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Cheeger inequalities for unbounded graph Laplacians

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Abstract. We use the concept of intrinsic metrics to give a new definition for an isoperimetric constant of a graph. We use this novel isoperimetric constant to prove a Cheeger-type estimate for the bottom of the spectrum which is nontrivial even if the vertex degrees are unbounded.

Keywords. Isoperimetric inequality, intrinsic metric, Schrödinger operators, weighted graphs, curvature, volume growth

1. Introduction

In 1984 Dodziuk [9] proved a lower bound on the spectrum of the Laplacian on infinite graphs in terms of an isoperimetric constant. Dodziuk's bound is an analogue of Cheeger's inequality for manifolds [6] except for the fact that Dodziuk's estimate also contains an upper bound for the vertex degrees in the denominator. In a later paper [12] Dodziuk and Kendall wrote that it would be desirable to have an estimate without the rather unnatural vertex degree bound. They overcame this problem in [12] by considering the normalized Laplace operator, which is always a bounded operator, instead. However, the original problem of finding a lower bound on the spectrum of unbounded graph Laplace operators that only depends on an isoperimetric constant has remained open until today.

In this paper, we solve this problem by using the concept of intrinsic metrics. More precisely, for a given weighted Laplacian, we use an intrinsic metric to redefine the boundary measure of a set. This leads to a modified definition of the isoperimetric constant for which we obtain a lower bound on the spectrum that depends solely on the constant. These

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estimates hold for all weighted Laplacians (including bounded and unbounded Laplace operators). The strategy of proof is not surprising, as it does not differ much from the one of [9, 12]. However, the main contribution of this note is to provide the right definition of an isoperimetric constant to solve the open problem mentioned above.

To this day, there is a vital interest in estimates of isoperimetric constants and in Cheeger-type inequalities. For example, rather classical estimates for isoperimetric constants in terms of the vertex degree can be found in [2, 11, 43, 44]; for relations to random walks, see [21, 48]. While, for regular planar tessellations, isoperimetric constants can be computed explicitly [26, 28]; for arbitrary planar tessellations there are estimates in terms of curvature [27, 35, 40, 47]. For Cheeger inequalities on simplicial complexes, there is recent work found in [45]; for general weighted graphs, see [10, 37]. Moreover, Cheeger estimates for the bottom of the essential spectrum and criteria for discreteness of spectrum are given in [19, 34, 49, 50]. Upper bounds for the top of the (essential) spectrum and another criterion for the concentration of the essential spectrum in terms of the dual Cheeger constant are given in [3]. Finally, let us mention works connecting discrete and continuous Cheeger estimates [1, 8, 33, 41].

The paper is structured as follows. The set up is introduced in the next section. The Cheeger inequalities are presented and proven in Section 3. Moreover, upper bounds are discussed. A technique to incorporate nonnegative potentials into the estimate is discussed in Section 4. Section 5 is dedicated to relating the exponential volume growth of a graph to the isoperimetric constant via upper bounds while lower bounds on the isoperimetric constant in terms of curvature are presented in Section 6. These lower bounds allow us to give examples where our estimate yields better results than all estimates known before.

2. The set up

2.1. Graphs

Let X be a countably infinite set equipped with the discrete topology. A function $m : X \rightarrow (0, \infty)$ gives a Radon measure on X of full support via $m(A) = \sum_{x \in A} m(x)$ for $A \subseteq X$, so that (X, m) becomes a discrete measure space.

A *graph* over (X, m) is a symmetric function $b : X \times X \rightarrow [0, \infty)$ with zero diagonal that satisfies

$$\sum_{y \in X} b(x, y) < \infty \quad \text{for } x \in X.$$

We can think of x and y as *neighbors*, i.e., being connected by an edge, if $b(x, y) > 0$; we then write $x \sim y$. In this case, $b(x, y)$ is the strength of the bond interaction between x and y . For convenience we assume that there are no isolated vertices, i.e., every vertex has a neighbor. We call b *locally finite* if each vertex has only finitely many neighbors.

The measure $n : X \rightarrow (0, \infty)$ given by

$$n(x) = \sum_{y \in X} b(x, y) \quad \text{for } x \in X.$$

plays a distinguished role in the proof of classical Cheeger inequalities. In the case when $b : X \times X \rightarrow \{0, 1\}$, $n(x)$ gives the number of neighbors of a vertex x .

