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Elchanan Mossel · Joe Neeman

Robust optimality of Gaussian noise stability

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Abstract. We prove that under the Gaussian measure, half-spaces are uniquely the most noise stable sets. We also prove a quantitative version of uniqueness, showing that a set which is almost optimally noise stable must be close to a half-space. This extends a theorem of Borell, who proved the same result but without uniqueness, and it also answers a question of Ledoux, who asked whether it was possible to prove Borell's theorem using a direct semigroup argument. Our quantitative uniqueness result has various applications in diverse fields.

Keywords. Gaussian noise sensitivity, isoperimetry, influence, Max-Cut

1. Introduction

Gaussian stability theory is a rich extension of Gaussian isoperimetric theory. As such it connects numerous areas of mathematics including probability, geometry [9], concentration and high dimensional phenomena [32], re-arrangement inequalities [10, 18] and more. On the other hand, this theory has recently found fascinating applications in combinatorics and theoretical computer science. It was essential in [36] for proving the "majority is stablest" conjecture [19, 26], the "it ain't over until it's over" conjecture [21], and for establishing the unique games computational hardness [25] of numerous optimization problems including, for example, constraint satisfaction problems [2, 14, 27, 40].

The standard measure of stability of a set is the probability that positively correlated standard Gaussian vectors both lie in the set. The main result in this area, which is used in all of the applications mentioned above, is that half-spaces have optimal stability among all sets with a given Gaussian measure. This fact was originally proved by Borell [9], in a difficult proof using Ehrhard symmetrization. Recently, two different proofs of Borell's result have emerged. First, Isaksson and the first author [18] applied some recent advances in spherical symmetrization [10] to give an proof that also generalizes to a problem involving more than two Gaussian vectors. Then Kindler and O'Donnell [28], using the subadditivity idea of Kane [22], gave a short and elegant proof, but only for sets of measure $1/2$ and for some special values of the correlation.

E. Mossel: University of Pennsylvania and University of California at Berkeley;
e-mail: mossel@wharton.upenn.edu

J. Neeman: University of Texas at Austin; e-mail: jneeman@stat.berkeley.edu

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In this paper, we will give a novel proof of Borell's result. In doing so, we answer a question posed 18 years ago by Ledoux [30], who used semigroup methods to show that Borell's inequality implies the Gaussian isoperimetric inequality and then asked whether similar methods could be used to give a short and direct proof of Borell's inequality. Moreover, our proof will allow us to strengthen Borell's result and its discrete applications. First, we will demonstrate that half-spaces are the *unique* optimizers of Gaussian stability (up to almost sure equality). Then we will quantify this statement, by showing that if the stability of a set is close to optimal given its measure, then the set must be close to a half-space.

The questions of equality and robustness of isoperimetric inequalities can be rather more subtle than the inequalities themselves. In the case of the standard Gaussian isoperimetric result, it took about 25 years from the time the inequality was established [8, 41] before the equality cases were fully characterized [11] (although the equality cases among sufficiently nice sets were known earlier [15]). Robust versions of the standard Gaussian isoperimetric result were first established only recently [12, 35]. Here, for the first time since Borell's original proof [9] more than 25 years ago, we establish both that half-spaces are the unique maximizers *and* that a robust version of this statement is also true.

1.1. Discrete applications

From our Gaussian results, we derive robust versions of some of the main discrete applications of Borell's result, including a robust version of the "majority is stablest" theorem [36]. The "majority is stablest" theorem concerns subsets A of the discrete cube $\{-1, 1\}^n$ with the property that each coordinate x_i has only a small *influence* on whether $x \in A$ (see [36] for a precise definition); the theorem says that over all such sets A, the ones that are most noise stable take the form $\{x : \sum a_i x_i \leq b\}$. From the results we prove here, it is possible to obtain a robust version of this, which says that any sets $A \subset \{-1, 1\}^n$ with small coordinate influences and almost optimal noise sensitivity must be close to some set of the form $\{x : \sum a_i x_i \leq b\}$.

A robust form of the "majority is stablest" theorem immediately implies a robust version of the quantitative Arrow theorem. In economics, Arrow's theorem [1] says that any non-dictatorial election system between three candidates which satisfies two natural properties (namely, the "independence of irrelevant alternatives" and "neutrality") has a chance of producing a non-rational outcome. (By non-rational outcome, we mean that there are three candidates, A, B and C say, such that candidate A is preferred to candidate B, B is preferred to C and C is preferred to A.) Kalai [19, 20] showed that if the election system is such that each voter has only a small influence on the outcome, then the probability of a non-rational outcome is substantial; moreover, the "majority is stablest" theorem [36] implies that the probability of a non-rational outcome can be minimized by using a simple majority vote to decide, for each pair of candidates, which one is preferred. A robust version of the "majority is stablest" theorem implies immediately that (weighted) majority-based voting methods are essentially the only low-influence methods that minimize the probability of a non-rational outcome.

In a different direction, our robust noise stability result has an application in hardness of approximation, specifically in the analysis of the well-known Max-Cut optimization problem. The Max-Cut problem seeks a partition of a graph G into two pieces such that the number of edges from one piece to the other is maximal. This problem is NP-hard [24] but Goemans and Williamson [17] gave an approximation algorithm with an approximation ratio of about 0.878. Their algorithm works by embedding the graph G in a high-dimensional sphere and then cutting it using a random hyperplane. Feige and Schechtman [16] showed that a random hyperplane is the optimal way to cut this embedded graph; with our robust noise stability theorem, we can show that any almost-optimal cutting procedure is almost the same as using a random hyperplane. The latter result is derived via a novel isoperimetric result for spheres in high dimensions where two points are connected if their inner product is exactly some prescribed number ρ.

1.2. Borell's theorem and a functional variant

Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. For $-1 < \rho < 1$ let X and Y be jointly Gaussian random vectors on $\mathbb{R}^n$ such that X and Y are standard Gaussian vectors and $\mathbb{E}X_iY_j = \delta_{ij}\rho$. We will write $\Pr_\rho$ for the joint probability distribution of X and Y. We will also write ϕ for the density of γ_1 and Φ for its distribution function:

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad \Phi(x) = \int_{-\infty}^{x} \phi(y)\,dy.$$

Theorem 1.1 (Borell [9]). *For any $0 < \rho < 1$ and any measurable $A_1, A_2 \subset \mathbb{R}^n$,*

$$\Pr_\rho(X \in A_1,\ Y \in A_2) \le \Pr_\rho(X \in B_1,\ Y \in B_2) \tag{1.1}$$

where

$$B_1 = \{x \in \mathbb{R}^n : x_1 \le \Phi^{-1}(\gamma_n(A_1))\}, \qquad B_2 = \{x \in \mathbb{R}^n : x_1 \le \Phi^{-1}(\gamma_n(A_2))\}$$

are parallel half-spaces with the same volumes as A_1 and A_2 respectively. If $-1 < \rho < 0$ then the inequality (1.1) is reversed.

Like many other inequalities about sets, Theorem 1.1 has a functional analogue. To state it, we define the function

$$J(x, y) = J(x, y; \rho) = \Pr_\rho(X_1 \le \Phi^{-1}(x),\ Y_1 \le \Phi^{-1}(y)).$$

Theorem 1.2. *For any measurable functions $f, g : \mathbb{R}^n \to [0, 1]$ and any $0 < \rho < 1$,*

$$\mathbb{E}_\rho J(f(X), g(Y); \rho) \le J(\mathbb{E}f, \mathbb{E}g; \rho) \tag{1.2}$$

If $-1 < \rho < 0$ then the inequality (1.2) is reversed.

To see that Theorem 1.2 generalizes Theorem 1.1, consider $f = 1_{A_1}$ and $g = 1_{A_2}$. Note that $J(0, 0) = J(1, 0) = J(0, 1) = 0$, while $J(1, 1) = 1$. Thus, $J(f(X), g(Y)) = 1_{X \in A_1, Y \in A_2}$ and so the left hand side (resp. right hand side) of Theorem 1.2 is the same as the left hand side (resp. right hand side) of Theorem 1.1.

In fact, we can also go in the other direction and prove Theorem 1.2 from Theorem 1.1: given $f, g : \mathbb{R}^n \to [0, 1]$, define $A_1, A_2 \subset \mathbb{R}^{n+1}$ to be the subgraphs of $\Phi^{-1} \circ f$ and $\Phi^{-1} \circ g$ respectively. It can be easily checked, then, that

$$\mathbb{E}_\rho J(f(X), g(Y); \rho) = \Pr_\rho(\tilde{X} \in A_1, \; \tilde{Y} \in A_2)$$

where $\tilde{X}$ and $\tilde{Y}$ are standard Gaussian vectors on $\mathbb{R}^{n+1}$ with $\mathbb{E}\tilde{X}_i \tilde{Y}_i = \delta_{ij}\rho$. On the other hand, $\mathbb{E}f = \gamma_{n+1}(A_1)$ and $\mathbb{E}g = \gamma_{n+1}(A_2)$ and so the definition of J implies that

$$J(\mathbb{E}f, \mathbb{E}g; \rho) = \Pr_\rho(\tilde{X} \in B_1, \; \tilde{Y} \in B_2)$$

where B_1 and B_2 are parallel half-spaces with the same volumes as A_1 and A_2. Thus, Theorem 1.1 in $n + 1$ dimensions implies Theorem 1.2 in n dimensions.

However, we will give a proof of Theorem 1.2 that does not rely on Theorem 1.1. We do this for two reasons: first, we believe that our proof of Theorem 1.2 is simpler than existing proofs of Theorem 1.1. More importantly, our proof of Theorem 1.2 is a good starting point for the main results of the paper. In particular, it allows us to characterize the cases of equality and near-equality. As we mentioned earlier, it is not known how to get such results from existing proofs of Theorem 1.1.

1.3. New results: Equality

In our first main result, we get a complete characterization of the functions for which equality in Theorem 1.2 is attained.

Theorem 1.3. *For any measurable functions $f, g : \mathbb{R}^n \to [0, 1]$ and any $-1 < \rho < 1$ with $\rho \neq 0$, if equality is attained in (1.2) then there exist $a, b, d \in \mathbb{R}^n$ such that either*

$$f(x) = \Phi(\langle a, x - b\rangle) \quad a.s., \qquad g(x) = \Phi(\langle a, x - d\rangle) \quad a.s.,$$

or

$$f(x) = 1_{\{\langle a, x-b\rangle \geq 0\}} \quad a.s., \qquad g(x) = 1_{\{\langle a, x-d\rangle \geq 0\}} \quad a.s.$$

In particular, the second case of Theorem 1.3 implies that if A_1 and A_2 achieve equality in Theorem 1.1 then A_1 and A_2 must be almost surely equal to parallel half-spaces.

1.4. New results: Robustness

Once we know the cases of equality, the next natural thing to ask is whether they are robust: if f and g almost achieve equality in (1.2)—in the sense that $\mathbb{E}_\rho J(f(X), g(Y)) \geq J(\mathbb{E}f, \mathbb{E}g) - \delta$—does it follow that f and g must be close to some functions of the form $\Phi(\langle a, x - b\rangle)$? In the case of the Gaussian isoperimetric inequality, which can be viewed as a limiting form of Borell's theorem, the question of robustness was first addressed by Cianchi et al. [12], who showed that the answer was "yes," and gave a bound that depended on both δ and n. The authors [35] then proved a similar result which had no dependence on n, but a worse (logarithmic, instead of polynomial) dependence on δ. The arguments we will apply here are similar to those used in [35], but with some improvements. In particular, we establish a result with no dependence on the dimension, and with a polynomial dependence on δ (although we suspect that the exponent is not optimal).

Theorem 1.4. *For measurable functions* $f, g : \mathbb{R}^n \to [0, 1]$, *define*

$$\delta = \delta(f, g) = J(\mathbb{E}f, \mathbb{E}g) - \mathbb{E}_\rho J(f(X), g(Y)) \tag{1.3}$$

and let

$$m = m(f, g) = \mathbb{E}f(1 - \mathbb{E}f)\mathbb{E}g(1 - \mathbb{E}g).$$

For any $0 < \rho < 1$, *there exists* $C(\rho) < \infty$ *such that for any* $f, g : \mathbb{R}^n \to [0, 1]$ *there exist* $a, b, d \in \mathbb{R}^n$ *such that*

$$\mathbb{E}|f(X) - \Phi(\langle a, X - b\rangle)| \le C(\rho)m^{-C(\rho)}\delta^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}},$$
$$\mathbb{E}|g(X) - \Phi(\langle a, X - d\rangle)| \le C(\rho)m^{-C(\rho)}\delta^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}}.$$

We should mention that a more careful tracking of constants in our proof would improve the exponent of δ slightly. However, this improvement would not bring the exponent above $1/4$ and it would not prevent the exponent from approaching zero as $\rho \to 1$.

Although Theorem 1.4 is stated only for $0 < \rho < 1$, the same result for $-1 < \rho < 0$ follows from certain symmetries. Indeed, one can easily check from the definition of J that $J(x, y; \rho) = x - J(x, 1 - y; -\rho)$. Taking expectations yields

$$\begin{aligned}\mathbb{E}_\rho J(f(X), g(Y); \rho) &= \mathbb{E}f - \mathbb{E}_\rho J(f(X), 1 - g(Y); -\rho)\\ &= \mathbb{E}f - \mathbb{E}_{-\rho} J(f(X), 1 - g(-Y); -\rho).\end{aligned}$$

Now, suppose that $-1 < \rho < 0$ and that f, g almost attain equality in Theorem 1.2:

$$\mathbb{E}_\rho J(f(X), g(Y); \rho) \le J(\mathbb{E}f, \mathbb{E}g; \rho) + \delta.$$

Setting $\tilde{g}(y) = 1 - g(-y)$ implies that

$$\mathbb{E}_{-\rho} J(f(X), \tilde{g}(Y); -\rho) \ge J(\mathbb{E}f, \mathbb{E}\tilde{g}; -\rho) - \delta.$$

Since $0 < -\rho < 1$, we can apply Theorem 1.4 to f and $\tilde{g}$ to conclude that f and $\tilde{g}$ are close to the equality cases of Theorem 1.3, and it follows that f and g are also close to one of these equality cases. Therefore, we will concentrate for the rest of this article on the case $0 < \rho < 1$.

