J. Eur. Math. Soc. 16, 2693–2708

DOI 10.4171/JEMS/496

Dmitri I. Panyushev

Abelian ideals of a Borel subalgebra and root systems

Received May 24, 2012 and in revised form January 28, 2013

Abstract. Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{Ab}^o$ the poset of non-trivial abelian ideals of a fixed Borel subalgebra of $\mathfrak{g}$. In [8], we constructed a partition $\mathfrak{Ab}^o = \bigsqcup_\mu \mathfrak{Ab}_\mu$ parameterised by the long positive roots of $\mathfrak{g}$ and studied the subposets $\mathfrak{Ab}_\mu$. In this note, we show that this partition is compatible with intersections, relate it to the Kostant–Peterson parameterisation and to the centralisers of abelian ideals. We also prove that the poset of positive roots of $\mathfrak{g}$ is a join-semilattice.

Keywords. Root system, Borel subalgebra, minuscule element, abelian ideal

Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$. Here $\mathfrak{t}$ is a fixed Cartan subalgebra and $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ is a fixed Borel subalgebra. Accordingly, Δ is the set of roots of $(\mathfrak{g}, \mathfrak{t})$, Δ^+ is the set of positive roots corresponding to $\mathfrak{u}$, and Π is the set of simple roots in Δ^+. Write θ for the highest root in Δ^+.

A subspace $\mathfrak{a} \subset \mathfrak{u}$ is an *abelian ideal* (of $\mathfrak{b}$) if $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{a}] = 0$. The set of abelian ideals of $\mathfrak{b}$ is denoted by $\mathfrak{Ab}$. In the landmark paper [7], Kostant elaborated on Dale Peterson's theory of abelian ideals (in particular, the astounding result that $\#\mathfrak{Ab} = 2^{\mathrm{rk}\,\mathfrak{g}}$) and related abelian ideals with problems in representation theory. Since then, abelian ideals have attracted a lot of attention: see e.g. [2, 3, 4, 8, 11, 12, 15]. We think of $\mathfrak{Ab}$ as a poset with respect to inclusion. As $\mathfrak{a} \in \mathfrak{Ab}$ is a sum of certain root spaces, we may (and will) identify such an $\mathfrak{a}$ with the corresponding subset $I = I_\mathfrak{a}$ of Δ^+.

Let $\mathfrak{Ab}^o = \mathfrak{Ab}^o(\mathfrak{g})$ denote the set of non-zero abelian ideals and Δ_l^+ the set of long positive roots. In the simply-laced case, all roots are assumed to be long. In [8, Sect. 2], we defined a surjective mapping $\tau : \mathfrak{Ab}^o \to \Delta_l^+$ and studied its fibres. If $\mathfrak{a} \in \mathfrak{Ab}^o$ and $\tau(\mathfrak{a}) = \mu$, then μ is called the *rootlet* of $\mathfrak{a}$, also denoted by $\mathrm{rt}(\mathfrak{a})$ or $\mathrm{rt}(I_\mathfrak{a})$. Letting $\mathfrak{Ab}_\mu = \tau^{-1}(\mu)$, we get a partition of $\mathfrak{Ab}^o$ parameterised by Δ_l^+. Each fibre $\mathfrak{Ab}_\mu$ is regarded as a subposet of $\mathfrak{Ab}$. It is known that, for any $\mu \in \Delta_l^+$, $\mathfrak{Ab}_\mu$ has a unique minimal and unique maximal element [8, Sect. 3]. Regarding abelian ideals as subsets of Δ^+, we write

D. I. Panyushev: Independent University of Moscow, Bol'shoi Vlasevskii per. 11, 119002 Moscow, Russia, and Institute for Information Transmission Problems of the R.A.S., B. Karetnyi per. 19, 127994 Moscow, Russia; e-mail: panyushev@iitp.ru

Mathematics Subject Classification (2010): 17B20, 17B22, 20F55

$I(\mu)_{\min}$ (resp. $I(\mu)_{\max}$) for the minimal (resp. maximal) element of $\mathfrak{Ab}_\mu$. We also say that $I(\mu)_{\min}$ is the μ-*minimal* and $I(\mu)_{\max}$ is the μ-*maximal* ideal. Various properties of the μ-minimal ideals are obtained in [8, Sect. 4]. For instance, if $(\ ,\ )$ is a Weyl group invariant scalar product, then

- $\#I(\mu)_{\min} = (\rho, \theta^\vee - \mu^\vee) + 1$, where $\rho = \frac{1}{2}\sum_{\gamma\in\Delta^+}\gamma$ and $\mu^\vee = 2\mu/(\mu,\mu)$;
- $I = I(\mu)_{\min}$ for some $\mu \in \Delta_l^+$ if and only if $I \subset \mathcal{H} := \{\gamma \in \Delta^+ \mid (\gamma,\theta) \ne 0\}$;
- $I(\mu)_{\min} \subset I(\mu')_{\min}$ if and only if $\mu' \preccurlyeq \mu$, where '$\preccurlyeq$' is the usual *root order* on Δ^+.
- $I(\mu)_{\min} = I(\mu)_{\max}$ if and only if $(\mu,\theta) = 0$ [8, Thm. 5.1].

If $\mathsf{rt}(I) \notin \Pi$, then there is $I' \in \mathfrak{Ab}$ such that $I' \supset I$, $\#I' = \#I + 1$ and $\mathsf{rt}(I') \prec \mathsf{rt}(I)$. This is implicit in [8, Thm. 2.6] (cf. also Proposition 1.1). This implies that the (globally) maximal ideals of $\mathfrak{Ab}$ are precisely the maximal elements of the posets $\mathfrak{Ab}_\alpha$ for $\alpha \in \Pi \cap \Delta_l^+ =: \Pi_l$ (see [8, Cor. 3.8]). A closed formula for the dimension of all maximal abelian ideals is proved in [4, Sect. 8], [15]. In this paper, we elaborate on further properties of the partition

$$\mathfrak{Ab}^o = \bigsqcup_{\mu\in\Delta_l^+} \mathfrak{Ab}_\mu \tag{0.1}$$

and related properties of abelian ideals and root systems.

In Section 2, we show that partition (0.1) behaves well with respect to intersections.

Theorem 0.1. *Let $\mu, \mu' \in \Delta_l^+$.*

(i) *If $I \in \mathfrak{Ab}_\mu$ and $I' \in \mathfrak{Ab}_{\mu'}$, then $I \cap I'$ belongs to $\mathfrak{Ab}_\nu$, where ν does not depend on the choice of I and I'. Actually, ν is the unique smallest long positive root such that $\nu \succcurlyeq \mu$ and $\nu \succcurlyeq \mu'$. In particular, such a ν always exists.*
(ii) *Furthermore, $I(\mu)_{\min} \cap I(\mu')_{\min} = I(\nu)_{\min}$, $I(\mu)_{\max} \cap I(\mu')_{\max} = I(\nu)_{\max}$, and every ideal in $\mathfrak{Ab}_\nu$ occurs as intersection of two ideals from $\mathfrak{Ab}_\mu$ and $\mathfrak{Ab}_{\mu'}$.*

The root ν occurring in (i) is denoted by $\mu \vee \mu'$. In our approach, the existence of $\mu \vee \mu'$ ($\mu, \mu' \in \Delta_l^+$) comes up as a byproduct of our theory of the posets $\mathfrak{Ab}_\mu$. This prompts the natural question of whether $\vee$ is well-defined for *all* pairs of positive roots, not necessarily long. The corresponding general assertion is proved in the Appendix (see Theorem A.1). It seems that this property of root systems has not been noticed before.

In Section 3, we give a characterisation of μ-minimal abelian ideals that relates two different approaches to $\mathfrak{Ab}$. We have associated the rootlet $\mathsf{rt}(I) \in \Delta_l^+$ to a non-zero abelian ideal I. On the other hand, there is a bijection between $\mathfrak{Ab}$ and certain elements in the coroot lattice $Q^\vee$, which is due to Kostant and Peterson [7]. Namely,

$$\mathfrak{Ab} \stackrel{1:1}{\longleftrightarrow} \mathcal{Z}_1 = \{z \in Q^\vee \mid -1 \le (z,\gamma) \le 2 \text{ for all } \gamma \in \Delta^+\}.$$

The element $z \in Q^\vee$ corresponding to $I \in \mathfrak{Ab}$ is denoted by z_I. Our result is

Theorem 0.2. *For an abelian ideal I, we have*

$$I = I(\mu)_{\min} \textit{ for } \mu = \mathsf{rt}(I) \quad \textit{if and only if} \quad \mathsf{rt}(I)^\vee = z_I.$$

We also prove that

- an abelian ideal I belongs to $\mathfrak{Ab}_\mu$ if and only if $I \cap \mathcal{H} = I(\mu)_{\min}$;
- $I(\mu)_{\max} \subset \{\nu \in \Delta^+ \mid \nu \succcurlyeq \mu\}$.

In Section 4, we consider the centralisers of abelian ideals. If $\mathfrak{a} \in \mathfrak{Ab}$, then the centraliser $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is a $\mathfrak{b}$-stable subspace of $\mathfrak{g}$. However, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is not always contained in $\mathfrak{b}$. We give criteria for $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ to be a nilpotent subalgebra or a sum of abelian ideals. We also prove

Theorem 0.3. *Let $\mathfrak{a} \in \mathfrak{Ab}$. Then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is again an abelian ideal if and only if $\mathrm{rt}(\mathfrak{a}) \in \Pi_l$. In particular, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}$ if and only if $\mathfrak{a}$ is a maximal ideal in $\mathfrak{Ab}$.*

In fact, Theorem 0.3 is closely related to the following interesting observation. For any $\mathcal{S} \subset \Delta^+$, let $\min(\mathcal{S})$ and $\max(\mathcal{S})$ denote the sets of minimal and maximal elements of $\mathcal{S}$, respectively.

