

# Central Characters for Smooth Irreducible Modular Representations of $\mathrm{GL}_2(\mathbf{Q}_p)$

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*To Francesco Baldassarri, on the occasion of his 60th birthday*

ABSTRACT - We prove that every smooth irreducible  $\overline{\mathbf{F}}_p$ -linear representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$  admits a central character.

## Introduction

Let  $\Pi$  be a representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$ . We say that  $\Pi$  is smooth, if the stabilizer of any  $v \in \Pi$  is an open subgroup of  $\mathrm{GL}_2(\mathbf{Q}_p)$ . We say that  $\Pi$  admits a central character, if every  $z \in Z(\mathrm{GL}_2(\mathbf{Q}_p))$ , the center of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , acts on  $\Pi$  by a scalar. The smooth irreducible representations of  $\mathrm{GL}_2(\mathbf{Q}_p)$  over an algebraically closed field of characteristic  $p$ , admitting a central character, have been studied by Barthel–Livné in [BL94, BL95] and by Breuil in [Bre03]. The purpose of this note is to prove the following theorem.

**THEOREM A.** *If  $\Pi$  is a smooth irreducible  $\overline{\mathbf{F}}_p$ -linear representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , then  $\Pi$  admits a central character.*

The idea of the proof of theorem A is as follows. If  $\Pi$  does not admit a central character, and if  $f = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ , then for any nonzero polynomial  $Q(X) \in \overline{\mathbf{F}}_p[X]$ , the map  $Q(f) : \Pi \rightarrow \Pi$  is bijective, so that  $\Pi$  has the

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structure of a  $\overline{\mathbf{F}}_p(X)$ -vector space. The representation  $\Pi$  is therefore a smooth irreducible  $\overline{\mathbf{F}}_p(X)$ -linear representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , which now admits a central character, since  $f$  acts by multiplication by  $X$ . It remains to apply Barthel–Livné and Breuil’s classification, which gives the structure of the components of  $\Pi$  after extending scalars to a finite extension  $K$  of  $\overline{\mathbf{F}}_p(X)$ . A corollary of this classification is that these components are all “defined” over a subring  $R$  of  $K$ , where  $R$  is a finitely generated  $\overline{\mathbf{F}}_p$ -algebra. This can be used to show that  $\Pi$  is not of finite length, a contradiction.

Note that it is customary to ask that smooth irreducible representations of  $\mathrm{GL}_2(\mathbf{Q}_p)$  also be admissible (meaning that  $\Pi^U$  is finite-dimensional for every open compact subgroup  $U$  of  $G$ ). A corollary of Barthel–Livné and Breuil’s classification is that every smooth irreducible  $\overline{\mathbf{F}}_p$ -linear representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$  that admits a central character is admissible, and hence theorem A implies that every smooth irreducible  $\overline{\mathbf{F}}_p$ -linear representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$  is admissible. In particular, such a representation also satisfies Schur’s lemma: every  $\mathrm{GL}_2(\mathbf{Q}_p)$ -equivariant map is a scalar. Our theorem A can also be seen as a special case of Schur’s lemma, since  $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$  is a  $\mathrm{GL}_2(\mathbf{Q}_p)$ -equivariant map.

There are (at least) two standard ways of proving Schur’s lemma: one way uses admissibility, and the other works for smooth irreducible  $E$ -linear representations of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , but only if  $E$  is uncountable (see proposition 2.11 of [BZ76]). In order to prove theorem A, we cannot simply extend scalars to an uncountable extension of  $\overline{\mathbf{F}}_p$ , as we do not know whether the resulting representation will still be irreducible.

We finish this introduction by pointing out that a few years ago, Henriart had sketched a different (and more complicated) argument for the proof of theorem A.

## 1. Barthel–Livné and Breuil’s classification

Let  $E$  be a field of characteristic  $p$ . In this section, we recall the explicit classification of smooth irreducible  $E$ -linear representations of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , admitting a central character.

