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Approximate Controllability of Systems Determined by Almost Sectorial Operators

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Abstract. In this work we study the approximate controllability of first order linear and semi-linear abstract control systems described by an almost sectorial operator.

Keywords. Distributed control systems, approximate controllability, semigroup of operators, almost sectorial operators

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1. Introduction.

In recent years there has been a growing interest in studying the abstract Cauchy problem (in short, ACP) described by an almost sectorial operator. Specifically, let X be a complex Banach space. Several authors have studied the existence and regularity of solutions of the Cauchy problem

$$x'(t) = -Ax(t) + f(t), \quad t \geq 0 \tag{1}$$
$$x(0) = x_0, \tag{2}$$

in the Banach space X. In this description $f : [0, \infty) \to X$ is a locally integrable function, and $A : D(A) \subseteq X \to X$ is a closed linear operator whose spectrum $\sigma(A)$ is included in the sector $S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \omega\} \cup \{0\}$ for some $0 < \omega < \frac{\pi}{2}$ and the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ satisfies the estimate

$$\|R(\lambda, A)\| \leq C_\eta |\lambda|^\gamma$$

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for all $\lambda \in \mathbb{C} \setminus S_\eta$, where η is such that $\omega < \eta < \pi$ and $\gamma \in (-1, 0)$. These operators have been called almost sectorial ([1, 4, 11, 12]).

It was shown in [12] that $-A$ is the generator of an analytic semigroup $(T(t))_{t>0}$ of growth order $\gamma + 1$, i.e.,

$$\|T(t)\| \leq Mt^{-(\gamma+1)}, \quad t > 0, \tag{3}$$

and that for $f \in L^1((0, a], X)$ the mild solution of (1), (2) is given by the variation of constants formula

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds.$$

Moreover, this function need not be continuous at $t = 0$.

The main objective of this paper is to extend some properties of controllability of linear distributed control systems modeled by analytic semigroups in the classic sense (C_0-semigroups) to systems modeled by analytic semigroups generated by almost sectorial operators. Exact and approximate controllability of linear distributed control systems modeled by C_0-semigroups has been studied by many authors. Related with our objectives we only mention here [2, 3, 7, 8].

In what follows X and U will denote Banach spaces endowed with a norm $\| \cdot \|$. We consider a linear operator $A : D(A) \subseteq X \to X$ that satisfies the conditions specified previously. Throughout this work we assume that X is an infinite dimensional space and that $D(A) \neq X$. In Section 2 we will be concerned with linear systems of first order modeled with states $x(t) \in X$ and controls $u(t) \in U$. More specifically, we consider first order systems governed by the equation

$$x'(t) = -Ax(t) + Bu(t), \quad t \geq 0, \tag{4}$$

with initial condition (2), where $B : U \to X$ is a bounded linear operator which represents the control action, and the control function $u(\cdot)$ is at least locally integrable. In Section 3, we establish some criteria of approximate controllability for nonlinear systems. Finally, in Section 4, we present an application.

The terminology and notations are those generally used in functional analysis. In particular, for Banach spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the space formed by the bounded linear maps from X into Y endowed with the norm of operators. We abbreviate this notation to $\mathcal{L}(X)$ when $X = Y$. We will say that a set $D \subseteq X$ is total in X if $\mathrm{Span}(D)$ is dense in X. We denote the dual space of a Banach space Z by $Z^* = \mathcal{L}(Z, \mathbb{K})$. In addition, if C is a linear operator, we denote by $\mathcal{R}(C)$ the range space of C, and if C is a linear operator defined in a dense subspace $D(C)$, then C^* will represent the adjoint operator of C. Finally, if $x^* \in X^*$ and M is a subspace of X, we will write $x^* \perp M$ if $\langle x^*, x \rangle = 0$, for all $x \in M$.

2. Approximate controllability of linear control systems

Throughout this section, we will assume that the operator A is almost sectorial. This implies that $-A$ generates an analytic semigroup of bounded linear operators $(T(t))_{t>0}$ on X . For the properties of this type of semigroups the reader can consult for example [4]. In particular, there are constants $M > 0$ and $\gamma \in (-1, 0)$ such that the estimate (3) holds. Moreover, the unique mild solution of (4), with initial condition $x(0) = x_0$, is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds.$$

In particular, consider the initial condition $x_0 = 0$ leads us to define, for each $a > 0$ and $1 \leq p \leq \infty$, the map $G_a : L^p([0, a], U) \to X$ by

$$G_a(u) = \int_0^a T(a-s)Bu(s)ds.$$

In what follows we consider a fixed $-\frac{1}{\gamma} < p \leq \infty$. It is clear that G_a is a bounded linear operator. We will use the following terminology.

Definition 2.1. System (4) is called exactly controllable on $[0, a]$ if the space $\mathcal{R}(G_a) = X$. The system (4) will be called exactly controllable in finite time if the space $\cup_{a>0}\mathcal{R}(G_a) = X$.

Since ([12, p. 56]) the space $\mathcal{R}(T(t)) \subseteq D(A)$ for $t > 0$, proceeding as usual ([2]) we can establish the following property of negative character.

Proposition 2.2. *System* (4) *is not exactly controllable in finite time.*

The lack of exact controllability motivates us to introduce a weaker notion of controllability.