Theorem 0.4. *For every $\alpha \in \Pi_l$, there is a one-to-one correspondence between $\min(I(\alpha)_{\min})$ and $\max(\Delta^+ \setminus I(\alpha)_{\max})$. Namely, if $\nu \in \min(I(\alpha)_{\min})$, then $\theta - \nu$ is in $\max(\Delta^+ \setminus I(\alpha)_{\max})$; and vice versa. In particular, $\max(\Delta^+ \setminus I(\alpha)_{\max}) \subset \mathcal{H}$.*

An analogous statement for arbitrary long roots (in place of $\alpha \in \Pi_l$) is not true. However, there is a modification of Theorem 0.4 that applies to the connected subsets of Π_l (see Theorem 4.9).

We refer to [1, 5] for standard results on root systems and (affine) Weyl groups.

1. Preliminaries on abelian ideals and minuscule elements

Throughout this paper, Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$ with positive roots Δ^+ corresponding to $\mathfrak{u}$, simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, and Weyl group W. Set $\Pi_l := \Pi \cap \Delta_l^+$. We equip Δ^+ with the usual partial ordering $\preccurlyeq$. This means that $\mu \preccurlyeq \nu$ if $\nu - \mu$ is a non-negative integral linear combination of simple roots. Write $\mu \prec \nu$ if $\mu \preccurlyeq \nu$ and $\mu \neq \nu$.

If $\mathfrak{a}$ is an abelian ideal of $\mathfrak{b}$, then $\mathfrak{a}$ is a sum of certain root spaces in $\mathfrak{u}$, i.e., $\mathfrak{a} = \bigoplus_{\gamma \in I_{\mathfrak{a}}} \mathfrak{g}_\gamma$. The relation $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{a}$ is equivalent to $I = I_{\mathfrak{a}}$ being an *upper ideal* of the poset $(\Delta^+, \preccurlyeq)$, i.e., if $\nu \in I$, $\gamma \in \Delta^+$, and $\nu \preccurlyeq \gamma$, then $\gamma \in I$. The property of being abelian means that $\gamma' + \gamma'' \notin \Delta^+$ for all $\gamma', \gamma'' \in I$. We often work in the setting of root systems, so that a $\mathfrak{b}$-ideal $\mathfrak{a} \subset \mathfrak{u}$ is being identified with the corresponding subset I of positive roots.

The theory of abelian ideals relies on the relationship, due to Peterson, between the abelian ideals and the so-called *minuscule elements* of the affine Weyl group of Δ. Recall the necessary setup.

We have the vector space $V = \bigoplus_{i=1}^n \mathbb{R}\alpha_i$, the Weyl group W generated by simple reflections $s_1, \dots, s_n$, and a W-invariant inner product $(\ ,\ )$ on V. Let $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$. We extend the inner product $(\ ,\ )$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$. Set $\alpha_0 = \delta - \theta$, where θ is the highest root in Δ^+. Then

- $\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ is the set of affine (real) roots;
- $\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\}$ is the set of positive affine roots;
- $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots;
- $\mu^\vee = 2\mu/(\mu, \mu)$ is the coroot corresponding to $\mu \in \widehat{\Delta}$;
- $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ is the *root lattice* and $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ is the *coroot lattice* in V.

For each $\alpha_i \in \widehat{\Pi}$, let s_i denote the corresponding reflection in $GL(\widehat{V})$. That is, $s_i(x) = x - (x, \alpha_i)\alpha_i^\vee$ for any $x \in \widehat{V}$. The affine Weyl group, $\widehat{W}$, is the subgroup of $GL(\widehat{V})$ generated by the reflections $s_0, s_1, \dots, s_n$. The extended inner product $(\ ,\ )$ on $\widehat{V}$ is $\widehat{W}$-invariant. The *inversion set* of $w \in \widehat{W}$ is $\mathcal{N}(w) = \{\nu \in \widehat{\Delta}^+ \mid w(\nu) \in -\widehat{\Delta}^+\}$.

Following Peterson, we say that $w \in \widehat{W}$ is *minuscule* if $\mathcal{N}(w) = \{-\gamma + \delta \mid \gamma \in I_w\}$ for some subset $I_w \subset \Delta$. One then proves that (i) $I_w \subset \Delta^+$, (ii) I_w is an abelian ideal, and (iii) the assignment $w \mapsto I_w$ yields a bijection between the minuscule elements of $\widehat{W}$ and the abelian ideals (see [7], [2, Prop. 2.8]). Accordingly, if $I \in \mathfrak{Ab}$, then w_I denotes the corresponding minuscule element of $\widehat{W}$. Obviously, $\#I = \#\mathcal{N}(w_I) = \ell(w_I)$, where ℓ is the usual length function on $\widehat{W}$.

Using minuscule elements of $\widehat{W}$, one can assign an element of $Q^\vee$ to any abelian ideal [7]. In fact, one can associate an element of $Q^\vee$ to any $w \in \widehat{W}$. The following appears in [9, Sect. 2] in a more comprehensive form.

Recall that $\widehat{W}$ is a semidirect product of W and $Q^\vee$, and it can also be regarded as a group of affine-linear transformations of V [5, 4.2]. For any $w \in \widehat{W}$, there is a unique decomposition

$$w = v \cdot t_r, \tag{1.1}$$

where $v \in W$ and t_r is the translation of V corresponding to $r \in Q^\vee$, i.e., $t_r * x = x + r$ for all $x \in V$. Then we assign the element $v(r) \in Q^\vee$ to $w \in \widehat{W}$. An alternative way for doing so, which does not explicitly use the semidirect product structure, is based on the relation between the linear $\widehat{W}$-action on $\widehat{V}$ and decomposition (1.1). Given $w \in \widehat{W}$, define the integers k_i, $i = 1, \dots, n$, by the formula $w^{-1}(\alpha_i) = \mu_i + k_i\delta$ ($\mu_i \in \Delta$). Then $v(r) \in Q^\vee$ is determined by the conditions that $(v(r), \alpha_i) = k_i$. The reason is that $w^{-1} = v^{-1} \cdot t_{-v(r)}$ and the linear $\widehat{W}$-action on $\widehat{V}$ satisfies the following relation:

$$w^{-1}(x) = v^{-1}(x) + (x, v(r))\delta \qquad \forall x \in V \oplus \mathbb{R}\delta. \tag{1.2}$$

It suffices to verify that $t_r(x) = x - (x, r)\delta$.

If $w = w_I$ is minuscule, then we also write z_I for the resulting element of $Q^\vee$. By [7, Theorem 2.5], the mapping $I \mapsto z_I \in V$ sets up a bijection between $\mathfrak{Ab}$ and $\mathcal{Z}_1 = \{z \in Q^\vee \mid (z, \gamma) \in \{-1, 0, 1, 2\}\ \forall\gamma \in \Delta^+\}$. A proof of this result is given in [11, Appendix A].

Given $I \in \mathfrak{Ab}^o$ and the corresponding non-trivial minuscule element $w_I \in \widehat{W}$, the *rootlet* of I is defined by

$$\mathsf{rt}(I) = w_I(\alpha_0) + \delta = w_I(2\delta - \theta).$$

By [8, Prop. 2.5], we have $\mathsf{rt}(I) \in \Delta_l^+$. The next result describes a procedure for extensions of abelian ideals. Namely, if the rootlet of $I = I_w$ is not simple, then one can construct a larger ideal I' such that $\#I' = \#I + 1$ and $\mathrm{rt}(I') = s_\alpha(\mathrm{rt}(I)) \prec \mathrm{rt}(I)$ for some $\alpha \in \Pi$.

Proposition 1.1. *Let $w \in \widehat{W}$ be minuscule and $\mu = \mathrm{rt}(I_w)$. Suppose that $\mu \notin \Pi$ and take any $\alpha \in \Pi$ such that $(\alpha, \mu) > 0$. Then $s_\alpha w$ is again minuscule. Moreover, the only root in $I_{s_\alpha w} \setminus I_w$ belongs to $\mathcal{H}$.*

Proof. Set $\mu' = s_\alpha(\mu) = s_\alpha w(2\delta - \theta)$ and $\mu'' = \mu - \alpha$. (Note that $\mu' = \mu''$ if and only if $\alpha \in \Pi_l$.) Then $w(2\delta - \theta) = \mu'' + \alpha$ and $w^{-1}(\mu'') + w^{-1}(\alpha) = 2\delta - \theta$. Therefore,

$$\begin{cases} w^{-1}(\mu'') = k\delta - \mu_1, \\ w^{-1}(\alpha) = (2-k)\delta - \mu_2, \end{cases} \qquad \text{where } \mu_1, \mu_2 \in \Delta \text{ and } \mu_1 + \mu_2 = \theta.$$

This clearly implies that both μ_1 and μ_2 are positive and hence $\mu_1, \mu_2 \in \mathcal{H}$. Furthermore, since w is minuscule, both $w^{-1}(\mu'')$ and $w^{-1}(\alpha)$ must be positive. [Indeed, if, say, $w^{-1}(\mu'')$ is negative, then $k \le 0$. Hence $w(\mu_1) = k\delta - \mu''$ is negative and $\mu_1 \in \mathcal{N}(w)$, which contradicts the definition of minuscule elements.] Therefore, one must have $k = 1$. Then $w(\delta - \mu_2) = \alpha \in \Pi$. Since $\mathcal{N}(s_\alpha w) = \mathcal{N}(w) \cup \{w^{-1}(\alpha)\}$, we then conclude that $s_\alpha w$ is minuscule and the corresponding abelian ideal is $I_{s_\alpha w} = I_w \cup \{\mu_2\}$.