2.2. Intrinsic metrics

We call a pseudometric d for a graph b on (X, m) an *intrinsic metric* if

$$\sum_{y \in X} b(x, y) d(x, y)^2 \leq m(x) \quad \text{for all } x \in X.$$

The concept of intrinsic metrics was first studied systematically by Sturm [46] for strongly local regular Dirichlet forms and it was generalized to all regular Dirichlet forms by Frank–Lenz–Wingert [17]. By [17, Lemma 4.7, Theorem 7.3] it can be seen that our definition coincides with the one of [17]. A possible choice for d is the path pseudometric induced by the edge weights $w(x, y) = ((m/n)(x) \wedge (m/n)(y))^{1/2}$ for $x \sim y$ (see e.g. [30]). Moreover, the combinatorial graph metric (i.e., the path metric with weights $w(x, y) = 1$ for $x \sim y$) is intrinsic if $m \geq n$. Intrinsic metrics for graphs were recently discovered independently in various contexts—see e.g. [14, 15, 16, 23, 25, 30, 31, 32, 42], where certain variations of the concept also go under the name of adapted metrics.

2.3. Isoperimetric constant

In this section we use the concept of intrinsic metrics to give a refined definition of the isoperimetric constant. It turns out that this novel isoperimetric constant is more suitable than the classical one if $n \geq m$. Let $W \subseteq X$. We define the *boundary* ∂W of W by

$$\partial W = \{(x, y) \in W \times X \setminus W \mid b(x, y) > 0\}.$$

For a given intrinsic metric d we set the measure of the boundary as

$$|\partial W| = \sum_{(x, y) \in \partial W} b(x, y) d(x, y).$$

Note that $|\partial W| < \infty$ for finite $W \subseteq X$ by the Cauchy–Schwarz inequality and the assumption that $\sum_y b(x, y) < \infty$. We define the *isoperimetric constant* or *Cheeger constant* $\alpha(U) = \alpha_{d, m}(U)$ for $U \subseteq X$ as

$$\alpha(U) = \inf_{W \subseteq U \text{ finite}} \frac{|\partial W|}{m(W)}.$$

If $U = X$, we write

$$\alpha = \alpha(X).$$

For $b : X \times X \rightarrow \{0, 1\}$ and d the combinatorial graph metric the measure of the boundary $|\partial W|$ is the number of edges leaving W . If additionally $m = n$, then our definition of α coincides with the classical one from [12].

2.4. Graph Laplacians

Denote by $C_c(X)$ the space of real valued functions on X with compact support. Let $\ell^2(X, m)$ be the space of square summable real valued functions on X with respect to the measure m which comes equipped with the scalar product $\langle u, v \rangle = \sum_{x \in X} u(x)v(x)m(x)$ and the norm $\|u\| = \|u\|_m = \langle u, u \rangle^{1/2}$. Let the form $Q = Q_b$ with domain D be given by

$$Q(u) = \frac{1}{2} \sum_{x, y \in X} b(x, y)(u(x) - u(y))^2, \quad D = \overline{C_c(X)}^{\|\cdot\|_Q},$$

where $\|\cdot\|_Q = (Q(\cdot) + \|\cdot\|^2)^{1/2}$. The form Q defines a regular Dirichlet form on $\ell^2(X, m)$ (see [20, 36]). The corresponding positive selfadjoint operator L can be seen to act as

$$Lf(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y))$$

(cf. [36, Theorem 9]). Let \tilde{L} be the extension of L to $\tilde{F} = \{f : X \rightarrow \mathbb{R} \mid \sum_{y \in X} b(x, y)|f(y)| < \infty \text{ for all } x \in X\}$. We have $C_c(X) \subseteq D(L)$ if (and only if) $\tilde{L}C_c(X) \subseteq \ell^2(X, m)$ (see [36, Theorem 6]). In particular, this is easily seen to be the case if the graph is locally finite or if $\inf_{x \in X} m(x) > 0$. Note that L becomes a bounded operator if and only if $Cm \geq n$ for some $C > 0$ (cf. [24, Theorem 9.3]). In particular, if $m = n$, then L is referred to as the *normalized Laplacian*.

We denote the bottom of the spectrum $\sigma(L)$ and of the essential spectrum $\sigma_{\text{ess}}(L)$ of L by

$$\lambda_0(L) = \inf \sigma(L) \quad \text{and} \quad \lambda_0^{\text{ess}}(L) = \inf \sigma_{\text{ess}}(L).$$

3. Cheeger inequalities

Let b be a graph over (X, m) and d be an intrinsic metric. In this section we prove the main results of the paper.

