

Connectivity of complexes of separating curves

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In memory of Fritz Grunewald

Abstract. We prove that the separating curve complex of a closed orientable surface of genus g is $(g - 3)$ -connected. We also obtain a connectivity property for a separating curve complex of the open surface that is obtained by removing a finite set from a closed one, where it is assumed that the removed set is endowed with a partition and that the separating curves respect that partition. These connectivity statements have implications for the algebraic topology of the moduli space of curves.

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1. Statements of the results

Let S be a connected oriented surface of genus g with finite first Betti number $2g + n$ (i.e., a closed surface with n punctures) and make the customary assumption that S has negative Euler characteristic: if $g = 0$, then $n \geq 3$ and if $g = 1$, then $n \geq 1$. We recall that the *curve complex* $\mathcal{C}(S)$ of S is the simplicial complex whose vertex set consists of the isotopy classes of embedded (unoriented) circles in S which do not bound in S a disk or a cylinder. A finite set of vertices spans a simplex precisely when its elements can be represented by embedded circles that are pairwise disjoint. Thus, a closed 1-dimensional submanifold A of S with $k + 1$ connected components such that every connected component of its complement has negative Euler characteristic defines a k -simplex σ_A of $\mathcal{C}(S)$ and every simplex of $\mathcal{C}(S)$ is thus obtained.

This complex has proven to be quite useful in the study of the mapping class group of S . For the purposes of studying the Torelli group of S a subcomplex $\mathcal{C}_{\text{sep}}(S)$ of $\mathcal{C}(S)$ can render a similar service. It is defined as the full subcomplex of $\mathcal{C}(S)$ spanned by the separating vertices of $\mathcal{C}(S)$, where a vertex is called *separating* if a representative embedded circle separates S into two components. Our main result for the case when S is closed is contained in the following theorem.

Theorem 1.1 (A_g). *If $n \leq 1$, then the simplicial complex $\mathcal{C}_{\text{sep}}(S)$ is $(g - 3)$ -connected.*

Previous work on this topic that we are aware of concerns the case $n = 0$. Farb and Ivanov announced in 2005 [1], Theorem 4, that $\mathcal{C}_{\text{sep}}(S)$ is connected for $g \geq 3$. Putman gave in [6], Theorem 1.4, another proof of this and showed that $\mathcal{C}_{\text{sep}}(S)$ is simply connected for $g \geq 4$ (op. cit., Theorem 1.11). In that paper he also mentions that Hatcher and Vogtmann have proved that $\mathcal{C}_{\text{sep}}(S)$ is $\lfloor \frac{1}{2}(g - 3) \rfloor$ -connected for all g (unpublished).

Remark 1.2. Possibly the connectivity bound in Theorem 1.1 is the best possible for every positive genus. In a paper with Van der Kallen [3] we showed that the quotient of $\mathcal{C}_{\text{sep}}(S)$ by the action of the Torelli group of S has the homotopy type of a bouquet of $(g - 2)$ -spheres. In the situation of the theorem, $\mathcal{C}_{\text{sep}}(S)$ has dimension $2g - 4 + n$ so that its connectivity is half the dimension (as $n \leq 1$). In particular, we cannot conclude that $\mathcal{C}_{\text{sep}}(S)$ is spherical.