Note also that $\mathrm{rt}(I_{s_\alpha w}) = \mu' \prec \mu$. □

2. Intersections of abelian ideals and posets $\mathfrak{Ab}_\mu$

In this section, we prove that taking intersection of abelian ideals is compatible with partition (0.1).

First of all, we notice that for any collection of non-empty abelian ideals (subsets of Δ^+) their intersection is non-empty, since all these ideals contain the highest root θ. In particular, if $\mu_1, \dots, \mu_s \in \Delta_l^+$, then

$$I = \bigcap_{i=1}^{s} I(\mu_i)_{\min}$$

is again an abelian ideal. Since $I(\mu_i)_{\min} \subset \mathcal{H}$ for all i, we have $I \subset \mathcal{H}$, and therefore $I = I(\mu)_{\min}$ for certain $\mu \in \Delta_l^+$ [8, Thm. 4.3]. Since $I(\mu)_{\min} \subset I(\mu_i)_{\min}$, we conclude that $\mu \succcurlyeq \mu_i$ [8, Cor. 3.3].

On the other hand, if $\gamma \in \Delta_l^+$ and $\gamma \succcurlyeq \mu_i$ for all i, then $I(\gamma)_{\min} \subset I(\mu_i)_{\min}$ [8, Thm. 4.5]. Therefore, $I(\gamma)_{\min} \subset I(\mu)_{\min}$, i.e., $\gamma \succcurlyeq \mu$. Thus, we have proved

Theorem 2.1. *For any collection $\mu_1, \dots, \mu_s \in \Delta_l^+$,*

(i) *there exists a unique long root μ such that $\mu \succcurlyeq \mu_i$ for all i, and if $\gamma \in \Delta_l^+$ and $\gamma \succcurlyeq \mu_i$ for all i, then $\gamma \succcurlyeq \mu$;*

(ii) $\bigcap_{i=1}^{s} I(\mu_i)_{\min} = I(\mu)_{\min}$.

The root μ occurring in part (i) is denoted by $\mu_1 \vee \cdots \vee \mu_s = \bigvee_{i=1}^{s} \mu_i$. We also say that μ is the *least upper bound* or *join* of $\mu_1, \dots, \mu_s$.

Remark 2.2. Clearly, the operation $\vee$ is associative, and it suffices to describe the least upper bound for only two (long) roots. In Appendix A, we prove directly that the join exists for *all* pairs of roots, not necessarily long ones, and give an explicit formula for it.

We are going to play the same game with arbitrary ideals in $\mathfrak{Ab}_{\mu_i}$. To this end, we need an analogue of [8, Thm. 4.5] for the μ-maximal ideals (see Corollary 2.4(i) below). This can be achieved as follows.

Proposition 2.3. *Let μ, μ' be long roots such that $\mu' \prec \mu$. Then*

(i) *for every $I \in \mathfrak{Ab}_\mu$, there exists $I' \subset \mathfrak{Ab}_{\mu'}$ such that $I' \supset I$ and $\#I' = \#I + (\rho, \mu^\vee - \mu'^\vee)$;*

(ii) *moreover, if $I = I_0 \subset I_1 \subset \cdots \subset I_m = I'$ is any chain of ideals with $m = (\rho, \mu^\vee - \mu'^\vee)$ and $\#I_j = \#I_{j-1} + 1$, then $\mathsf{rt}(I_j) \neq \mathsf{rt}(I_{j-1})$ for all j.*

Proof. If $\mu \notin \Pi_l$ and $\alpha \in \Pi$ with $(\alpha, \mu) > 0$, then a direct calculation shows that $(\rho, \mu^\vee - s_\alpha(\mu)^\vee) = 1$. [Use the relations $(\rho, \alpha^\vee) = 1$ and $(\alpha, \mu^\vee) = 1$.]

(i) Arguing by induction, one readily proves that if μ, μ' are both long and $\mu' \prec \mu$, then μ' can be reached from μ by a sequence of simple reflections:

$$\mu = \mu_0 \to s_{\gamma_1}(\mu_0) = \mu_1 \to s_{\gamma_2}(\mu_1) = \mu_2 \to \cdots \to s_{\gamma_m}(\mu_{m-1}) = \mu_m = \mu',$$

where $\gamma_i \in \Pi$ and $(\gamma_i, \mu_{i-1}) > 0$. The number of steps m equals $(\rho, \mu^\vee - \mu'^\vee)$. If $I \in \mathfrak{Ab}_\mu$ is arbitrary and w_I is the corresponding minuscule element, then the repeated application of Proposition 1.1 shows that $w' := s_{\gamma_1} \cdots s_{\gamma_m} w_I$ is again minuscule and $I' = I_{w'}$ is the required ideal.

(ii) Let $w_j \in \widehat{W}$ be the minuscule element corresponding to I_j. Then $w_j = s_{i_j} w_{j-1}$ for a sequence $(\alpha_{i_1}, \ldots, \alpha_{i_m})$ of affine simple roots. The corresponding sequence of rootlets is

$$\mu = \mu_0 \to s_{i_1}\mu_0 = \mu_1 \to s_{i_2}\mu_1 = \mu_2 \to \cdots \to \mu_m = \mu'.$$

If $i_j = 0$, i.e., the j-th step is the reflection with respect to $\alpha_0 = \delta - \theta$, then $\mu_{j-1} = \mu_j$ (see [8, Prop. 3.2]). For the steps corresponding to $\alpha_{i_j} \in \Pi$, the value of $(\rho, \mu_j^\vee)$ is reduced by at most 1. Consequently, the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_m})$ does not contain α_0 and the value of $(\rho, \mu_j^\vee)$ decreases by 1 at each step, i.e., all these rootlets are different. □

Corollary 2.4. *If μ, μ' are long roots such that $\mu' \preccurlyeq \mu$, then*

(i) $I(\mu)_{\max} \subset I(\mu')_{\max}$*;*

(ii) $\#\mathfrak{Ab}_{\mu'} \geq \#\mathfrak{Ab}_\mu$.

Proof. (i) This readily follows from Proposition 2.3(i) applied to $I = I(\mu)_{\max}$.

(ii) Argue by induction on $m = (\rho, \mu^\vee - \mu'^\vee)$. For $m = 1$, the assertion follows from Proposition 1.1. □

Theorem 2.5. *For any set $\{\mu_1, \ldots, \mu_s\} \subset \Delta_l^+$ and $\mu = \bigvee_{i=1}^s \mu_i$, we have*

(i) $\bigcap_{i=1}^s I(\mu_i)_{\max} = I(\mu)_{\max}$*;*

(ii) *if $I_i \in \mathfrak{Ab}_{\mu_i}$ for $i = 1, \ldots, s$, then $\bigcap_{i=1}^s I_i \in \mathfrak{Ab}_\mu$;*

(iii) *for every $I \in \mathfrak{Ab}_\mu$, there exist $I_i \in \mathfrak{Ab}_{\mu_i}$ such that $I = \bigcap_{i=1}^s I_i$.*

Proof. (i) Consider the abelian ideal $I = \bigcap_{i=1}^{s} I(\mu_i)_{\max}$. Since $I \subset I(\mu_i)_{\max}$, we have $\mathsf{rt}(I) \succcurlyeq \mu_i$ for all i, hence $\mathsf{rt}(I) \succcurlyeq \bigvee_{i=1}^{s} \mu_i = \mu$. We also have $I \supset \bigcap_{i=1}^{s} I(\mu_i)_{\min} = I(\mu)_{\min}$, hence $\mathsf{rt}(I) \preccurlyeq \mu$ by [8, Cor. 3.3]. It follows that $\mathsf{rt}(I) = \mu$ and $I \subset I(\mu)_{\max}$.

Since $\mu \succcurlyeq \mu_i$, by Corollary 2.4(i), we have $I(\mu)_{\max} \subset I(\mu_i)_{\max}$ for all i, and $I(\mu)_{\max} \subset I$.

Thus, $I = I(\mu)_{\max}$.

(ii) It follows from Theorem 2.1(ii) and part (i) that $I(\mu)_{\min} \subset \bigcap_{i=1}^{s} I_i \subset I(\mu)_{\max}$. By [8, Thm. 3.1(iii)], the intermediate ideal $\bigcap_{i=1}^{s} I_i$ also belongs to $\mathfrak{Ab}_{\mu}$.

(iii) Given $I \in \mathfrak{Ab}_{\mu}$, we construct the ideals $I_i \in \mathfrak{Ab}_{\mu_i}$, $i = 1, \dots, s$, as prescribed in Proposition 2.3(i). Then $I \subset \bigcap_{i=1}^{s} I_i =: J$ and $\mathsf{rt}(J) = \bigvee_{i=1}^{s} \mu_i = \mu$. That is, $\mathsf{rt}(I) = \mathsf{rt}(J)$. By Proposition 2.3(ii), this is only possible if $J = I$. □

Combining Theorems 2.1 and 2.5 yields Theorem 0.1 in the Introduction.

