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Generalized Euler-Frobenius Polynomials

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Abstract. An initial value problem for the two-dimensional difference equation $a_{n+1,\nu+1} = a_{n+1,\nu} + (1-z)a_{n\nu}$ is solved by means of the generating function and their functional equation. Special values of the solution are the well known Euler-Frobenius polynomials.

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Let k be an arbitrary but fixed non-negative integer. We start from the two-dimensional difference equation

$$a_{n+1,\nu+1} = a_{n+1,\nu} + (1-z)a_{n\nu} \tag{1}$$

for $0 \le k \le n$ and $0 \le \nu \le n$ with $n, k, \nu \in \boldsymbol{Z}$, a real parameter z, and the initial conditions

$$a_{k0} = a_{k1} = \ldots = a_{kk} = 1 \tag{2}$$

as well as

$$a_{n+1,0} = z \sum_{\nu=0}^{n} a_{n\nu} \tag{3}$$

for $n \ge k$. Obviously, the solutions of this initial value problem are uniquely determined polynomials $a_{n\nu} = a_{n\nu}(z)$ of order $n-k$, in particular

$$a_{k+1,\nu}(z) = \nu + (k+1-\nu)z \tag{4}$$

for $\nu = 0, \ldots, k+1$. The general solution is given by

Theorem 1: *For $0 \le k \le n$ and $0 \le \nu \le n$, the difference equation* (1) *with the initial conditions* (2) *and* (3) *has the solution*

$$a_{n\nu}(z) = (1-z)^{n-k+1} \sum_{m=0}^{\infty} z^m \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \binom{n-\nu}{k-\mu} (m+1)^{\nu-\mu} m^{n+\mu-k-\nu}. \tag{5}$$

Note that the binomial coefficients $\binom{n-\nu}{k-\mu}$ vanish for $k < \mu$. For the proof of this theorem we need some preliminaries.

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Lemma 1: *For $n \geq k+1$, we have*

$$a_{n0}(z) = z a_{nn}(z). \tag{6}$$

Proof: By summation, (1) implies

$$a_{n+1,n+1} - a_{n+1,0} = (1-z) \sum_{\nu=0}^{n} a_{n\nu}$$

for $n \geq k$, and by (3) we obtain

$$z a_{n+1,n+1} - z a_{n+1,0} = (1-z) a_{n+1,0} ,$$

i.e. (6) ∎

Next, we introduce the *generating function* of $a_{n\nu}$ by the formal power series

$$F(x,y,z) = \sum_{n=k}^{\infty} \sum_{\nu=0}^{n} a_{n\nu}(z)\, y^{\nu} x^{n}. \tag{7}$$

From (1), we obtain

$$\sum_{n=k+1}^{\infty} \sum_{\nu=1}^{n} a_{n\nu}(z)\, y^{\nu-1} x^{n-1} = \sum_{n=k+1}^{\infty} \sum_{\nu=0}^{n-1} a_{n\nu}(z)\, y^{\nu} x^{n-1} + (1-z) F(x,y,z).$$

In view of (2), the left-hand side is equal to

$$\frac{1}{xy} \left(F(x,y,z) - F(x,0,z) - \sum_{\nu=1}^{k} y^{\nu} x^{k} \right),$$

and the sum on the right-hand side equals to

$$\frac{1}{x} \left(F(x,y,z) - \sum_{n=k+1}^{\infty} a_{nn}(z)\, y^{n} x^{n} - \sum_{\nu=0}^{k} y^{\nu} x^{k} \right),$$

where (6) implies

$$\sum_{n=k+1}^{\infty} a_{nn}(z)\, y^{n} x^{n} = \frac{1}{z} \left(F(xy,0,z) - y^{k} x^{k} \right).$$

Using the abbreviations $F = F(x,y,z)$ and $F(x) = F(x,0,z)$, we obtain the equation

$$F - F(x) - \sum_{\nu=1}^{k} y^{\nu} x^{k} = y \left(F - \frac{1}{z}(F(xy) - y^{k} x^{k}) - \sum_{\nu=0}^{k} y^{\nu} x^{k} \right) + xy(1-z)F,$$

and finally, the functional equation

$$\big(1 - y - xy(1-z)\big)F = F(x) - \frac{y}{z}F(xy) + y^{k+1}x^k\left(\frac{1}{z} - 1\right). \tag{8}$$