1.5. Optimal dependence on ρ *in the case* $f = g$

The dependence on ρ in Theorem 1.4 is particularly interesting as $\rho \to 1$, since it is in that limit that Borell's inequality recovers the Gaussian isoperimetric inequality. As it is stated, however, Theorem 1.4 does not recover a robust version of the Gaussian isoperimetric inequality because of its poor dependence on ρ as $\rho \to 1$. In particular, as $\rho \to 1$, the exponent of δ tends to zero and the constant $C(\rho)$ tends to infinity.

It turns out that a poor dependence on ρ is necessary in some sense. To see this, take $n = 1$, $A = [2, \infty)$ and $B = [-1, 0] \cup [1, \infty)$. If $B' = [0, \infty)$ then B' is a half-space with the same measure as B; hence,

$$\delta(A, B) = \Pr_\rho(X \in A,\ Y \in B') - \Pr(X \in A,\ Y \in B) \le \Pr(X \in A,\ Y \notin B).$$

Now, if $X \in A$ and $Y \notin B$ then $X - Y \geq 1$. But $X - Y$ is a mean-zero Gaussian variable with variance $2(1-\rho)$, and so

$$\delta(A, B) \leq \Pr_\rho(X - Y \geq 1) \leq e^{-c/(1-\rho)^2}.$$

On the other hand, the distance between B and the nearest half-space is some fixed constant. Hence, either the exponent of δ must decay like $(1-\rho)^2$ as $\rho \to 1$, or the constant in front of δ must grow like $e^{c/(1-\rho)^2}$.

We can, however, obtain much a much better dependence on ρ if we restrict to the case $f = g$. In this case, it turns out that $\delta(f, f)$ grows only like $(1-\rho)^{-1/2}$ as $\rho \to 1$, which is exactly the right rate for recovering the Gaussian isoperimetric inequality.

Theorem 1.5. *For every $\epsilon > 0$, there is a $\rho_0 < 1$ and a $C(\epsilon)$ such that for any $\rho_0 < \rho < 1$ and any $f : \mathbb{R}^n \to [0, 1]$ with $\mathbb{E} f = 1/2$, there exists $a \in \mathbb{R}^n$ such that*

$$\mathbb{E}|f(X) - \Phi(\langle a, X\rangle)| \leq C(\epsilon) \left(\frac{\delta(f, f)}{\sqrt{1-\rho}}\right)^{1/4-\epsilon}.$$

The requirement $\mathbb{E} f = 1/2$ is there for technical reasons, and we do not believe that it is necessary (see Conjecture 6.9).

By applying Ledoux's result [31] connecting Borell's inequality with the Gaussian isoperimetric inequality, Theorem 1.5 has the following corollary (for the definition of Gaussian surface area, see [35]):

Corollary 1.6. *For every $\epsilon > 0$, there is a $C(\epsilon) < \infty$ such that for every set $A \subset \mathbb{R}^n$ such that $\Pr(A) = 1/2$ and A has Gaussian surface area less than $1/\sqrt{2\pi} + \delta$, there is a half-space B such that*

$$\Pr(A \,\triangle\, B) \leq C(\epsilon)\delta^{1/4-\epsilon}.$$

This should be compared with the work of Cianchi et al. [12], who gave the best possible dependence on δ, but suffered some unspecified dependence on n:

Theorem 1.7. *For every n and every $a \in (0, 1)$, there is a constant $C(n, a)$ such that for every set $A \subset \mathbb{R}^n$ such that $\Pr(A) = a$ and A has Gaussian surface area less than $\phi(\Phi^{-1}(a)) + \delta$, there is a half-space B such that*

$$\Pr(A \,\triangle\, B) \leq C(n, a)\delta^{1/2}.$$

Note that Theorem 1.7 is stronger than Corollary 1.6 in two senses, but weaker in one. Theorem 1.7 is stronger since it applies to sets of all volumes and because it has a better dependence on δ (in fact, Cianchi et al. show that $\delta^{1/2}$ is the best possible dependence on δ). However, Corollary 1.6 is stronger in the sense that it—like the rest of our robustness results—has no dependence on the dimension. For the applications we have in mind, this dimension independence is more important than having optimal rates. Nevertheless, we conjecture that it is possible to have both at the same time:

Conjecture 1.8. *There is a universal constant C such that for every $A \subset \mathbb{R}^n$ with Gaussian surface area less than $\phi(\Phi^{-1}(\Pr(A))) + \delta$, there is a half-space B such that*

$$\Pr(A \,\triangle\, B) \leq C \Pr(A)^{-C}\delta^{1/2}.$$

1.6. *On highly correlated functions*

Let us mention one more corollary of Theorem 1.5. We have used $\mathbb{E}_\rho J(f(X), f(Y))$ as a functional generalization of $\Pr_\rho(X \in A,\ Y \in A)$. However, $\mathbb{E}_\rho f(X) f(Y)$ is another commonly used functional generalization of $\Pr_\rho(X \in A,\ Y \in A)$ which appeared, for example, in [31]. Since $xy \le J(x, y)$ for $0 < \rho < 1$, we see immediately that Theorem 1.2 holds when the left hand side is replaced by $\mathbb{E}_\rho f(X) f(Y)$. The equality case, however, turns out to be different: whereas equality in Theorem 1.2 holds for $f(x) = \Phi(\langle a, x-b\rangle)$, there is equality in

$$\mathbb{E}_\rho f(X) f(Y) \le J(\mathbb{E} f, \mathbb{E} f; \rho) \tag{1.4}$$

only when f is the indicator of a half-space. Moreover, a robustness result for (1.4) follows fairly easily from Theorems 1.4 and 1.5.

Corollary 1.9. *For any $0 < \rho < 1$, there is a constant $C(\rho) < \infty$ such that if a function $f : \mathbb{R}^n \to [0, 1]$ satisfies $\mathbb{E} f = 1/2$ and*

$$\mathbb{E} f(X) f(Y) \ge \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho - \delta$$

then there is a half-space B such that

$$\mathbb{E}|f(X) - 1_B(X)| \le C(\rho)\delta^c,$$

where $c > 0$ is a universal constant.

1.7. *Discrete applications*

Corollary 1.9 implies a robust version of the "majority is stablest" theorem [36], which concerns functions of low influence and high noise stability; for a function $f : \{-1, 1\}^n \to \{-1, 1\}$, we define the *influence* of the ith coordinate by

$$\mathrm{Inf}_i(f) = \Pr\big(f(x_1, \ldots, x_n) \ne f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)\big)$$

and the *noise stability* of f by

$$\mathbb{S}_\rho(f) = \mathbb{E}_\rho f(\xi) f(\sigma)$$

where $(\xi, \sigma) = ((\xi_1, \ldots, \xi_n), (\sigma_1, \ldots, \sigma_n)) \in \{-1, 1\}^n \times \{-1, 1\}^n$ is chosen so that $(\xi_i, \sigma_i) \in \{-1, 1\}^2$ are independent random variables with $\mathbb{E}\xi_i = \mathbb{E}\sigma_i = 0$ and $\mathbb{E}_\rho \xi_i \sigma_i = \rho$.

The "majority is stablest" theorem [36] informally states that low-influence, balanced functions cannot be essentially more noise stable than the majority function. This was first explicitly conjectured by Khot, Kindler, Mossel, and O'Donnell [27] in a paper studying the hardness of approximation of Max-Cut. It was used to show that approximating the maximum cut in a graph to within a factor of about 0.87856 is unique-games hard. This result is optimal, since the famous efficient algorithm of Goemans and Williamson [17] is guaranteed to find a cut that is within a 0.87856 factor of the maximum cut. A special case of the "majority is stablest" theorem was conjectured earlier by Kalai [19] in the context of his quantitative version of Arrow's theorem.

Combining our Gaussian results with the original proof from [36], we obtain a robust version of the "majority is stablest" theorem:

Theorem 1.10. *For every $\delta > 0$, there is a $\tau > 0$ such that the following holds: suppose that $f : \{-1, 1\}^n \to [0, 1]$ is a function with $\mathrm{Inf}_i(f) \le \tau$ for every i. Then for every $0 < \rho < 1$,*

$$\mathbb{S}_\rho(f) \le J(\mathbb{E}f, \mathbb{E}f; \rho) + \delta. \tag{1.5}$$

If, moreover, there is some $0 < \rho < 1$ such that

$$\mathbb{S}_\rho(f) \ge J(\mathbb{E}f, \mathbb{E}f; \rho) - \delta \tag{1.6}$$

then there exist $a, b \in \mathbb{R}^n$ such that

$$\mathbb{E}|f(\xi) - 1_{\{\langle a, \xi - b\rangle \ge 0\}}| \le C(\rho)\delta^{c(\rho)},$$

where $c(\rho), C(\rho) > 0$ are constants depending only on ρ.

If we set $a_n = (1/\sqrt{n})(1, \dots, 1)$ and $b_n = \Phi^{-1}(\mathbb{E}f)a_n$, then the central limit theorem implies that $\mathbb{E}1_{\{\langle a_n, \xi - b_n\rangle \ge 0\}} \to \mathbb{E}f$ and $\mathbb{S}_\rho(1_{\{\langle a_n, \xi - b_n\rangle \ge 0\}}) \to J(\mathbb{E}f, \mathbb{E}f; \rho)$. In the case $\mathbb{E}f = 1/2$ and $b_n = 0$, (1.5) says, therefore, that no low-influence function can be much more noise stable than the simple majority function—this is the content of the "majority is stablest" theorem from [36]. Our contribution is (1.6), which says that the only low-influence functions which come close to this bound are close to weighted majority functions.

We remark that Theorem 1.10 is not stated in the most general possible form that we can prove. In particular, we could state a two-function version of Theorem 1.10, or a version that uses the functional $\mathbb{E}_\rho J(f(\xi), f(\sigma); \rho)$ in place of $\mathbb{S}_\rho(f)$. All of these variations, however, are proved in essentially the same way, namely by combining the ideas from [36] with the appropriate Gaussian robustness result. In order to avoid repetition, therefore, we will only state and prove one version.

1.8. Spherical noise stability and the Max-Cut problem

The well-known similarity between a Gaussian vector and a uniformly random vector on a high-dimensional sphere suggests that there might be a spherical analogue of our Gaussian noise sensitivity result. The correlation structure on the sphere that is most useful is the uniform measure over all pairs of points (x, y) whose inner product $\langle x, y\rangle$ is exactly ρ. Under this model of noise, we can use robust Gaussian noise sensitivity to show, asymptotically in the dimension, robustness for spherical noise sensitivity. This uses the theory of spherical harmonics and has applications to rounding semidefinite programs (in particular, the Goemans–Williamson algorithm for Max-Cut). Our proof uses and generalizes the work of Klartag and Regev [29], in which a related problem was studied in the context of one-way communication complexity.

Our spherical noise stability result mostly follows from Theorem 1.4, by replacing X and Y by $X/|X|$ and $Y/|Y|$. When n is large, these renormalized Gaussian vectors are uniformly distributed on the sphere and their inner product is tightly concentrated around ρ. The fact that their inner product is not *exactly* ρ causes some difficulty, particularly because Q_ρ is actually orthogonal to the joint distribution of two normalized Gaussians. Working through this difficulty with some properties of spherical harmonics, we obtain the following spherical analogue of Theorem 1.4:

Theorem 1.11. *Let* $0 < \rho < 1$ *and write* Q_ρ *for the measure of* (X, Y) *on the sphere* S^{n-1} *where the pair* (X, Y) *is uniformly distributed in*

$$\{(x, y) \in S^{n-1} \times S^{n-1} : \langle x, y\rangle = \rho\}.$$

For measurable $A_1, A_2 \subset S^{n-1}$*, define*

$$\delta = \delta(A_1, A_2) = Q_\rho(X \in B_1,\ Y \in B_2) - Q_\rho(X \in A_1,\ Y \in A_2),$$

where B_1 *and* B_2 *are parallel spherical caps with the same volumes as* A_1 *and* A_2 *respectively. Define also*

$$m(A_1, A_2) = p(1-p)q(1-q),$$

where $p = \Pr(X \in A_1)$ *and* $q = \Pr(Y \in A_2)$.

For any $A_1, A_2 \subset S^{n-1}$*, there exist parallel spherical caps* B_1 *and* B_2 *such that*

$$Q(A_1 \triangle B_1) \le C(\rho) m^{-C(\rho)} \delta_*^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}},$$

$$Q(A_2 \triangle B_2) \le C(\rho) m^{-C(\rho)} \delta_*^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}},$$

where $\delta_* = \max(\delta, n^{-1/2}\log n)$.

The case $\rho = 0$ of the above theorem is related to work by Klartag and Regev [29]. In this case one expects that X and Y should behave as independent random variables on S^{n-1} and that therefore for *all* A_1, A_2, $Q_0(X \in A_1,\ Y \in A_2)$ should be close to $Q(X \in A_1)Q(Y \in A_2)$. Indeed the main technical statement of Klartag and Regev (Theorem 5.2) says that for any two sets A_1, A_2,

$$|Q_0(X \in A_1,\ Y \in A_2) - Q(X \in A_1)Q(Y \in A_2)| \le C/n.$$

In other words the results of Klartag and Regev show that in the case $\rho = 0$, a uniform orthogonal pair (X, Y) on the sphere behaves like a pair of independent random variables up to an error of order n^{-1}, while our results show that for $0 < \rho < 1$, (X, Y) that are ρ correlated behave like Gaussians with the same correlation.