Definition 2.3. System (4) is called approximately controllable on $[0, a]$ if the space $\mathcal{R}(G_a)$ is dense in X. System (4) will be called approximately controllable in finite time if the space $\cup_{a>0}\mathcal{R}(G_a)$ is dense in X.

We next will show that the approximate controllability is independent of the interval $[0, a]$. The following property was originally established by Fattorini [5].

Proposition 2.4. *Under the previous conditions the following statements are satisfied:*

(a) *System* (4) *is approximately controllable on* $[0, a]$ *if and only if for every* $x^* \in X^*$ *such that* $B^*T(t)^*x^* = 0$, *for all* $0 < t \leq a$, *we have that* $x^* = 0$.

(b) *System* (4) *is approximately controllable in finite time if and only if for every* $x^* \in X^*$ *such that* $B^*T(t)^*x^* = 0$, *for all* $t > 0$, *we have that* $x^* = 0$.

Corollary 2.5. *System* (4) *is approximately controllable on* $[0, a]$ *for some* $a > 0$ *if and only if it is approximately controllable in finite time.*

We next will show that the well known Kalman's controllability criterion for lumped systems can be extended to distributed systems of type (4) (see also [8]). Henceforth we use the notations

$$\begin{aligned} D_\infty(A) &= \cap_{n=1}^{\infty} D(A^n), \\ U_1 &= \{u \in U : Bu \in D(A)\}, \\ U_\infty &= \{u \in U : Bu \in D_\infty(A)\}. \end{aligned}$$

Theorem 2.6. *Under the previous conditions the following statements are satisfied:*

(a) *If* $\mathrm{Span}\{A^n BU_\infty : n \in \mathbb{N}_0\}$ *is dense in* X, *then the system* (4) *is approximately controllable in finite time.*

(b) *If the system* (4) *is approximately controllable in finite time, then the space* $\mathrm{Span}\{A^n T(t)BU : n \in \mathbb{N}_0\}$ *is dense in* X, *for all* $t > 0$.

Proof. (a) Let $x^* \in X^*$ be such that $B^*T(t)^*x^* = 0$ for all $t > 0$. We choose $u \in U_\infty$ and define the function $f(t) = \langle x^*, T(t)Bu\rangle$ for $t > 0$. Since $Bu \in D_\infty(A)$, it follows from [4, Observation 1] that

$$f^{(n)}(t) = \langle x^*, A^n T(t)Bu\rangle = \langle x^*, T(t)A^n Bu\rangle = 0.$$

Turning to apply [4, Observation 1], the function $T(t)A^n Bu \to A^n Bu$ as $t \to 0^+$. This implies that $\langle x^*, A^n Bu\rangle = 0$ and $x^* \perp \mathrm{Span}\{A^n BU_\infty : n \in \mathbb{N}_0\}$. It follows from the hypothesis that $x^* = 0$. Our assertion is now a direct consequence of Proposition 2.4.

(b) Let now $s > 0$ be fixed and $x^* \perp \mathrm{Span}\{A^n T(s)BU : n \in \mathbb{N}_0\}$. Since $T(s)(X) \subseteq D(A)$ and $T(s) = T(\frac{s}{n})^n$, using [4, Observation 1] again we get that $T(s)(X) \subseteq D(A^n)$ for all $n \in \mathbb{N}_0$. Proceeding as in the proof of (a) for the function $f(t) = \langle x^*, T(t)Bu\rangle$ for $t > 0$ and $u \in U$, we obtain that $f^{(n)}(s) = \langle x^*, A^n T(s)Bu\rangle = 0$. Since f is analytic on $(0, \infty)$, it follows that $f = 0$. As $u \in U$ was chosen arbitrarily, we get that $B^*T(t)^*x^* = 0$, for all $t > 0$. Using both the fact that the system (4) is approximately controllable in finite time as the Proposition 2.4, we can affirm that $x^* = 0$, which implies that the space $\mathrm{Span}\{A^n T(s)BU : n \in \mathbb{N}_0\}$ is dense in X. □

In [2, Theorem 3.15] has been shown that the study of approximate controllability in finite time of a linear distributed systems modeled by a C_0-semigroup can be reduced to the study of approximate controllability in finite time of a linear distributed systems where the system operator is bounded. Next we will show that, at least when U_1 is dense in U, this result remains valid in the context of almost sectorial operators. Specifically, let $-\lambda_0 \in \mathbb{C} \setminus S_\omega$. Hence, $\lambda_0 \in \rho(-A)$, and we may consider the control system

$$x'(t) = R(\lambda_0, -A)x(t) + Bu(t), \quad t \geq 0. \tag{5}$$

Theorem 2.7. *Assume that U_1 is dense in U. Then the system* (4) *is approximately controllable in finite time if, and only if, it is the system* (5).