For any $\gamma \in \Delta^+$, set $I\langle \succcurlyeq\gamma \rangle = \{\nu \in \Delta^+ \mid \nu \succcurlyeq \gamma\}$. We also say that $I\langle \succcurlyeq\gamma \rangle$ is the *principal* upper ideal of Δ^+ *generated* by γ. It is not necessarily abelian.

Example 2.6. Let $\alpha_1, \dots, \alpha_s$ be the set of all long simple roots. Then $\bigvee_{i=1}^{s} \alpha_i = \sum_{i=1}^{s} \alpha_i = |\Pi_l|$ and $\{I(\alpha_i)_{\max} \mid i = 1, \dots, s\}$ is the set of all maximal abelian ideals in $\mathfrak{Ab}$. Hence $\bigcap_{i=1}^{s} I(\alpha_i)_{\max}$ is an ideal with rootlet $|\Pi_l|$. Inspecting the list of root systems, we notice that the ideal $\bigcap_{i=1}^{s} I(\alpha_i)_{\min} = I(|\Pi_l|)_{\min}$ has a nice uniform description. For any $\gamma = \sum_{i=1}^{n} a_i\alpha_i \in \Delta^+$, we set $[\gamma/2] = \sum_{i=1}^{n} [a_i/2]\alpha_i$. Then $I(|\Pi_l|)_{\min}$ is the upper ideal of Δ^+ generated by the root $\theta - [\theta/2]$. (It is true that $\theta - [\theta/2]$ is always a root in $\mathcal{H}$.)

In the **A-D-E** case, we have $|\Pi_l| = |\Pi|$ and hence $(\theta, |\Pi_l|) \neq 0$. In fact, $(\theta, |\Pi_l|) \neq 0$ for all simple Lie algebras except those of type $\mathbf{C}_n$, $n \geq 2$. The condition $(\theta, |\Pi_l|) \neq 0$ implies that $\#\mathfrak{Ab}_{|\Pi_l|} = 1$ [8, Thm. 5.1], i.e., $I(|\Pi_l|)_{\min} = I(|\Pi_l|)_{\max}$ if $\mathfrak{g}$ is not of type $\mathbf{C}_n$.

Remark 2.7. The interest in $[\theta/2]$ is also justified by the following observations. As in [10], we say that $\gamma \in \Delta^+$ is *commutative* if the $\mathfrak{b}$-submodule of $\mathfrak{g}$ generated by $\mathfrak{g}_{\gamma}$ is an abelian ideal; equivalently, if the upper ideal $I\langle \succcurlyeq\gamma \rangle$ is abelian. Let Δ^+_{com} denote the set of all commutative roots. Clearly, $\Delta^+_{\mathsf{com}} = \bigcup_{\alpha_i \in \Pi_l} I(\alpha_i)_{\max}$. It was noticed in [10, Thm. 4.4] that $\Delta^+ \setminus \Delta^+_{\mathsf{com}}$ has a unique maximal element, and this maximal element is $[\theta/2]$.

For any $\gamma \in \Delta^+$, it appears to be true that $[\gamma/2] \in \Delta^+ \cup \{0\}$ and $\gamma - [\gamma/2] \in \Delta^+$. It would be interesting to have a conceptual explanation for this.

3. Some properties of the posets $\mathfrak{Ab}_{\mu}$

Let $I \subset \Delta^+$ be an abelian ideal and $w_I = v \cdot t_r \in \widehat{W}$ the corresponding minuscule element. Recall that $v \in W$ and $r \in Q^{\vee}$. We have associated two objects to these data: the rootlet $\mathsf{rt}(I) = w_I(2\delta - \theta) \in \Delta_l^+ \subset Q$ and the element $z_I := v(r) \in Q^{\vee}$.

Theorem 3.1. *For an abelian ideal I, the following conditions are equivalent:*

(i) $\mathsf{rt}(I)^{\vee} = z_I$;

(ii) $I = I(\mu)_{\min}$ *for* $\mu = \mathsf{rt}(I)$.

Proof. 1) Suppose that $I = I(\mu)_{\min}$. By [8, Thm. 4.3], $w_I = v_\mu s_0$, where $v_\mu \in W$ is the unique element of minimal length such that $v_\mu(\theta) = \mu$. Here $\ell(v_\mu) = (\rho, \theta^\vee - \nu^\vee)$. It is easily seen that for $w = s_0$ decomposition (1.1) is $s_0 = s_\theta \cdot t_{-\theta^\vee}$, where $s_\theta \in W$ is the reflection with respect to θ. Hence the linear part of w_I is $v_\mu s_\theta$ and $r = -\theta^\vee$. Therefore, $v_\mu s_\theta(-\theta^\vee) = v_\mu(\theta^\vee) = \mu^\vee$, as required.

2) Conversely, if $\mathsf{rt}(I) = \mu$ and $I \ne I(\mu)_{\min}$, then $z_I \ne z_{I(\mu)_{\min}}$. By the first part, we have $z_{I(\mu)_{\min}} = \mu^\vee$. Thus, $z_I \ne \mathsf{rt}(I)^\vee$. □

Applying formulae (1.1) and (1.2) to an arbitrary minuscule w_I, we obtain

$$w_I(2\delta - \theta) = v \cdot t_r(2\delta - \theta) = v(2\delta + (\theta, r)\delta - \theta) = -v(\theta) + (2 + (\theta, r))\delta.$$

As we know that $\mathsf{rt}(I) \in \Delta^+$, one must have $(\theta, r) = -2$ and $-v(\theta) \in \Delta^+$. Therefore, the equality $\mathsf{rt}(I)^\vee = z_I$ is equivalent to $v(r) = -v(\theta^\vee)$, i.e., $r = -\theta^\vee$.

This can be summarised as follows:

If $w_I = v \cdot t_r \in \widehat{W}$ is minuscule, then $(\theta, r) = -2$. Moreover, $\mathsf{rt}(I)^\vee = z_I$ if and only if $r = -\theta^\vee$, i.e., r is the shortest element in the affine hyperplane $\{x \in V \mid (x, \theta) = -2\}$.

The theory developed in [8, Sect. 4] yields, in principle, a very good understanding of μ-minimal ideals. In particular, an abelian ideal I is minimal in some $\mathfrak{Ab}_\mu$ if and only if $I \subset \mathcal{H} = \{\nu \in \Delta^+ \mid (\nu, \theta) \ne 0\}$ [8, Thm. 4.3]. The other ideals in $\mathfrak{Ab}_\mu$ can be characterised as follows.

Proposition 3.2. *For $\mu \in \Delta_l^+$ and $I \in \mathfrak{Ab}$, we have $I \in \mathfrak{Ab}_\mu$ if and only if $I \cap \mathcal{H} = I(\mu)_{\min}$.*

Proof. ($\Rightarrow$) Since $I(\mu)_{\min} \subset \mathcal{H}$, we have $I(\mu)_{\min} \subset I \cap \mathcal{H}$. Moreover, $I \cap \mathcal{H} = I(\mu')_{\min}$ for some $\mu' \in \Delta_l^+$. Then we have $I(\mu)_{\min} \subset I(\mu')_{\min} \subset I$. By [8, Cor. 3.3], this yields opposite inequalities for the rootlets, i.e., $\mu \succcurlyeq \mu' \succcurlyeq \mu$.

($\Leftarrow$) If $\mu' = \mathsf{rt}(I)$, then $I \cap \mathcal{H} = I(\mu')_{\min}$ according to the previous part. Hence $\mu' = \mu$. □

This implies that all ideals in $\mathfrak{Ab}_\mu$ can be obtained from $I(\mu)_{\min}$ by adding suitable roots outside $\mathcal{H}$. In particular, $I(\mu)_{\max}$ is maximal among all abelian ideals having the prescribed intersection, $I(\mu)_{\min}$, with $\mathcal{H}$.

For future use, we provide a property of the ideals $I(\alpha)_{\min}$ with $\alpha \in \Pi_l$. Recall that the integer $1 + (\rho, \theta^\vee) = h^*$ is called the *dual Coxeter number* of Δ. By [14, Prop. 1], $\#\mathcal{H} = 2h^* - 3$. If $\gamma \in \mathcal{H} \setminus \{\theta\}$, then $\theta - \gamma \in \mathcal{H} \setminus \{\theta\}$ as well. Hence $\mathcal{H}$ can be represented as the disjoint union of θ and $h^* - 2$ pairs of the form $\{\gamma_i, \theta - \gamma_i\}$. By [8, Theorem 3.1], we have $\# I(\alpha)_{\min} = (\rho, \theta^\vee - \alpha^\vee) + 1 = (\rho, \theta^\vee) = h^* - 1$. Because $I(\alpha)_{\min} \subset \mathcal{H}$, we see that $I(\alpha)_{\min}$ must contain θ and exactly one element from each pair $\{\gamma_i, \theta - \gamma_i\}$ (cf. [4, Prop. 7.2]). In particular,

Lemma 3.3. *For $\alpha \in \Pi_l$ and $\gamma \in \mathcal{H} \setminus \{\theta\}$, we have*

$$\gamma \in I(\alpha)_{\min} \quad \textit{if and only if} \quad \theta - \gamma \notin I(\alpha)_{\min}.$$

Our next goal is to compare the upper ideals $I\langle \succcurlyeq \mu \rangle$ and $I(\mu)_{\max}$ ($\mu \in \Delta_l^+$). This will be achieved in two steps.