That spherical caps minimize the quantity $Q_\rho(X \in A_1, Y \in A_2)$ over all sets A_1 and A_2 with some prescribed volumes is originally due to Baernstein and Taylor [3], while a similar result for a different noise model is due to Beckner [5]. Their results do not follow from ours because of the dependence on n in Theorem 1.11, and so one could ask for a sharper version of Theorem 1.11 that does imply these earlier results. One obstacle is that we do not know a proof of Beckner's inequality that gives control of the deficit.

1.8.1. Rounding the Goemans–Williamson algorithm. Let $G = (V, E)$ be a graph and recall that the *Max-Cut problem* is to find a set $A \subset V$ such that the number of edges between A and $V \setminus A$ is maximal. It is of course equivalent to look for a function $f : V \to \{-1, 1\}$ such that $\sum_{(u,v)\in E} |f(u) - f(v)|^2$ is maximal. Goemans and Williamson's breakthrough was to realize that this combinatorial optimization problem can be efficiently solved if we relax the range $\{-1, 1\}$ to S^{n-1}. Let us say, therefore, that an embedding f of a graph $G = (V, E)$ into the sphere S^{n-1} is *optimal* if

$$\sum_{(u,v)\in E} |f(u) - f(v)|^2$$

is maximal. An oblivious rounding procedure is a (possibly random) function $R : S^{n-1} \to \{-1, 1\}$ (we call it "oblivious" because it does not look at the graph G). We will then denote by $\mathrm{Cut}(G, R)$ the expected value of the cut produced by rounding the worst possible optimal spherical embedding of G:

$$\mathrm{Cut}(G, R) = \frac{1}{2} \min_f \mathbb{E} \sum_{(u,v)\in E} |R(f(u)) - R(f(v))|,$$

where the minimum is over all optimal embeddings f. If MaxCut denotes the maximum cut in G, then Goemans and Williamson [17] showed that when $R(x) = \mathrm{sgn}(\langle X, x\rangle)$ for a standard Gaussian vector X, then for every graph G,

$$\mathrm{Cut}(G, R) \geq \mathrm{MaxCut}(G) \min_\theta \alpha_\theta,$$

where $\alpha_\theta = \frac{2}{\pi} \frac{\theta}{1-\cos\theta}$. In the other direction, Feige and Schechtman [16] showed that for every oblivious rounding scheme R and every $\epsilon > 0$, there is a graph G such that

$$\mathrm{Cut}(G, R) \leq \mathrm{MaxCut}(G)\Big(\epsilon + \min_\theta \alpha_\theta\Big).$$

In other words, no rounding scheme is better than the half-space rounding scheme. Using Theorem 1.4, we can go further:

Theorem 1.12. *Suppose R is a rounding scheme on S^{n-1} such that for every graph G with n vertices,*

$$\mathrm{Cut}(G, R) \geq \mathrm{MaxCut}(G)\Big(\min_\theta \alpha_\theta - \epsilon\Big).$$

Then there is a hyperplane rounding scheme $\tilde{R}$ such that

$$\mathbb{E}|R(Y) - \tilde{R}(Y)| \leq C\epsilon_\star^c,$$

where Y is a uniform (independent of R and $\tilde{R}$) random vector on S^{n-1}, C and c are absolute constants, and $\epsilon_\star = \max\{\epsilon, n^{-1/2}\log n\}$.

In other words, any rounding scheme that is almost optimal is essentially the same as rounding by a random half-space.

1.9. Testing half-spaces

We quickly sketch an application of Theorems 1.4 and 1.10 to testing. Suppose we are given oracle access to a set $A \subset \mathbb{R}^n$ (meaning that we are not given an explicit representation of the set, but we can query whether points belong to A), and we want to design an algorithm that (1) will answer "yes" with high probability if A is a half-space and (2) will answer "no" with high probability if $\Pr(A \mathbin{\triangle} B) > \epsilon$ for all half-spaces B.

An efficient test for this problem was found in [34]. We note that Theorem 1.5 provides a simpler and very direct test just by sampling ϵ^{-5} pairs (X_i, Y_i) and counting the number of times that $X_i \in A$ and the number of times that $1_A(X_i) = 1_A(Y_i)$. By doing so, we obtain accurate estimates of $\Pr(A)$ and $\Pr(X \in A,\ Y \in A)$ and so by Theorem 1.5, we can tell whether A is close to a half-space.

By Theorem 1.10, this algorithm also applies to linear threshold functions with low influences on the discrete cube (such functions are called regular in [34]). (By the more general arguments in [36], the algorithm also applies to other discrete spaces such as half-spaces in biased cubes or cubes of the form $[q]^n$ for some $q \geq 3$.) Using the arguments of [34] it is then possible to extend the testing algorithm to general linear threshold functions on the discrete cube.

1.10. Proof techniques

1.10.1. Borell's theorem. We prove Theorem 1.2 by differentiating along the Ornstein–Uhlenbeck semigroup. This technique was used by Bakry and Ledoux [4] in their proof of the Gaussian isoperimetric inequality and, more generally, a Gaussian version of the Lévy–Gromov comparison theorem. Recall that the Ornstein–Uhlenbeck semigroup can be specified by defining, for every $t \geq 0$, the operator

$$(P_t f)(x) = \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1-e^{-2t}}\, y\big)\, d\gamma_n(y). \tag{1.7}$$

Note that $P_t f \to f$ as $t \to 0$ (pointwise, and also in L_p), while $P_t f \to \mathbb{E} f$ as $t \to \infty$.

Let $f_t = P_t f$, $g_t = P_t g$, and consider the quantity

$$R_t := \mathbb{E}_\rho J(f_t(X), g_t(Y)). \tag{1.8}$$

As $t \to 0$, R_t converges to the right hand side of (1.2); as $t \to \infty$, R_t converges to the left hand side of (1.2). We will prove Theorem 1.2 by showing that $dR_t/dt \geq 0$ for all $t > 0$.

1.10.2. The equality case. The equality case almost comes for free from our proof of Theorem 1.2. Indeed, Lemma 2.2 exprsses dR_t/dt as the expectation of a strictly positive quantity times

$$|(\nabla(\Phi^{-1} \circ f_t))(X) - (\nabla(\Phi^{-1} \circ g_t))(Y)|,$$

where $|\cdot|$ denotes the Euclidean norm. Now, if there is equality in Theorem 1.2 then dR_t/dt must be zero for all t, which implies that the expression above must be zero almost surely. This implies that $\nabla(\Phi^{-1} \circ f_t)$ and $\nabla(\Phi^{-1} \circ g_t)$ are almost surely equal to the same constant, and therefore f_t and g_t can be written as Φ composed with a linear function. We can then infer the same statement for f and g because P_t is one-to-one.

1.10.3. Robustness. Our approach to robustness begins similarly to the approach in our recent work [35]. If $\delta(f, g)$ is small then dR_t/dt must also be small for most $t > 0$.

Looking at the expression in Lemma 2.2 we first concentrate on the main term: $|\nabla v_t(X) - \nabla w_t(Y)|^2$ where $v_t = \Phi^{-1} \circ f_t$ and $w_t = \Phi^{-1} \circ g_t$. Using an analogue of Poincaré's inequality, we argue that if the expected value of $|\nabla v_t(X) - \nabla w_t(Y)|^2$ is small then v_t and w_t are close to linear functions.

Considerable effort goes into controlling the "secondary terms" of the expression in Lemma 2.2. This control is established in a sequence of analytic results, which rely heavily on the smoothness of the semigroup P_t, concentration of Gaussian vectors and L_p interpolation inequalities. In the end, we show that if $\delta = \delta(f, g)$ is small then for every $t > 0$, v_t is $\epsilon(\delta, t)$-close to a linear function. Since Φ is a contraction, this implies that f_t must be close to a function of the form $\Phi(\langle x, a\rangle - b)$.

We would like to then conclude the proof by applying P_t^{-1}, and saying that f must be close to $P_t^{-1}\Phi(\langle x, a\rangle - b)$, which also has the form $\Phi(\langle x, a'\rangle - b')$. The obvious problem here is that P_t^{-1} is not a bounded operator, but we work around this by arguing that it acts boundedly on the functions that we care about. This part of the argument marks a substantial departure from [35], where our argument used smoothness and spectral information. Here, we will use a geometric argument to say that if $h = 1_A - 1_B$ where B is a half-space, then $\mathbb{E}|h|$ can be bounded in terms of $\mathbb{E}|P_t h|$. This improved argument is essentially the reason that the rates in Theorem 1.4 are polynomial, while the rates in [35] were logarithmic.

1.11. Subsequent work

A quite different study of the functional $\mathbb{E}_\rho J(f(X), g(Y); \rho)$ turns out to yield yet another proof of Borell's inequality: in a subsequent work with De [13], the authors give a proof of Borell's inequality by first proving a four-point inequality for J which tensorizes to the discrete cube. Applying the central limit theorem then recovers Borell's inequality. That approach is similar to Bobkov's elementary proof of the Gaussian isoperimetric inequality [7]. The proof in [13] has an advantage and a disadvantage compared to the one presented here. The advantage of the tensorization argument is that it directly yields some interesting inequalities on the cube (in particular, one obtains a direct proof of the "majority is stablest" theorem), while the proof we present here has the advantage of giving control over the deficit. In particular, we do not know how to prove Theorem 1.4 using the techniques in [13].

2. Proof of Borell's theorem

Recall the definition of P_t and R_t from (1.7) and (1.8). In this section, we will compute dR_t/dt and show that it is non-negative, thereby proving Theorem 1.2. First, define $v_t = \Phi^{-1} \circ f_t$, $w_t = \Phi^{-1} \circ g_t$, and $K(x, y; \rho) = \Pr_\rho(X \le x,\ Y \le b)$. Then

$$J(f_t(X), g_t(Y)) = K(v_t(X), w_t(Y)).$$

Lemma 2.1.

$$\frac{\partial K(x, y)}{\partial x} = \phi(x)\Phi\left(\frac{y - \rho x}{\sqrt{1 - \rho^2}}\right), \qquad \frac{\partial K(x, y)}{\partial y} = \phi(y)\Phi\left(\frac{x - \rho y}{\sqrt{1 - \rho^2}}\right).$$

Proof. Note that Y can be written as $\rho X + \sqrt{1-\rho^2}\,Z$, where X and Z independent standard Gaussian vectors. Then $\{X \le x,\ Y \le y\} = \bigl\{X \le x,\ Z \le \frac{y-\rho X}{\sqrt{1-\rho^2}}\bigr\}$, and so

$$K(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{\frac{y-\rho s}{\sqrt{1-\rho^2}}} \phi(s)\phi(t)\,dt\,ds.$$

Differentiating in x gives

$$\frac{\partial K(x,y)}{\partial x} = \int_{-\infty}^{\frac{y-\rho x}{\sqrt{1-\rho^2}}} \phi(x)\phi(t)\,dt = \phi(x)\Phi\Bigl(\frac{y-\rho x}{\sqrt{1-\rho^2}}\Bigr).$$

This proves the first claim. The second claim follows because $K(x,y)$ is symmetric in x and y. □

Lemma 2.2.

$$\frac{dR_t}{dt} = \frac{\rho}{2\pi\sqrt{1-\rho^2}}\mathbb{E}_\rho \exp\Bigl(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\Bigr)|\nabla v_t - \nabla w_t|^2.$$

Before we prove Lemma 2.2, note that it immediately implies Theorem 1.2 because the right hand side in Lemma 2.2 is clearly non-negative.

Proof. Set $L = \Delta - \langle x, \nabla\rangle$; it is well-known (and easy to check by direct computation) that $df_t/dt = Lf_t$ for all $t \ge 0$. The integration by parts formula

$$\mathbb{E} f(X) Lg(X) = -\mathbb{E}\langle \nabla f(X), \nabla g(X)\rangle \tag{2.1}$$

for bounded smooth functions f and g is also standard and easily checked. Thus,

$$\frac{dR_t}{dt} = \mathbb{E}_\rho\Bigl(K_x(v_t(X), w_t(Y))\frac{dv_t(X)}{dt}\Bigr) + \mathbb{E}_\rho\Bigl(K_y(v_t(X), w_t(Y))\frac{dw_t(X)}{dt}\Bigr). \tag{2.2}$$

Now, the chain rule implies that $\frac{dv_t}{dt} = \frac{Lf_t}{\phi(v_t)}$. Hence, the first term of (2.2) is

$$\mathbb{E}_\rho\Bigl(\frac{K_x(v_t(X), w_t(Y)}{\phi(v_t(X))}Lf_t(X)\Bigr) = \mathbb{E}_\rho\Phi\Bigl(\frac{w_t(Y) - \rho v_t(X)}{\sqrt{1-\rho^2}}\Bigr)Lf_t(X), \tag{2.3}$$

where we have used Lemma 2.1. Now write $Y = \rho X + \sqrt{1-\rho^2}\,Z$ (with X and Z independent); conditioning on Z and applying the integration by parts (2.1) with respect to X, we have

$$\begin{aligned}(2.3) &= -\frac{\rho}{\sqrt{1-\rho^2}}\mathbb{E}_\rho\phi\Bigl(\frac{w_t - \rho v_t}{\sqrt{1-\rho^2}}\Bigr)\langle \nabla w_t - \nabla v_t, \nabla f_t\rangle \\ &= \frac{\rho}{\sqrt{1-\rho^2}}\mathbb{E}_\rho\phi\Bigl(\frac{w_t - \rho v_t}{\sqrt{1-\rho^2}}\Bigr)\phi(v_t)\langle \nabla v_t - \nabla w_t, \nabla v_t\rangle, \end{aligned} \tag{2.4}$$

where we have written, for brevity, v_t and w_t instead of $v_t(X)$ and $w_t(Y)$. Since K is symmetric in its arguments, there is a similar computation for the second term of (2.2):

$$\mathbb{E}\left(K_y(v_t(X), w_t(Y))\frac{dw_t(X)}{dt}\right) = -\frac{\rho}{\sqrt{1-\rho^2}}\mathbb{E}_\rho\phi\left(\frac{v_t - \rho w_t}{\sqrt{1-\rho^2}}\right)\phi(w_t)\langle\nabla v_t - \nabla w_t, \nabla w_t\rangle. \quad (2.5)$$

Note that

$$\phi\left(\frac{w_t - \rho v_t}{\sqrt{1-\rho^2}}\right)\phi(v_t) = \phi\left(\frac{v_t - \rho w_t}{\sqrt{1-\rho^2}}\right)\phi(w_t) = \frac{1}{2\pi}\exp\left(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\right);$$

hence, we can plug (2.4) and (2.5) into (2.2) to obtain

$$\frac{dR_t}{dt} = \frac{\rho}{2\pi\sqrt{1-\rho^2}}\mathbb{E}\exp\left(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\right)|\nabla v_t - \nabla w_t|^2. \qquad \square$$

3. The equality case

Lemma 2.2 allows us to analyze the equality case (Theorem 1.3), with very little additional effort. Similar ideas were used by Carlen and Kerce [11] to analyze the equality case in the standard Gaussian isoperimetric problem. Clearly, Lemma 2.2 implies that if for every t, v_t and w_t are linear functions with the same slope, then equality is attained in Theorem 1.2. To prove Theorem 1.3, we will show that the converse also holds (i.e. if equality is attained then v_t and w_t are linear functions with the same slope). Then we will take $t \to 0$ to obtain the desired conclusion regarding f and g.