Theorem 2: *Equation* (8) *has the solution*

$$\begin{aligned} F(x,y,z) &= (1-z)x^k \sum_{m=0}^{\infty} z^m \\ &\quad \times \sum_{\mu=0}^{k} \big(1-(m+1)xy(1-z)\big)^{-\mu-1}\big(1-mx(1-z)\big)^{\mu-k-1} y^{\mu} \end{aligned} \tag{9}$$

for $|z| < 1$ *and* $x(1-z) \neq \frac{1}{n}$, $xy(1-z) \neq \frac{1}{n}$ $(n \in \mathbb{N})$.

Proof: In order to solve (8), it is necessary to determine $F(x)$. The easiest way would be to put $y = 0$ in (8), however, then we get an identity. Hence, we introduce a new variable u with $u(1-z) \neq \frac{1}{n}$ $(n \in \mathbb{N})$, and choose

$$x = \frac{u}{1-u(1-z)} \qquad \text{and} \qquad y = 1 - u(1-z).$$

Then $1 - y - xy(1-z) = 0$, and (8) turns into

$$F(u) = (1-z)u^k + \frac{z}{1-u(1-z)}F\left(\frac{u}{1-u(1-z)}\right). \tag{10}$$

By iteration, we find the series

$$F(u) = (1-z)u^k \sum_{m=0}^{\infty} \frac{z^m}{\big(1-mu(1-z)\big)^{k+1}}, \tag{11}$$

which converges for $|z| < 1$. Since

$$F\left(\frac{u}{1-u(1-z)}\right) = (1-z)u^k \sum_{m=0}^{\infty} \frac{z^m\big(1-u(1-z)\big)}{\big(1-(m+1)u(1-z)\big)^{k+1}},$$

we see that (11) is indeed a solution of (10).

Now, the right-hand side of (8) can be written as

$$\begin{aligned} &(1-z)x^k\left(\sum_{m=0}^{\infty} \frac{z^m}{\big(1-mx(1-z)\big)^{k+1}} - \sum_{m=1}^{\infty} \frac{z^{m-1}y^{k+1}}{\big(1-mxy(1-z)\big)^{k+1}}\right) \\ &= (1-z)x^k \sum_{m=0}^{\infty} \frac{\big(1-(m+1)xy(1-z)\big)^{k+1} - \big(y-mxy(1-z)\big)^{k+1}}{\big((1-mx(1-z)\big)\big(1-(m+1)xy(1-z)\big)^{k+1}} z^m, \end{aligned}$$

and after division by $1 - y - xy(1-z)$, we easily find (9) ∎

Corollary: *For $k = 0$, (9) simplifies to*

$$F(x,y,z) = (1-z)\sum_{m=0}^{\infty} \frac{z^m}{\big(1-mx(1-z)\big)\big(1-(m+1)xy(1-z)\big)}. \tag{12}$$

Proof of Theorem 1: By means of binomial series and

$$(-1)^i \binom{-\mu-1}{i} = \binom{\mu+i}{i} \quad \text{and} \quad (-1)^j \binom{\mu-k-1}{j} = \binom{k+j-\mu}{j},$$

we obtain from (9) the formal power series

$$F(x,y,z) = \sum_{m=0}^{\infty} z^m \sum_{\mu=0}^{k} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{\mu+i}{i} \times (m+1)^i \binom{k+j-\mu}{j} m^j (1-z)^{i+j+1} y^{\mu+i} x^{i+j+k}.$$

Choosing $n = i+j+k$ and $\nu = \mu + i$, we find by comparison with (7) that equation (5) holds ∎

In particular, for $\nu = 0$ we have

$$a_{n0}(z) = (1-z)^{n-k+1} \binom{n}{k} \sum_{m=0}^{\infty} m^{n-k} z^m. \tag{13}$$

For $n > k$, these functions can be expressed by the Euler-Frobenius polynomials

$$E_n(z) = (1-z)^{n+1} \sum_{m=1}^{\infty} m^n z^{m-1} = \sum_{m=0}^{n-1} \sum_{\nu=0}^{m} \binom{n+1}{\nu} (-1)^\nu (m+1-\nu)^n z^m$$

of degree $n-1$ for $n \geq 1$ (cf. Chui [2] and Schoenberg [3]), namely

$$a_{n0}(z) = \binom{n}{k} z E_{n-k}(z).$$

The polynomial character of (13) implies that the functions (5) are also polynomials, which we call *generalized* Euler-Frobenius polynomials.

