim_{\epsilon\to 0+} \mu_{j,\epsilon} = \mu_j|_{\{(t,z,\zeta);\, \zeta=(0,1)\}}. \tag{4.11}$$

We now recall the following normal-form result given in [8, Proposition 2.3 and Corollary 2.4]:

Proposition 4.4. *Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is given by (4.9) and that $V \in C^\infty(\mathbb{R}^2)$ is periodic with respect to $A\mathbb{Z} \times B\mathbb{Z}$. Let a, b and γ be as in (4.10). Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 0 in a neighbourhood of $\eta = 0$. Suppose that $V_j(x, y) \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^1)$. Then there exist operators*

$$Q_j(x, y, hD_y) \in C^\infty(\mathbb{R}) \otimes \Psi^0(\mathbb{R}), \qquad R_j(x, y, hD_x, hD_y) \in \Psi^0(\mathbb{R}^2),$$

(where Ψ^0 denotes the space of semiclassical pseudodifferential operators of order 0) such that $(F^{-1})^* Q F^*$ and $(F^{-1})^* R F^*$ preserve $A\mathbb{Z} \times B\mathbb{Z}$ periodicity, and

$$\begin{aligned}(I + hQ_j)(D_y^2 + F^* V_j(x, y))\chi(hD_x, hD_y)& \\ = (D_y^2 + W_j(x))(I + hQ_j)\chi(hD_x, hD_y) + hR_j&,\end{aligned} \tag{4.12}$$

where $W_j(x) = (1/b)\int_0^b F^* V_j(x, y)\, dy$ satisfies $W_j(x + a) = W_j(x)$.

Moreover, there exist operators $P_j = P_j(x, y, hD_x, hD_y) \in \Psi^0(\mathbb{R}^2)$ *such that (with properties as above)*

$$(I + hQ_j)(D_x^2 + D_y^2 + F^* V_j(x, y))\chi(hD_x, hD_y) \\ = \big((D_x^2 + D_y^2 + W_j(x))(I + hQ_j) + P_j\big)\chi(hD_x, hD_y) + hR_j, \tag{4.13}$$

$$P_j(x, y, x, \eta) = \frac{2}{i}\xi\partial_x q_j(x, y, \eta)\widetilde{\chi}_\epsilon(\xi, \eta), \qquad q_j = \sigma(Q_j), \tag{4.14}$$

where $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ *is equal to* 1 *on the support of* χ.

Using Proposition 4.4 we define

$$v_{n,j,\epsilon} = (1 + hQ_j)u_{n,j,\epsilon}, \qquad h = h_{n,j}.$$

Since the operator Q_j is bounded on L^2, the semiclassical defect measures associated to $v_{n,j,\epsilon}$ and $u_{n,j,\epsilon}$ are equal. We now consider the time dependent Schrödinger equation satisfied by $v_{n,j,\epsilon}$. With

$$\begin{aligned} Q_{n,j} &:= Q_j(x, y, h_{n,j}D_y), \qquad R_{n,j} := R(x, y, h_{n,j}D_x, h_{n,j}D_y), \\ P_{n,j} &:= P_j(x, y, h_{n,j}D_x, h_{n,j}D_y), \end{aligned} \tag{4.15}$$

given in Proposition 4.4 and $\chi_{n,j,\epsilon} := \chi(h_{n,j}D_z)$, we have

$$\begin{aligned} &(i\partial_t + \Delta - W_j(x))v_{n,j} \\ &= (I + h_{n,j}Q_{n,j})(i\partial_t + \Delta - V_j(x, y))\chi_{n,j,\epsilon}u_{n,j} - P_{n,j}\chi_{n,j,\epsilon}u_{n,j} - h_{n,j}R_{n,j,\epsilon}u_{n,j} \\ &= -P_{n,j}\chi_{n,j,\epsilon}u_{n,j} + [V, \chi_{n,j,\epsilon}]u_{n,j} + o_{L^2}(1) = -P_{n,j}\chi_{n,j,\epsilon}u_{n,j} + o_{L^2_{x,y}}(1). \end{aligned} \tag{4.16}$$

We also recall that according to (4.14), on the support of $\mu_{j,\epsilon}$, the symbol of the operator W is smaller than $C\epsilon$. This implies that

$$(i\partial_t + \Delta - W_j(x))v_{n,j,\epsilon} = f_{n,j,\epsilon} \tag{4.17}$$

with

$$\limsup_{n\to\infty} \|f_{n,j,\epsilon}\|^2_{L^2((0,T)\times\mathbb{T}^2)} = \langle \mu_{j,\epsilon}, |P_{n,j}|^2\rangle \leq C_j\epsilon^2. \tag{4.18}$$