Proof. Assume initially that the system (4) is approximately controllable in finite time. Proceeding as in [2, Theorem 3.15], if $B^* e^{R(\lambda_0,-A)^* t} x^* = 0$, for all $t > 0$, using that $R(\cdot, -A)$ is a holomorphic function on $\rho(-A)$, we obtain that $B^* R(\lambda, -A)^* x^* = 0$, for all $\lambda \in \mathbb{C} \setminus (-S_\omega)$. Applying now the expression for $T(t)$ given by (2.1) in [4], it follows that $\langle x^*, T(t)Bu \rangle = 0$ for all $t > 0$. Consequently, from the Proposition 2.4 follows that $x^* = 0$, and using again the Proposition 2.4 we can affirm that the system (5) is approximately controllable in finite time. Conversely, we assume that the system (5) is approximately controllable in finite time, and we choose $x^* \in X^*$ such that $B^* T(t)^* x^* = 0$ for all $t > 0$. For $\mu \in \mathbb{C}$ such that $Re(\mu) > 0$ we have that

$$R(\mu, -A) = \int_0^\infty e^{-\mu t} T(t) dt \tag{6}$$

which implies that $B^* R(\mu, -A)^* x^* = 0$. Using again that $R(\cdot, -A)$ is a holomorphic function, it follows that $B^*(R(\lambda_0, -A)^*)^n x^* = 0$ for all $n \in \mathbb{N}$. Furthermore, for $u \in U_1$ the function $T(\cdot)Bu$ is continuous on $[0, \infty)$. Since

$$\langle x^*, \mu R(\mu, -A)Bu \rangle = \langle \mu B^* R(\mu, -A)^* x^*, u \rangle = 0,$$

using (6) and the properties of the Laplace transformation, we have that

$$\lim_{\mu \to +\infty} \langle x^*, \mu R(\mu, -A)Bu \rangle = \langle x^*, Bu \rangle$$

which implies that $B^* x^* = 0$. Hence we deduce that $B^* e^{R(\lambda_0, A)^* t} x^* = 0$, for all $t \geq 0$. Therefore, using again repeatedly the Proposition 2.4, we infer that $x^* = 0$, and that the system (4) is approximately controllable in finite time. □

In the control theory for distributed systems modeled by C_0-semigroups is also frequently used the weaker notion of controllability to the zero state. Here we introduce this concept in the form given in [2, Definition 3.19].

Definition 2.8. The system (4) is called exactly null controllable on $[0, a]$ if $\mathcal{R}(T(a)) \subseteq \mathcal{R}(G_a)$. The system (4) will be called approximately null controllable on $[0, a]$ if $\mathcal{R}(T(a)) \subseteq \overline{\mathcal{R}(G_a)}$.

Our first result shows that the concept of exact null controllability is only apparently weaker than the concept of exact controllability. To establish this result we will use the following general property of continuous linear operators.

Lemma 2.9. *Let X, Y, Z be Banach spaces, and let $F \in \mathcal{L}(X, Z)$ and $G \in \mathcal{L}(Y, Z)$ be such that $\mathcal{R}(F) \subseteq \mathcal{R}(G)$. Then there exists a constant $\beta > 0$ such that for every $x \in X$, there is $y \in Y$ with $\|y\| \leq \beta\|x\|$ such that $F(x) = G(y)$. Furthermore, $\overline{\mathcal{R}(F)} \subseteq \mathcal{R}(G)$.*

Proof. For completeness, we indicate briefly the main steps of the proof. The first assertion is a consequence of the construction carried out in the proof of [2, Theorem 3.3]. To establish the second assertion we proceed as follows. Let $z \in \overline{\mathcal{R}(F)}$. We can choose a sequence $(x_n)_n$ in X such that $F(x_n) \to z$ as $n \to \infty$. Let $y_n \in Y$ such that $F(x_n) = G(y_n)$. If G is injective, then $G^{-1}F$ is a bounded linear map, which implies that $(y_n)_n$ is a Cauchy sequence, and therefore convergent to some $y \in Y$. Consequently,

$$z = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(y_n) = G(y) \in \mathcal{R}(G).$$

In the case when G is not injective, we replace G by $\widetilde{G}$, where $\widetilde{G} : Y/\ker(G) \to Z$ is the map induced by G in the quotient space. □

Proposition 2.10. *If $D(A)$ is dense in X, then the system* (4) *is not exactly null controllable on* $[0, a]$.

Proof. If we assume that $\mathcal{R}(T(a)) \subseteq \mathcal{R}(G_a)$, applying the Lemma 2.9 follows that $\overline{\mathcal{R}(T(a))} \subseteq \mathcal{R}(G_a)$. On the other hand, if $x^* \perp \mathcal{R}(T(a))$, then $\langle x^*, T(t)x\rangle = 0$ for all $t > 0$ and $x \in X$. In particular, if $x \in D(A)$, since in this case $T(t)x \to x$ as $t \to 0^+$, we obtain that $\langle x^*, x\rangle = 0$. Consequently, applying the Hahn-Banach theorem we get $\overline{D(A)} \subseteq \overline{\mathcal{R}(T(a))} \subseteq \mathcal{R}(G_a)$, and this inclusion implies that the system (4) is exactly controllable on $[0, a]$, which is absurd. □

Similar to what we have stated in Proposition 2.4 and Theorem 2.6, for the concept of approximate null controllability we obtain the following characterizations.

Proposition 2.11. *System* (4) *is approximately null controllable on* $[0, a]$ *if and only if for every $x^* \in X^*$ such that $B^*T(t)^*x^* = 0$, for all $0 < t \leq a$, we have that $T(a)^*x^* = 0$.*

The following result easily follows from the fact that the function $T(\cdot)x$ is analytic on $(0, \infty)$ for all $x \in X$.