3.1. Main results

Theorem 3.1. $\lambda_0(L) \geq \alpha^2/2$.

Remark. (a) A similar statement can be proven for weighted Laplacians on finite graphs for the first nonzero eigenvalue. In this case, the infimum in the definition of the isoperimetric constant has to be taken over all sets that have at most half of the measure of the whole graph. The main part of the proof works similarly; for details of the adaptation to the finite graph case, see [7, proof of Theorem 2.2].

(b) We also obtain a similar statement for the selfadjoint operator that is related to the maximal form (cf. Section 5) which is discussed under the name of Neumann Laplacian in [24]. Here one has to take the infimum in the definition of the isoperimetric constant over all sets of finite measure. With this choice, all of our proofs work analogously.

Under the additional assumption that L is bounded with operator norm 1, we recover the classical Cheeger inequality from [19, 43] which can be seen to be stronger than the one of [12] by the Taylor expansion. We say that $d \geq 1$ (respectively, $d \leq 1$) for neighbors if $d(x, y) \geq 1$ (respectively, $d(x, y) \leq 1$) for all $x \sim y$.

Theorem 3.2. *If $m \geq n$ and $d \geq 1$ or $d \leq 1$ for neighbors, then*

$$\lambda_0(L) \geq 1 - \sqrt{1 - \alpha^2}.$$

In order to estimate the essential spectrum let the *isoperimetric constant at infinity* be given by

$$\alpha_\infty = \sup_{K \subseteq X \text{ finite}} \alpha(X \setminus K),$$

which coincides with the one of [37] in the case of the combinatorial graph metric and with the one of [19, 34] if additionally $b : X \times X \rightarrow \{0, 1\}$ and $m = n$. Note that the assumptions of the following theorem are in particular fulfilled if the graph is locally finite or if $\inf_{x \in X} m(x) > 0$.

Theorem 3.3. *Assume $C_c(X) \subseteq D(L)$. Then $\lambda_0^{\text{ess}}(L) \geq \alpha_\infty^2/2$.*

3.2. Co-area formulae

Among the key ingredients for the proof are the following well-known area and co-area formulae. For example, they are already found in [37] (see also [22]). We include a short proof for the sake of convenience. Let $\ell^1(X, m) = \{f : X \rightarrow \mathbb{R} \mid \sum_{x \in X} |f(x)|m(x) < \infty\}$.

Lemma 3.4. *Let $f \in \ell^1(X, m)$, $f \geq 0$ and $\Omega_t := \{x \in X \mid f(x) > t\}$. Then*

$$\frac{1}{2} \sum_{x, y \in X} b(x, y)d(x, y)|f(x) - f(y)| = \int_0^\infty |\partial\Omega_t| dt,$$

where the value ∞ on both sides of the equation is allowed, and

$$\sum_{x \in X} f(x)m(x) = \int_0^\infty m(\Omega_t) dt.$$

Proof. For $x, y \in X$, $x \sim y$ with $f(x) \neq f(y)$, let $I_{x,y}$ be the interval $[f(x) \wedge f(y), f(x) \vee f(y))$ and let $|I_{x,y}| = |f(x) - f(y)|$ be its length. Then $(x, y) \in \partial\Omega_t$ or $(y, x) \in \partial\Omega_t$ if and only if $t \in I_{x,y}$. Hence, by Fubini's theorem,

$$\begin{aligned} \int_0^\infty |\partial\Omega_t| dt &= \frac{1}{2} \int_0^\infty \sum_{x, y \in X} b(x, y)d(x, y)1_{I_{x,y}}(t) dt \\ &= \frac{1}{2} \sum_{x, y \in X} b(x, y)d(x, y) \int_0^\infty 1_{I_{x,y}}(t) dt \\ &= \frac{1}{2} \sum_{x, y \in X} b(x, y)d(x, y)|f(x) - f(y)|. \end{aligned}$$

Note that $x \in \Omega_t$ if and only if $1_{(t,\infty)}(f(x)) = 1$. Again, by Fubini's theorem,

$$\begin{aligned} \int_0^\infty m(\Omega_t) dt &= \int_0^\infty \sum_{x \in X} m(x) 1_{(t,\infty)}(f(x)) dt \\ &= \sum_{x \in X} m(x) \int_0^\infty 1_{(t,\infty)}(f(x)) dt = \sum_{x \in X} m(x) f(x). \quad \square \end{aligned}$$