The simple observation that

$$e^{it(\partial_y^2 + \partial_x^2 - W_j(x))} = e^{it\partial_y^2}e^{it(\partial_x^2 - W_j(x))}$$

shows that we can write

$$v_{n,j,\epsilon}(t, x, y) = \sum_{k\in\mathbb{Z}} e^{-i(tk^2+ky)}v_{n,j,\epsilon,k}(t, x), \qquad f_{n,j,\epsilon}(t, x, y) = \sum_{k\in\mathbb{Z}} e^{-iky}f_{n,j,\epsilon,k}(t, x),$$

where

$$(i\partial_t + \partial_x^2 - W_j(x))v_{n,j,\epsilon,k} = f_{n,j,\epsilon,k}$$

and the coefficients satisfy the Floquet condition (see [8, proof of Proposition 2.2])

$$\begin{aligned} v_{n,j,\epsilon,k}(t, x + a) &= e^{2\pi i\gamma k/b}v_{n,j,\epsilon,k}(t, x) = e^{2\pi i\gamma_k}v_{n,j,\epsilon,k}(t, x), \\ f_{n,j,\epsilon,k}(t, x + a) &= e^{2\pi i\gamma_k}f_{n,j,\epsilon,k}(t, x), \qquad \gamma_k := \gamma k/b = [\gamma k/b] \in [0, 1). \end{aligned}$$

Since $W_j(x+a) = W_j(x)$ and

$$\|W - W_j\|^2_{L^2([0,a]_x)} = \int_0^a \left(\frac{1}{b}\int_0^b \int (F^*V(x,y) - F^*V_j(x,y))\,dy\right)^2 dx$$
$$\leq \|F^*(V - V_j)\|^2_{L^2([0,a]_x\times[0,b]_y)}$$
$$\leq C_{\Xi_0}\|V - V_j\|_{L^2(\mathbb{T}^2)} \to 0, \qquad j \to \infty,$$

we can apply Proposition 3.4 to $u_{n,j,\epsilon,k}(t,x) = e^{-2i\pi\gamma kx/(ab)}v_{n,j,\epsilon,k}(t,x)$, which is periodic on the torus $\mathbb{R}/a\mathbb{Z}$. For that we fix a domain $\omega \subset [0,a]_x$ such that for any $x \in \overline{\omega}$, the line $\{x\}\times[0,b]_y$ encounters Ω. The estimate (3.10) gives the following non-geometric estimate (it is here that the dependence on the potential enters):

$$\|v_{n,j,\epsilon,k}\|^2_{L^\infty((0,T);L^2([0,a]_x))} \leq 2\|v_{n,j,\epsilon,k}|_{t=0}\|_{L^2([0,a]_x)} + 2\|f_{n,j,\epsilon,k}\|^2_{L^1((0,T);L^2([0,a]_x))}$$
$$\leq K_0\int_0^T \|e^{it(\partial_x^2 - W_j(x))}v_{n,j,\epsilon,k}|_{t=0}\|^2_{L^2(\omega)}\,dt + C\|f_{n,j,\epsilon,k}\|^2_{L^2((0,T)\times[0,a]_x)}$$
$$\leq K_0\int_0^T \|v_{n,j,\epsilon,k}\|^2_{L^2(\omega)}\,dt + C\|f_{n,j,\epsilon,k}\|^2_{L^2((0,T)\times[0,a]_x)}.$$

Summing over $k \in \mathbb{Z}$ gives

$$\|v_{n,j,\epsilon}\|^2_{L^\infty((0,T);L^2([0,a]_x\times[0,b]_y)}$$
$$\leq K_0\int_0^T \|v_{n,j,\epsilon}|_{t=0}\|^2_{L^2(\omega)}\,dt + C\|f_{n,j,\epsilon}\|^2_{L^2((0,T)\times[0,a]_x)}.$$

Taking first the limit as $n \to \infty$, we obtain, according to (4.18),

$$\mu_{j,\epsilon}\big((0,T)\times([0,a]\times[0,b]_y)\times\mathbb{R}^2\big) \leq K_0\mu_{j,\epsilon}\big((0,T)\times\omega\times[0,b]_y\times\mathbb{R}^2\big) + C_j\epsilon.$$

Then taking the limit as $\epsilon \to 0$, we conclude that, according to (4.11),

$$\mu_j\big((0,T)\times([0,a]_x\times[0,b]_y)\times\{(0,1)\}\big) \leq K_0\mu_j\big((0,T)\times\omega\times[0,b]_y\times\{(0,1)\}\big). \tag{4.19}$$

Since any vertical line over $\overline{\omega}$ encounters the open set Ω, we have

$$\min_{x\in\overline{\omega}}\int_{\Omega\cap(\{x\}\times[0,b]_y)} dy > \delta_0 > 0.$$

This and the invariance of the measure under the flow (which is now just the translation in the y direction) imply that

$$\mu_j\big((0,T)\times\omega\times[0,b]_y\times\{(0,1)\}\big) \leq \delta_0\mu_j\big((0,T)\times\Omega\times\{(0,1)\}\big).$$