Proposition 3.4. *For any* $\mu \in \Delta_l^+$*, we have* $I(\mu)_{\min} \subset I\langle \succcurlyeq \mu \rangle$.

Proof. As above, $v_\mu \in W$ is the element of minimal length such that $v_\mu(\theta) = \mu$ and $w = v_\mu s_0$ is the minuscule element for $I(\mu)_{\min}$. Then we have $I(\mu)_{\min} = \{\gamma \in \Delta^+ \mid -\gamma + \delta \in \mathcal{N}(w)\}$ and $\mathcal{N}(w) = \{\alpha_0\} \cup s_0(\mathcal{N}(v_\mu))$. Therefore, if $\gamma \in I(\mu)_{\min}$, then either $\gamma = \theta$, or $-\gamma + \delta \in s_0(\mathcal{N}(v_\mu))$, i.e., $\theta - \gamma \in \mathcal{N}(v_\mu)$. Clearly, $\mathcal{N}(v_\mu^{-1}) = -v_\mu(\mathcal{N}(v_\mu))$. Hence $\theta - \gamma \in \mathcal{N}(v_\mu)$ if and only if $-v_\mu(\theta - \gamma) = v_\mu(\gamma) - \mu \in \mathcal{N}(v_\mu^{-1})$. Consequently,

$$\gamma \in I(\mu)_{\min} \;\&\; \gamma \neq \theta \;\Leftrightarrow\; v_\mu(\gamma) - \mu \in \mathcal{N}(v_\mu^{-1}).$$

Set $\nu = v_\mu(\gamma) - \mu$. Then $\gamma = v_\mu^{-1}(\nu + \mu) = \theta + v_\mu^{-1}(\nu)$. Hence our goal is to prove that

$$\theta + v_\mu^{-1}(\nu) \succcurlyeq \mu \quad \textit{for any } \nu \in \mathcal{N}(v_\mu^{-1}). \tag{$*$}$$

We will argue by induction on $\ell(v_\mu) = (\rho, \theta^\vee - \mu^\vee)$. For the induction step, assume that $\mu \notin \Pi_l$ and $(*)$ is satisfied. Take any $\alpha \in \Pi$ such that $(\alpha, \mu) > 0$ and set $\mu' := s_\alpha(\mu) \prec \mu$. Consider $v_{\mu'} = s_\alpha v_\mu$, which corresponds to the minuscule element $w' = s_\alpha w = v_{\mu'} s_0$ (Proposition 1.1) and the larger abelian ideal $I(\mu')_{\min}$. Then

$$\mathcal{N}(v_{\mu'}^{-1}) = \{\alpha\} \cup s_\alpha(\mathcal{N}(v_\mu^{-1})).$$

Thus, to prove the analogue of $(*)$ for $\nu' \in \mathcal{N}(v_{\mu'}^{-1})$, we have to handle two possibilities:

a) $\nu' = s_\alpha(\nu)$ for $\nu \in \mathcal{N}(v_\mu^{-1})$.

Then $\theta + v_{\mu'}^{-1}(\nu') = \theta + v_\mu^{-1}(\nu) \succcurlyeq \mu \succ \mu'$, as required.

b) $\nu' = \alpha$.

We have to prove here that $\theta + v_{\mu'}^{-1}(\alpha) = \theta - v_\mu^{-1}(\alpha) \succcurlyeq \mu' = s_\alpha(\mu)$. To this end, take a reduced decomposition $v_\mu^{-1} = s_{\gamma_k} \cdots s_{\gamma_1}$, where $\{\gamma_1, \dots, \gamma_k\}$ is a multiset of simple roots. Recall that $v_\mu^{-1}(\mu) = \theta$ and $k = (\rho, \theta^\vee - \mu^\vee)$. Since $(\rho, s_\alpha(\nu)^\vee - \nu^\vee) \in \{-1, 0, 1\}$ for any $\nu \in \Delta_l^+$ and $\alpha \in \Pi$, the chain of roots

$$\mu_0 = \mu, \quad \mu_1 = s_{\gamma_1}(\mu), \quad \mu_2 = s_{\gamma_2} s_{\gamma_1}(\mu), \; \dots, \; \mu_k = \theta,$$

has the property that $\mu_i \prec \mu_{i+1}$ and each simple reflection s_{γ_i} increases the "level" $(\rho, (\cdot)^\vee)$ by 1. Then we must have $\theta = \mu + \sum_{i=1}^k n_i \gamma_i$, where

$$n_i = \begin{cases} 1 & \text{if } \gamma_i \text{ is long}, \\ \|\text{long}\|^2/\|\text{short}\|^2 & \text{if } \gamma_i \text{ is short}. \end{cases}$$

We also have $s_\alpha(\mu) = \mu - (\mu, \alpha^\vee)\alpha$ and $v_\mu^{-1}(\alpha) \preccurlyeq \alpha + \sum_{i=1}^k n_i \gamma_i$, whence

$$v_\mu^{-1}(\alpha) + s_\alpha(\mu) \preccurlyeq \mu + \sum_{i=1}^k n_i \gamma_i + (1 - (\mu, \alpha^\vee))\alpha \preccurlyeq \theta.$$

This completes the induction step and the proof of the proposition. □

Theorem 3.5. *For any $\mu \in \Delta_l^+$, we have $I(\mu)_{\max} \subset I\langle\succcurlyeq\mu\rangle$. In particular, if $I \in \mathfrak{Ab}_\mu$, then $I \subset I\langle\succcurlyeq\mu\rangle$.*

Proof. Suppose that $\gamma \in I(\mu)_{\max}$. In particular, γ is a commutative root.

If $\gamma \in \Delta_l^+$, then the ideal $I(\gamma)_{\min}$ is well-defined and

$$I(\gamma)_{\min} \underset{\text{Prop. 3.4}}{\subset} I\langle\succcurlyeq\gamma\rangle \cap \mathcal{H} \subset I(\mu)_{\max} \cap \mathcal{H} \underset{\text{Prop. 3.2}}{=} I(\mu)_{\min}.$$

By [8, Thm. 4.5], we conclude that $\gamma \succcurlyeq \mu$. (This completes the proof in the **A-D-E** case!)

If γ is short and $\gamma \in \mathcal{H}$, then $\gamma \in I(\mu)_{\min} \subset I\langle\succcurlyeq\mu\rangle$ by Propositions 3.2 and 3.4.

The remaining possibility is that γ is short and $\gamma \notin \mathcal{H}$. But no such commutative roots exist for $\mathbf{B}_n, \mathbf{F}_4, \mathbf{G}_2$. (For $\mathbf{B}_n$, the only short commutative root is ε_1 and $\theta = \varepsilon_1 + \varepsilon_2$.) For $\mathbf{C}_n$, such commutative roots are of the form $\gamma = \varepsilon_i + \varepsilon_j$ with $2 \le i < j \le n$. Here $\mathcal{H} = \{\varepsilon_1 \pm \varepsilon_j \mid 2 \le j \le n\} \cup \{2\varepsilon_1\}$ and $I\langle\succcurlyeq\varepsilon_i + \varepsilon_j\rangle \cap \mathcal{H} = I\langle\succcurlyeq\varepsilon_1 + \varepsilon_j\rangle$. Then Proposition 3.2 shows that $\mathsf{rt}(I\langle\succcurlyeq\varepsilon_i + \varepsilon_j\rangle) = 2\varepsilon_j$. Clearly, we have $\varepsilon_i + \varepsilon_j \succcurlyeq 2\varepsilon_j$. (As usual, the simple roots of $\mathbf{C}_n$ are $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$.) □

Remark 3.6. If $\mathfrak{g}$ is of type $\mathbf{A}_n$ or $\mathbf{C}_n$, then $I(\mu)_{\max} = I\langle\succcurlyeq\mu\rangle$ for all $\mu \in \Delta_l^+$. For other types, this is not always the case.

4. Centralisers of abelian ideals

In this section, we mostly regard abelian ideals as subspaces $\mathfrak{a}$ of $\mathfrak{u}$. Accordingly, for $\mu \in \Delta_l^+$, the minimal and maximal elements of $\mathfrak{Ab}_\mu$ are denoted by $\mathfrak{a}(\mu)_{\min}$ and $\mathfrak{a}(\mu)_{\max}$, respectively.

If $\mathfrak{c} \subset \mathfrak{g}$ is a subspace, then $\mathfrak{z}_\mathfrak{g}(\mathfrak{c})$ denotes the *centraliser* of $\mathfrak{c}$ in $\mathfrak{g}$. If $\mathfrak{c}$ is $\mathfrak{b}$-stable, then so is $\mathfrak{z}_\mathfrak{g}(\mathfrak{c})$. If $\mathfrak{a} \in \mathfrak{Ab}$, then $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ is a $\mathfrak{b}$-stable subalgebra of $\mathfrak{g}$ and $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \supset \mathfrak{a}$. However, $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ may contain semisimple elements and/or it may happen that $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \not\subset \mathfrak{b}$.

Consider the following properties of abelian ideals:

(P1): $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \subset \mathfrak{u}$; (P2): $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ is a sum of abelian ideals; (P3): $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ is an abelian ideal.

Clearly, (P3)⇒(P2)⇒(P1).

We say that $\mathfrak{a}$ is of *full rank* if $I_\mathfrak{a}$ contains n linearly independent roots ($n = \mathsf{rk}\,\mathfrak{g}$).