3.3. Form estimates

Lemma 3.5. For $U \subseteq X$ and $u \in D$ with support in U and $\|u\|_m = 1$,

$$Q(u) \geq \alpha(U)^2/2.$$

Moreover, if $m \geq n$ and $d \geq 1$ or $d \leq 1$ for neighbors, then

$$Q(u)^2 - 2Q(u) + \alpha(U)^2 \leq 0.$$

Proof. Let $u \in C_c(X)$. We calculate, using the co-area formulae above with $f = u^2$,

$$\begin{aligned} \alpha \|u\|_m^2 &= \alpha \int_0^\infty m(\Omega_t) dt \leq \int_0^\infty |\partial \Omega_t| dt = \frac{1}{2} \sum_{x,y \in X} b(x,y) d(x,y) |u(x)^2 - u(y)^2| \\ &\leq Q(u)^{1/2} \left(\frac{1}{2} \sum_{x,y \in X} b(x,y) d(x,y)^2 (u(x) + u(y))^2 \right)^{1/2} \\ &\leq Q(u)^{1/2} \left(2 \sum_{x \in X} u(x)^2 \sum_{y \in X} b(x,y) d(x,y)^2 \right)^{1/2} \leq 2^{1/2} Q(u)^{1/2} \|u\|_m, \end{aligned}$$

where the final estimate follows from the intrinsic metric property. The second statement follows if we use in the above estimates

$$\begin{aligned} \frac{1}{2} \sum_{x,y \in X} b(x,y) d(x,y)^2 (u(x) + u(y))^2 \\ &= 2 \sum_{x,y \in X} b(x,y) d(x,y)^2 u(x)^2 - \frac{1}{2} \sum_{x,y \in X} b(x,y) d(x,y)^2 (u(x) - u(y))^2 \\ &\leq 2 \|u\|_m^2 - Q(u), \end{aligned}$$

where we distinguish the cases $d \geq 1$ and $d \leq 1$: For the first case we use that d is intrinsic and that $-d(x,y)^2 \leq -1$. For the second case, we estimate $d(x,y) \leq 1$ in the first line and then use $n \leq m$. The statement follows by the density of $C_c(X)$ in D . \square

3.4. Proofs of the theorems

Proof of Theorems 3.1 and 3.2. By Lemma 3.5, the statements follow from the variational characterization of λ_0 via the Rayleigh–Ritz quotient: $\lambda_0 = \inf_{u \in D, \|u\|=1} Q(u)$. \square

Proof of Theorem 3.3. Let $Q_U, U \subseteq X$, be the restriction of Q to $\overline{C_c(U)}^{\|\cdot\|_Q}$ and L_U be the corresponding operator. Note that $Q_U = Q$ on $C_c(U)$. The assumption $C_c(X) \subseteq D(L)$ clearly implies $\overline{LC_c(X)} \subseteq \ell^2(X, m)$, which is equivalent to the fact that the functions $y \mapsto b(x, y)/m(y)$ are in $\ell^2(X, m)$ for all $x \in X$ (see [36, Proposition 3.3]). This implies that for any finite set $K \subseteq X$ the operator $L_{X \setminus K}$ is a compact perturbation of L . Thus, from Lemma 3.5 we conclude that

$$\lambda_0^{\text{ess}}(L) = \lambda_0^{\text{ess}}(L_{X \setminus K}) \geq \lambda_0(L_{X \setminus K}) = \inf_{u \in C_c(X \setminus K), \|u\|=1} Q(u) \geq \alpha(X \setminus K)^2/2.$$

This implies the statement of Theorem 3.3. \square

3.5. Upper bounds for the bottom of spectrum

In this section we show an upper bound for $\lambda_0(L)$ by α as in [9, 11, 12] for uniformly discrete metric spaces.

Theorem 3.6. *Let d be an intrinsic metric such that (X, d) is uniformly discrete with lower bound $\delta > 0$. Then $\lambda_0(L) \leq \alpha/\delta$.*

Proof. By assumption we have $d \geq \delta > 0$ away from the diagonal. It follows that $|\partial W| \geq \delta \sum_{(x,y) \in \partial W} b(x, y) = \delta Q(1_W)$ for all $W \subseteq X$ finite. By the inequality $\delta \lambda_0(L) \leq \delta Q(1_W)/\|1_W\|^2 \leq |\partial W|/m(W)$, we deduce the statement. \square

The example below shows that, in general, there is no upper bound by α only.