First of all, if $f(x) = 1_{\{\langle a, x-b\rangle \geq 0\}}$, then a direct computation gives

$$f_t(x) = \Phi\left(k_t\frac{\langle a, x - e^t b\rangle}{|a|}\right), \quad (3.1)$$

where $k_t = (e^{2t} - 1)^{-1/2}$. Since P_t is injective, it follows that whenever $f_t = \Phi(\langle a, x - b'\rangle)$ for some a, b with $|a| = k_t$, f must have the form $f(x) = 1_{\{\langle a, x-b\rangle \geq 0\}}$. Since, moreover, k_t is decreasing in t, we have the following lemma:

Lemma 3.1. *If $f_t(x) = \Phi(\langle a, x - b'\rangle)$ for some $a, b' \in \mathbb{R}^n$ with $|a| \leq k_t$, then there exists $b \in \mathbb{R}^n$ such that if $\tilde{f}(x) = 1_{\{\langle a, x-b\rangle \geq 0\}}$ then $f = P_s\tilde{f}$, where s solves $|a| = k_{s+t}$.*

In order to apply Lemma 3.1, we will use the following pointwise bound on ∇v_t, whose proof can be found in [4]. Note that the bound is sharp because, according to (3.1), equality is attained when f is the indicator function of a half-space.

Lemma 3.2. *For any function $f : \mathbb{R}^n \to [0, 1]$, any $t > 0$, and any $x \in \mathbb{R}^n$,*

$$|\nabla v_t(x)| \leq k_t.$$

Proof of Theorem 1.3. Suppose that equality is attained in (1.2). Since dR_t/dt is non-negative, it must be zero for almost every $t > 0$. In particular, we may fix some $t > 0$ such that $dR_t/dt = 0$. Note that everything in Lemma 2.2 is strictly positive, except for the last term, which can be zero. Therefore, $dR_t/dt = 0$ implies that $\nabla v_t(X) = \nabla w_t(Y)$ almost surely. Since the conditional distribution of Y given X is fully supported, ∇v_t and ∇w_t must be almost surely equal to some constant $a \in \mathbb{R}^n$. Moreover, v_t and w_t are smooth functions (because f_t, g_t and Φ^{-1} are smooth); hence, $v_t(x) = \langle a, x - b'\rangle$ and $w_t(x) = \langle a, x - d'\rangle$ for some $b', d' \in \mathbb{R}^n$, and so

$$f_t(x) = \Phi(\langle a, x - b'\rangle), \qquad g_t(x) = \Phi(\langle a, x - d'\rangle).$$

Now, Lemma 3.2 asserts that $|a| = |\nabla v_t| \le k_t$. Hence, Lemma 3.1 implies that there are $b \in \mathbb{R}$ and $s \ge 0$ such that if $\tilde{f}(x) = 1_{\{\langle a, x-b\rangle \ge 0\}}$ then $f = P_s \tilde{f}$, where s solves $|a| = k_{s+t}$. In particular, f takes one of the two forms indicated in Theorem 1.3: if $s = 0$ then $f(x) = \tilde{f}(x) = 1_{\{\langle a, x-b\rangle \ge 0\}}$. On the other hand, $s > 0$ implies, by (3.1), that $f_s = \Phi(k_s\langle a/|a|, x - e^s b\rangle)$, which we can write in the form $\Phi(\langle a, x - b\rangle)$ by replacing $k_s a/|a|$ with a and $k_s e^s b$ with b. We complete the proof by applying the same argument to g. ◻

4. Robustness: approximation for large t

The proof of Theorem 1.4 follows the same general lines as the one in [35]. Our starting point is Lemma 2.2, and the observation that if (1.2) is close to an equality then dR_t/dt must be small for most t. For such t, using Lemma 2.2, we will argue that v_t must be close to linear for that t; it then follows that f_t must be close to one of the equality cases in Theorem 1.3. Finally, we use a time-reversal argument to show that f must be close to one of those equality cases also.

Our proof will be divided into two main parts. In this section, we will show that v_t is close to linear; we will give the time-reversal argument in Section 5. The main result in this section, therefore, is Proposition 4.1, which says that f_t must be close to one of the equality cases of Theorem 1.3. Recall the definition of δ from (1.3), and recall that $k_t = (e^{2t} - 1)^{1/2}$.

Proposition 4.1. *For any $0 < \rho < 1$, and for any $t > 0$, there exists $C(t, \rho)$ such that for any f, g and for any $0 < \alpha < 1$, there exist $b, d \in \mathbb{R}$ and $a \in \mathbb{R}^n$ with $|a| \le k_t$ such that*

$$\mathbb{E}\big(f_t(X) - \Phi(\langle a, X\rangle - b)\big)^2 + \mathbb{E}\big(g_t(X) - \Phi(\langle a, X\rangle - d)\big)^2 \\ \le C(t, \rho) m(f, g)^{C(t,\rho)} \left(\frac{\delta}{\alpha}\right)^{\frac{1}{1+4k_t^2/(1-\rho)} \frac{1}{1+\alpha}}$$

where $m(f, g) = \mathbb{E}f(1 - \mathbb{E}f)\mathbb{E}g(1 - \mathbb{E}g)$.

Let us observe—and this will be important when we apply Proposition 4.1—that by Lemma 3.1, $|a| \le k_t$ implies that $\Phi(\langle a, \cdot\rangle - b)$ can be written in the form $P_{t+s} 1_B$ for some $s > 0$ and some half-space B.

The main goal of this section is to prove Proposition 4.1. The proof proceeds according to the following steps:

- First, using a Poincaré-like inequality (Lemma 4.2) we show that if the quantity $\mathbb{E}_\rho |\nabla v(X) - \nabla w(Y)|^2$ is small then v and w are close to linear functions (with the same slope).
- In Proposition 4.3, we use the reverse Hölder inequality and some concentration properties to show that if dR_t/dt is small, then $\mathbb{E}_\rho |\nabla v_t(X) - \nabla w_t(Y)|^{2p}$ must be small for some $p < 1$.
- Using Lemma 3.2, we argue that if the quantity $\mathbb{E}_\rho |\nabla v_t(X) - \nabla w_t(Y)|^{2p}$ is small then $\mathbb{E}_\rho |\nabla v_t(X) - \nabla w_t(Y)|^2$ is also small. Thus, we can apply the Poincaré inequality mentioned in the first bullet point, and so we obtain linear approximations for v_t and w_t.

4.1. A Poincaré-like inequality

Recall that we proved the equality case by arguing that if $dR_t/dt = 0$ then $|\nabla v_t(X) - \nabla w_t(Y)|$ is identically zero, so ∇v_t and ∇w_t must be constant and thus v_t and w_t must be linear. The first step towards a robustness result is to show that if $|\nabla v_t(X) - \nabla w_t(Y)|$ is small, then v_t and w_t must be almost linear, and with the same slope.

Lemma 4.2. *For any smooth functions $v, w \in L_2(\mathbb{R}^n, \gamma_n)$, if we set $a = \frac{1}{2}(\mathbb{E}\nabla v + \mathbb{E}\nabla w)$ then for any $0 < \rho < 1$,*

$$\mathbb{E}\big(v(X) - \langle X, a\rangle - \mathbb{E}v\big)^2 + \mathbb{E}\big(w(X) - \langle X, a\rangle - \mathbb{E}w\big)^2 \le \frac{\mathbb{E}_\rho |\nabla v(X) - \nabla w(Y)|^2}{2(1-\rho)}.$$

We remark that Lemma 4.2 achieves equality when v and w are quadratic polynomials which differ only in the constant term.

In order to prove Lemma 4.2, we recall the Hermite polynomials: for $k \in \mathbb{N}$, define $H_k(x) = (k!)^{-1/2} e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$. It is well-known that the H_k form an orthonormal basis of $L_2(\mathbb{R}, \gamma_1)$. For a multiindex $\alpha \in \mathbb{N}^n$, let

$$H_\alpha(x) = \prod_{i=1}^n H_{\alpha_i}(x_i).$$

Then the H_α form an orthonormal basis of $L^2(\mathbb{R}^n, \gamma_n)$. Define $|\alpha| = \sum_i \alpha_i$; note that H_α is linear if and only if $|\alpha| = 1$, and $\alpha_i = 0$ implies that $\frac{\partial}{\partial x_i} H_\alpha = 0$. If $\alpha_i > 0$, define $S_i\alpha$ by $(S_i\alpha)_i = \alpha_i - 1$ and $(S_i\alpha)_j = \alpha_j$ for $j \ne i$. Then a well-known recurrence for Hermite polynomials states that

$$\frac{\partial}{\partial x_i} H_\alpha = \begin{cases} \sqrt{\alpha_i}\, H_{S_i\alpha} & \text{if } \alpha_i > 0,\\ 0 & \text{if } \alpha_i = 0.\end{cases}$$

In particular,

$$\mathbb{E}\left(\frac{\partial}{\partial x_i} H_\alpha\right)^2 = \alpha_i. \tag{4.1}$$

It will be convenient for us to reparametrize the Ornstein–Uhlenbeck semigroup P_t: for $0 < \rho < 1$, let $T_\rho = P_{\log(1/\rho)}$. It is then easily checked that for any $v \in L_1(\mathbb{R}^n, \gamma_n)$, $\mathbb{E}_\rho(v(Y) \mid X) = (T_\rho v)(X)$.

The final piece of background that we need before proving Lemma 4.2 is the fact that T_ρ acts diagonally on the Hermite basis, with

$$T_\rho H_\alpha = \rho^{|\alpha|} H_\alpha. \tag{4.2}$$

Proof of Lemma 4.2. First, consider two arbitrary functions $b(x), c(x) \in L_2(\mathbb{R}^n, \gamma_n)$ and suppose that their expansions in the Hermite basis are $b = \sum_\alpha b_\alpha H_\alpha$ and $c = \sum_\alpha c_\alpha H_\alpha$. Then

$$\begin{aligned}\mathbb{E}_\rho(b(X) - c(Y))^2 &= \mathbb{E}b^2 + \mathbb{E}c^2 - 2\mathbb{E}_\rho b(X)c(Y) = \mathbb{E}b^2 + \mathbb{E}c^2 - 2\mathbb{E}b(X)(T_\rho c)(X) \\ &= \sum_\alpha (b_\alpha^2 + c_\alpha^2 - 2\rho^{|\alpha|} b_\alpha c_\alpha),\end{aligned}$$

where we have used (4.2) in the last line to compute the Hermite expansion of $T_\rho c$. Now, $2b_\alpha c_\alpha \le b_\alpha^2 + c_\alpha^2$ and so

$$\begin{aligned}\mathbb{E}_\rho(b(X) - c(Y))^2 &= (b_0 - c_0)^2 + \sum_{|\alpha|\ge 1} (b_\alpha^2 + c_\alpha^2 - 2\rho^{|\alpha|} b_\alpha c_\alpha) \\ &\ge (b_0 - c_0)^2 + \sum_{|\alpha|\ge 1} (b_\alpha^2 + c_\alpha^2)(1 - \rho^{|\alpha|}) \\ &\ge (b_0 - c_0)^2 + (1 - \rho) \sum_{|\alpha|\ge 1} (b_\alpha^2 + c_\alpha^2).\end{aligned} \tag{4.3}$$

Now write v and w in the Hermite basis as $v = \sum v_\alpha H_\alpha$ and $w = \sum w_\alpha H_\alpha$. Then, by (4.1),

$$\frac{\partial v}{\partial x_i} = \sum_{\alpha_i \ge 1} v_\alpha \sqrt{\alpha_i}\, H_{S_i\alpha}, \qquad \frac{\partial w}{\partial x_i} = \sum_{\alpha_i \ge 1} w_\alpha \sqrt{\alpha_i}\, H_{S_i\alpha}.$$

In particular, if we set $b = \partial v / \partial x_i$, then $b_{S_i\alpha} = \sqrt{\alpha_i}\, v_\alpha$ for any α with $\alpha_i \ge 1$. Specifically, $b_0 = v_{e_i}$ (where e_i is the multi-index with 1 in position i and 0 elsewhere) and

$$\sum_{|\alpha|\ge 1} b_\alpha^2 = \sum_{|\alpha|\ge 2,\, \alpha_i \ge 1} b_{S_i\alpha}^2 = \sum_{|\alpha|\ge 2,\, \alpha_i\ge 1} \alpha_i v_\alpha^2$$