Combining this with (4.19) we find that there exists a constant $K_{(0,1)}$, independent of j, such that

$$\mu_j\big((0,T)\times([0,a]_x\times[0,b]_y)\times\{(0,1)\}\big) \leq K_{(0,1)}\mu_j\big((0,T)\times\Omega\times\{(0,1)\}\big).$$

Returning to an arbitrary rational direction

$$\zeta_{p,q} = \frac{(p,q)}{\sqrt{A^2p^2 + B^2q^2}}, \qquad \sqrt{p^2+q^2} \le N,$$

we conclude that there exists a constant $K_{p,q}$ such that

$$\mu_j((0,T) \times \mathbb{T}^2 \times \zeta_{p,q}) \le K_{p,q}\mu_j((0,T) \times \Omega \times \zeta_{p,q}). \tag{4.20}$$

4.4. Conclusion of the proof of Proposition 4.1

If the constant K in the statement of the proposition is chosen so that, with δ in (4.7),

$$\frac{K}{T} > \max\left(\frac{\mathrm{vol}(\mathbb{T}^2)}{\mathrm{vol}(\Omega)}, \frac{1}{\delta}, \max_{\sqrt{p^2+q^2} \le N} K_{p,q}\right),$$

then, according to (4.6) and (4.7), we must have

$$\mu((0,T) \times \mathbb{T}^2 \times \mathbb{R}^2) < T,$$

which contradicts (4.3) and completes the proof of Proposition 4.1.

5. From smooth to rough potentials

Proposition 4.1 was proved under the assumptions that $V_j \in C^\infty(\mathbb{T}^2)$ converge to $V \in L^2(\mathbb{T}^2)$. To pass to L^2 potentials we will now use the results of §2.2.

5.1. Classical observation estimate for smooth potentials

The first proposition is the analogue of [8, Proposition 4.1] but with constants described by Proposition 4.1.

Proposition 5.1. *Suppose that $V_j \in C^\infty(\mathbb{T}^2; \mathbb{R})$ converge to V in the $L^2(\mathbb{T}^2)$ topology. Then for any non-empty open subset Ω of $\mathbb{T}^2$ and $T > 0$, there exists $C > 0$ such that for any $j \in \mathbb{N}$ there exists C_j such that for any $u_0 \in L^2(\mathbb{T}^2)$, we have*

$$\|u_0\|_{L^2(\mathbb{T}^2)} \le C\|e^{it(\Delta - V_j)}u_0\|_{L^2((0,T)\times\Omega)} + C_j\|u_0\|_{H^{-2}(\mathbb{T}^2)}. \tag{5.1}$$

Proof. To obtain the estimate (5.1) from Proposition 4.1, we apply pseudodifferential calculus in the time variable. This was already performed in [8], but since we need a precise dependence on the constants we recall the argument. Consider a j-dependent partition of unity

$$1 = \varphi_{0,j}(r)^2 + \sum_{k=1}^{\infty} \varphi_{k,j}(r)^2, \qquad \varphi_{k,j}(r) := \varphi(R_j^{-k}|r|), \qquad R > 1,$$
$$\varphi \in C_0^\infty((R_j^{-1}, R_j); [0,1]), \qquad (R_j^{-1}, R_j) \subset \{r;\ \chi(r/\rho_j) \ge 1/2\},$$

where χ and ρ_j come from Proposition 4.1. Then, we decompose u_0 dyadically:

$$\|u_0\|_{L^2}^2 = \sum_{k=0}^{\infty} \|\varphi_{k,j}(P_{V_j})u_0\|_{L^2}^2, \qquad P_{V_j} := -\Delta + V_j.$$

Let $\psi \in C_0^\infty((0,T);[0,1])$ satisfy $\psi(t) > 1/2$ for $T/3 < t < 2T/3$. We first observe (using the time translation invariance of the Schrödinger equation) that in Proposition 4.1 we have actually proved that

$$\|\Pi_{h,\rho_j,j}u_0\|_{L^2}^2 \le K \int_{\mathbb{R}} \psi(t)^2 \|e^{-it(-\Delta+V_j)}\Pi_{h,\rho_j,j}u_0\|_{L^2(\Omega)}^2\, dt, \quad 0 < h < h_0, \tag{5.2}$$

which is the version we will use.