Example 3.7. Let $b_0 : X \times X \rightarrow \{0, 1\}$ be a k -regular rooted tree for $k > 1$ with root $x_0 \in X$ (that is, each vertex has k forward neighbors). Furthermore, let $b_1 : X \times X \rightarrow \{0, 1\}$ be such that $b_1(x, y) = 1$ if and only if x and y have the same distance to x_0 with respect to the combinatorial graph distance in b_0 . Now, let $b = b_0 + b_1$, $m \equiv 1$ and let d be given by the path metric with weights $w(x, y) = (n(x) \vee n(y))^{-1/2}$ for $x \sim y$. Then $\alpha = \alpha_{d,m} = 0$, which can be seen from $|\partial B_r|/m(B_r) \leq k^{-(r-1)/2} \rightarrow 0$ as $r \rightarrow \infty$, where B_r is the set of vertices that have distance $\leq r$ to x_0 with respect to the combinatorial graph metric.

On the other hand, by [39, Theorem 2] the heat kernel $p_t(x_0, \cdot)$ of the graph b equals the corresponding heat kernel on the k -regular tree b_0 . Hence, by a Li type theorem (see [24, Theorem 8.1] or [38, Corollary 5.6]), we get $\frac{1}{t} \log p_t(x_0, y) \rightarrow -\lambda_0(L) = -(k+1 - 2\sqrt{k})$ for any $y \in X$ as $t \rightarrow \infty$ (see also [39, Corollary 6.7]). As $\alpha = 0$, this shows that α can yield no upper bound without further assumptions.

Nevertheless, the example does not exclude the possibility that there is a different intrinsic metric which might yield a reasonable upper bound. Hence, one might ask whether there exist examples for which every intrinsic metric fails to give an upper bound or, otherwise, if one can always find an intrinsic metric that yields an upper bound.

4. Potentials

In this section we briefly discuss how the strategy proposed in [37] to incorporate potentials into the inequalities can be applied to the new definition of the Cheeger constant. This yields a Cheeger estimate for all regular Dirichlet forms on discrete sets (cf. [36, Theorem 7]).

Let b be a graph over a discrete measure space (X, m) . Furthermore, let $c : X \rightarrow [0, \infty)$ be a potential and define

$$Q_{b,c}(u) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(u(x) - u(y))^2 + \sum_{x \in X} c(x)u(x)^2$$

on $D(Q_{b,c}) = \overline{C_c(X)}^{\|\cdot\|_{Q_{b,c}}}$ and let $L_{b,c}$ be the corresponding operator.

Let (X', b', m') be a copy of (X, b, m) . Let $\dot{X} = X \cup X'$, let $\dot{m} : \dot{X} \rightarrow (0, \infty)$ be such that $\dot{m}|_X = m$, $\dot{m}|_{X'} = m'$, and let $\dot{b} : \dot{X} \times \dot{X} \rightarrow [0, \infty)$ be given by $\dot{b}|_{X \times X} = b$, $\dot{b}|_{X' \times X'} = b'$, $\dot{b}(x, x') = c(x) = c'(x')$ for corresponding vertices $x \in X$ and $x' \in X'$, and $\dot{b} \equiv 0$ otherwise. Then the restriction $Q_{\dot{b}, X}$ of the form $Q_{\dot{b}}$ on $\ell^2(\dot{X}, \dot{m})$ to $D(Q_{\dot{b}, X}) = \overline{C_c(X)}^{\|\cdot\|_{Q_{\dot{b}}}}$ satisfies

$$D(Q_{b,c}) = D(Q_{\dot{b}, X}) \quad \text{and} \quad Q_{b,c} = Q_{\dot{b}, X}.$$

Let $d : X \times X \rightarrow [0, \infty)$ be an intrinsic metric for b over (X, m) and assume there is a function $\delta : X \rightarrow [0, \infty)$ such that

$$\sum_{y \in X} b(x, y)d(x, y)^2 + c(x)\delta(x)^2 \leq m(x) \quad \text{for all } x \in X.$$

Example 4.1. (1) For a given intrinsic metric d a possible choice for the function δ is $\delta(x) = ((m(x) - \sum_{y \in X} b(x, y)d(x, y)^2)/c(x))^{1/2}$ if $c(x) > 0$, and 0 otherwise.