(Setting $c = \partial w / \partial x_i$, there is of course an analogous inequality for c and w.) Applying this to (4.3), we have

$$\mathbb{E}_\rho\left(\frac{\partial v}{\partial x_i}(X) - \frac{\partial w}{\partial x_i}(Y)\right)^2 \ge (v_{e_i} - w_{e_i})^2 + (1 - \rho) \sum_{|\alpha|\ge 2,\, \alpha_i \ge 1} \alpha_i (v_\alpha^2 + w_\alpha^2). \tag{4.4}$$

Now if we apply (4.4) for each i and sum the resulting inequalities, we obtain

$$\mathbb{E}_\rho|\nabla v(X) - \nabla w(Y)|^2 \geq \sum_{|\alpha|=1} (v_\alpha - w_\alpha)^2 + 2(1-\rho) \sum_{|\alpha|\geq 2} (v_\alpha^2 + w_\alpha^2). \tag{4.5}$$

On the other hand, let $a = \frac{1}{2}(\mathbb{E}\nabla v + \mathbb{E}\nabla w)$. Since $\mathbb{E}\frac{\partial v}{\partial x_i} = v_{e_i}$ and $H_{e_i}(x) = x_i$, it follows that

$$\langle x, a\rangle = \frac{1}{2} \sum_{|\alpha|=1} (v_\alpha + w_\alpha) H_\alpha(x).$$

Since $\mathbb{E}v = v_0$, we have

$$\mathbb{E}(v(X) - \langle X, a\rangle - \mathbb{E}v)^2 = \sum_{|\alpha|=1} \left(\frac{v_\alpha - w_\alpha}{2}\right)^2 + \sum_{|\alpha|\geq 2} v_\alpha^2.$$

Adding to this the analogous expression for w, we obtain

$$\begin{aligned} 2(1-\rho)\big(&\mathbb{E}(v(X) - \langle X, a\rangle - \mathbb{E}v)^2 + \mathbb{E}(w(X) - \langle X, a\rangle - \mathbb{E}w)^2\big) \\ &= (1-\rho) \sum_{|\alpha|=1} (v_\alpha - w_\alpha)^2 + 2(1-\rho) \sum_{|\alpha|\geq 2} (v_\alpha^2 + w_\alpha^2). \end{aligned}$$

Noting that $1 - \rho \leq 1$, we see that this is smaller than (4.5). Hence

$$\mathbb{E}\big(v(X) - \langle X, a\rangle - \mathbb{E}v\big)^2 + \mathbb{E}\big(w(X) - \langle X, a\rangle - \mathbb{E}w\big)^2 \leq \frac{\mathbb{E}_\rho|\nabla v(X) - \nabla w(Y)|^2}{2(1-\rho)}. \quad \square$$

4.2. A lower bound on dR_t/dt

Recall the formula for dR_t/dt given in Lemma 2.2. In this section, we will use the reverse-Hölder inequality to split this formula into an exponential term and a term depending on $|\nabla v_t(X) - \nabla w_t(Y)|$. We will then use the smoothness of v_t and w_t to bound the exponential term, with the following result:

Proposition 4.3. *For any $0 < \rho < 1$ and any $t > 0$, there is a $c(t, \rho) > 0$ such that for any $r \leq \frac{1}{1+4k_t^2/(1-\rho)}$ and for any f and g,*

$$\frac{dR_t}{dt} \geq c(t,\rho) m^{4\frac{k_t^2(1+k_t)^2}{1-\rho}} \left(\mathbb{E}|\nabla v_t(X) - \nabla w_t(Y)|^{2r}\right)^{1/r}.$$

There are three main ingredients in the proof of Proposition 4.3. The first is the reverse-Hölder inequality, which states that for any functions $f > 0$ and $g \geq 0$ and for any $\beta > 0$ and $0 < r < 1$ with $1/r - 1/\beta = 1$,

$$\mathbb{E}fg \geq (\mathbb{E}f^{-\beta})^{-1/\beta} (\mathbb{E}g^r)^{1/r}. \tag{4.6}$$

The second ingredient involves a well-known property of the Gaussian measure: the concentration of Lipschitz functions (see, e.g., [33]).

Lemma 4.4. *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is* 1*-Lipschitz with median* M *then for any* $t > 0$*,*

$$\Pr(f(X) \geq M + t) \leq \Phi(-t), \qquad \Pr(f(X) \leq M - t) \leq \Phi(-t).$$

By integrating out Lemma 4.4 in t, one obtains the following bound:

Lemma 4.5. *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is* 1*-Lipschitz with median* M *then for any* $\lambda < 1$*,*

$$\mathbb{E}\exp(\lambda f^2(X)/2) \leq \frac{2}{\sqrt{1-\lambda}} e^{\frac{\lambda}{2(1-\lambda)}M^2}.$$

Proof. Suppose without loss of generality that $M \geq 0$. By Lemma 4.4, $|f - M|$ is stochastically dominated by the absolute value of a Gaussian variable. That is, if Z is a standard Gaussian variable and $\Psi : [0, \infty) \to \mathbb{R}$ is an increasing, convex function then

$$\mathbb{E}\Psi(f) \leq \mathbb{E}\Psi(|f - M| + M) \leq \mathbb{E}\Psi(|Z| + M).$$

Now, if $\Psi(x) = e^{\lambda x^2/2}$—which is convex, symmetric in x, and increasing on $[0, \infty)$—then

$$\begin{aligned}\mathbb{E}\Psi(|Z| + M) &= \mathbb{E}\Psi(\max\{Z, -Z\} + M) = \mathbb{E}\max\{\Psi(Z + M), \Psi(-Z + M)\} \\ &\leq 2\mathbb{E}\Psi(Z + M).\end{aligned}$$

That is, we have $\mathbb{E}\Psi(f) \leq 2\mathbb{E}\Psi(Z + M)$. But for a Gaussian variable, we have

$$\mathbb{E}e^{\lambda(Z+M)^2/2} = \frac{1}{\sqrt{1-\lambda}} e^{\frac{\lambda}{2(1-\lambda)}M^2}.$$ □

The third and final ingredient in the proof of Proposition 4.3 is a relationship between the mean of f and the median of v_t.

Lemma 4.6. *If* N_t *is a median of* v_t *then*

$$m(f) = \mathbb{E}f(1 - \mathbb{E}f) \leq 2\exp\left(-\frac{N_t^2}{2(1 + k_t)^2}\right).$$

Proof. The proof essentially follows the one of Lemma 3.8 of [35]. Note that it is enough to show that if $N_t \leq 0$ then

$$\mathbb{E}f \leq 2\Phi\left(\frac{N_t}{1 + k_t}\right). \tag{4.7}$$

Indeed, since $m(f) \leq \mathbb{E}f$, the inequality $\Phi(x) \leq e^{-x^2/2}$ for $x \leq 0$ will complete the proof as long as $N_t \leq 0$; on the other hand, if $N_t > 0$ then we apply (4.7) to $1 - f$ instead of f. Note that replacing f by $1 - f$ changes the sign of N_t; also, the definition of $m(f)$ ensures that $m(f) \leq \mathbb{E}(1 - f)$.

To prove (4.7), recall that $v_t = \Phi^{-1}(f_t)$. Let $M_t = \Phi(N_t)$, so that M_t is a median of f_t. Then for any $\alpha < 1$, $\Pr(f_t \geq \Phi(\alpha N_t)) = \Pr(v_t \geq \alpha N_t)$. Recall from Lemma 3.2 that v_t is k_t-Lipschitz. Thus, by Lemma 4.4,

$$\Pr(f_t \geq \Phi(\alpha N_t)) = \Pr(v_t \geq \alpha N_t) = \Pr(v_t \geq N_t + (1 - \alpha)N_t) \leq \Phi\left(\frac{(1 - \alpha)N_t}{k_t}\right).$$

Setting $\alpha = 1/(1 + k_t)$, we have $(1 - \alpha)/k_t = \alpha$. Thus, $\Pr(f_t \geq \Phi(\alpha N_t)) \leq \Phi(\alpha N_t)$. Since $f_t \leq 1$, Markov's inequality implies that $\mathbb{E}f_t \leq 2\Phi(\alpha N_t)$. Recalling that $\alpha = 1/(1 + k_t)$, this proves (4.7). □

Proof of Proposition 4.3. We begin by applying the reverse-Hölder inequality (4.6) to the equation in Lemma 2.2:

$$\frac{dR_t}{dt} \geq \frac{\rho}{2\pi\sqrt{1-\rho^2}}\left(\mathbb{E}_\rho \exp\left(\beta\frac{v_t^2+w_t^2-2\rho v_t w_t}{2(1-\rho^2)}\right)\right)^{-1/\beta}(\mathbb{E}_\rho|\nabla v_t - \nabla w_t|^{2r})^{1/r} \tag{4.8}$$

with β and r yet to be determined. Let us first consider the exponential term in (4.8). Since $2|v_t w_t| \leq v_t^2 + w_t^2$, we have

$$\begin{aligned}\mathbb{E}_\rho \exp\left(\beta\frac{v_t^2+w_t^2-2\rho v_t w_t}{2(1-\rho^2)}\right) &\leq \mathbb{E}_\rho \exp\left(\beta\frac{v_t^2+w_t^2}{2(1-\rho)}\right)\\ &\leq \left(\mathbb{E}\exp\left(\beta\frac{v_t^2}{1-\rho}\right)\mathbb{E}\exp\left(\beta\frac{w_t^2}{1-\rho}\right)\right)^{1/2},\end{aligned} \tag{4.9}$$

where we used the Cauchy–Schwarz inequality in the last line. Recall from Lemma 3.2 that v_t and w_t are both k_t-Lipschitz. Thus, we can apply Lemma 4.5 with $f = v_t/k_t$ and $\lambda = 2\beta k_t^2/(1-\rho)$; we see that if $\lambda = 2\beta k_t^2/(1-\rho) \leq 1/2$, then

$$\mathbb{E}\exp\left(\beta\frac{v_t^2}{1-\rho}\right) \leq Ce^{\lambda M_t^2},$$

where M_t is a median of v_t. Applying the same argument to w_t and plugging the result into (4.9), we have

$$\mathbb{E}_\rho \exp\left(\beta\frac{v_t^2+w_t^2-2\rho v_t w_t}{2(1-\rho^2)}\right) \leq Ce^{\lambda(M_t^2+N_t^2)},$$

where N_t is a median of w_t. Going back to (4.8), we have

$$\frac{dR_t}{dt} \geq \frac{c^{-1/\beta}\rho}{\sqrt{1-\rho^2}}e^{-\frac{\lambda}{\beta}(M_t^2+N_t^2)}(\mathbb{E}_\rho|\nabla v_t - \nabla w_t|^{2r})^{1/r}, \tag{4.10}$$

with (recall) $\lambda = 2\beta k_t^2/(1-\rho) \leq 1/2$; hence, $\beta \leq \frac{1}{4}(1-\rho)/k_t^2$. Recalling that $1/r - 1/\beta = 1$, we see that (4.10) holds for any $r \leq \frac{1}{1+4k_t^2/(1-\rho)}$. Finally, we invoke Lemma 4.6 to show that

$$\exp\left(-\frac{\lambda}{\beta}M_t^2\right) = \exp\left(-\frac{2k_t^2M_t^2}{1-\rho}\right) \geq (c\mathbb{E}f(1-\mathbb{E}f))^{4\frac{k_t^2(1+k_t)^2}{1-\rho}}$$

(and similarly for g and N_t). Plugging this into (4.10) completes the proof. □

4.3. *Proof of Proposition 4.1*

We are now prepared to prove Proposition 4.1 by combining Proposition 4.3 with Lemmas 3.2 and 4.2. Besides combining these three results, there is a small technical obstacle: we know only that the integral of dR_t/dt is small; we do not know anything about dR_t/dt

at specific values of t. So instead of showing that v_t is close to linear for every t, we will show that for every t, there is a nearby t^* such that v_{t^*} is close to linear. By ensuring that t^* is close to t, we will then be able to argue that v_t is also close to linear.

Proof of Proposition 4.1. For any $0 < r < 1$, Lemma 3.2 implies that

$$(\mathbb{E}_\rho|\nabla v_t - \nabla w_t|^{2r})^{1/r} \geq \frac{(\mathbb{E}|\nabla v_t - \nabla w_t|^2)^{1/r}}{(2k_t)^{2(1-r)/r}}.$$

By Lemma 4.2 applied to v_t and w_t, if we set $a = \frac{1}{2}(\mathbb{E}\nabla v_t + \mathbb{E}\nabla w_t)$ and we define $\epsilon(v_t) = \mathbb{E}(v_t(X) - \langle X, a\rangle - \mathbb{E}v)^2$ (and similarly for $\epsilon(w_t)$), then

$$(\epsilon(v_t) + \epsilon(w_t))^{1/r} \leq \frac{(2k_t)^{2(1-r)/r}}{(2(1-\rho))^{1/r}}(\mathbb{E}_\rho|\nabla v_t - \nabla w_t|^{2r})^{1/r}.$$

Now we plug this into Proposition 4.3 to obtain

$$(\epsilon(v_t) + \epsilon(w_t))^{1/r} \leq C(t,\rho)m^{-4\frac{k_t^2(1+k_t)^2}{1-\rho}}\frac{dR_t}{dt} \leq C(t,\rho)m^{-C(t,\rho)}\frac{dR_t}{dt}. \tag{4.11}$$

Recall that $\delta(f,g) = \int_0^\infty \frac{dR_s}{ds}\,ds$. In particular,

$$\alpha t \min_{t\leq s\leq t(1+\alpha)} \frac{dR_t}{dt}\bigg|_s \leq \int_t^{t(1+\alpha)} \frac{dR_s}{ds}\,ds \leq \delta(f,g)$$

and so there is some $s \in [t, t(1+\alpha)]$ such that $\frac{dR_t}{dt}\big|_s \leq \frac{\delta}{\alpha t}$. If we apply this to (4.11) with t replaced by s and with $r = \frac{1}{1+4k_t^2/(1-\rho)} \leq \frac{1}{1+4k_s^2/(1-\rho)}$, we obtain

$$\epsilon(v_s) + \epsilon(w_s) \leq C(t,\rho)m^{-rC(t,\rho)}(\delta/\alpha)^r.$$

Since Φ is Lipschitz, if we denote $\mathbb{E}(f_s - \Phi(\langle X, a\rangle - \mathbb{E}v_s))^2$ by $\epsilon(f_s)$ (and similarly for g_s), then we have

$$\epsilon(f_s) + \epsilon(g_s) \leq C(t,\rho)m^{-rC(t,\rho)}(\delta/\alpha)^r \leq C(t,\rho)m^{-C(t,\rho)}(\delta/\alpha)^r, \tag{4.12}$$

where the second inequality follows because r depends only on t and ρ, so it can be absorbed into the constant $C(t,\rho)$.