Before we state a version for the case $n \geq 2$, we point out a consequence that pertains to the moduli space of curves. Consider the Teichmüller space $\mathcal{T}(S)$ of S , a contractible manifold on which acts the mapping class group $\Gamma(S)$. The action is proper and the orbit space may be identified with the moduli space \mathcal{M}_g of curves of genus g . The *Harvey bordification* of $\mathcal{T}(S)$, here denoted by $\mathcal{T}(S)^+ \supset \mathcal{T}(S)$, is a (noncompact) manifold with boundary with corners to which the action of $\Gamma(S)$ naturally extends. This action is also proper and according to [4] the orbit space $\mathcal{M}_g^+ := \Gamma(S) \backslash \mathcal{T}(S)^+$ is a compactification of \mathcal{M}_g that can also be obtained from the Deligne–Mumford compactification $\tilde{\mathcal{M}}_g \supset \mathcal{M}_g$ as a ‘real oriented blowup’ of its boundary $\Delta_g := \tilde{\mathcal{M}}_g - \mathcal{M}_g$. The walls of $\mathcal{T}(S)^+$ define a closed covering of the boundary $\partial\mathcal{T}(S)^+$ and any nonempty corner closure is an intersection of walls. As is well-known, the curve complex $\mathcal{C}(S)$ can be identified with the nerve of this covering of $\partial\mathcal{T}(S)^+$. Since the corner closures are contractible, Weil’s nerve theorem implies that $\partial\mathcal{T}(S)^+$ has the same homotopy type as $\mathcal{C}(S)$.

Let $\Delta_{g,0} \subset \Delta_g$ denote the irreducible component of the Deligne–Mumford boundary whose generic point parameterizes irreducible curves with one singular point. We may understand $\mathcal{M}_g^c := \tilde{\mathcal{M}}_g - \Delta_{g,0}$ as the moduli space of stable genus g curves all of whose nodes are separating (which is equivalent to the irreducible components of the curve being smooth and with their genera summing up to g) and $\Delta_g^c := \Delta_g - \Delta_{g,0}$ as the locus in \mathcal{M}_g^c that parameterizes the singular ones among them. If Γ is a subgroup of $\Gamma(S)$ with the property that every Dehn twist along a separating curve in S has a positive power lying in Γ , then this defines a (not necessarily finite) cover $\tilde{\mathcal{M}}_g^c \rightarrow \mathcal{M}_g^c$.

Corollary 1.3. *Suppose that $\Gamma \subset \Gamma(S)$ is as above and is in addition torsion-free. If we denote by $\tilde{\Delta}_g^c \subset \tilde{\mathcal{M}}_g^c$ the preimage of Δ_g^c , then the pair $(\tilde{\mathcal{M}}_g^c, \tilde{\Delta}_g^c)$ is $(g - 2)$ -connected. Moreover, $H_k(\mathcal{M}_g^c, \Delta_g^c; \mathbb{Q}) = 0$ for $k \leq g - 2$.*

Proof. Let $\mathcal{T}(S)_{\text{sep}}^+$ be obtained from $\mathcal{T}(S)^+$ by removing the walls that correspond to the nonseparating vertices of $\mathcal{C}(S)$. Then $\mathcal{T}(S)_{\text{sep}}^+$ is the preimage of \mathcal{M}_g^c in $\mathcal{T}(S)^+$. The same reasoning as above shows that $\partial\mathcal{T}(S)_{\text{sep}}^+$ is homotopy equivalent to $\mathcal{C}(S)_{\text{sep}}$ and so $\partial\mathcal{T}(S)_{\text{sep}}^+$ is $(g - 3)$ -connected. It follows that we can construct a relative CW complex $(Z, \partial\mathcal{T}(S)_{\text{sep}}^+)$ obtained from $\partial\mathcal{T}(S)_{\text{sep}}^+$ by attaching cells of dimension $\geq g - 1$ in a $\Gamma(S)$ -equivariant manner as to ensure that Z is contractible and no nontrivial element of $\Gamma(S)$ fixes a cell. Then Γ acts freely on Z (as it does on the contractible space $\mathcal{T}(S)_{\text{sep}}^+$) and so there is a Γ -equivariant homotopy equivalence $Z \rightarrow \mathcal{T}(S)_{\text{sep}}^+$ relative to $\partial\mathcal{T}(S)_{\text{sep}}^+$. It follows that we also have a homotopy equivalence $\Gamma \backslash Z \rightarrow \tilde{\mathcal{M}}_g^c$ relative to $\tilde{\Delta}_g^c$ and we conclude that $(\tilde{\mathcal{M}}_g^c, \tilde{\Delta}_g^c)$ is $(g - 2)$ -connected.