Taking K_j large enough so that $R^{-K_j} \le h_{0,j}$, where $h_{0,j}$ is as in Proposition 4.1, we apply (5.2) to the dyadic pieces:

$$\begin{aligned}
\|u_0\|_{L^2}^2 &= \sum_{k\in\mathbb{Z}} \|\varphi_{k,j}(P_{V_j})u_0\|_{L^2}^2 \\
&\le \sum_{k=0}^{K_j} \|\varphi_{k,j}(P_{V_j})u_0\|_{L^2}^2 + C \sum_{k=K_j+1}^{\infty} \int_0^T \psi(t)^2 \|\varphi_{k,j}(P_{V_j})\, e^{-itP_{V_j}}u_0\|_{L^2(\Omega)}^2\, dt \\
&= \sum_{k=0}^{K_j} \|\varphi_{k,j}(P_{V_j})u_0\|_{L^2}^2 + C \sum_{k=K_j+1}^{\infty} \int_{\mathbb{R}} \|\psi(t)\varphi_{k,j}(P_{V_j})\, e^{-itP_{V_j}}u_0\|_{L^2(\Omega)}^2\, dt.
\end{aligned}$$

Using the equation we can replace $\varphi(P_{V_j})$ by $\varphi(D_t)$, which means that we do not change the domain of z integration. We need to consider the commutator of $\psi \in C_0^\infty((0,T))$ and $\varphi_{k,j}(D_t) = \varphi(R^{-j}D_t)$. If $\widetilde{\psi} \in C_0^\infty((0,T))$ is equal to 1 on supp ψ then the semiclassical pseudodifferential calculus with $h = R_j^{-k}$ (see for instance [23, Chapter 4]) gives

$$\psi(t)\varphi_{k,j}(D_t) = \psi(t)\varphi_{k,j}(D_t)\widetilde{\psi}(t) + E_j(t,D_t), \quad \partial^\alpha E_j = \mathcal{O}(\langle t\rangle^{-N}\langle\tau\rangle^{-N}R_j^{-Nk}), \tag{5.3}$$

for all N and uniformly in k.

The errors obtained from E_k can be absorbed into the $\|u_0\|_{H^{-2}(\mathbb{T}^2)}$ term on the right-hand side (with a constant depending on j). Hence we obtain

$$\begin{aligned}
\|u_0\|_{L^2}^2 &\le C_j\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + C\sum_{k=0}^{\infty}\int_0^T \|\psi(t)\varphi_{k,j}(D_t)\, e^{-itP_{V_j}}u_0\|_{L^2(\Omega)}^2\, dt \\
&\le \widetilde{C}_j\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + K\sum_{k=0}^{\infty}\langle \varphi_{k,j}(D_t)^2\widetilde{\psi}(t)\, e^{-itP_{V_j}}u_0, \widetilde{\psi}(t)\, e^{-itP_{V_j}}u_0\rangle_{L^2(\mathbb{R}_t\times\Omega)} \\
&= \widetilde{C}_j\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + K\int_{\mathbb{R}} \|\widetilde{\psi}(t)\, e^{-itP_V}u_0\|_{L^2(\Omega)}^2\, dt \\
&\le \widetilde{C}_j\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + K\int_0^T \|e^{-itP_V}u_0\|_{L^2(\Omega)}^2\, dt,
\end{aligned}$$

where the last inequality is the statement of the proposition. □

5.2. *Proof of Theorem 2*

We can now deduce Theorem 2 from Proposition 5.1. For that we consider a sequence V_j of smooth potentials converging to V in $L^2(\mathbb{T}^2)$ (to construct such a sequence, consider the Littlewood–Paley cut-off $V_j = \chi(2^{-2j}\Delta)V$ with $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 near 0). We now have, according to Proposition 5.1,

$$\|u_0\|_{L^2(\mathbb{T}^2)} \le C\|e^{it(\Delta - V_j)}u_0\|_{L^2((0,T)\times\Omega)} + D_j\|u_0\|_{H^{-2}(\mathbb{T}^2)}.$$

On the other hand, according to (2.21), we have

$$\|e^{it(\Delta - V_j)}u_0\|_{L^2((0,T)\times\Omega)} \le \|e^{it(\Delta - V)}u_0\|_{L^2((0,T)\times\Omega)} + C\|V - V_j\|_{L^2(\mathbb{T}^2)}\|u_0\|_{L^2(\mathbb{T}^2_x)},$$

hence, we deduce

$$\|u_0\|_{L^2(\mathbb{T}^2)} \le C\|e^{it(\Delta - V)}u_0\|_{L^2((0,T)\times\Omega)} + C\|V - V_j\|_{L^2(\mathbb{T}^2)}\|u_0\|_{L^2(\mathbb{T}^2_x)} + D_j\|u_0\|_{H^{-2}(\mathbb{T}^2)},$$

and consequently, taking j large enough so that $C\|V - V_j\|_{L^2(\mathbb{T}^2)} \le 1/2$, we conclude that

$$\|u_0\|_{L^2(\mathbb{T}^2)} \le 2C\|e^{it(\Delta - V)}u_0\|_{L^2((0,T)\times\Omega)} + 2D_j\|u_0\|_{H^{-2}(\mathbb{T}^2)}.$$

It remains to eliminate the last term on the right-hand side. For this we use again the classical uniqueness-compactness argument of Bardos–Lebeau–Rauch [2] (see also [8, §4]) or the direct argument presented in the Appendix. The needed unique continuation result for L^2 potentials in $\mathbb{R}^2$ follows, as it did in §2.1, from the results of [21].