Now we will need a lemma to show that $\epsilon(f_t)$ and $\epsilon(f_t)$ are small. We will prove the lemma after this proof is complete.

Lemma 4.7. *For any $t < s$ and any $h \in L_2(\mathbb{R}^n, \gamma_n)$,*

$$\mathbb{E}(P_t h)^2 \leq (\mathbb{E}(P_s h)^2)^{t/s}(\mathbb{E}h^2)^{1-t/s}.$$

To complete the proof of Proposition 4.1, we apply Lemma 4.7 with $h = f - P_s^{-1}\Phi(\langle X, a\rangle - \mathbb{E}v_s)$ (note that $P_s^{-1}\Phi(\langle X, a\rangle - \mathbb{E}v_s)$ exists by Lemma 3.1, because $|a| \le k_s$). Since $\mathbb{E}h^2 \le \sup|h| \le 1$ and $s \le (1+\alpha)t$, we see that

$$\epsilon(f_t) = \mathbb{E}(P_t h)^2 \le (\mathbb{E}(P_s h)^2)^{t/s} = \epsilon(f_s)^{1/(1+\alpha)}.$$

Applying this (and the equivalent inequality for g) to (4.12), we have

$$\epsilon(f_t) + \epsilon(g_t) \le C(t,\rho)^{1/(1+\alpha)} m^{-C(t,\rho)/(1+\alpha)} (\delta/\alpha)^{r/(1+\alpha)},$$

where $\epsilon(f_t)$ means $\mathbb{E}(f_t - P_{s-t}^{-1}\Phi(\langle X, a\rangle - \mathbb{E}v_s))^2$ and similarly for $\epsilon(g_t)$. Since $\alpha < 1$, we have $1/2 \le 1/(1+\alpha) \le 1$ and so we can absorb the power $1/(1+\alpha)$ into the constant $C(t,\rho)$. □

Proof of Lemma 4.7. Expand $P_s h$ in the Hermite basis as $P_s h = \sum b_\alpha H_\alpha$. Then

$$\mathbb{E}(P_s h)^2 = \sum b_\alpha^2, \quad \mathbb{E}(P_t h)^2 = \sum b_\alpha^2 e^{2(s-t)|\alpha|}, \quad \mathbb{E}h^2 = \sum b_\alpha^2 e^{2s|\alpha|}.$$

By Hölder's inequality applied with the exponents s/t and $s/(s-t)$,

$$\begin{aligned}\mathbb{E}(P_t h)^2 &= \sum b_\alpha^{(s-t)/s} e^{2(s-t)|\alpha|} b_\alpha^{t/s} \le \Big(\sum b_\alpha^2 e^{2s|\alpha|}\Big)^{(s-t)/s} \Big(\sum b_\alpha^2\Big)^{t/s} \\ &= (\mathbb{E}h^2)^{(s-t)/s} (\mathbb{E}(P_s h)^2)^{t/s}.\end{aligned}$$

□

5. Robustness: time-reversal

The final step in proving Theorem 1.4 is to show that the conclusion of Proposition 4.1 implies that f and g are close to one of the equality cases. In [35], the authors used a spectral argument. However, that spectral argument was responsible for the logarithmically slow rates (in δ) that [35] showed. Here, we use a better time-reversal argument that gives polynomial rates. The argument here will need the function f to take values only in $\{0, 1\}$. Thus, we will first establish Theorem 1.4 for sets; having done so, it is not difficult to extend it to functions using the equivalence, described in Section 1.4, between the set and functional forms of Borell's theorem.

The main goal of a time-reversal argument is to bound $\mathbb{E}|h|$ from above in terms of $\mathbb{E}|P_t h|$, for some function h. The difficulty is that such bounds are not possible for general h. An illuminating example is the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \operatorname{sgn}(\sin(kx))$: on the one hand, $\mathbb{E}|h| = 1$; on the other, $\mathbb{E}|P_t h|$ can be made arbitrarily small by taking k large.

The example above is problematic because there is a lot of cancellation in $P_t h$. The essence of this section is that for the functions h we are interested in, there is a geometric reason which disallows too much cancellation. Indeed, we are interested in functions h of the form $1_A - 1_B$ where B is a half-space. The negative part of such a function is supported on B, while the positive part is supported on B^c. As we will see, this fact allows us to bound the amount of cancellation that occurs, and thus obtain a time-reversal result:

Proposition 5.1. *Let B be a half-space and A be any other set. There is an absolute constant C such that for any $t > 0$,*

$$\gamma(A \triangle B) \le C \max\bigl\{\mathbb{E}|P_t 1_A - P_t 1_B|, (e^{2t}-1)^{1/4}\sqrt{\mathbb{E}|P_t 1_A - P_t 1_B|}\bigr\},$$

The main idea in Proposition 5.1 is in the following lemma, which states that if a non-negative function is supported on a half-space then P_t will push strictly less than half of its mass onto the complementary half-space.

Lemma 5.2. *There is a constant $c > 0$ such that for any $b \in \mathbb{R}$, if $f : \mathbb{R}^n \to [0, 1]$ is supported on $\{x_1 \le b\}$ then for any $t > 0$,*

$$\mathbb{E}(P_t f) 1_{\{X_1 \ge e^{-t} b\}} \le \max\left\{\frac{1}{2}\mathbb{E}f - c\frac{(\mathbb{E}f)^2}{\sqrt{e^{2t}-1}}, \frac{3}{8}\mathbb{E}f\right\}.$$

Proof. Because P_t is self-adjoint,

$$\mathbb{E}(P_t f) 1_{\{X_1 \ge e^{-t} b\}} = \mathbb{E} f P_t 1_{\{X_1 \ge e^{-t} b\}} = \mathbb{E} f \Phi\left(\frac{X_1 - b}{\sqrt{e^{2t}-1}}\right).$$

Now, the set $\{b - \mathbb{E}f \le x_1 \le b\}$ has measure at most $\phi(0)\mathbb{E}f$. In particular, $\mathbb{E}f 1_{\{b-\mathbb{E}f \le x_1 \le b\}} \le \phi(0)\mathbb{E}f \le \frac{1}{2}\mathbb{E}f$.

Let $A = \{x_1 \le b - \mathbb{E}f\}$ and $B = \{b - \mathbb{E}f \le x_1 \le b\}$ and recall that f is supported on $\{x_1 \le b\}$, so that $f = f(1_A + 1_B)$. Now,

$$\Phi\left(\frac{x_1 - b}{\sqrt{e^{2t}-1}}\right) \le \begin{cases} \Phi\bigl(-\frac{\mathbb{E}f}{\sqrt{e^{2t}-1}}\bigr), & x \in A, \\ \frac{1}{2}, & x \in B, \end{cases}$$

and so

$$\begin{aligned} \mathbb{E} f \Phi\left(\frac{X_1 - b}{\sqrt{e^{2t}-1}}\right) &= \mathbb{E} 1_A f \Phi\left(\frac{X_1 - b}{\sqrt{e^{2t}-1}}\right) + \mathbb{E} 1_B f \Phi\left(\frac{X_1 - b}{\sqrt{e^{2t}-1}}\right) \\ &\le \Phi\left(-\frac{\mathbb{E}f}{\sqrt{e^{2t}-1}}\right)\mathbb{E}1_A f + \frac{1}{2}\mathbb{E}1_B f \\ &= \frac{1}{2}\mathbb{E}f - \left(\frac{1}{2} - \Phi\left(-\frac{\mathbb{E}f}{\sqrt{e^{2t}-1}}\right)\right)\mathbb{E}f 1_A. \end{aligned} \tag{5.1}$$

There is a constant $c > 0$ such that $\Phi(-x) \le \max\{1/2 - cx, 1/4\}$ for all $x \ge 0$. Applying this with $x = \frac{\mathbb{E}f}{\sqrt{e^{2t}-1}}$, we have

$$(5.1) \le \frac{1}{2}\mathbb{E}f - \mathbb{E}f 1_A \min\left\{c\frac{\mathbb{E}f}{\sqrt{e^{2t}-1}}, \frac{1}{4}\right\} \le \max\left\{\frac{1}{2}\mathbb{E}f - c\frac{(\mathbb{E}f)^2}{\sqrt{e^{2t}-1}}, \frac{3}{8}\mathbb{E}f\right\}$$

where for the last inequality, recall that $\mathbb{E}f 1_A \ge \frac{1}{2}\mathbb{E}f$. □

Proof of Proposition 5.1. Without loss of generality, B is the half-space $\{x_1 \le b\}$. Let f be the positive part of $1_A - 1_B$ and let g be the negative part, so that $\gamma(A \triangle B) = \mathbb{E}f + \mathbb{E}g$. Note that f is supported on B^c and g is supported on B.

Without loss of generality, $\mathbb{E}f \geq \mathbb{E}g$; Lemma 5.2 implies that if $\mathbb{E}f \leq C\sqrt{e^{2t}-1}$ then

$$2\mathbb{E}(1_B P_t f + 1_{B^c} P_t g) \leq \mathbb{E}f + \mathbb{E}g - c\frac{(\mathbb{E}f + \mathbb{E}g)^2}{\sqrt{e^{2t}-1}}. \tag{5.2}$$

On the other hand, if $\mathbb{E}f \geq C\sqrt{e^{2t}-1}$ then

$$2\mathbb{E}(1_B P_t f + 1_{B^c} P_t g) \leq \frac{3}{4}\mathbb{E}f + \mathbb{E}g \leq \frac{7}{8}(\mathbb{E}f + \mathbb{E}g). \tag{5.3}$$

Thus,

$$\begin{aligned}\mathbb{E}|P_t f - P_t g| &= \mathbb{E}P_t f + \mathbb{E}P_t g - 2\mathbb{E}\min\{P_t f, P_t g\} = \mathbb{E}f + \mathbb{E}g - 2\mathbb{E}\min\{P_t f, P_t g\} \\ &\geq \mathbb{E}f + \mathbb{E}g - 2\mathbb{E}(1_B P_t f + 1_{B^c} P_t g) \geq \min\left\{c\frac{(\mathbb{E}f + \mathbb{E}g)^2}{\sqrt{e^{2t}-1}}, \frac{\mathbb{E}f + \mathbb{E}g}{8}\right\},\end{aligned}$$

where we have applied (5.2) and (5.3) in the last inequality. Now there are two cases, depending on which term in the minimum is smaller: if the first term is smaller then

$$\mathbb{E}f + \mathbb{E}g \leq C(e^{2t}-1)^{1/4}\sqrt{\mathbb{E}|P_t f - P_t g|};$$

otherwise, the second term in the minimum is smaller and

$$\mathbb{E}f + \mathbb{E}g \leq 8\mathbb{E}|P_t f - P_t g|.$$

In either case,

$$\gamma(A \triangle B) = \mathbb{E}f + \mathbb{E}g \leq C\max\left\{\mathbb{E}|P_t f - P_t g|, (e^{2t}-1)^{1/4}\sqrt{\mathbb{E}|P_t f - P_t g|}\right\},$$

as claimed. □

5.1. Synchronizing the time-reversal

Proposition 5.1 would be enough if we knew that $\mathbb{E}(P_t 1_A - P_t 1_B)^2$ were small. Now, Proposition 4.1 and Lemma 3.1 imply that $\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2$ is small, for some $s \geq 0$. In this section, we will show that if $e^{-t} = \rho$ then s must be small. Now, this is not necessarily the case for arbitrary sets A; in fact, for any $s > 0$ one can find A such that $\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2$ is arbitrarily small. Fortunately, we have some extra information on A: we know that it is almost optimally noise stable with parameter ρ. In particular, if $e^{-t} = \rho$ then $\mathbb{E}1_A P_t 1_A$ is close to $\mathbb{E}1_B P_t 1_B$.

Using this extra information, the proof of robustness proceeds as follows: since $\mathbb{E}1_A P_t 1_A$ is close to $\mathbb{E}1_B P_t 1_B$ and $P_t 1_A$ is close to $P_{t+s} 1_B$, we will show that $\mathbb{E}1_B P_{t+s} 1_B$ is close to $\mathbb{E}1_B P_t 1_B$. But we know all about B: it is a half-space. Therefore, we can find explicit and accurate estimates for $\mathbb{E}1_B P_{t+s} 1_B$ and $\mathbb{E}1_B P_t 1_B$ in terms of t, s and $\gamma_n(B)$; using them, we can conclude that s is small. Now, if s is small then we can show (again, using explicit estimates) that $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ is small. Since $\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2$ is small (this was our starting point, remember), we can apply the triangle inequality to conclude that $\mathbb{E}(P_t 1_A - P_t 1_B)^2$ is small. Finally, we can apply Proposition 5.1 to show that $\mathbb{E}|1_A - 1_B|$ is small.