Lemma 4.1. *Let $\mathfrak{a} \in \mathfrak{Ab}$. Then $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \subset \mathfrak{u}$ if and only if $\mathfrak{a}$ is of full rank.*

Proof. If $\mathfrak{a}$ is not of full rank, then $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{t} \ne 0$. If $\mathfrak{a}$ is of full rank, then $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{t} = 0$ and $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ is $\mathfrak{b}$-stable. Therefore, $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ cannot contain root spaces corresponding to negative roots. □

Lemma 4.2. *$\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ is a sum of abelian ideals if and only if $\mathfrak{a}$ is of full rank and $\theta - [\theta/2]$ is in $I_\mathfrak{a}$.*

Proof. The root space $\mathfrak{g}_{[\theta/2]}$ is contained in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ if and only if $\theta - [\theta/2] \notin I_{\mathfrak{a}}$. The rest follows from the fact that $[\theta/2]$ is the unique maximal non-commutative root. □

Recall that $\{\mathfrak{a}(\alpha)_{\max} \mid \alpha \in \Pi_l\}$ is the complete set of maximal abelian ideals. For any $\mathfrak{a} \in \mathfrak{Ab}$, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ contains the sum of all maximal abelian ideals that contain $\mathfrak{a}$. Therefore, if $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is an abelian ideal, then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}(\alpha)_{\max}$ for some $\alpha \in \Pi_l$ and $\mathfrak{a}(\alpha)_{\max}$ is the only maximal abelian ideal containing $\mathfrak{a}$.

Lemma 4.3. *An abelian ideal* $\mathfrak{a}$ *is contained in a unique maximal abelian ideal if and only if there is a unique* $\alpha \in \Pi_l$ *such that* $\mathsf{rt}(\mathfrak{a}) \succcurlyeq \alpha$. *In particular, in the simply-laced case, the last condition means precisely that* $\mathsf{rt}(\mathfrak{a}) \in \Pi_l$.

Proof. Follows from the fact that the inclusion $\mathfrak{a} \subset \tilde{\mathfrak{a}}$ implies that $\mathsf{rt}(\mathfrak{a}) \succcurlyeq \mathsf{rt}(\tilde{\mathfrak{a}})$ (see [8, Cor. 3.3]; cf. also [8, Thm. 2.6(3)]). □

Note that if $\mathfrak{a}$ is a maximal abelian ideal, then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}$ and in particular $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is an abelian ideal. Indeed, if $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \supsetneq \mathfrak{a}$ and γ is a maximal element in $I_{\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})} \setminus I_{\mathfrak{a}}$, then $\mathfrak{a} \oplus \mathfrak{g}_\gamma$ would be a larger abelian ideal! To get a general answer, we need some preparatory results.

Lemma 4.4. *For any* $\alpha \in \Pi_l$*, the ideal* $\mathfrak{a}(\alpha)_{\min}$ *is of full rank.*

Proof. By [8, Thm. 4.3], the corresponding minuscule element $w \in \widehat{W}$ equals $v_\alpha s_0$, where $v_\alpha \in W$ is the unique element of minimal length taking θ to α. Since $v_\alpha(\theta) = \alpha$, any reduced decomposition of v_α contains all simple reflections corresponding to $\Pi \setminus \{\alpha\}$. Therefore w contains reflections corresponding to $n = \#\Pi$ linearly independent roots. This easily implies that the inversion set $\mathcal{N}(w)$ contains n linearly independent affine roots. Hence $\mathfrak{a}(\alpha)_{\min}$ is of full rank. □

Lemma 4.5. *For any* $\alpha \in \Pi_l$*, we have* $\max(\Delta^+ \setminus I(\alpha)_{\max}) \subset \mathcal{H} \setminus \{\theta\}$.

Proof. If $\gamma \in \max(\Delta^+ \setminus I(\alpha)_{\max})$, then $I(\alpha)_{\max} \cup \{\gamma\}$ determines a $\mathfrak{b}$-stable subspace of $\mathfrak{u}$, which is no longer abelian. That is, there exists $\xi \in I(\alpha)_{\max}$ such that $\xi + \gamma \in \Delta^+$. Then there are $\xi' \succcurlyeq \xi$ and $\gamma' \succcurlyeq \gamma$ such that $\xi' + \gamma' = \theta$ (see [8, p. 1897]). Since $I(\alpha)_{\max}$ is abelian, this clearly implies that $\gamma' = \gamma$, hence $\gamma \in \mathcal{H} \setminus \{\theta\}$. □

Theorem 4.6. *Let* $\mathfrak{a} \in \mathfrak{Ab}$. *The following conditions are equivalent:*

(1) $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ *is an abelian ideal;*
(2) $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}(\alpha)_{\max}$ *for some* $\alpha \in \Pi_l$*;*
(3) $\mathsf{rt}(\mathfrak{a}) \in \Pi_l$.

Proof. (1)⇒(2): See the paragraph preceding Lemma 4.3.

(2)⇒(1): Obvious.

(2)⇒(3): Here $\mathfrak{a}(\alpha)_{\max}$ is the only maximal abelian ideal that contains $\mathfrak{a}$. Therefore, in the simply-laced case, the assertion follows from Lemma 4.3.

For the non-simply-laced case, assume that $\mathsf{rt}(\mathfrak{a}) = \gamma \notin \Pi_l$, but still γ majorises a unique long simple root. Then γ also majorises a short simple root, whence $\gamma \not\preccurlyeq |\Pi_l|$. We claim that $\theta - [\theta/2] \notin I_{\mathfrak{a}}$, and so $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is not a sum of abelian ideals, in view of

Lemma 4.2. Indeed, assume that $\theta - [\theta/2] \in I_{\mathfrak{a}}$. Then $I_{\mathfrak{a}}$ contains the upper ideal of Δ^+ generated by $\theta - [\theta/2]$, which is exactly $\bigcap_{\alpha\in\Pi_l} I(\alpha)_{\min} = I(|\Pi_l|)_{\min}$ (see Example 2.6). Then the inclusion $I_{\mathfrak{a}} \supset I(|\Pi_l|)_{\min}$ implies that $\gamma = \mathsf{rt}(\mathfrak{a}) \preccurlyeq |\Pi_l|$, a contradiction.

(3)⇒(2): It suffices to prove that the centraliser of $\mathfrak{a}(\alpha)_{\min}$ equals $\mathfrak{a}(\alpha)_{\max}$ for any α in Π_l. To this end, we have to check that:

(i) $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}(\alpha)_{\min})$ contains no semisimple elements of $\mathfrak{g}$ (i.e., $\mathfrak{a}(\alpha)_{\min}$ is of full rank), and
(ii) the nilpotent subalgebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}(\alpha)_{\min})$ cannot be larger than $\mathfrak{a}(\alpha)_{\max}$, i.e., for every $\gamma \in \max(\Delta^+ \setminus I(\alpha)_{\max})$, there exists a $\nu \in I(\alpha)_{\min}$ such that $\gamma + \nu \in \Delta^+$.

For (i): This is Lemma 4.4.

For (ii): If $\gamma \in \max(\Delta^+ \setminus I(\alpha)_{\max})$, then $\gamma \in \mathcal{H} \setminus \{\theta\}$ (Lemma 4.5). Consequently, $\theta - \gamma \in I(\alpha)_{\min}$ in view of Lemma 3.3. □

Theorem 4.7. *For $\alpha \in \Pi_l$, there is a one-to-one correspondence between* $\min(I(\alpha)_{\min})$ *and* $\max(\Delta^+ \setminus I(\alpha)_{\max})$*. Namely, for every $\eta \in \max(\Delta^+ \setminus I(\alpha)_{\max})$, there exists $\eta' \in \min(I(\alpha)_{\min})$ such that $\eta + \eta' = \theta$; and vice versa.*

Proof. We assume that $\mathsf{rk}\,\Delta > 1$, hence $\theta \neq \alpha$, $I(\alpha)_{\min} \neq \{\theta\}$, and $\mathcal{H} \setminus \{\theta\} \neq \emptyset$.

1) If $\eta \in \max(\Delta^+ \setminus I(\alpha)_{\max})$, then $\eta \in \mathcal{H} \setminus \{\theta\}$ (Lemma 4.5) and also $\eta \notin I(\alpha)_{\min}$. Hence $\eta' = \theta - \eta \in I(\alpha)_{\min}$ (Lemma 3.3). Actually, η' is a minimal element of $I(\alpha)_{\min}$: if $\xi' \in I(\alpha)_{\min}$ and $\eta' \succ \xi'$, then $\theta - \xi' \succ \eta$ and hence $\theta - \xi' \in I(\alpha)_{\max}$; as $I(\alpha)_{\max} \cap \mathcal{H} = I(\alpha)_{\min}$, one would obtain $\theta - \xi' \in I(\alpha)_{\min}$, which contradicts Lemma 3.3.

2) If $\eta' \in I(\alpha)_{\min}$, then $\eta := \theta - \eta' \in \mathcal{H} \setminus I(\alpha)_{\min}$. Hence $\eta \notin I(\alpha)_{\max}$. Assume that η is not maximal in $\Delta^+ \setminus I(\alpha)_{\max}$ and $\xi \succ \eta$ with $\xi \notin I(\alpha)_{\max}$. Then $\theta - \xi \prec \eta'$ and $\theta - \xi \in I(\alpha)_{\min}$, which contradicts the choice of η'. □

Remark. Lemma 4.5 and the above proof of Theorem 4.7 (= Theorem 0.4) are based on the suggestion of the anonymous referee. This uniform proof replaces our initial case-by-case considerations.