Appendix A. A quantitative version of the uniqueness-compactness argument

We present an abstract result which eliminates the low-frequency contributions in observability estimates.

Let P be an unbounded self-adjoint operator on a Hilbert space $\mathcal{H}$. We assume that the spectrum of P is discrete:

$$P\varphi_n = \lambda_n\varphi_n, \quad \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_n \ge n^\delta/C_0, \quad \delta > 0,$$

where $\{\varphi\}_{n=1}^\infty$ form an orthonormal basis of $\mathcal{H}$.

We define P-based Sobolev spaces using the norms

$$\|\varphi\|^2_{\mathcal{H}^s_P} := \sum_{n=1}^\infty \langle\lambda_n\rangle^{2s}|\langle\varphi, \varphi_n\rangle|^2. \tag{A.1}$$

The Schrödinger group for P is formed by the following unitary operators on $\mathcal{H}$:

$$U(t)\varphi = \exp(-itP)\varphi = \sum_{n=1}^\infty \langle\varphi, \varphi_n\rangle e^{-it\lambda_n}\varphi_n.$$

We have the following general result:

Theorem 4. *Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded operator with the property that for any $\lambda \in \mathbb{R}$ there exists a constant $C(\lambda)$ such that for $\varphi \in \mathcal{H}_P^2$,*

$$\|\varphi\|_{\mathcal{H}} \leq C(\lambda)(\|(P-\lambda)\varphi\|_{\mathcal{H}} + \|A\varphi\|_{\mathcal{H}}). \tag{A.2}$$

Suppose also that for some $\epsilon > 0$, $T > 0$, C_1 and C_2,

$$\|\varphi\|_{\mathcal{H}}^2 \leq C_1 \int_0^t \|AU(s)\varphi\|_{\mathcal{H}}^2 \, ds + C_2 \|\varphi\|_{\mathcal{H}_P^{-\epsilon}}^2, \qquad T/4 \leq t \leq T. \tag{A.3}$$

Then there exists an explicitly computable constant K such that

$$\|\varphi\|_{\mathcal{H}}^2 \leq K \int_0^T \|AU(t)\varphi\|_{\mathcal{H}}^2 \, dt. \tag{A.4}$$

Remarks. 1. We do not compute the constant explicitly but the construction in the proof certainly allows that.

2. In the applications in this paper,

$$P = -\Delta + V, \quad \mathcal{H} = L^2(\mathbb{T}^2), \quad A = \mathbb{1}_\Omega, \quad \Omega \subset \mathbb{T}^2 \text{ open},$$

or

$$P = -(\partial_x + ik)^2 + W, \quad \mathcal{H} = L^2(\mathbb{T}^1), \quad A = \mathbb{1}_\omega, \quad \omega \subset \mathbb{T}^1 \text{ open}.$$

Proof. We start by observing that (A.3) and the definition (A.1) imply that for $N > (2C_2)^{1/\epsilon}$,

$$\begin{gathered} \|(I-\Pi)\varphi\|^2 \leq 2C_1 \int_0^t \|AU(s)(I-\Pi)\varphi\|^2 \, ds, \qquad T/4 \leq t \leq T, \\ \Pi\varphi := \sum_{\lambda_n \leq N} \langle \varphi, \varphi_n \rangle \varphi_n. \end{gathered} \tag{A.5}$$

For reasons which will be explained below we will use this inequality for $t = T/4$ and apply it with φ replaced by $U(T/2)\varphi$:

$$\|(I-\Pi)\varphi\|^2 \leq 2C_1 \int_{T/2}^{3T/4} \|AU(t)(I-\Pi)\varphi\|^2 \, dt. \tag{A.6}$$

We will show that the same estimate is true for $\Pi\varphi$. For that let $\mu_1 < \cdots < \mu_{r_1}$ be the enumeration of $\{\lambda_n\}_{n=1}^{K_1}$ and define

$$\psi_r := \sum_{\lambda_n = \mu_r} \langle \varphi, \varphi_n \rangle \varphi_n,$$

so that

$$U(t)\Pi\varphi = \sum_{n \leq K_1} e^{-i\lambda_n t} \langle \varphi, \varphi_n \rangle \varphi_n = \sum_{r=1}^{r_1} e^{i\mu_r t} \psi_r.$$

Since $(P-\mu_r)\psi_r = 0$, we can apply (A.2) to obtain

$$\|\psi_r\| \leq K_2 \|A\psi_r\|, \qquad K_2 = \max_{n\leq K_1} C(\lambda_n). \tag{A.7}$$

The functions $t \mapsto e^{i\mu_r t}$, $r = 1, \ldots, r_1$, are linearly independent and there exists a constant $K_3 = K_3(\mu_1, \ldots, \mu_{r_1}, T)$ such that for any $f_1, \ldots, f_{r_1} \in \mathcal{H}$,

$$\int_{T/2}^{3T/4} \Big\| \sum_{r=1}^{r_1} e^{i\mu_r t} f_r \Big\|^2 dt \geq K_3 \sum_{r=1}^{r_1} \|f_r\|^2, \tag{A.8}$$

as both sides provide equivalent norms on $\times_{r=1}^{r_1}\mathcal{H}$.