The last assertion follows from the existence of a normal subgroup $\Gamma \subset \Gamma(S)$ of finite index that is torsion-free. For if Γ is such a group, then $H_k(\mathcal{M}_g^c, \Delta_g^c; \mathbb{Q}) \cong H_k(\tilde{\mathcal{M}}_g^c, \tilde{\Delta}_g^c; \mathbb{Q})^{\Gamma(S)/\Gamma} = 0$ for $k \leq g - 2$. □

A similar statement holds for the universal curve $\mathcal{M}_{g,1}$.

When $n > 1$, we need to come to terms with the fact that the separability notion has no good hereditary properties: if T is a closed surface, $A \subset T$ a compact 1-dimensional submanifold representing a simplex of $\mathcal{C}(T)$ and S a connected component of $T - A$, then a vertex of $\mathcal{C}(S)$ may split S , but not T . This happens precisely when the vertex in question separates two boundary components of ∂S that lie on the same connected component of $T - S$. So the basic object should be, what Andy Putman calls in [5], a partitioned surface: a closed surface minus a finite set, for which the removed set comes with a partition. This leads to the following definition.

Definition 1.4. Let N be the set of points of S at infinity (the cusps) and let P be a partition of N . We call a vertex of $\mathcal{C}(S)$ *separating relative to P* if a representative embedded circle $\alpha \subset S$ has the property that $S - \alpha$ has two connected components each of which meets N in a union of parts of P . We denote by $\mathcal{C}(S, P)$ the full subcomplex of $\mathcal{C}(S)$ spanned by such vertices.

One might also understand $\mathcal{C}(S, P)$ as the full subcomplex of $\mathcal{C}(S)$ spanned by the isotopy classes of embedded cycles which are separating on the surface S^P that is obtained by capping off for each part of P the corresponding set of cusps by a sphere with that many holes. Notice that $\mathcal{C}(S, P) \subset \mathcal{C}_{\text{sep}}(S)$ and that we have equality when P is discrete or N is empty.

We shall prove Theorem 1.1 by induction and simultaneously with

Theorem 1.5 ($A_{g,n}$). *Suppose that $g > 0$ and $n = |N| > 1$. Let P be a partition of N . Then $\mathcal{C}(S, P)$ is $(g - 2)$ -connected.*

To be precise, the induction starts with $g = 0$, where the statements (A_g) and ($A_{0,n}$) are trivially true and the induction strategy will be to show that

- (i) $(A_{h,n})$ for $h < g$ implies (A_g) , and
- (ii) (A_g) and $(A_{h,k})$ for $(h, k) < (g, n)$ (for the lexicographic ordering) imply $(A_{g,n})$.

I am indebted to Allen Hatcher for pointing out that the stronger version of Theorem 1.5 that I stated in a previous version was incorrect. Yet it may be that some such statement might hold. For instance, if $r(P)$ denotes the number of nonempty parts of P and $s(P)$ the number of parts with at least two elements, is it true that $\mathcal{C}(S, P)$ is $(g + r(P) + s(P) - 4)$ -connected when $g > 0$ (as I claimed in the earlier version)? In case $g = 0$, $\mathcal{C}(S, P)$ is a complex of dimension $r(P) + s(P) - 4$. Is this $(r(P) + s(P) - 5)$ -connected? In other words, is this complex spherical?

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2. Proofs

Before we start off, we mention the following elementary fact that we will frequently use.