Corollary 2.12. *If the system* (4) *is approximately null controllable on* $[0, a]$ *for some $a > 0$, then it is approximately null controllable on* $[0, b]$ *for all $b > 0$.*

Theorem 2.13. *Under the previous conditions the following statements are satisfied:*

(a) *If* $\mathcal{R}(T(a)) \subseteq \overline{\mathrm{Span}\{A^n BU_\infty : n \in \mathbb{N}_0\}}$, *then the system* (4) *is approximately null controllable on* $[0, a]$.

(b) *If the system* (4) *is approximately null controllable on* $[0, a]$, *then*

$$\mathcal{R}(T(a)) \subseteq \overline{\mathrm{Span}\{A^n T(t)BU : n \in \mathbb{N}_0\}}, \quad \textit{for all } t > 0.$$

Proof. The proof is very similar to that we carried out to establish the Theorem 2.6 so we will omit it.

Definition 2.14. The system (4) is called exactly null controllable in finite time if for each $x \in X$, there is $t > 0$ such that $T(t)x \in \mathcal{R}(G_t)$.

The following result relates these concepts.

Proposition 2.15. *If the system* (4) *is exactly null controllable in finite time, then it is approximately null controllable on* $[0, a]$ *for all* $a > 0$.

Proof. For $t > 0$, we define the space $E_t = \{x \in X : T(t)x \in \mathcal{R}(G_t)\}$. Clearly $X = \cup_{t>0} E_t$. Furthermore, if $x \in E_t$ and $h > 0$, there is $u \in L^p([0, t], U)$ such that $T(t)x = \int_0^t T(t-s)Bu(s)ds$. Consequently,

$$T(t+h)x = \int_0^t T(t+h-s)Bu(s)ds = \int_0^{t+h} T(t+h-s)B\tilde{u}(s)ds = G_{t+h}(\tilde{u}),$$

where $\tilde{u}$ is the function given by $\tilde{u}(s) = u(s)$ for $0 < s \leq t$ and $\tilde{u}(s) = 0$ for $t < s \leq t+h$. This shows that the family of subspaces $(E_t)_{t>0}$ is nondecreasing. Therefore, there is $a > 0$ such that $\overline{E_a} = X$. Let $x^* \in X^*$ be such that $B^*T(t)^*x^* = 0$, for all $0 < t \leq a$. This implies that $x^* \perp \mathcal{R}(G_a)$. Therefore, for all $x \in E_a$ we have that $\langle T(a)x, x^* \rangle = 0$. That is, $T(a)^*x^* \perp E_a$. Hence we get that $T(a)^*x^* \perp \overline{E_a} = X$, which implies that $T(a)^*x^* = 0$. We complete the proof combining the assertions in Proposition 2.11 and Corollary 2.12. □

3. Approximate controllability of nonlinear control systems

In this section we apply the preceding results to study the approximate controllability of nonlinear systems modeled by the equation

$$x'(t) = -Ax(t) + f(t, x(t), u(t)) + Bu(t), \quad 0 \leq t \leq a. \tag{7}$$

In this section the operators A and B satisfy all conditions considered in previous sections, and $f : [0, a] \times X \times U \to X$ is a function which satisfies appropriate conditions. Specifically, we will assume that for any $u \in L^p([0, a], U)$, there exists a unique mild solution of (7) with initial condition $x(0) = 0$. For completeness, we next establish some results to ensure that this property holds.

Definition 3.1. A continuous function $x : [0, a] \to X$ is said to be a mild solution of (7) with initial condition $x(0) = 0$ if $x(\cdot)$ satisfies the integral equation

$$x(t) = \int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)f(s, x(s), u(s))ds, \quad \text{for } 0 \le t \le a.$$

Proceeding as in [10, Lemma 5.6.7] we can establish the following property.

Lemma 3.2. *Let $r, w : [0, a] \to [0, \infty)$ be continuous functions. If w is nondecreasing and there are constants $\beta > 0$ and $0 < \alpha < 1$ such that*

$$r(t) \le w(t) + \beta \int_0^t \frac{r(s)}{(t-s)^{1-\alpha}} ds,$$

then $r(t) \le Cw(t)$, where C is a constant independent of $r(\cdot)$.

We next consider a fixed $p > -\frac{1}{\gamma}$ and q denotes the conjugate exponent of p. Moreover, we set $\alpha = -\gamma$.

We say that a function f verifies a local Lipschitz condition at x if for each $R \ge 0$ there exists a constant $L(R) \ge 0$ such that

$$\|f(t, x_1, u) - f(t, x_2, u)\| \le L(R)\|x_1 - x_2\|, \tag{8}$$

for $0 \le t \le a$, $u \in U$ and all $x_1, x_2 \in X$ with $\|x_1\|$, $\|x_2\| \le R$.

Theorem 3.3. *Assume that $T(t)$ is a compact operator for every $t > 0$. Assume further that f is continuous, verifies the local Lipschitz condition* (8) *and there is a constant $\eta > 0$ such that*

$$\|f(t, x, u)\| \le \eta(1 + \|x\| + \|u\|)$$

for all $x \in X$ and $u \in U$. Then for each $u \in L^p([0, a], U)$ there exists a unique mild solution of (7) *defined on* $[0, a]$.