Example 4.8. We describe the corresponding minimal and maximal elements in the two extreme cases—the most classical ($\mathbf{A}_n$) and most exceptional ($\mathbf{E}_8$).

As usual, $\Delta^+(\mathbf{A}_n) = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n+1\}$, and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Here

$$\min(I(\alpha_i)_{\max}) = \{\alpha_i\} \quad \text{and} \quad \mathcal{H} = \{\varepsilon_i - \varepsilon_j \mid i = 1 \text{ or } j = n+1\}.$$

Therefore

$$\max(\Delta^+ \setminus I(\alpha_i)_{\max}) = \{\varepsilon_1 - \varepsilon_i, \varepsilon_{i+1} - \varepsilon_{n+1}\}, \qquad \min(I(\alpha_i)_{\min}) = \{\varepsilon_i - \varepsilon_{n+1}, \varepsilon_1 - \varepsilon_{i+1}\}.$$

The respective roots in the previous row sum to $\theta = \varepsilon_1 - \varepsilon_{n+1}$.

For $\mathbf{E}_8$, we use the natural numbering of Π, i.e., $\begin{pmatrix} \text{1-2-3-4-5-6-7} \\ \text{8} \end{pmatrix}$. The root $\gamma = \sum_{i=1}^{8} n_i\alpha_i$ is denoted by $n_1 n_2 \dots n_8$. Here $\theta = 23456423$ and $\gamma \in \mathcal{H}$ if and only if $n_1 \neq 0$. The respective maximal and minimal elements are gathered in Table 1.

Table 1. Data for the root system of type $\mathbf{E}_8$

i	$\min(I(\alpha_i)_{\max})$	$\min(I(\alpha_i)_{\min})$	$\max(\Delta^+ \setminus I(\alpha_i)_{\max})$
1	12222101	12222101	11234322
2	12222111 01234322	12222111 11234322	11234312 12222101
3	12222211 01234312	12222211 11234312	11234212 12222111
4	12223211 01234212	12223211 11234212	11233212 12222211
5	12223212 12233211 01233212	12223212 12233211 11233212	11233211 11223212 12223211
6	12333211 01223212	12333211 11223212	11123212 12233211
7	00123212	11123212	12333211
8	01233211	11233211	12223212

Theorem 4.7 is not true for arbitrary long roots in place of $\alpha \in \Pi_l$. However, it can be extended as follows.

Theorem 4.9. *Let $\mathcal{S}$ be any connected subset of Π_l. Then there is a one-to-one correspondence between* $\min(\bigcap_{\alpha_i \in \mathcal{S}} I(\alpha_i)_{\min})$ *and* $\max(\bigcap_{\alpha_i \in \mathcal{S}} (\Delta^+ \setminus I(\alpha_i)_{\max}))$. *Namely, for every $\nu \in \min(\bigcap_{\alpha_i \in \mathcal{S}} I(\alpha_i)_{\min})$, there is $\nu' \in \max(\bigcap_{\alpha_i \in \mathcal{S}} (\Delta^+ \setminus I(\alpha_i)_{\max}))$ such that $\nu + \nu' = \theta$.*

Our proof is based on direct calculations, which are omitted. It would be interesting to find a conceptual argument.

Example 4.10. For $\#\mathcal{S} = 1$, we have Theorem 4.7. At the other extreme, if $\mathcal{S} = \Pi_l$, then $\bigcap_{\alpha_i \in \Pi_l} (\Delta^+ \setminus I(\alpha_i)_{\max}) = \Delta^+ \setminus \bigcup_{\alpha_i \in \Pi_l} I(\alpha_i)_{\max} = \Delta^+ \setminus \Delta^+_{\mathsf{com}}$. Therefore,

$$\max\Big(\bigcap_{\alpha_i \in \Pi_l} (\Delta^+ \setminus I(\alpha_i)_{\max})\Big) = \{[\theta/2]\}.$$

Also, $\bigcap_{\alpha_i \in \Pi_l} I(\alpha_i)_{\min} = I(|\Pi_l|)_{\min}$ and the unique minimal element of this ideal is $\theta - [\theta/2]$ (see Example 2.6). Thus, an a priori proof of Theorem 4.9 would provide an explanation of properties of abelian ideals with rootlet $|\Pi_l|$ (cf. Example 2.6 and Remark 2.7).

Appendix A. A property of root systems

Let Δ be a reduced irreducible root system, with a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$.

Definition. Let $\eta, \beta \in \Delta^+$. The root κ is the *least upper bound* (or *join*) of η and β if

- $\kappa \succcurlyeq \eta,\ \kappa \succcurlyeq \beta$;
- if $\kappa' \succcurlyeq \eta,\ \kappa' \succcurlyeq \beta$, then $\kappa' \succcurlyeq \kappa$.

The join of η and β is denoted by $\eta \vee \beta$.

Our goal is to prove that $\eta \vee \beta$ exists for all pairs (η, β), i.e., $(\Delta^+, \preccurlyeq)$ is a join-semilattice (see [13, 3.3] for lattices). We actually prove a more precise assertion. For any pair $\eta, \beta \in \Delta^+$, we define an element $\eta \vee \beta \in Q$ and then prove that it is always a root. The very construction of $\eta \vee \beta$ will make it clear that this root satisfies the conditions of the definition above. We also prove that $\eta \vee \beta \in \Delta_l$ whenever $\eta, \beta \in \Delta_l$, so that this general setup is compatible with that of Section 2. The argument goes as follows. If $\eta = \sum_{i=1}^n a_i\alpha_i$, then $\mathsf{ht}(\eta) = \sum_i a_i$ and the *support* of η is $\mathsf{supp}(\eta) = \{\alpha_i \mid a_i \ne 0\}$. We regard $\mathsf{supp}(\eta)$ as a subset of the Dynkin diagram $\mathcal{D}(\Delta)$. As is well known, $\mathsf{supp}(\eta)$ is a connected subset of $\mathcal{D}(\Delta)$ for all $\eta \in \Delta$ [1, Ch. VI, §1, n. 6]. If $\beta = \sum_{i=1}^n b_i\alpha_i$, then $\max\{\eta, \beta\} := \sum_{i=1}^n \max\{a_i, b_i\}\alpha_i$. In general, it is merely an element of Q.

Say that $\mathsf{supp}(\eta)$ and $\mathsf{supp}(\beta)$ are *disjoint* if $\mathsf{supp}(\eta) \cup \mathsf{supp}(\beta)$ is disconnected. Then there is a unique chain in $\mathcal{D}(\Delta)$ connecting both supports, since $\mathcal{D}(\Delta)$ is a tree. If this chain consists of simple roots $\{\alpha_{i_1}, \dots, \alpha_{i_s}\}$, then, by definition, the *connecting root* is $\alpha_{i_1} + \dots + \alpha_{i_s}$. By [1, Ch. VI, §1, n. 6, Cor. 3], it is indeed a root.

Theorem A.1. (i) *If* $\mathsf{supp}(\eta) \cup \mathsf{supp}(\beta)$ *is a connected subset of* $\mathcal{D}(\Delta)$, *then* $\eta \vee \beta = \max\{\eta, \beta\}$.

(ii) *If* $\mathsf{supp}(\eta)$ *and* $\mathsf{supp}(\beta)$ *are disjoint, then* $\eta \vee \beta = \eta + \beta + ($*connecting root*$)$.

Proof. 1) Obviously, if $\kappa \succcurlyeq \eta,\ \kappa \succcurlyeq \beta$, then $\kappa \succcurlyeq \max\{\eta, \beta\}$. Hence it suffices to prove that here $\max\{\eta, \beta\}$ is a root.

If $\mathsf{supp}(\eta) \cap \mathsf{supp}(\beta) = \emptyset$, then $\max\{\eta, \beta\} = \eta + \beta$. Since $\mathsf{supp}(\eta) \cup \mathsf{supp}(\beta)$ is connected, we have $(\eta, \beta) < 0$. Hence $\eta + \beta$ is a root, and we are done.

Assume that $\mathsf{supp}(\eta) \cap \mathsf{supp}(\beta) \ne \emptyset$. Without loss of generality, we may also assume that $\mathsf{ht}(\eta) \ge \mathsf{ht}(\beta)$. Then we will argue by induction on $\mathsf{ht}(\beta)$.

If $\mathsf{ht}(\beta) = 1$, then $\beta \in \mathsf{supp}(\eta)$ and $\max\{\eta, \beta\} = \eta$.

Suppose that $\mathsf{ht}(\beta) > 1$ and the assertion is true for all pairs of positive roots such that one of them has height strictly less than $\mathsf{ht}(\beta)$.