Lemma 2.1. *Let X_i be a d_i -connected space ($d_i = -1$ means $X_i \neq \emptyset$), where $i = 1, \dots, k$. Then the iterated join $X_1 * \dots * X_k$ is $(-2 + \sum_{i=1}^k (d_i + 2))$ -connected.*

Proof that $(A_{h,n})$ for $h < g$, all n , implies (A_g) . So here $n \leq 1$. We must show that $\mathcal{C}_{\text{sep}}(S)$ is $(g - 3)$ -connected. For $g < 2$, there is nothing to show and so we may assume that $g \geq 2$. A theorem of Harer [2, Thm. 1.2] asserts that $\mathcal{C}(S)$ is $(2g - 3)$ -connected. So it is certainly $(g - 3)$ -connected. Let \mathcal{C}_k be the subcomplex of $\mathcal{C}(S)$ that is the union of $\mathcal{C}_{\text{sep}}(S)$ and the k -skeleton of $\mathcal{C}(S)$. So $\mathcal{C}_{-1} = \mathcal{C}_{\text{sep}}(S)$ and $\mathcal{C}_k = \mathcal{C}(S)$ for k large. Notice that a finite set of vertices of $\mathcal{C}(S)$ spans a simplex of \mathcal{C}_k if and only if no more than $k + 1$ of these are nonseparating. Hence a minimal simplex of $\mathcal{C}_k - \mathcal{C}_{k-1}$ is represented by a compact 1-dimensional submanifold $A \subset S$ with $k + 1$ connected components, each of which is nonseparating. We prove that the boundary of the star of such a simplex in \mathcal{C}_k is a $(g - 3)$ -connected subcomplex of \mathcal{C}_{k-1} . This property implies that the pair $(\mathcal{C}(S), \mathcal{C}_{\text{sep}}(S))$ is $(g - 2)$ -connected. Since $\mathcal{C}(S)$ is $(g - 3)$ -connected, it then follows that $\mathcal{C}_{\text{sep}}(S)$ is. Let $\{S_i\}_{i \in I}$ be the set of connected components of $S - A$. Notice that if g_i is the genus of S_i , then $g_i < g$. An Euler characteristic argument shows that

$$g - 1 = k + 1 + \sum_{i \in I} (g_i - 1).$$

We denote by N_i the set of ‘cusps’ of S_i , i.e., the finitely many points needed to make S_i a closed surface. So an element of N_i is given by possibly a cusp of S (if it exists and if it is also a cusp of S_i) or by a connected component of A in the boundary of S_i *endowed with the orientation it receives as such*. The set N_i comes with an evident partition P_i : if S has a cusp and N_i contains it, then this cusp makes up a singleton part of P_i and any other two elements of N_i belong to the same part of P_i if and only if they come from connected components of A that lie on the same connected component of $S - S_i$. (NB: beware that a connected component of $S - S_i$ could be simply a connected component A_o of A ; then its two orientations define a 2-element part of P_i .) connected components of A that bound S_i . Note that since the connected components of A are nonseparating, we always have $|N_i| \geq 2$. By our induction hypothesis $\mathcal{C}(S_i, P_i)$ is then $(g_i - 2)$ -connected. The boundary of the star of the k -simplex σ_A defined by A in \mathcal{C}_k lies in \mathcal{C}_{k-1} and can be identified with the $(|I| + 1)$ -fold join

$$\partial\sigma_A * \left(\bigstar_{i \in I} \mathcal{C}(S_i, P_i) \right).$$

Since $\partial\sigma_A$ is a combinatorial $(k - 1)$ -sphere, this join has by Lemma 2.1 connectivity at least $-2 + k + \sum_{i \in I} g_i$. By the displayed formula above this is equal to $g - 4 + |I|$ and is therefore $\geq g - 3$. \square

The proof of $A_{g,n}$ begins with a discussion. We now assume that $g > 0$ and $n \geq 2$. Let $x \in N$. Notice that $S' := S \cup \{x\}$ has still negative Euler characteristic. We put $N' := N - \{x\}$ and $P' := P|N'$. The goal is to compare $\mathcal{C}(S', P')$ with $\mathcal{C}(S, P)$. There is in general no forgetful map $\mathcal{C}(S, P) \rightarrow \mathcal{C}(S', P')$ because there will be vertices of $\mathcal{C}(S, P)$ that do not give vertices of $\mathcal{C}(S', P')$. Let us first identify this set of vertices.