Proof. It follows from [9] that for any function $u \in L^p([0,a],U)$ and $x \in C([0,a],X)$ the function $T(t-s)f(s, x(s), u(s))$ is integrable on $[0, t]$. Hence we may define $F : C([0, a], X) \to C([0, a], X)$ by

$$Fx(t) = \int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)f(s, x(s), u(s))ds, \quad \text{for } t \in [0, a]. \tag{9}$$

It follows from the Lebesgue dominated convergence theorem that F is continuous. Furthermore, it is easy to see that F is completely continuous. Let

$0 < \lambda < 1$. If $x = \lambda F x$, then combining the estimate

$$\|x(t)\| \leq \frac{M\|B\|t^{\alpha-\frac{1}{p}}}{(1+(\alpha-1)q)^{\frac{1}{q}}}\|u\|_p + M\int_0^t (t-s)^{-(1+\gamma)}\|f(s,x(s),u(s))\|ds$$
$$\leq \frac{M(\eta+\|B\|)t^{\alpha-\frac{1}{p}}}{(1+(\alpha-1)q)^{\frac{1}{q}}}\|u\|_p + \frac{M\eta t^{\alpha}}{\alpha} + \eta M\int_0^t (t-s)^{-(1+\gamma)}\|x(s)\|ds$$

with the Lemma 3.2 we get that $\|x(t)\| \leq C_1$, where C_1 is a positive constant independent of x. Consequently, the set $\{x \in C([0,a],X) \colon x = \lambda F x,\ 0 < \lambda < 1\}$ is bounded. Applying the Leray-Schauder alternative theorem ([6]) we get that F has at least a fixed point x, which is a mild solution of equation (7). The uniqueness of x is an immediate consequence of (8) and Lemma 3.2. □

As usual, we can avoid the condition of compactness of the semigroup $T(\cdot)$ demanding instead that f verifies a stronger Lipschitz condition.

Theorem 3.4. *Assume that $f(t,x,u)$ satisfies the conditions:*

(a) *$f(t,x,u)$ is continuous at t for all $x \in X$ and $u \in U$.*
(b) *$f(t,x,u)$ is continuous at u for all $t \in [0,a]$ and $x \in X$.*
(c) *There is $L > 0$ such that $f(t,x,u)$ satisfies the uniform Lipschitz condition*

$$\|f(t,x_1,u) - f(t,x_2,u)\| \leq L\|x_1 - x_2\|, \tag{10}$$

for $0 \leq t \leq a$, $u \in U$ and all $x_1, x_2 \in X$.

Then for each $u \in L^p([0,a],U)$ there exists a unique mild solution of (7) *defined on $[0,a]$.*

Proof. We define F as in (9). For $x(\cdot), y(\cdot) \in C([a,b],X)$, it follows from (10) that

$$\|Fx(t) - Fy(t)\| \leq LM\int_0^t (t-s)^{\alpha-1}\|x(s)-y(s)\|ds \leq \frac{LM}{\alpha}t^{\alpha}\max_{0\leq s\leq t}\|x(s)-y(s)\|.$$

Arguing inductively we get that

$$\|F^n x(t) - F^n y(t)\| \leq \frac{1}{\Gamma(\alpha+1)}\frac{(LM\Gamma(\alpha)t^{\alpha})^n}{\Gamma(n\alpha+1)}\max_{0\leq s\leq t}\|x(s)-y(s)\|,$$

which implies that F^n is a contraction for n large enough. The fixed point of F is the unique mild solution of equation (7). □

We next assume that for each $u \in L^p([0,a],U)$ there exists a unique mild solution of (7), which we will denote $x(t,u)$. The set $\mathcal{R}(a) = \{x(a,u) : u \in L^p([0,a],U)\}$ is called the reachable set of system (7).

Definition 3.5. The system (7) is called approximately controllable on $[0, a]$ if $\overline{\mathcal{R}(a)} = X$.

In what follows, we present two results of approximate controllability of system (7). Our next result extends a property of approximate controllability studied previously in [14].

Theorem 3.6. *Assume that f is uniformly bounded. If the system* (4) *is approximately controllable on* $[0, a]$*, then so is the system* (7).

Proof. There is a constant $C > 0$ such that $\|f(t, x, u)\| \leq C$ for $(t, x, u) \in [0, a] \times X \times U$. We choose $b_n \to a$ as $n \to \infty$. It is clear that $\int_{b_n}^{a} (a - s)^{-(1+\gamma)} ds \to 0$ as $n \to \infty$.

We consider a fixed $z \in X$. Let $x_n = x(b_n, 0)$. It follows from Corollary 2.5 that the system (4) is approximately controllable on $[0, a - b_n]$. Consequently, there exists a control function $w_n \in L^p([0, a - b_n], U)$ such that

$$G_{a-b_n}(w_n) + T(a - b_n)x_n - z = \int_{b_n}^{a} T(a - s)Bv_n(s)ds + T(a - b_n)x_n - z$$

converges to zero as $n \to \infty$, where $v_n(s) = w_n(s - b_n)$ for $s \in [b_n, a]$.