Applying (A.8) with $f_r = A\psi_r$ and (A.7) gives

$$\begin{aligned}\|AU(t)\Pi\varphi\|^2_{L^2((T/2,3T/4);\mathcal{H})} &= \int_{T/2}^{3T/4} \Big\| \sum_{r=1}^{r_1} A\psi_r e^{i\mu_r t} \Big\|^2 dt \geq K_2 \sum_{r=1}^{r_1} \|A\psi_r\|^2 \\ &\geq K_2 K_3 \sum_{r=1}^{r_1} \|\psi_r\|^2 = K_2 K_3 \|\Pi\varphi\|. \end{aligned} \tag{A.9}$$

The combination of (A.6) and (A.9) does not yet provide the estimate (A.4). However, if

$$\Pi_M \varphi := \sum_{\lambda_n \leq M} \langle \varphi, \varphi_n\rangle \varphi_n,$$

then for M sufficiently large we have

$$\begin{aligned}&\|AU(t)(I-\Pi_M+\Pi)\varphi\|^2_{L^2((0,T);\mathcal{H})} \\ &\qquad \geq K_2^2 K_3^2 \|\Pi\varphi\|^2 + (1/4C_1^2)\|(I-\Pi_M)\varphi\|^2 - K_4 M^{-1}\|\varphi\|^2, \end{aligned} \tag{A.10}$$

where K_4 will be defined below. In fact, if we choose $\eta \in C_0^\infty((0,T))$ equal to 1 on $[T/2, 3T/4]$, then the left-hand side in (A.10) is estimated from below by

$$\begin{aligned}\int \|AU(t)(I-\Pi_M+\Pi)\varphi\|^2 \eta(t)\, dt &= \int \|AU(t)(I-\Pi_M)\varphi\|^2 \eta(t)\, dt \\ &+ \int \|AU(t)\Pi\varphi\|^2 \eta(t)\, dt - 2\operatorname{Re} \int \langle AU(t)(I-\Pi_M)\varphi, AU(t)\Pi\varphi\rangle \eta(t)\, dt. \end{aligned}$$

We can apply (A.5) and (A.9) to estimate the first two terms from below. Since

$$\begin{aligned}&2\operatorname{Re}\int \langle AU(t)(I-\Pi_M)\varphi, AU(t)\Pi\varphi\rangle \eta(t)\, dt \\ &\qquad = 2\operatorname{Re} \sum_{\lambda_n<N}\sum_{\lambda_m>M} \langle \varphi, \varphi_n\rangle\langle \varphi_m, \varphi\rangle \langle A\varphi_n, A\varphi_m\rangle \int e^{i(\lambda_n-\lambda_m)t}\eta(t)\, dt \\ &\qquad \leq C_P \|A\|^2 \sum_{\lambda_n<N}\sum_{\lambda_m>M} |\lambda_n - \lambda_m|^{-P} \|\varphi\|^2 \leq K_4 M^{-1}\|\varphi\|^2 \end{aligned}$$

if we choose P sufficiently large, we obtain (A.10).

We now have to deal with the remaining eigenfunctions corresponding to $N \le \lambda_n < M$. Let $\mu_{r_1+1} < \cdots < \mu_{r_2}$ be the enumeration of these eigenvalues. Put

$$\tau = \frac{T}{10r_2}. \tag{A.11}$$

The Vandermonde matrix $(e^{i\mu_r p\tau})_{1\le r\le r_2, 1\le p\le r_2}$ is non-singular and hence we can find scalars σ_p with $\max|\sigma_p| = 1$ satisfying

$$\sum_{p=1}^{r_2} \sigma_p e^{i\mu_r p\tau} = 0 \quad \text{for } r \le r_1, \qquad \Big|\sum_{p=1}^{r_2} \sigma_p e^{i\mu_r p\tau}\Big| \ge K_5 \quad \text{for } r_1 < r \le r_2, \tag{A.12}$$

with a constant $K_5 = K_5(\mu_1, \dots, \mu_{r_2}, T)$. (Note the implicit dependence on M.)