We denote the center of  $\mathrm{GL}_2(\mathbf{Q}_p)$  by  $Z$ . If  $r \geq 0$ , then  $\mathrm{Sym}^r E^2$  is a representation of  $\mathrm{GL}_2(\mathbf{F}_p)$  which gives rise, by inflation, to a representation of  $\mathrm{GL}_2(\mathbf{Z}_p)$ . We extend it to  $\mathrm{GL}_2(\mathbf{Z}_p)Z$  by letting  $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$  act trivially.

Consider the representation

$$\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2.$$

The Hecke algebra

$$\mathrm{End}_{E[\mathrm{GL}_2(\mathbf{Q}_p)]} \left( \mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2 \right)$$

is isomorphic to  $E[T]$  where  $T$  is a Hecke operator, which corresponds to the double class  $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Z} \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{GL}_2(\mathbf{Z}_p)$ . If  $\chi : \mathbf{Q}_p^\times \rightarrow E^\times$  is a smooth character, and if  $\lambda \in E$ , then let

$$\pi(r, \lambda, \chi) = \frac{\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2}{T - \lambda} \otimes (\chi \circ \det).$$

This is a smooth representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , with central character  $\omega^r \chi^2$  (where  $\omega : \mathbf{Q}_p^\times \rightarrow \mathbf{F}_p^\times$  is given by  $p^n x_0 \mapsto \bar{x}_0$ , with  $x_0 \in \mathbf{Z}_p^\times$ ). Let  $\mu_\lambda : \mathbf{Q}_p^\times \rightarrow E^\times$  be given by  $\mu_\lambda|_{\mathbf{Z}_p^\times} = 1$ , and  $\mu_\lambda(p) = \lambda$ . If  $\lambda = \pm 1$ , then we have two exact sequences:

$$0 \rightarrow \mathrm{Sp}_E \otimes (\chi \mu_\lambda \circ \det) \rightarrow \pi(0, \lambda, \chi) \rightarrow \chi \mu_\lambda \circ \det \rightarrow 0,$$

$$0 \rightarrow \chi \mu_\lambda \circ \det \rightarrow \pi(p-1, \lambda, \chi) \rightarrow \mathrm{Sp}_E \otimes (\chi \mu_\lambda \circ \det) \rightarrow 0,$$

where the representation  $\mathrm{Sp}_E$  is the “special” representation with coefficients in  $E$ .

**THEOREM 1.1.** *If  $E$  is algebraically closed, then the smooth irreducible  $E$ -linear representations of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , admitting a central character, are as follows:*

- (1)  $\chi \circ \det$ ;
- (2)  $\mathrm{Sp}_E \otimes (\chi \circ \det)$ ;
- (3)  $\pi(r, \lambda, \chi)$ , where  $r \in \{0, \dots, p-1\}$  and  $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$ .

This theorem is proved in [BL95] and [BL94], which treat the case  $\lambda \neq 0$ , and in [Bre03], which treats the case  $\lambda = 0$ .

We now explain what happens if  $E$  is not algebraically closed.

**PROPOSITION 1.2.** *If  $\Pi$  is a smooth irreducible  $E$ -linear representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , admitting a central character, then there exists a finite*

extension  $K/E$  such that  $(\Pi \otimes_E K)^{\text{ss}}$  is a direct sum of  $K$ -linear representations of the type described in theorem 1.1.

PROOF. Barthel and Livné's methods show (as is observed in § 5.3 of [Paš10]) that  $\Pi$  is a quotient of

$$\Sigma = \frac{\text{ind}_{\text{GL}_2(\mathbf{Z}_p)\mathbf{Z}}^{\text{GL}_2(\mathbf{Q}_p)} \text{Sym}^r E^2}{P(T)} \otimes (\chi \circ \det),$$

for some integer  $r \in \{0, \dots, p-1\}$ , character  $\chi: \mathbf{Q}_p^\times \rightarrow E^\times$ , and polynomial  $P(Y) \in E[Y]$ . Let  $K$  be a splitting field of  $P(Y)$ , write  $P(Y) = (Y - \lambda_1) \cdots (Y - \lambda_d)$ , and let  $P_i(Y) = (Y - \lambda_1) \cdots (Y - \lambda_i)$  for  $i = 0, \dots, d$ . The representations  $P_{i-1}(T)\Sigma/P_i(T)\Sigma$  are then subquotients of the  $\pi(r, \lambda_i, \chi)$ , for  $i = 1, \dots, d$ .  $\square$