We define $u_n(s) = 0$, for $0 \leq s \leq b_n$ and $u_n(s) = v_n(s)$, for $s \in [b_n, a]$. Therefore, the mild solution of (7) corresponding to control $u_n(s)$ verifies

$$\begin{aligned} x(a, u_n) &= G_a(u_n) + \int_0^a T(a - s)f(s, x(s, u_n), u_n)ds \\ &= T(a - b_n) \int_0^{b_n} T(b_n - s)f(s, x(s, u_n), 0)ds + \int_{b_n}^{a} T(a - s)Bv_n(s)ds \\ &\quad + \int_{b_n}^{a} T(a - s)f(s, x(s, u_n), v_n)ds. \end{aligned}$$

Since $x_n = \int_0^{b_n} T(b_n - s)f(s, x(s, u_n), 0)ds$ and $\int_{b_n}^{a} T(a - s)f(s, x(s, u_n), v_n)ds \to 0$ as $n \to \infty$, we get that $x(a, u_n) \to z$ as $n \to \infty$, which implies that $z \in \overline{\mathcal{R}(a)}$. Since z was arbitrarily chosen, this completes the proof. □

The condition that f is uniformly bounded used in Theorem 3.6 is something demanding. To avoid this condition we can generalize the results of approximate controllability established in [13].

We define the map $\Lambda(t) : L^p([0, a], X) \to X$ by

$$\Lambda(t)(x(\cdot)) = \int_0^t T(t - s)x(s)ds.$$

It is clear that $\Lambda(t)$ is a bounded linear map for $p > -\frac{1}{\gamma}$. Moreover, if q denotes the conjugate exponent of p, then

$$\|\Lambda(t)(x(\cdot))\| \leq M \int_0^t (t-s)^{-(\gamma+1)} \|x(s)\| ds \leq \frac{M}{(1-(\gamma+1)q)^{\frac{1}{q}}} t^{\frac{1}{q}-(\gamma+1)} \|x\|_p,$$

which implies that $\|\Lambda(t)\| \leq M(1+(1-\alpha)q)^{-\frac{1}{q}} t^{\alpha-\frac{1}{p}}$. In addition, it is clear that $G_a(u) = \Lambda(a)(Bu)$. Let $\widetilde{G_a} : L^p([0,a],U)/\ker(G_a) \to \mathcal{R}(G_a)$ be the map induced by G_a on the quotient. We consider the space $\mathcal{R}(G_a)$ endowed with the norm

$$|||z||| = \|\widetilde{G_a}^{-1}(z)\|, \quad z \in \mathcal{R}(G_a).$$

It is well known that $\mathcal{R}(G_a)$ endowed with the norm $|||\cdot|||$ is a Banach space. Let $\Psi : C([0,a],X) \times L^p([0,a],U) \to X$ be given by

$$\Psi(x(\cdot), u(\cdot)) = \Lambda(a)(f(\cdot, x, u)).$$

We consider the following conditions:

(H1) The space U is reflexive and $-\frac{1}{\gamma} < p < \infty$.

(H2) There exists a Banach space $(Z, \|\cdot\|_Z)$ continuously included in $\mathcal{R}(G_a)$ for the norm $|||\cdot|||$ such that $\mathcal{R}(\Psi) \subseteq Z$.

In this case, we denote by $k > 0$ a constant such that $|||z||| \leq k\|z\|_Z$ for all $z \in Z$.

If condition (H1) holds, then $L^p([0,a],U)$ is a reflexive Banach space and the map $J : L^p([0,a],U)/\ker(G_a) \to L^p([0,a],U)$ defined by

$$J(\overline{u}) = u_0,$$

where $\|u_0\|_p = \inf\{\|v\|_p : v \in \overline{u}\}$, is a bounded linear map with $\|J(\overline{u})\|_p = \|\overline{u}\| \leq \|u\|_p$ for all $u \in L^p([0,a],U)$.

Theorem 3.7. *Assume that conditions* (H1) *and* (H2) *hold, and* $f(t,x,u)$ *is continuous at* t *for each* $x \in X$ *and* $u \in U$*, and satisfies the Lipschitz condition*

$$\|f(t,x_1,u_1) - f(t,x_2,u_2)\| \leq L_1\|x_1 - x_2\| + L_2\|u_1 - u_2\|.$$

If

$$\beta = M\left(\max\{1,k\}\left(\frac{L_1 a^\alpha}{\alpha} + \frac{L_2 a^{\alpha-\frac{1}{p}}}{(1+q(\alpha-1))^{\frac{1}{q}}}\right) + \frac{\|B\| a^{\alpha-\frac{1}{p}}}{(1+q(\alpha-1))^{\frac{1}{q}}}\right) < 1,$$

then $Z \subseteq \mathcal{R}(a)$*. In particular, if* Z *is dense in* X*, then system* (7) *is approximately controllable on* $[0,a]$*.*