Proposition 5.3. *For every t, there is a $C(t)$ such that the following holds. For sets $A, A' \subset \mathbb{R}^n$, suppose that $B, B' \subset \mathbb{R}^n$ are parallel half-spaces with $\gamma(A) = \gamma(B)$ and $\gamma(A') = \gamma(B')$. If there exist $s, \epsilon_1, \epsilon_2 > 0$ such that*

$$\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2 \le \epsilon_1^2 \quad \text{and} \quad \mathbb{E} 1_A P_t 1_{A'} \ge \mathbb{E} 1_B P_t 1_{B'} - \epsilon_2$$

then

$$\big(\mathbb{E}(P_t 1_A - P_t 1_B)^2\big)^{1/2} \le C(t) \frac{\epsilon_1 + \epsilon_2}{\big(I(\gamma(A)) I(\gamma(A'))\big)^{C(t)}},$$

where $I(x) = \phi(\Phi^{-1}(x))$.

Rather than prove Proposition 5.3 all at once, we have split the part relating the quantities $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ and $\mathbb{E} 1_B (P_t 1_{B'} - P_{t+s} 1_{B'})$ into a separate lemma.

Lemma 5.4. *For every t there is a $C(t)$ such that for any parallel half-spaces B and B', and for every $s > 0$,*

$$\big(\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2\big)^{1/2} \le C(t) \frac{\mathbb{E} 1_B (P_t 1_{B'} - P_{t+s} 1_{B'})}{\big(I(\gamma(B)) I(\gamma(B'))\big)^{C(t)}}.$$

Proof. First of all, one can easily check through integration by parts that for a smooth function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_b^\infty \phi(x)(Lf)(x)\, dx = -f'(b)\phi(b). \tag{5.4}$$

By rotating B and B', we can assume that $B = \{x_1 \ge a\}$ and $B' = \{x_1 \ge b\}$. Let $F_{ab}(t) = \mathbb{E} 1_B P_t 1_{B'} = \int_a^\infty \phi(x) \Phi\big(\frac{e^{-t}x - b}{\sqrt{1-e^{-2t}}}\big)\, dx$ and consider its derivative: by (5.4),

$$\begin{aligned} F'_{ab}(t) &= \int_a^\infty \phi(x) L\Phi\bigg(\frac{e^{-t}x - b}{\sqrt{1-e^{-2t}}}\bigg) dx = -k_t \phi(a) \phi\bigg(\frac{e^{-t}a - b}{\sqrt{1-e^{-2t}}}\bigg) \\ &= -\frac{k_t}{2\pi} \exp\bigg(-\frac{a^2 + b^2 - 2e^{-t}ab}{2(1-e^{-2t})}\bigg) \le -\frac{k_t}{2\pi} \exp\bigg(-\frac{a^2+b^2}{1-e^{-2t}}\bigg). \end{aligned}$$

Now, k_t is decreasing in t and $\exp(-x/(1-e^{-2t}))$ is increasing in t. In particular, for any $\tau \in [t, t+s]$,

$$F'_{ab}(\tau) \le -\frac{k_{t+s}}{2\pi} \exp\bigg(-\frac{a^2+b^2}{1-e^{-2t}}\bigg).$$

Hence,

$$F_{ab}(t) - F_{ab}(t+s) \ge -s \max_{t \le \tau \le t+s} F'_{ab}(\tau) \ge \frac{s k_{t+s}}{2\pi} \exp\bigg(-\frac{a^2+b^2}{1-e^{-2t}}\bigg). \tag{5.5}$$

If s is large, this is a poor bound because $s k_{t+s}$ decreases exponentially in s. However, when $s \ge 1$ we can instead use

$$F_{ab}(t) - F_{ab}(t+s) \ge F_{ab}(t) - F_{ab}(t+1) \ge \frac{k_{t+1}}{2\pi} \exp\bigg(-\frac{a^2+b^2}{1-e^{-2t}}\bigg). \tag{5.6}$$

Equations (5.5) and (5.6) show that if $\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})$ is small then s must be small. The next step, therefore, is to control $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ in terms of s. Now,

$$\begin{aligned}\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2 &= \mathbb{E}\big((P_t 1_B)^2 + (P_{t+s} 1_B)^2 - 2(P_t 1_B)(P_{t+s} 1_B)\big)\\ &= \mathbb{E}1_B(P_{2t} 1_B + P_{2(t+s)} 1_B - 2P_{2t+s} 1_B)\\ &= \big(F_{aa}(2t) - F_{aa}(2t+s)\big) - \big(F_{aa}(2t+s) - F_{aa}(2t+2s)\big)\\ &\le s\big(F'_{aa}(2t) - F'_{aa}(2t+2s)\big), \end{aligned} \tag{5.7}$$

where the inequality follows because

$$F'_{aa}(t) = -\frac{k_t}{2\pi} \exp\left(-\frac{(1-e^{-t})a^2}{1-e^{-2t}}\right) = -\frac{k_t}{2\pi} \exp\left(-\frac{a^2}{1+e^{-t}}\right)$$

and so F'_{aa} is an increasing function. To control the right hand side of (5.7), we go to the second derivative of F_{aa}:

$$F''_{aa}(t) = \frac{e^{2t}}{2\pi(e^{2t}-1)^{3/2}} \exp\left(-\frac{a^2}{1+e^{-t}}\right) + \frac{1}{2\pi\sqrt{e^{2t}-1}} \frac{a^2 e^{-t}}{(1+e^{-t})^2} \exp\left(-\frac{a^2}{1+e^{-t}}\right)$$

This is decreasing in t; hence

$$\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2 \le s\big(F'_{aa}(2t) - F'_{aa}(2t+2s)\big) \le 2s^2 F''_{aa}(2t). \tag{5.8}$$

We will now complete the proof by combining our upper bound on $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ with our lower bounds on $\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})$. First, assume $s \le 1$. Then $k_{t+s} \ge k_{t+1}$ and so (5.5) plus (5.8) implies that

$$\begin{aligned}\big(\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2\big)^{1/2} &\le 2\pi \exp\left(\frac{a^2+b^2}{1-e^{-2t}}\right) \frac{\sqrt{2F''_{aa}(2t)}}{k_{t+1}} \mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})\\ &= 2\pi^{1-\frac{2}{1-e^{-2t}}} \frac{\sqrt{2F''_{aa}(2t)}}{k_{t+1}} \frac{\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})}{\big(I(\gamma(B)) I(\gamma(B'))\big)^{\frac{2}{1-e^{-2t}}}}.\end{aligned}$$

If we take $C(t) \ge \max\{\sqrt{2F''_{aa}(2t)}/k_{t+1}, 2/(1-e^{-2t})\}$ then the lemma holds in this case. On the other hand, if $s > 1$ then (5.6) implies that

$$\frac{2\pi^{1-\frac{2}{1-e^{-2t}}}}{k_{t+1}} \frac{\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})}{\big(I(\gamma(B)) I(\gamma(B'))\big)^{\frac{2}{1-e^{-2t}}}} \ge 1.$$

Since $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2 \le 1$ trivially, the lemma holds in this case provided that

$$C(t) \ge \max\{1/k_{t+1}, 2/(1-e^{-2t})\}. \qquad \square$$

Proof of Proposition 5.3. By the Cauchy–Schwarz inequality,

$$\mathbb{E}1_A P_t 1_A \le \mathbb{E}1_A P_{t+s} 1_B + \sqrt{\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2} \le \mathbb{E}1_A P_{t+s} 1_B + \epsilon_1.$$

Moreover, $\mathbb{E}1_A P_{t+s} 1_{B'} \le \mathbb{E}1_B P_{t+s} 1_{B'}$ since B is a super-level set of $P_{t+s}1_{B'}$ with the same volume as A. Thus,

$$\mathbb{E}1_B P_t 1_{B'} - \epsilon_2 \le \mathbb{E}1_A P_t 1_{A'} \le \mathbb{E}1_A P_{t+s} 1_{B'} + \epsilon_1 \le \mathbb{E}1_B P_{t+s} 1_{B'} + \epsilon_1.$$

By Lemma 5.4,

$$(\mathbb{E}(P_t 1_B - P_{t+s}1_B)^2)^{1/2} \le \frac{C(t)\mathbb{E}1_B(P_t 1_{B'} - P_{t+s}1_{B'})}{(I(\gamma_n(A))I(\gamma_n(A')))^{C(t)}} \le \frac{C(t)(\epsilon_1 + \epsilon_2)}{(I(\gamma_n(A))I(\gamma_n(A')))^{C(t)}}.$$

Finally, the triangle inequality gives

$$\begin{aligned}(\mathbb{E}(P_t 1_A - P_t 1_B)^2)^{1/2} &\le \frac{(\mathbb{E}(P_t 1_A - P_{t+s}1_B)^2)^{1/2} + (\mathbb{E}(P_t 1_B - P_{t+s}1_B)^2)^{1/2}}{(I(\gamma_n(A))I(\gamma_n(A')))^{C(t)}} \\ &\le \frac{\epsilon_1 + C(t)(\epsilon_1 + \epsilon_2)}{(I(\gamma_n(A))I(\gamma_n(A')))^{C(t)}}.\end{aligned}$$

Of course, 1 can be absorbed into the constant $C(t)$. □

5.2. Proof of robustness

Proof of Theorem 1.4. First, define t by $e^{-t} = \rho$. We then have $k_t^2 = \rho^2/(1-\rho^2)$ and so the exponent of δ in Proposition 4.1 becomes

$$\frac{1}{1 + 4\frac{\rho^2}{(1-\rho^2)(1-\rho)}} \cdot \frac{1}{1+\alpha} = \frac{(1-\rho^2)(1-\rho)}{1-\rho+3\rho^2+\rho^3} \cdot \frac{1}{1+\alpha}. \tag{5.9}$$

Of course, we can define $\alpha > 0$ (depending on ρ) so that (5.9) is

$$\eta := \frac{(1-\rho^2)(1-\rho)}{1+3\rho}.$$

Now suppose that $f = 1_A$ and $g = 1_{A'}$ for some $A, A' \subset \mathbb{R}^n$. Proposition 4.1 implies that there are $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $|a| \le k_t$ and

$$\mathbb{E}\big((P_t 1_A)(X) - \Phi(\langle a, X\rangle - b)\big)^2 \le C(\rho) m^{-C(\rho)} \delta^\eta.$$

Since $|a| \le k_t$, Lemma 3.1 implies that we can find some $s > 0$ and a half-space B such that $\Phi(\langle a, x\rangle - b) = (P_{t+s}1_B)(x)$; then

$$\mathbb{E}(P_t 1_A - P_{t+s}1_B)^2 \le C(\rho) m^{-C(\rho)} \delta^\eta. \tag{5.10}$$

At this point, it is not clear that $\gamma(A) = \gamma(B)$; however, we can ensure this by modifying B slightly:

$$\mathbb{E}(P_t 1_A - P_{t+s}1_B)^2 \ge (\mathbb{E}P_t 1_A - \mathbb{E}P_{t+s}1_B)^2 = (\gamma(A) - \gamma(B))^2.$$

Therefore let $\tilde{B}$ be a translation of B so that $\gamma(\tilde{B}) = \gamma(A)$. By the triangle inequality,

$$\begin{aligned}(\mathbb{E}(P_t 1_A - P_{t+s}1_{\tilde{B}})^2)^{1/2} &\le (\mathbb{E}(P_t 1_A - P_{t+s}1_B)^2)^{1/2} + (\mathbb{E}(P_{t+s}1_B - P_{t+s}1_{\tilde{B}})^2)^{1/2} \\ &\le (\mathbb{E}(P_t 1_A - P_{t+s}1_B)^2)^{1/2} + |\gamma(B) - \gamma(\tilde{B})|^{1/2} \\ &\le 2(\mathbb{E}(P_t 1_A - P_{t+s}1_B)^2)^{1/2}.\end{aligned}$$

By replacing B with $\tilde{B}$, we can assume in (5.10) that $\gamma(A) = \gamma(B)$ (at the cost of increasing $C(\rho)$ by a factor of 2).

Now we apply Proposition 5.3 with $\epsilon_1^2 = C(\rho)m^{-C(\rho)}\delta^\eta$ and $\epsilon_2 = \delta$. The conclusion of Proposition 5.3 leaves us with

$$(\mathbb{E}(P_t 1_A - P_t 1_B)^2)^{1/2} \le C(\rho)m^{-C(\rho)}(\epsilon_1 + \epsilon_2) \le C(\rho)m^{-C(\rho)}\delta^{\eta/2},$$

where we have absorbed the constant $C(t)$ from Proposition 5.1 into $C(\rho)$. Since $\mathbb{E}|X| \le (\mathbb{E}X^2)^{1/2}$ for any random variable X, we may apply Proposition 5.1:

$$\gamma(A \triangle B) \le C(\rho)\sqrt{\mathbb{E}|P_t 1_A - P_t 1_B|} \le C(\rho)(\mathbb{E}(P_t 1_A - P_t 1_B)^2)^{1/4} \le C(\rho)m^{-C(\rho)}\delta^{\eta/4}.$$

By applying the same argument to A' and B', this establishes Theorem 1.4 in the case that f and g are indicator functions.