We finish this section by recalling that if  $\lambda \neq 0$ , then the representations  $\pi(r, \lambda, \chi)$  are parabolic inductions (when  $\lambda = 0$ , they are called supersingular). Let  $B_2(\mathbf{Q}_p)$  be the upper triangular Borel subgroup of  $\text{GL}_2(\mathbf{Q}_p)$ , let  $\chi_1$  and  $\chi_2: \mathbf{Q}_p^\times \rightarrow E^\times$  be two smooth characters, and consider the parabolic induction  $\text{ind}_{B_2(\mathbf{Q}_p)}^{\text{GL}_2(\mathbf{Q}_p)}(\chi_1 \otimes \chi_2)$ . The following result is proved in [BL94] and [BL95].

**THEOREM 1.3.** *If  $\lambda \in E \setminus \{0; \pm 1\}$ , and if  $r \in \{0, \dots, p-1\}$ , then  $\pi(r, \lambda, \chi)$  is isomorphic to  $\text{ind}_{B_2(\mathbf{Q}_p)}^{\text{GL}_2(\mathbf{Q}_p)}(\chi\mu_{1/\lambda} \otimes \chi\omega^r\mu_\lambda)$ .*

## 2. Proof of the theorem

We now give the proof of theorem A. Let  $\Pi$  be a smooth irreducible  $\overline{\mathbf{F}}_p$ -linear representation of  $\text{GL}_2(\mathbf{Q}_p)$ . We have  $\Pi^{(1+p\mathbf{Z}_p) \cdot \text{Id}} \neq 0$  (since a  $p$ -group acting on a  $\mathbf{F}_p$ -vector space always has nontrivial fixed points), so that if  $\Pi$  is irreducible, then  $(1+p\mathbf{Z}_p) \cdot \text{Id}$  acts trivially on  $\Pi$ . If  $g \in \mathbf{Z}_p^\times \cdot \text{Id}$ , then  $g^{p-1} = \text{Id}$  on  $\Pi$ , so that  $\Pi = \bigoplus_{\omega \in \mathbf{F}_p^\times} \Pi^{g=\omega \cdot \text{Id}}$ . Since  $\Pi$  is irreducible, this implies that the elements of  $\mathbf{Z}_p^\times \cdot \text{Id}$  act by scalars.

If  $f = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ , then for any nonzero polynomial  $Q(X) \in \overline{\mathbf{F}}_p[X]$ , the kernel and image of the map  $Q(f): \Pi \rightarrow \Pi$  are subrepresentations of  $\Pi$ . If  $Q(f) = 0$  on a nontrivial subspace of  $\Pi$ , then  $f$  admits an eigenvector for an eigenvalue  $\lambda \in \overline{\mathbf{F}}_p^\times$ . This implies that  $\Pi = \Pi^{f=\lambda \cdot \text{Id}}$ , so that  $\Pi$  does admit a central character. If this is not

the case, then  $Q(f)$  is bijective for every nonzero polynomial  $Q(X) \in \overline{\mathbf{F}}_p[X]$ , so that  $\Pi$  has the structure of a  $\overline{\mathbf{F}}_p(X)$ -vector space, and is a  $\overline{\mathbf{F}}_p(X)$ -linear smooth irreducible representation of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , admitting a central character.

Let  $E = \overline{\mathbf{F}}_p(X)$ . Proposition 1.2 gives us a finite extension  $K$  of  $E$ , such that  $(\Pi \otimes_E K)^{\mathrm{ss}}$  is a direct sum of  $K$ -linear representations of the type described in theorem 1.1. The  $\overline{\mathbf{F}}_p$ -linear representation underlying  $(\Pi \otimes_E K)^{\mathrm{ss}}$  is isomorphic to  $\Pi^{[K:E]}$ , and hence of length  $[K:E]$ . We now prove that none of the  $K$ -linear representations of the type described in theorem 1.1 are of finite length, when viewed as  $\overline{\mathbf{F}}_p$ -linear representations.