Proof. It is clear that hypotheses of Theorem 3.4 are fulfilled. Let $z \in Z$. Following [13] we define the map $\Phi : C([0,a],X) \times L^p([0,a],U) \to C([0,a],X) \times L^p([0,a],U)$ by $\Phi(x,u)(t) = (y(t), v(t))$, where

$$y(t) = \Lambda(t)(f(\cdot,x,u)) + \Lambda(t)(Bu), \quad v = J(\widetilde{G_a}^{-1}(z - \Lambda(a)(f(\cdot,x,u)))),$$

for $t \in [0,a]$ and the product space $C([0,a],X) \times L^p([0,a],U)$ is provided with the norm $\|(x(\cdot),u(\cdot))\| = \max\{\|x(\cdot)\|_\infty, \|u(\cdot)\|_p\}$. It follows from (H1) and (H2) that Φ is well defined. Moreover, Φ is a contraction. In fact, for $x_1, x_2 \in C([0,a],X)$ and $u_1, u_2 \in L^p([0,a],U)$, let $v_i = J(\widetilde{G_a}^{-1}(z - \Lambda(a)(f(\cdot,x_i,u_i))))$ and $y_i(t) = \Lambda(t)(f(\cdot,x_i,u_i)) + \Lambda(t)(Bu_i)$ for $i = 1,2$. Then

$$\begin{aligned}
\|v_2 - v_1\|_p &= \|J(\widetilde{G_a}^{-1}(\Lambda(a)(f(\cdot,x_2,u_2) - f(\cdot,x_1,u_1))))\|_p \\
&\le k\|\Lambda(a)(f(\cdot,x_2,u_2) - f(\cdot,x_1,u_1))\| \\
&\le kM\int_0^a (a-s)^{\alpha-1}(L_1\|x_2(s) - x_1(s)\| + L_2\|u_2(s) - u_1(s)\|)ds \\
&\le \frac{kML_1a^\alpha}{\alpha}\|x_2 - x_1\|_\infty + \frac{kML_2a^{\alpha-\frac{1}{p}}}{(1+q(\alpha-1))^{\frac{1}{q}}}\|u_2 - u_1\|_p.
\end{aligned}$$

A similar argument shows that

$$\|y_2 - y_1\|_\infty \le \frac{ML_1a^\alpha}{\alpha}\|x_2 - x_1\|_\infty + \frac{M(L_2 + \|B\|)a^{\alpha-\frac{1}{p}}}{(1+q(\alpha-1))^{\frac{1}{q}}}\|u_2 - u_1\|_p.$$

Therefore, $\|\Phi(x_2,u_2) - \Phi(x_1,u_1)\| \le \beta\|(x_2 - x_1, u_2 - u_1)\|$ which shows that Φ is a contraction. Let $(x(\cdot),u(\cdot))$ be the fixed point of Φ. It follows directly from the definition of Φ that $x(a) = z$. Hence, $z \in \mathcal{R}(a)$, which completes the proof of our assertion. □

4. Applications

In this section we present some applications of our abstract results.

Application 1. We are concerned with the approximate controllability for the heat conduction. To simplify the exposition, we consider only a simple model with one-dimensional domain.

Initially we study the approximate controllability of a system governed by the heat equation

$$\frac{\partial w(t,\xi)}{\partial t} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + Bu(t), \quad 0 \le t \le a,\ 0 < \xi < \pi \tag{11}$$

$$w(t,0) = w(t,\pi) = 0. \tag{12}$$

To model this system we consider $X = C^{\alpha}([0,\pi])$, $0 < \alpha < 1$, the space of α-Hölder-continuous functions endowed with the norm

$$|||z|||_{\alpha} = \|z\|_{\infty} + \sup_{0 \le \xi < \xi' \le \pi} \frac{|z(\xi) - z(\xi')|}{|\xi - \xi'|^{\alpha}}.$$

We define the operator A by

$$(Az)(\xi) = -\frac{d^2 z(\xi)}{d\xi^2} \tag{13}$$

with domain

$$D(A) = \left\{ z \in C^{2+\alpha}([0,\pi]) : z(0) = z(\pi) = 0 \right\}.$$

It follows from [12, Example 2.3] that $-A$ is an almost sectorial operator that generates an analytic semigroup $(T(t))_{t>0}$ of growth order $\frac{\alpha}{2}$ in X. Let U be a Banach space and let $B : U \to X$ be a bounded linear map. Therefore, using the notation $x(t) = w(t, \cdot)$, problem (11), (12) can be modeled as the abstract control system (4). Furthermore, since $T(t)(X) \subseteq D(A)$ and $D(A)$ is not dense in X, then there exists $0 \ne x^* \in X^*$ such that $\langle x^*, T(t)Bu \rangle = 0$ for all $u \in U$. This implies that system (11), (12) is not approximately controllable on $[0, a]$. That is, superficially speaking, we can not regulate the temperature with α-Hölder-continuous functions.

We next study the approximate controllability of the system

$$\frac{\partial w(t,\xi)}{\partial t} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + u(t,\xi), \quad 0 \le t \le a,\ 0 < \xi < \pi \tag{14}$$

$$w(t,0) = w(t,\pi) = 0, \tag{15}$$

in the space $\overline{D(A)}$. To model this system, we take $U = \overline{D(A)}$ and $B = I$. Let A_D be the part of A in $\overline{D(A)}$. It follows from [12] that A_D is a densely defined operator that generates an analytic semigroup $(T(t))_{t>0}$ of growth order $\frac{\alpha}{2}$. Hence, using again the notation $x(t) = w(t, \cdot)$, system (14), (15) can be modeled as the abstract system (4). Let $x^* \in \overline{D(A)}^*$ such that

$$\langle x^*, T(t)Bu \rangle = \langle x^*, T(t)x \rangle = 0$$

for all $t > 0$ and $x \in \overline{D(A)}$. Since $T(t)x \to x$ as $t \to 0^+$ for $x \in D(A)$, then $\langle x^*, x \rangle = 0$ for all $x \in D(A)$. Using that $D(A)$ is dense in $\overline{D(A)}$, we infer that $x^* = 0$. Therefore, system (14), (15) is approximately controllable on $[0, a]$ in the space $\overline{D(A)}$.