(2) Choose d as the path metric induced by the edge weights $w(x, y) = ((\frac{m}{n+c})(x) \wedge (\frac{m}{n+c})(y))^{1/2}$ for $x \sim y$ and δ as in (1). If $c > 0$, then $\delta > 0$.

We next define \dot{d} . Since we are only interested in the subgraph X of \dot{X} , we do not need to extend d to all of \dot{X} but only set $\dot{d}|_{X \times X} = d$ and $\dot{d}(x, x') = \delta(x)$ for the corresponding vertex $x' \in X'$ of x . Defining $\dot{\alpha}(X) = \dot{\alpha}_{\dot{d}, m}(X)$ by

$$\dot{\alpha}(X) = \inf_{W \subseteq X \text{ finite}} \frac{|\partial W|_{\dot{d}}}{m(W)}$$

with $|\partial W|_{\dot{d}} = \sum_{(x,y) \in \partial W} \dot{b}(x, y)\dot{d}(x, y) = \sum_{(x,y) \in \partial W} b(x, y)d(x, y) + \sum_{x \in W} c(x)\delta(x)$ implies that $\dot{\alpha}(X) = \alpha_{\dot{d}, \dot{m}}(X)$, where the right hand side is the Cheeger constant of the subgraph $X \subseteq \dot{X}$ as in Section 2.3. Hence, we get

$$\lambda_0(L_{b,c}) \geq \dot{\alpha}(X)^2/2$$

by Lemma 3.5 and the arguments from the proof of Theorem 3.1.

5. Upper bounds by volume growth

In this section we relate the isoperimetric constant to the exponential volume growth of the graph. Let b be a graph over a discrete measure space (X, m) and let d be an intrinsic metric. We let $B_r(x) = \{y \in X \mid d(x, y) \leq r\}$ and define the *exponential volume growth* $\mu = \mu_{d,m}$ by

$$\mu = \liminf_{r \rightarrow \infty} \inf_{x \in X} \frac{1}{r} \log \frac{m(B_r(x))}{m(B_1(x))}.$$

Other than for the classical notions of isoperimetric constants and exponential volume growth on graphs (see [5, 11, 18, 43]), it is, geometrically, not obvious that $\alpha = \alpha_{d,m}$ and $\mu = \mu_{d,m}$ can be related. However, given a Brooks-type theorem, the proof is rather immediate. Therefore, let the *maximal form domain* be given by

$$D^{\max} := \left\{ u \in \ell^2(X, m) \mid Q^{\max}(u) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(u(x) - u(y))^2 < \infty \right\}.$$

Theorem 5.1. *If $D = D^{\max}$, then $2\alpha \leq \mu$. In particular, this holds if one of the following assumptions is satisfied:*

- (a) *The graph b is locally finite and d is an intrinsic path metric such that (X, d) is metrically complete.*
- (b) *Every infinite path of vertices has infinite measure.*

Proof. Under the assumption $D = D^{\max}$ we have $\lambda_0(L) \leq \mu^2/8$ by [25, Theorem 4.2]. (Note that the 8 in the denominator as opposed to the 4 found in [25] is explained in [25, Remark 4.3].) Thus, the statement follows from Theorem 3.1. Note that, by [32, Theorem 2] and [24, Corollary 6.3], (a) implies $D = D^{\max}$, and by [36, Theorem 6], (b) implies $D = D^{\max}$. \square

6. Lower bounds by curvature

In this section we give a lower bound on the isoperimetric constant by a quantity that is sometimes interpreted as a curvature [11, 29, 39]. Let b be a graph over (X, m) and let d be an intrinsic metric.

6.1. The lower bound

We fix an orientation on a subset of edges, that is, we choose $E_+, E_- \subset X \times X$ with $E_+ \cap E_- = \emptyset$ such that $(x, y) \in E_+$ if and only if $(y, x) \in E_-$. We define the *curvature* $K : X \rightarrow \mathbb{R}$ with respect to this orientation by

$$K(x) = \frac{1}{m(x)} \left(\sum_{(x,y) \in E_-} b(x, y)d(x, y) - \sum_{(x,y) \in E_+} b(x, y)d(x, y) \right).$$

Let us give an example for a choice of E_{\pm} .