Let  $\Sigma$  be one such representation, and let  $\lambda \in K$  be the corresponding Hecke eigenvalue. We now construct a subring  $R$  of  $K$ , which is a finitely generated  $\overline{\mathbf{F}}_p$ -algebra, and an  $R$ -linear representation  $\Sigma_R$  of  $\mathrm{GL}_2(\mathbf{Q}_p)$ , such that  $\Sigma = \Sigma_R \otimes_R K$ .

If  $\lambda \in \overline{\mathbf{F}}_p$ , then theorem 1.1 shows that

$$\Sigma = \frac{\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2}{T - \lambda} \otimes_{\overline{\mathbf{F}}_p} K(\chi \circ \det), \text{ or } \mathrm{Sp}_{\overline{\mathbf{F}}_p} \otimes_{\overline{\mathbf{F}}_p} K(\chi \circ \det), \text{ or } K(\chi \circ \det).$$

We can then take  $R = \overline{\mathbf{F}}_p[\chi(p)^{\pm 1}]$ , and  $\Sigma_R = (\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2 / (T - \lambda)) \otimes_{\overline{\mathbf{F}}_p} R(\chi \circ \det)$ , or  $\mathrm{Sp}_{\overline{\mathbf{F}}_p} \otimes_{\overline{\mathbf{F}}_p} R(\chi \circ \det)$ , or  $R(\chi \circ \det)$ , respectively.

If  $\lambda \notin \overline{\mathbf{F}}_p$ , then by theorem 1.3, we have

$$\Sigma = \mathrm{ind}_{\mathrm{B}_2(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} (\chi \mu_{1/\lambda}, \chi \omega^r \mu_\lambda).$$

We can take  $R = \overline{\mathbf{F}}_p[\lambda^{\pm 1}, \chi(p)^{\pm 1}]$ , and let  $\Sigma_R$  be the set of functions  $f \in \Sigma$  with values in  $R$ .

In the first case,  $\Sigma_R$  is a free  $R$ -module, while in the second case,  $\Sigma_R$  is isomorphic as an  $R$ -module to  $C^\infty(\mathbf{P}^1(\mathbf{Q}_p), R)$  and hence also free. In either case, if  $f \in R$  is nonzero and not a unit and  $j \in \mathbf{Z}$ , then  $f^{j+1} \cdot \Sigma_R$  is a proper  $\overline{\mathbf{F}}_p$ -linear subrepresentation of  $f^j \cdot \Sigma_R$ , so that the underlying  $\overline{\mathbf{F}}_p$ -linear representation of  $\Sigma_R$  is not of finite length. Since  $\Sigma_R \subset \Sigma$ , the underlying  $\overline{\mathbf{F}}_p$ -linear representation of  $\Sigma$  is not of finite length, which is a contradiction. This finishes the proof of theorem A.

*Acknowledgments.* I am grateful to C. Breuil, G. Chenevier, P. Colmez, G. Henniart, F. Herzig, M. Schein, M.-F. Vignéras and the referee for helpful comments.

## REFERENCES

- [BL94] L. BARTHEL - R. LIVNÉ, *Irreducible modular representations of  $GL_2$  of a local field*, Duke Math. J., **75**, no. 2 (1994), pp. 261–292.
- [BL95] L. BARTHEL - R. LIVNÉ, *Modular representations of  $GL_2$  of a local field: the ordinary, unramified case*, J. Number Theory, **55**, no. 1 (1995), pp. 1–27.
- [BZ76] I. BERNŠTEĪN - A. ZELEVINSKIĪ, *Representations of the group  $GL(n, F)$ , where  $F$  is a local non-Archimedean field*, Russian Math. Surveys, **31**, no. 3 (1976), pp. 1–68.
- [Bre03] C. BREUIL, *Sur quelques représentations modulaires et  $p$ -adiques de  $GL_2(\mathbb{Q}_p)$ : I*, Compositio Math., **138**, no. 2, (2003), pp. 165–188.
- [Paš10] V. PAŠKŪNAS, *The image of Colmez's Montréal functor*, preprint, 2010.

Manoscritto pervenuto in redazione il 23 agosto 2011.