Application 2. In this application we exhibit a general method for constructing analytic semigroups of growth order α with $0 < \alpha < 1$, and defining approximately controllable systems defined by this kind of semigroups.

For each $n \in \mathbb{N}$, we introduce the group

$$T_n(t) = e^{-nt} \begin{pmatrix} 1 & n(n+1)^\alpha t \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},$$

on $\mathbb{R}^2$. We consider $\mathbb{R}^2$ endowed with the norm $\|(a,b)\| = \max\{|a|,|b|\}$. Then

$$\|T_n(t)\| = e^{-nt}(1 + n(n+1)^\alpha t), \quad t \geq 0.$$

It follows that

$$\sup_{t\geq 0} \|T_n(t)\| = \|T_n(t_n)\| = e^{-nt_n}(n+1)^\alpha, \tag{16}$$

where $t_n = \frac{1}{n}\big(1 - \frac{1}{(n+1)^\alpha}\big)$. Hence we deduce that $\|T_n(t)\| \leq Ct^{-\alpha}$, $t > 0$, where C denotes a generic constant independent of t and n. In fact, for $0 < t \leq t_n$ we have that $(n+1)^\alpha \leq Ct^{-\alpha}$ for a positive constant C, and the assertion follows from (16). For $t \geq t_n$ we have that

$$e^{-nt} n(n+1)^\alpha t \leq \frac{2}{n^2t^2} n(n+1)^\alpha \leq \frac{2(n+1)^\alpha}{nt} \leq \frac{2C}{n^{1-\alpha}t} = \frac{2C}{t^\alpha (nt)^{1-\alpha}} \leq \frac{C}{t^\alpha}.$$

We next consider $X = \oplus_{n=1}^\infty X_n$, where $X_n = \mathbb{R}^2$, endowed with the norm

$$\|(x_n)_n\| = \sum_{n=1}^\infty \|x_n\|.$$

We define $T(t) : X \to X$ by

$$T(t)(x_n)_n = (T_n(t)x_n)_n, \quad t > 0.$$

It is easy to see that $T(t) \in \mathcal{L}(X)$ and $(T(t))_{t>0}$ is an analytic semigroup. Moreover,

$$\|T(t)\| = \sup_{n\geq 1} \|T_n(t)\| \leq Ct^{-\alpha}, \quad t > 0,$$

and

$$\|T(t_n)\| \geq \|T_n(t_n)\| \geq e^{-a}(n+1)^\alpha, \quad n \in \mathbb{N},$$

for some constant $a > 0$ independent of $n \in \mathbb{N}$. Combining these assertions we can affirm that $(T(t))_{t>0}$ is an analytic semigroup of growth order α.

On the other hand, it is not difficult to see that the infinitesimal generator A of $(T(t))_{t>0}$ is the operator given by $A(x_n)_n = (A_n x_n)_n$, where

$$A_n = \begin{pmatrix} -n & n(n+1)^\alpha \\ 0 & -n \end{pmatrix}.$$

Let $b = (b_n)_n$, where $b_n = \frac{1}{2^n}\,\mathrm{col}\,(0,1)$. We consider the control system

$$x'(t) = Ax(t) + bu(t), \quad t > 0. \tag{17}$$

This system is a particular case of system (4) with $U = \mathbb{R}$ and $B : U \to X$ the operator given by $Bu = bu$.

It is well known that $X^* = \oplus_{n=1}^{\infty} X_n^*$, endowed with the norm $\|(x_n^*)_n\| = \sup_{n\geq 1} \|x_n^*\|$. Hence $B^* : X^* \to \mathbb{R}$ is given by

$$B^*(x_n^*)_n = \sum_{n=1}^{\infty} \langle x_n^*, b_n \rangle.$$

We are in a position to establish the following property.

Corollary 4.1. *System* (17) *is approximately controllable in finite time.*

Proof. Following Proposition 2.4, we take $(x_n^*)_n \in X^*$ and we assume that

$$B^*T(t)^*(x_n^*)_n = \sum_{n=1}^{\infty} \langle x_n^*, e^{A_n t} b_n \rangle = 0. \tag{18}$$

We will prove inductively that $x_n^* = 0$. In fact, it follows from (18) that

$$\langle x_1^*, e^{A_1 t} b_1 \rangle = -\sum_{n=2}^{\infty} \langle x_n^*, e^{A_n t} b_n \rangle$$

which implies that $\langle x_1^*, e^t e^{A_1 t} b_1 \rangle = -\sum_{n=2}^{\infty} \langle x_n^*, e^t e^{A_n t} b_n \rangle \to 0,\ t \to \infty$. Since $e^t e^{A_1 t} b_1 = \frac{1}{2}\,\mathrm{col}\,(2^\alpha t, 1)$, we get that $x_1^* = 0$. We can repeat this argument to conclude that $x_n^* = 0$ for $n \in \mathbb{N}$. □

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