Example 6.1. Let b take values in $\{0, 1\}$, m be the vertex degree function n , and d be the combinatorial graph metric. For some fixed vertex $x_0 \in X$, let S_r be the sphere with respect to d around x_0 and set $|x| = r$ for $x \in S_r$. We choose E_{\pm} such that outward (resp., inward) oriented edges are in E_+ (resp., E_-), i.e., $(x, y) \in E_+$, $(y, x) \in E_-$ if $x \in S_{r-1}$ and $y \in S_r$ for some r and $x \sim y$. Then $K(x) = (n_-(x) - n_+(x))/n(x)$, where $n_{\pm}(x) = \#\{y \in S_{|x|\pm 1} \mid y \sim x\}$ and $\#A$ denotes the cardinality of A .

The following theorem is an analogue to [11, Lemma 1.15] and [13, Proposition 3.3] which was also used in [49, 50] to estimate the bottom of the essential spectrum.

Theorem 6.2. *If $-K \geq k \geq 0$, then $\alpha \geq k$.*

Proof. Let W be a finite set and denote by 1_W the corresponding characteristic function. Furthermore, let $\sigma(x, y) = \pm d(x, y)$ for $(x, y) \in E_{\pm}$ and zero otherwise. We calculate directly

$$\begin{aligned} km(W) &\leq - \sum_{x \in W} K(x)m(x) = \sum_{x \in X} 1_W(x) \sum_{y \in X} b(x, y)\sigma(x, y) \\ &= \frac{1}{2} \left(\sum_{x \in X} 1_W(x) \sum_{y \in X} b(x, y)\sigma(x, y) - \sum_{y \in X} 1_W(y) \sum_{x \in X} b(x, y)\sigma(x, y) \right) \\ &\leq \frac{1}{2} \sum_{x, y \in X} b(x, y)d(x, y)|1_W(x) - 1_W(y)| = |\partial W|, \end{aligned}$$

where we used $\sum_y b(x, y)d(x, y) < \infty$ and the antisymmetry of σ in the second step. This finishes the proof. \square

6.2. Example of antitrees

In the final subsection we give an example of an antitree for which Theorem 6.2 together with Theorem 3.1 yields a better estimate than the estimates known before. Recently, antitrees received some attention as they provide examples of graphs of polynomial volume growth (with respect to the combinatorial graph metric) that are stochastically incomplete and have a spectral gap (see [4, 23, 25, 31, 39, 51]).

For a given graph $b : X \times X \rightarrow \{0, 1\}$ with root $x_0 \in X$ and measure $m \equiv 1$, let S_r be the vertices that have combinatorial graph distance r to x_0 as above. We call a graph an *antitree* if every vertex in S_r is connected to all vertices in $S_{r+1} \cup S_{r-1}$ and to none in S_r .

In [25], it is shown that $\lambda_0(L) = 0$ whenever $\lim_{r \rightarrow \infty} \log \#S_r / \log r < 2$. It remains open by this result what happens in the case of an antitree with $\#S_{r-1} = r^2$. The classical Cheeger constant $\alpha_{\text{classical}} = \alpha_{1,n}$ for the normalized Laplacian with the combinatorial graph metric which is given as the infimum over $\#\partial W/n(W)$ (with $W \subseteq X$ finite) is zero. This can be easily checked by choosing distance balls $B_r = \bigcup_{j=0}^r S_j$ as test sets W . Hence, the estimate $\lambda_0(L) \geq (1 - \sqrt{1 - \alpha^2}) \inf_{x \in X} n(x)$ with $\alpha = \alpha_{1,n}$ found in [34] is trivial.

Likewise, the estimates presented in [39] and [50], which use an unweighted curvature, also give zero as a lower bound for the bottom of the spectrum in this case.

By Theorem 6.2 we obtain a positive estimate for the Cheeger constant $\alpha = \alpha_{d,1}$ with the path metric induced by the edge weights $w(x, y) = (n(x) \vee n(y))^{-1/2}$, $x \sim y$ for the antitree with $\#S_{r-1} = r^2$. We pick E_{\pm} as in Example 6.1 above and obtain a positive lower bound for $-K$. In particular, Theorem 6.2 shows that $\alpha > 0$, and thus $\lambda_0(L) > 0$ by Theorem 3.1 for the antitree satisfying $\#S_{r-1} = r^2$.

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