Denote by $\Sigma_x \subset N - \{x\}$ the set of $y \in N - \{x\}$ for which $\{x, y\}$ is a union of parts of P . In other words, if P_x denotes the part of P that contains x , then Σ_x is empty if P_x has more than 2 elements, equals $P_x - \{x\}$ if P_x is a 2-element set, and equals the set of $y \neq x$ for which P_y is a singleton in case $P_x = \{x\}$. Then the vertices of $\mathcal{C}(S, P)$ that have no image in $\mathcal{C}(S', P')$ are precisely the vertices α of $\mathcal{C}_{\text{sep}}(S)$ which for some $y \in \Sigma_x$ bound a disk neighborhood of $\{x, y\}$ in $S \cup \{x, y\}$. Such a disk neighborhood can be thought of as a regular neighborhood of an arc in $S \cup \{x, y\}$ connecting the two added cusps; this may help to explain why we have chosen to denote this set of vertices by $\text{arc}_{(S,P)}(x)$. Denote by $\mathcal{C}(S, P)_x$ the full subcomplex of $\mathcal{C}(S, P)$ spanned by the vertices not in $\text{arc}_{(S,P)}(x)$.

Observe that $\text{arc}_{(S,P)}(x)$ is empty (so that $\mathcal{C}(S, P)_x = \mathcal{C}(S, P)$) if Σ_x is.

Lemma 2.2. *The link in $\mathcal{C}(S, P)$ of every vertex of $\text{arc}_{(S,P)}(x)$ is a subcomplex of $\mathcal{C}(S, P)_x$ that projects isomorphically onto $\mathcal{C}(S', P')$.*

Proof. A vertex of $\text{arc}_{(S,P)}(x)$ defines a $y \in \Sigma_x$ and (up to isotopy) a closed disk D in $S \cup \{x, y\}$ that is a neighborhood of $\{x, y\}$. The inclusion $S - D \subset S'$ identifies the link in question with $\mathcal{C}(S', P')$. \square

Denote by \tilde{P} the refinement of P which coincides with P on $N - P_x$ and partitions P_x further into $\{x\}$ and $P_x - \{x\}$. So $\tilde{P}' = P'$. It is clear that $\mathcal{C}(S, P)$ is a subcomplex of $\mathcal{C}(S, \tilde{P})$. Notice that $\text{arc}_{(S, P)}(x) = \mathcal{C}(S, P) \cap \text{arc}_{(S, \tilde{P})}(x)$ (we have $\text{arc}_{(S, P)}(x) = \text{arc}_{(S, \tilde{P})}(x)$ unless $|P_x| = 2$) and $\mathcal{C}(S, P)_x = \mathcal{C}(S, P) \cap \mathcal{C}(S, \tilde{P})_x$. We denote by f the forgetful simplicial map $\mathcal{C}(S, \tilde{P})_x \rightarrow \mathcal{C}(S', P')$ so that we have the diagram

$$\begin{array}{ccc} C(S, P)_x & \subset & C(S, \tilde{P})_x & \xrightarrow{f} & C(S', P') \\ \cap & & \cap & & \\ C(S, P) & \subset & C(S, \tilde{P}). & & \end{array}$$

Lemma 2.3. *The map $f : \mathcal{C}(S, \tilde{P})_x \rightarrow \mathcal{C}(S', P')$ is a homotopy equivalence.*

Proof. Choose an arc γ which connects x with another point of N and defines a vertex of $\text{arc}_{(S, P)}(x)$ and observe that the full subcomplex $K \subset \mathcal{C}(S, \tilde{P})_x$ spanned by vertices that avoid γ defines a section of f (the inclusion $S - \gamma \subset S'$ is isotopic to a homeomorphism). We shall prove that $\mathcal{C}(S, \tilde{P})_x$ admits K as a deformation retract. (The proof will in fact show that each fiber of $|f|$ is a tree and essentially produces for every element of $|\mathcal{C}(S, \tilde{P})_x|$ the unique path in its $|f|$ -fibre that connects it to the point of $|K|$.)