If we define

$$\tilde{\varphi} = \sum_{\lambda_n > N} \Big(\sum_{r=1}^{r_2} \sigma_p e^{i\lambda_n p\tau}\Big) \langle \varphi, \varphi_n\rangle \varphi_n, \tag{A.13}$$

then

$$(I - \Pi)\tilde{\varphi} = \tilde{\varphi} \quad \text{and} \quad U(t)\tilde{\varphi} = \sum_{r=1}^{r_2} \sigma_p U(t + p\tau)\varphi. \tag{A.14}$$

Applying (A.5), (A.12) and the definition (A.13) gives

$$\begin{aligned} 4C_1^2 \|AU(t)\widetilde{\varphi}\|^2_{L^2((T/2,3T/4);\mathcal{H})} \ge \|\tilde{\varphi}\|^2 &\ge \sum_{N\le\lambda_n<M} \Big|\sum_{r=1}^{r_2} \sigma_p e^{i\lambda_n p\tau}\Big|^2 |\langle\varphi, \varphi_n\rangle|^2 \\ &\ge K_5^2 \|(\Pi_M - \Pi)\varphi\|^2. \end{aligned}$$

The choice of τ in (A.11) and (A.14) show that

$$\|AU(t)\varphi\| \ge \frac{K_5}{2C_1 r_2} \|(\Pi_M - \Pi)\varphi\|^2. \tag{A.15}$$

This gives

$$\begin{aligned} \|AU(t)(I - \Pi_M + \Pi)\varphi\|_{L^2((0,T);\mathcal{H})} &\le \|AU(t)\varphi\|_{L^2((0,T);\mathcal{H})} + \sqrt{T}\,\|(\Pi_M - \Pi)\varphi\| \\ &\le \Big(1 + \frac{2\sqrt{T}\,r_2 C_1}{K_5}\Big) \|AU(t)\varphi\|_{L^2((0,T);\mathcal{H})}, \end{aligned}$$

which combined with (A.10) and (A.15) produces

$$\begin{aligned} \Big(1 + \frac{2(\sqrt{T}+1) r_2 C_1}{K_5}\Big) \|AU(t)\varphi\|_{L^2((0,T);\mathcal{H})} &\ge K_2K_3\|\Pi\varphi\| + (1/2C_1)\|(I - \Pi_M)\varphi\| \\ &\quad + \|(\Pi_M - \Pi)\varphi\| - \sqrt{K_4/M}\,\|\varphi\|^2 \\ &\ge (K_6 - \sqrt{K_4/M})\|\varphi\|. \end{aligned}$$

A K_6 and K_4 are independent of M, we obtain (A.4) by choosing M large enough. □

Appendix B. Proof of Lemma 2.5

This is a purely geometric result which does not involve integer points. It is a consequence of the fact that the circle is curved but we prove it by explicit calculations.

We start with the case where $\gamma = 1$ (recall that in Lemma 2.5 the modulus is defined by $|(x_1, x_2)|^2 = x_1^2 + \gamma x_2^2$). We perform a change of variables $x \mapsto xh$, and denote $\epsilon = \kappa^2 h^2$. We are reduced to proving that for

$$\mathcal{B}_{\epsilon,\alpha} = \big\{z \in \mathbb{C};\ \operatorname{Re} z \geq 0,\ \operatorname{Im} z \geq 0,\ \big||z| - 1\big| \leq \epsilon,\ \arg(z) \in [\alpha\sqrt{\epsilon}, (\alpha+1)\sqrt{\epsilon})\big\}, \tag{B.1}$$

we have

Lemma B.1. *There exist* $\epsilon_0 > 0$ *and* $Q > 0$ *such that for any* $0 < \epsilon \leq \epsilon_0$*, we have*

$$\begin{aligned} &\forall \alpha_j \in \{0, 1, \ldots, N_\epsilon\},\ j = 1, \ldots 4,\ N_\epsilon := [\pi/2\sqrt{\epsilon}],\\ &(\mathcal{B}_{\epsilon,\alpha_1} + \mathcal{B}_{\epsilon,\alpha_2}) \cap (\mathcal{B}_{\epsilon,\alpha_3} + \mathcal{B}_{\epsilon,\alpha_4}) \neq \emptyset\\ &\qquad \Rightarrow\ |\alpha_1 - \alpha_3| + |\alpha_2 - \alpha_4| \leq Q \text{ or } |\alpha_1 - \alpha_4| + |\alpha_2 - \alpha_3| \leq Q. \end{aligned} \tag{B.2}$$

Proof. We first observe that it is enough to prove the lemma with the condition $\big||z| - 1\big| < \epsilon$ replaced by $0 \leq |z| - 1 \leq \epsilon$ in the definition of $\mathcal{B}_{\epsilon,\alpha}$: $0 \leq 1 - |z| \leq \epsilon$ is the same as $0 \leq |z|/(1-\epsilon) - 1 \leq \epsilon/(1-\epsilon)$.