By means of the notation $D = z\frac{d}{dz}$, the polynomials E_n have the representation

$$E_n(z) = \frac{1}{z}(1-z)^{n+1} D^n (1-z)^{-1},$$

which can be generalized in the following way.

Lemma 2: The polynomials (5) have the representation .

$$a_{n\nu}(z) = (1-z)^{n-k+1}\sum_{j=0}^{n}\binom{n-\nu}{j}\binom{\nu}{n-k-j}D^j(1+D)^{n-k-j}(1-z)^{-1}. \quad (14)$$

Proof: If we use the equations

$$(1+D)^{n-k-j} = \sum_{i=0}^{n-k-j}\binom{n-k-j}{i}D^i \qquad \text{and} \qquad D^{i+j}(1-z)^{-1} = \sum_{m=0}^{\infty} m^{i+j}z^m$$

and the substitution $\mu = k+\nu+j-n$, we obtain

$$\sum_{i=0}^{n-k-j}\binom{n-k-j}{i}m^i = (m+1)^{\nu-\mu},$$

and (14) turns into (5) ∎

Application: In [1] , there appear (in different notations) the $((\nu+1)\times(\nu+1))$-matrices

$$Y_\nu(z) = \begin{pmatrix} 1 & \dots & \dots & 1 \\ z & 1 & \dots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z & \dots & z & 1 \end{pmatrix}, \quad (15)$$

the direct sums $Y_\nu^n(z) = I_{n-\nu}\oplus Y_\nu(z)$ with

$$I_\mu \oplus Y_\nu = \begin{pmatrix} I_\mu & O^\top \\ O & Y_\nu \end{pmatrix},$$

where I_μ is the μ-dimensional unit matrix with dummy I_0, and the products

$$P_n(z) = Y_1^n(z)\cdots Y_n^n(z) \quad (16)$$

with $n \geq 1$. For clearness, we denote the functions (5) more precisely by $a_{n\nu}^k(z)$. Then for the $((n+1)\times(n+1))$-dimensional matrices $P_n(z)$, we obtain

Lemma 3: *The entries of $P_n(z)$ are the polynomials $a_{nj}^{n-i}(z)$ $(i,j = 0,\dots,n)$, where i is the row index and j the column index.*

Proof: The matrices (16) satisfy the recursion

$$P_{n+1}(z) = \big(I_1 \oplus P_n(z)\big)Y_{n+1}(z).$$

Denoting the entries of $P_n(z)$ by a_{nj}^{n-i}, this equation implies $a_{n+1,j}^{n+1} = 1$ for $j = 0,\dots,n+1$, i.e. (2), and

$$a_{n+1,j}^{n+1-i} = \sum_{\ell=0}^{j-1} a_{n\ell}^{n-(i-1)} + z\sum_{\ell=j}^{n} a_{n\ell}^{n-(i-1)}$$

for $i = 1, \ldots, n+1$. For $j = 0$, the last equation equals to (3). For $j = 0, \ldots, n$, we subtract it from

$$a_{n+1,j+1}^{n+1-i} = \sum_{\ell=0}^{j} a_{n\ell}^{n+1-i} + z \sum_{\ell=j+1}^{n} a_{n\ell}^{n+1-i}$$

and obtain

$$a_{n+1,j+1}^{n+1-i} = a_{n+1,j}^{n+1-i} + (1-z) a_{nj}^{n+1-i},$$

i.e. (1). In view of

$$P_1(z) = Y_1(z) = \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix}$$

or already $P_0(z) = (1)$, the lemma is proved by induction ∎

Further examples are

$$P_2(z) = \begin{pmatrix} 1 & 1 & 1 \\ 2z & z+1 & 2 \\ z^2+z & 2z & z+1 \end{pmatrix},$$

$$P_3(z) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3z & 2z+1 & z+2 & 3 \\ 3z^2+3z & z^2+5z & 5z+1 & 3z+3 \\ z^3+4z^2+z & 4z^2+2z & 2z^2+4z & z^2+4z+1 \end{pmatrix}.$$

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