Denote by $K_r \subset \mathcal{C}(S, \tilde{P})_x$ the subcomplex whose simplices can be represented by a closed submanifold $A \subset S$ which meets γ transversally in at most r points. This defines a filtration $K = K_0 \subset K_1 \subset K_2 \subset \dots$ whose union is $\mathcal{C}(S, \tilde{P})_x$. Although this filtration is infinite, it is enough to construct for every $r \geq 0$ a deformation retraction of $|K_{r+1}|$ onto $|K_r|$, for in the simplicial setting an infinite sequence of deformation retractions still gives a deformation retraction.

We do this per simplex: if σ is a simplex of K_{r+1} that is not in K_r and is minimal for this property, then its link in K_{r+1} lies in K_r and so it suffices to define for such a σ a deformation retraction h_σ of $|\text{Star}_{K_{r+1}}(\sigma)|$ onto $|\text{Link}_{K_{r+1}}(\sigma)|$.

The simplex σ is represented by a closed submanifold $A \subset S$ of which every connected component meets γ transversally and is such that $A \cap \gamma$ has cardinality $r + 1$ (a number that cannot be made smaller in its isotopy class). Let x_0 be the point of $A \cap \gamma$ closest to x . Denote by α_0 the connected component of A which contains x_0 and choose in S' a thin regular neighborhood of the union of α_0 and the subarc of γ which connects x_0 with x . The boundary of that neighborhood has two connected components. Both lie in S and only one of them is isotopic to α_0 . Denote by α'_0 the other boundary component. If τ is a simplex of K_{r+1} which contains σ , then adding α'_0 to τ gives also a simplex τ' of K_{r+1} and the codimension one face τ'' of τ' obtained by removing α_0 is contained in K_r . So if we regard $|\text{Star}_{K_{r+1}}(\sigma)|$ as the cone over $|\text{Link}_{K_{r+1}}(\sigma)|$ with the barycenter of σ as its vertex, then there is a simplicial map from this cone to its base which sends the barycenter to α'_0 and is the identity on the base. Its geometric realization yields the desired h_σ . □

Corollary 2.4. *The complex $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ is canonically homotopy equivalent to the join $\text{arc}_{(S,P)}(x) * \mathcal{C}(S', P')$ (where $\text{arc}_{(S,P)}(x)$ is discrete).*

Proof. The set of vertices of $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ not in $\mathcal{C}(S, \tilde{P})_x$ is $\text{arc}_{(S,P)}(x)$. The link of any such vertex is contained in $\mathcal{C}(S, \tilde{P})_x$ and by Lemma 2.2 that link projects isomorphically onto $\mathcal{C}(S', P')$. In view of Lemma 2.3 this implies that the inclusion of this link in $\mathcal{C}(S, \tilde{P})_x$ is also a homotopy equivalence. Hence the natural inclusion $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x \subset \text{arc}_{(S,P)}(x) * \mathcal{C}(S, \tilde{P})_x$ is a homotopy equivalence. The corollary follows. \square

From now on we assume that A_g holds and that $A_{h,k}$ holds for all (h, k) smaller than (g, n) for the lexicographic ordering. Our goal is to prove $A_{g,n}$.