Let $z_j = \rho_j e^{i\theta_j} \in \mathcal{B}_{\epsilon,\alpha_j}$, $1 \leq j \leq 4$, be such that $z_1 + z_2 = z_3 + z_4$. By possibly exchanging z_1 and z_2 we can assume $\theta_1 \geq \theta_2$ and similarly that $\theta_3 \geq \theta_4$. In particular,

$$(\theta_1 - \theta_2)/2 \in [0, \pi/4], \qquad (\theta_3 - \theta_4)/2 \in [0, \pi/4]. \tag{B.3}$$

Since $\rho_j \in [1, 1+\epsilon]$, we have

$$|e^{i\theta_1} + e^{i\theta_2} - e^{i\theta_3} - e^{i\theta_4}| \leq 4\epsilon,$$

which is the same as

$$\big|e^{i/2(\theta_1+\theta_2)} \cos\big(\tfrac{\theta_1-\theta_2}{2}\big) - e^{i/2(\theta_3+\theta_4)} \cos\big(\tfrac{\theta_3-\theta_4}{2}\big)\big| \leq 2\epsilon. \tag{B.4}$$

On the other hand,

$$\begin{aligned} &\big|e^{\frac{i}{2}(\theta_1+\theta_2)} \cos\big(\tfrac{\theta_1-\theta_2}{2}\big) - e^{\frac{i}{2}(\theta_3+\theta_4)} \cos\big(\tfrac{\theta_3-\theta_4}{2}\big)\big|\\ &\quad = \big|e^{i/2(\theta_1+\theta_2-\theta_3-\theta_4)} \cos\big(\tfrac{\theta_1-\theta_2}{2}\big) - \cos\big(\tfrac{\theta_3-\theta_4}{2}\big)\big| \geq \big|\sin\big(\tfrac{\theta_1+\theta_2-\theta_3-\theta_4}{2}\big) \cos\big(\tfrac{\theta_1-\theta_2}{2}\big)\big|. \end{aligned}$$

Since (B.3) implies that $\cos\big(\frac{\theta_1-\theta_2}{2}\big) \geq 1/\sqrt{2}$, we deduce from (B.4) that

$$\big|\sin\big(\tfrac{\theta_1+\theta_2-\theta_3-\theta_4}{2}\big)\big| \leq 2\sqrt{2}\,\epsilon.$$

We also have $(\theta_1 + \theta_2 - \theta_3 - \theta_4)/2 \in [-\pi/2, \pi/2]$ and as $|\sin\theta| \geq 2|\theta|/\pi$ for $-\pi/2 \leq \theta \leq \pi/2$, we conclude that

$$\big|\tfrac{\theta_1+\theta_2-\theta_3-\theta_4}{2}\big| \leq \pi\sqrt{2}\,\epsilon. \tag{B.5}$$

We assumed that $z_j = \rho_j e^{i\theta_j} \in \mathcal{B}_{\epsilon,\alpha_j}$ and that means that $0 \leq \theta_j - \sqrt{\epsilon}\,\alpha_j < \sqrt{\epsilon}$. Hence (B.5) gives

$$|\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4| \leq C\sqrt{\epsilon} + 2 \leq 3, \tag{B.6}$$

provided that $\epsilon > 0$ small enough.

Going back to (B.3) and (B.4) we get, with $p = \frac{\theta_1-\theta_2}{2}$, $q = \frac{\theta_3-\theta_4}{2}$,

$$|\cos p - \cos q| = 2\left|\sin\left(\tfrac{p+q}{2}\right)\sin\left(\tfrac{p-q}{2}\right)\right| \leq 2\epsilon. \tag{B.7}$$

As, $p, q \in [0, \pi/4]$ we get

$$\left|\tfrac{p+q}{2}\,\tfrac{p-q}{2}\right| \leq \tfrac{\pi^2}{4}\epsilon.$$

This is the same as (recall that $0 \leq \theta_1 - \theta_2$, $0 \leq \theta_3 - \theta_4$)

$$(|\theta_1 - \theta_2| - |\theta_3 - \theta_4|)(|\theta_1 - \theta_2| + |\theta_3 - \theta_4|) \leq 4\pi^2\epsilon, \tag{B.8}$$

and this gives

$$|(\theta_1-\theta_2)-(\theta_3-\theta_4)| \leq \big((|\theta_1-\theta_2|-|\theta_3-\theta_4|)(|\theta_1-\theta_2|+|\theta_3-\theta_4|)\big)^{1/2} \leq 2\pi\sqrt{\epsilon}. \tag{B.9}$$

Using again the fact that $0 \leq \theta_j - \sqrt{\epsilon}\,\alpha_j < \sqrt{\epsilon}$, this gives

$$|(\alpha_1 - \alpha_2) - (\alpha_3 - \alpha_4)| \leq 2\pi + 2. \tag{B.10}$$

Finally, from (B.6) and (B.10) we obtain

$$|\alpha_1 - \alpha_3| \leq \pi + 5/2, \qquad |\alpha_2 - \alpha_4| \leq \pi + 5/2,$$

which proves Lemma 2.5 in the case $\gamma = 1$ (notice that here only the first term in the alternative is possible which follows from the assumption $\theta_1 \geq \theta_2$, $\theta_3 \geq \theta_4$). The general case follows by applying the transformation $(x_1, x_2) \in \mathbb{R}^2 \mapsto (x_1, \sqrt{\gamma}\,x_2) \in \mathbb{R}^2$. □

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