Assume that there are different simple roots α', α'' such that $\beta - \alpha', \beta - \alpha'' \in \Delta^+$. Then $\max\{\beta - \alpha', \beta - \alpha''\} = \beta$, and by the induction assumption

$$\max\{\eta, \beta\} = \max\big\{\max\{\eta, \beta - \alpha'\}, \beta - \alpha''\big\} \in \Delta^+.$$

It remains to handle the case in which there is a unique $\alpha \in \Pi$ such that $\beta - \alpha \in \Delta^+$. Let $\mathsf{ht}_\alpha(\beta)$ denote the coefficient of α in the expression of β via simple roots. Set $\Delta_\alpha(i) = \{\nu \in \Delta^+ \mid \mathsf{ht}_\alpha(\nu) = i\}$. By a result of Kostant (see Joseph's exposition in [6, 2.1]), each $\Delta_\alpha(i)$ is the set of weights of a *simple* $\mathfrak{l}$-module, where $\mathfrak{l}$ is the Levi subalgebra of $\mathfrak{g}$ whose set of simple roots is $\Pi \setminus \{\alpha\}$. Therefore, $\Delta_\alpha(i)$ has unique minimal and unique maximal elements. Clearly, β is the minimal element in $\Delta_\alpha(j)$, where $\mathsf{ht}_\alpha(\beta) = j$. This also implies that if $\mathsf{ht}_\alpha(\nu) \ge j$, then $\nu \succcurlyeq \beta$. Therefore, if $\mathsf{ht}_\alpha(\eta) \ge j$, then $\max\{\eta, \beta\} = \eta$. Hence we may assume that $\mathsf{ht}_\alpha(\eta) \le j - 1$. Since $\mathsf{supp}(\eta) \cup \mathsf{supp}(\beta)$ is connected and

$\mathsf{supp}(\eta) \cap \mathsf{supp}(\beta) \neq \emptyset$, the union $\mathsf{supp}(\eta) \cup \mathsf{supp}(\beta - \alpha)$ is still connected. Therefore

$$\max\{\eta, \beta - \alpha\} \in \Delta^+ \quad \text{and} \quad \mathsf{ht}_\alpha(\max\{\eta, \beta - \alpha\}) = j - 1.$$

Hence $\max\{\eta, \beta - \alpha\} + \alpha = \max\{\eta, \beta\}$ and our task is to prove that, under these circumstances, $\max\{\eta, \beta - \alpha\} + \alpha$ is a root.

Since $\mathsf{ht}_\alpha(\max\{\eta, \beta - \alpha\}) = \mathsf{ht}_\alpha(\beta - \alpha)$, we have $(\max\{\eta, \beta - \alpha\}, \alpha) \le (\beta - \alpha, \alpha)$. If $\|\alpha\| \ge \|\beta\|$, then $\beta - \alpha = s_\alpha(\beta)$ and $(\beta - \alpha, \alpha) < 0$. This implies that $\max\{\eta, \beta - \alpha\} + \alpha \in \Delta^+$ (and completes the proof of part (i) if all the roots have the same length!)

Suppose that $\|\alpha\| < \|\beta\|$. We exclude the obvious case when Δ is of type $\mathbf{G}_2$ and assume that $\|\beta\|/\|\alpha\| = \sqrt{2}$. Then $s_\alpha(\beta) = \beta - 2\alpha$, $\beta - \alpha$ is short, and $(\beta - \alpha, \alpha) = 0$.

Now, if $(\max\{\eta, \beta - \alpha\}, \alpha) < (\beta - \alpha, \alpha)$, we again conclude that $\max\{\eta, \beta - \alpha\} + \alpha \in \Delta^+$. The other possibility is that $(\max\{\eta, \beta - \alpha\}, \alpha) = (\beta - \alpha, \alpha) = 0$. Because $\mathsf{ht}_\alpha(\max\{\eta, \beta - \alpha\}) = \mathsf{ht}_\alpha(\beta - \alpha)$, this means that $\max\{\eta, \beta - \alpha\}$ and $\beta - \alpha$ also have the same coefficients on the simple roots adjacent to α. That is, $\max\{\eta, \beta - \alpha\}$ is obtained from $\beta - \alpha$ by adding a sequence of simple roots that are *orthogonal* to α. Therefore, arguing by induction on $\mathsf{ht}(\max\{\eta, \beta - \alpha\}) - \mathsf{ht}(\beta - \alpha)$, we are left with the following problem:

Suppose that α, α' are orthogonal simple roots such that $\nu, \nu + \alpha, \nu + \alpha' \in \Delta^+$, both ν and α are short, and $(\nu, \alpha) = 0$. Prove that $\nu + \alpha + \alpha' \in \Delta^+$.

Now, if $(\alpha', \nu) < 0$, then $(\alpha', \nu - \alpha) < 0$ as well. Hence $\nu - \alpha + \alpha' \in \Delta$ and $s_\alpha(\nu - \alpha + \alpha') = \nu + \alpha + \alpha' \in \Delta^+$, as required. The remaining conceivable possibility is that ν, α, α' are pairwise orthogonal and short. A quick case-by-case argument shows that this is actually impossible.

2) In this case, at least one support, say $\mathsf{supp}(\beta)$, is a chain with all roots of the same length. Therefore, β equals the sum of all simple roots in its support. Hence $\tilde\beta = \beta + (\text{connecting root})$ is a root. Then $\mathsf{supp}(\eta) \cap \mathsf{supp}(\tilde\beta) = \emptyset$ and $\mathsf{supp}(\eta) \cup \mathsf{supp}(\tilde\beta)$ is connected. Hence $(\eta, \tilde\beta) < 0$ and $\eta + \tilde\beta \in \Delta^+$.

Obviously, $\eta + \tilde\beta$ is the minimal root that majorises both η and β. □

An equivalent formulation of Theorem A.1 is:

The intersection of two principal upper ideals in Δ^+ is again a principal ideal. That is, $I\langle \succcurlyeq \eta\rangle \cap I\langle \succcurlyeq \beta\rangle = I\langle \succcurlyeq (\eta \vee \beta)\rangle$.

Corollary A.2. *If $\eta, \beta \in \Delta_l^+$, then $\eta \vee \beta$ is also long.*

Proof. Let r be the squared ratio of lengths of long and short roots. (Hence $r \in \{1, 2, 3\}$.) Then $\eta = \sum_{i=1}^n a_i\alpha_i$ is long if and only if r divides n_i whenever α_i is short (see [1, Ch. VI, § 1, Ex. 20]). Obviously, the formulae of Theorem A.1 preserve this property. □

Recall that $\Delta_\alpha(i) = \{\nu \in \Delta^+ \mid \mathsf{ht}_\alpha(\nu) = i\}$ if $\alpha \in \Pi$. We regard it as a subposet of Δ^+.

Corollary A.3. *For any $\alpha \in \Pi$ and $i \in \mathbb{N}$, the poset $\Delta_\alpha(i)$ is a lattice.*

Proof. The formulae of Theorem A.1 imply that if $\eta, \beta \in \Delta_\alpha(i)$, then $\eta \vee \beta \in \Delta_\alpha(i)$. Therefore $\Delta_\alpha(i)$ is a finite join-semilattice having a unique minimal element. (The latter is a part of Kostant's result referred to above.) Hence $\Delta_\alpha(i)$ is a lattice by [13, Prop. 3.3.1]. □

Acknowledgements. This work was done during my visits to CRC 701 at the Universität Bielefeld and Max-Planck-Institut für Mathematik (Bonn). I thank both Institutions for the hospitality and support. I am grateful to E.B. Vinberg for fruitful discussions related to results in Appendix A. Thanks are also due to the anonymous referee for providing the outline of a uniform proof of Theorem 0.4.

References

[1] Bourbaki, N.: Groupes et algèbres de Lie, Chapitres 4, 5 et 6. Hermann, Paris (1968) Zbl 1120.17002 MR 0240238

[2] Cellini, P., Papi, P.: ad-nilpotent ideals of a Borel subalgebra. J. Algebra **225**, 130–141 (2000) Zbl 0951.17003 MR 1743654

[3] Cellini, P., Papi, P.: ad-nilpotent ideals of a Borel subalgebra II. J. Algebra **258**, 112–121 (2002) Zbl 1033.17008 MR 1958899

[4] Cellini, P., Papi, P.: Abelian ideals of Borel subalgebras and affine Weyl groups. Adv. Math. **187**, 320–361 (2004) Zbl 1112.17011 MR 2078340

[5] Humphreys, J. E.: Reflection Groups and Coxeter Groups. Cambridge Univ. Press (1992) Zbl 0768.20016

[6] Joseph, A.: Orbital varieties of the minimal orbit. Ann. Sci. École Norm. Sup. (4) **31**, 17–45 (1998) Zbl 0908.17002 MR 1604290

[7] Kostant, B.: The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations. Int. Math. Res. Notices **1998**, 225–252 Zbl 0896.17002 MR 1616913

[8] Panyushev, D.: Abelian ideals of a Borel subalgebra and long positive roots. Int. Math. Res. Notices **2003**, 1889–1913 Zbl 1067.17004 MR 1995141

[9] Panyushev, D.: Short antichains in root systems, semi-Catalan arrangements, and B-stable subspaces. Eur. J. Combin. **25**, 93–112 (2004) Zbl 1171.17301 MR 2031805

[10] Panyushev, D.: The poset of positive roots and its relatives. J. Algebraic Combin. **23**, 79–101 (2006) Zbl 1091.17008 MR 2218851

[11] Panyushev, D.: Abelian ideals of a Borel subalgebra and subsets of the Dynkin diagram. J. Algebra **344**, 197–204 (2011) Zbl 1259.17007 MR 2831936

[12] Panyushev, D., Röhrle, G.: Spherical orbits and abelian ideals. Adv. Math. **159**, 229–246 (2001) Zbl 0993.22016 MR 1825058

[13] Stanley, R. P.: Enumerative Combinatorics, Vol. 1. Cambridge Stud. Adv. Math. 49, Cambridge Univ. Press (1997) Zbl 0889.05001 MR 1442260

[14] Suter, R.: Coxeter and dual Coxeter numbers. Comm. Algebra **26**, 147–153 (1998) Zbl 0894.17004 MR 1600666

[15] Suter, R.: Abelian ideals in a Borel subalgebra of a complex simple Lie algebra. Invent. Math. **156**, 175–221 (2004) Zbl 1112.17012 MR 2047661