Lemma 2.5. *The pair $(\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x, \mathcal{C}(S, P))$ is $(g - 1)$ -connected.*

Proof. If $P_x = \{x\}$, then $\tilde{P} = P$ and there is nothing to show. We therefore assume that P_x has more than one element. Denote by \mathcal{C}_k the subcomplex of $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ that is the union of $\mathcal{C}(S, P)$ and the k -skeleton of $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$: a finite set of vertices of $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ spans a simplex of \mathcal{C}_k if and only if no more than $k + 1$ of these separate x from $P_x - \{x\}$. Notice that $\mathcal{C}_{-1} = \mathcal{C}(S, P)$ and $\mathcal{C}_k = \mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ for k large. A minimal simplex of $\mathcal{C}_k - \mathcal{C}_{k-1}$ is represented by a compact 1-dimensional submanifold $A \subset S$ with $k + 1$ connected components, each of which separates x from $P_x - \{x\}$ (the graph that is associated to A is then a string with $k + 2$ nodes). We prove that the boundary of the star of such a simplex in \mathcal{C}_{k-1} is $(g - 2)$ -connected if $g > 0$. We enumerate the connected components of A as $\alpha_0, \dots, \alpha_k$ and the connected components of $S - A$ as S_0, \dots, S_{k+1} such that α_i is a boundary component of S_i and S_{i+1} and so that S_0 resp. S_{k+1} is punctured by x resp. $P_x - \{x\}$. The cusps of $S - A$ are naturally indexed by $\hat{N} := N \sqcup \{i_{\pm}\}_{i=0}^k$, where i_- resp. i_+ corresponds to the cusp defined by α_i on S_i resp. S_{i+1} . Let $\hat{N}_i \subset \hat{N}$ index the set of cusps on S_i . Denote by P_i the partition of $(N - P_x) \cap S_i$ that is simply the restriction of P and denote by \hat{P}_i the partition of \hat{N}_i that on $(N - P_x) \cap S_i$ is equal to P_i and has what remains of \hat{N}_i as a single part. So this new part is $\{x, 0_+\}$ for $i = 0$, $\{(i - 1)_-, i_+\}$ for $0 < i < k + 1$ and $(P_x - \{x\}) \cup \{k_-\}$ for $i = k + 1$.

The reason for introducing these partitions is that we can now observe that the boundary of the star of the k -simplex σ_A defined by A in \mathcal{C}_k lies in \mathcal{C}_{k-1} and can be identified with the iterated join

$$\partial\sigma_A * \mathcal{C}(S_0, \hat{P}_0) * \dots * \mathcal{C}(S_{k+1}, \hat{P}_{k+1}).$$

It is then enough to show that this join is $(g - 2)$ -connected for $g > 0$. Since $\partial\sigma_A$ is a $(k - 1)$ -sphere, it is $(k - 2)$ -connected. The connectivity of a factor $\mathcal{C}(S_i, \hat{P}_i)$ with $g_i > 0$ is at least $g_i - 2$. So by Lemma 2.1 the connectivity of the above join is at least $-2 + k + \sum_{\{i:g_i>0\}} g_i = g + k - 2 \geq g - 2$. \square

Proof of $(A_{g,n})$. We must show that $\mathcal{C}(S, P)$ is $(g - 2)$ -connected. In view of Lemma 2.5 it suffices to show that $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ has that property.

If $\text{arc}_{(S,P)}(x) = \emptyset$, then $n > 2$ and so our induction hypothesis implies that $\mathcal{C}(S', P')$ is $(g - 2)$ -connected by $A_{g,n-1}$. It follows from Corollary 2.4 that $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ is homotopy equivalent to $\mathcal{C}(S', P')$ and hence is $(g - 2)$ -connected.

If $\text{arc}_{(S,P)}(x) \neq \emptyset$, then we may have $n = 2$. At least we know that $\mathcal{C}(S', P')$ is $(g - 3)$ -connected (invoke A_g if $n = 2$). But since $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ is homotopy equivalent to $\text{arc}_{(S,P)}(x) * \mathcal{C}(S', P')$ (by Corollary 2.4), it is $(g - 2)$ -connected. \square

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