To extend the result to other functions, note $\mathbb{E}J(f(X), g(Y)) = \mathbb{E}J(1_A(\tilde{X}), 1_{A'}(\tilde{Y}))$ where $\tilde{X}$ and $\tilde{Y}$ are ρ-correlated Gaussian vectors in $\mathbb{R}^{n+1}$, and

$$A = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \ge \Phi^{-1}(f(x))\},$$
$$A' = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \ge \Phi^{-1}(g(x))\}.$$

Moreover, $\mathbb{E}f = \gamma_{n+1}(A)$ and $\mathbb{E}g = \gamma_{n+1}(A')$. Applying Theorem 1.4 for indicator functions in dimension $n + 1$, we find a half-space B such that

$$\gamma_{n+1}(A \triangle B) \le C(\rho)m^{-C(\rho)}\delta^{\eta/4}. \tag{5.11}$$

By slightly perturbing B, we can assume that it does not take the form $\{x_i \ge b\}$ for any $1 \le i \le n$; in particular, this means that we can write B in the form

$$B = \{(x, x_{n+1}) \in \mathbb{R}^n : x_{n+1} \ge \langle a, x\rangle - b\}$$

for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. But then

$$\gamma_{n+1}(A \triangle B) = \mathbb{E}|f(X) - \Phi(\langle a, X\rangle - b)|;$$

combined with (5.11), this completes the proof. □

6. Optimal dependence on ρ

In this section, we will prove Theorem 1.5. To do so we need to improve the dependence on ρ that appeared in Theorem 1.4. Before we begin, let us list the places where the dependence on ρ can be improved:

1. In Proposition 4.3, we needed to control

$$\mathbb{E}_\rho \exp\left(\beta \frac{v_t^2(X) + w_t^2(Y) - 2\rho v_t(X) w_t(Y)}{2(1-\rho^2)}\right).$$

   Of course, the denominator of the exponent blows up as $\rho \to 1$. However, if $v_t = w_t$ then the numerator goes to zero (in law, at least) at the same rate. In this case, therefore, we are able to bound the above expectation by an expression not depending on ρ.

2. In the proof of Proposition 4.1, we used an L_∞ bound on $|\nabla v_t|$ and $|\nabla w_t|$ to show that for some $r < 1$,

$$\mathbb{E}_\rho\big(|\nabla v_t(X) - \nabla w_t(Y)|^2\big)^{1/r} \le C(t)\mathbb{E}_\rho\big(|\nabla v_t(X) - \nabla w_t(Y)|^{2r}\big)^{1/r}.$$

   This inequality is not sharp in its ρ-dependence because when $v_t = w_t$, the left hand side shrinks like $(1-\rho)^{1/r}$ as $\rho \to 1$, while the right hand side shrinks like $1-\rho$. We can get the right ρ-dependence by using an L_p bound on $|\nabla v_t(X) - \nabla v_t(Y)|$ when applying Hölder's inequality, instead of an L_∞ bound.
3. In applying Proposition 5.3, we were forced to take $e^{-t} = \rho$. Since most of our bounds have a (necessary) dependence on t, this causes a dependence on ρ which is not optimal. To get around this, we will use the subadditivity property of Kane [22] and Kindler and O'Donnell [28] to show that we can actually choose certain values of t such that e^{-t} is much smaller than ρ. In particular, we can take t to be quite large even when ρ is close to 1.

Once we have incorporated the first two improvements, we will obtain a better version of Proposition 4.1:

Proposition 6.1. *For any $\alpha, t > 0$, there is a constant $C(t,\alpha)$ such that for any $f : \mathbb{R}^n \to [0,1]$, there exist $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ with $|a| \le k_t$ such that*

$$\mathbb{E}\big(f_t(X) - \Phi(\langle X, a\rangle - b)\big)^2 \le C(t,\alpha) m(f)^{\frac{8k_t^2(1+k_t)^2}{1+8k_t^2} - \alpha} \left(\frac{\delta}{\rho\sqrt{1-\rho}}\right)^{\frac{1}{1+8k_t^2} - \alpha},$$

where $k_t = (e^{2t}-1)^{-1/2}$, $\delta(f) = \mathbb{E}_\rho J(f(X), f(Y)) - J(\mathbb{E}f, \mathbb{E}f)$, and $m(f) = \mathbb{E}f(1-\mathbb{E}f)$.

Moreover, this statement holds with a $C(t,\alpha)$ which, for any fixed α, is decreasing in t.

Once we have incorporated the third improvement above, we will use the arguments of Section 5 to prove Theorem 1.5.

6.1. A better bound on the auxiliary term

First, we will tackle item 1 above. Our improved bound leads to a version of Proposition 4.3 with the correct dependence on ρ.

Proposition 6.2. *Let $k_t = (e^{2t}-1)^{-1/2}$. There are constants $0 < c, C < \infty$ such that for any $t > 0$, if $r \le 1/(1+8k_t^2)$ then*

$$\frac{dR_t}{dt} \ge \frac{\rho}{\sqrt{1-\rho^2}} (cm(f))^{8k_t^2(1+k_t)^2} \big(\mathbb{E}|\nabla v_t(X) - \nabla v_t(Y)|^{2r}\big)^{1/r}$$

where $m(f) = \mathbb{E}f(1-\mathbb{E}f)$.

To obtain this improvement, we note that for a Lipschitz function v, the quotient $(v(X) - v(Y))/\sqrt{1-\rho}$ satisfies a Gaussian tail bound that does not depend on ρ:

Lemma 6.3. *If $v : \mathbb{R}^n \to \mathbb{R}$ is L-Lipschitz and $2\beta L^2 < 1$ then*

$$\mathbb{E}_\rho \exp\left(\beta \frac{(v(X) - v(Y))^2}{1-\rho}\right) \le \frac{2}{\sqrt{1 - 2\beta L^2}}.$$

Proof. Let $Z_1 = \frac{X+Y}{2}$ and $Z_2 = \frac{X-Y}{\sqrt{2(1-\rho)}}$. Then $\mathbb{E} Z_2^2 = 1$. Now we condition on Z_1: the function

$$v(X) - v(Y) = v\left(Z_1 + \sqrt{\frac{1-\rho}{2}}\, Z_2\right) - v\left(Z_1 - \sqrt{\frac{1-\rho}{2}}\, Z_2\right)$$

is $L\sqrt{2(1-\rho)}$-Lipschitz in Z_2 and has conditional median zero (because it is odd in Z_2). Now we apply Lemma 4.5 (conditionally on Z_1) with $f(Z_2) = \frac{1}{L\sqrt{2(1-\rho)}}(v(X) - v(Y))$ and $\lambda = 2L^2\beta$: provided that $\lambda = 2L^2\beta < 1$, we have

$$\mathbb{E}_\rho \exp\left(\beta \frac{(v(X) - v(Y))^2}{1-\rho} \,\middle|\, Z_1\right) \le \frac{1}{\sqrt{1 - 2L^2\beta}}.$$

Integrating out Z_1 proves the claim. □

Next, we use the estimate of Lemma 6.3 to prove a bound on

$$\mathbb{E}_\rho \exp\left(\beta \frac{v_t^2(X) + v_t^2(Y) - 2\rho v_t(X) w_t(Y)}{2(1-\rho^2)}\right)$$

that is better than the one from (4.9) which was used to derive Proposition 4.3.

Lemma 6.4. *There is a constant C such that for any $t > 0$, and for any $\beta > 0$ with $8\beta k_t^2 \le 1$,*

$$\mathbb{E}_\rho \exp\left(\beta \frac{v_t^2(X) + v_t^2(Y) - 2\rho v_t(X) v_t(Y)}{2(1-\rho^2)}\right) \le C e^{M_t^2/2},$$

where M_t is a median of v_t.

Proof. We begin with the Cauchy–Schwarz inequality:

$$\begin{aligned}
&\mathbb{E}_\rho \exp\left(\beta \frac{v_t^2(X) + v_t^2(Y) - 2\rho v_t(X) v_t(Y)}{2(1-\rho^2)}\right) \\
&\quad = \mathbb{E}_\rho \exp\left(\beta \frac{(v_t(X) - v_t(Y))^2}{2(1-\rho^2)}\right) \exp\left(\beta \frac{v_t(X) v_t(Y)}{1+\rho}\right) \\
&\quad \le \left(\mathbb{E}_\rho \exp\left(\beta \frac{(v_t(X) - v_t(Y))^2}{(1-\rho^2)}\right)\right)^{1/2} \left(\mathbb{E} \exp\left(2\beta \frac{v_t(X)^2}{1+\rho}\right)\right)^{1/2}. \qquad (6.1)
\end{aligned}$$

Now, recall from Lemma 3.2 that v_t is k_t-Lipschitz. In particular, Lemma 6.3 implies that if $8\beta k_t^2 \le 1$ then the first term of (6.1) is at most $2\sqrt{2}$. Finally, Lemma 4.5 implies that the second term of (6.1) is bounded by $C e^{M_t^2/2}$. □

Proof of Proposition 6.2. First, we follow the proof of Proposition 4.3 up until (4.8). At this point, we can apply Lemma 6.4 to obtain

$$\frac{dR_t}{dt} \geq c\frac{\rho}{\sqrt{1-\rho^2}}e^{-M_t^2/(2\beta)}\left(\mathbb{E}_\rho|\nabla v_t(X) - \nabla v_t(Y)|^{2r}\right)^{1/r},$$

and we conclude by applying Lemma 4.6, which implies that

$$e^{-M_t^2/(2\beta)} \geq (cm(f))^{(1+k_t)^2/\beta}.$$

Then set $\beta = 1/(8k_t^2)$. □

6.2. Higher moments of $|\nabla v_t(X) - \nabla v_t(Y)|$

Here, we will carry out the second step of the plan outlined at the beginning of Section 6. The main result is an upper bound on arbitrary moments of $|\nabla v_t(X) - \nabla v_t(Y)|$.

Proposition 6.5. *There is a constant C such that for any $t > 0$ and any $1 \leq q < \infty$,*

$$\left(\mathbb{E}_\rho|\nabla v_t(X) - \nabla v_t(Y)|^q\right)^{1/q} \leq Ck_t^2\sqrt{q(1-\rho)}\left((1+k_t)\sqrt{\log(1/m(f))} + \sqrt{q}\,k_t\right).$$

If we fix q and t, then the bound of Proposition 6.5 has the right dependence on ρ. In particular, we will use it instead of the uniform bound $|\nabla v_t| \leq k_t$, which does not improve as $\rho \to 1$.

There are two main tools in the proof of Proposition 6.5. The first is a moment bound on the Hessian of v_t, which was proved in [35] (see the last line in the proof of Proposition 3.6). In what follows, $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.

Proposition 6.6. *Let Hv_t denote the Hessian matrix of v_t. There is a constant C such that for all $t > 0$ and all $1 \leq q < \infty$,*

$$(\mathbb{E}\|Hv_t\|_F^q)^{1/q} \leq Ck_t^2\left((1+k_t)\sqrt{\log\frac{1}{m(f)}} + \sqrt{q}\,k_t\right).$$

The other tool in the proof of Proposition 6.5 is a result of Pinelis [39], which will allow us to relate moments of $|\nabla v_t(X) - \nabla v_t(Y)|$ to moments of $\|Hv_t\|_F$.

Proposition 6.7. *Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a C^1 function and let Dh be the $n \times k$ matrix of its partial derivatives. If Z_1 and Z_2 are independent, standard Gaussian vectors in $\mathbb{R}^n$ then*

$$(\mathbb{E}|h(Z_1) - h(Z_2)|^q)^{1/q} \leq C\sqrt{q}(\mathbb{E}\|Dh\|_F^q)^{1/q}$$

for every $1 \leq q < \infty$, where C is a universal constant.

Proof. Define $f : \mathbb{R}^{2n} \to \mathbb{R}^k$ by $f(Z) = h(Z_1) - h(Z_2)$ where $Z = (Z_1, Z_2)$. Pinelis [39] showed that if $\Psi : \mathbb{R}^k \to \mathbb{R}$ is a convex function then for any function $f : \mathbb{R}^{2n} \to \mathbb{R}^k$ with $\mathbb{E}f = 0$,

$$\mathbb{E}\Psi(f(Z)) \leq \mathbb{E}\Psi\left(\frac{\pi}{2}Df(Z)\cdot\tilde{Z}\right),$$

where $\tilde{Z}$ is an independent copy of Z. Applying this with $\Psi(x) = |x|^q$, and noting that $Df = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes Dh$, we obtain

$$\mathbb{E}|f(Z)|^q \le C^q \mathbb{E}|Dh(Z_1) \cdot Z_2|^q.$$

Now, $\mathbb{E}|AZ_2|^q \le (C\sqrt{q})^q \|A\|_F$ for any fixed matrix A; if we apply this fact conditionally on Z_1, then we obtain

$$\mathbb{E}|f(Z)|^q \le (C\sqrt{q})^q \mathbb{E}\|Dh\|_F^q.$$ □

Proof of Proposition 6.5. Let Z, Z_1 and Z_2 be independent standard Gaussians on $\mathbb{R}^n$; set $X = \sqrt{\rho}\, Z + \sqrt{1-\rho}\, Z_1$ and $Y = \sqrt{\rho}\, Z + \sqrt{1-\rho}\, Z_2$ so that X and Y are standard Gaussians with correlation ρ. Conditioned on Z, define the function

$$h(x) = \nabla v_t(\sqrt{Z} + \sqrt{1-\rho}\, x),$$

so that $h(Z_1) = \nabla v_t(X)$ and $h(Z_2) = \nabla v_t(Y)$. Note that

$$(Dh)(x) = \sqrt{1-\rho}(Hv_t)(\sqrt{\rho}\, Z + \sqrt{1-\rho}\, x);$$

thus Proposition 6.7 (conditioned on Z) implies that

$$\mathbb{E}\big(|\nabla v_t(X) - \nabla v_t(Y)|^q \;\big|\; Z\big) \le \big(C\sqrt{q(1-\rho)}\big)^q \mathbb{E}\big(\|Hv_t(X)\|_F^q \;\big|\; Z\big).$$

Integrating out Z and raising both sides to the power $1/q$, we have

$$\big(\mathbb{E}|\nabla v_t(X) - \nabla v_t(Y)|^q\big)^{1/p} \le C\sqrt{q(1-\rho)}(\mathbb{E}\|Hv_t\|_F^q)^{1/q}.$$

We conclude by applying Proposition 6.6 to the right hand side. □

With the first two steps of our outline complete, we are ready to prove Proposition 6.1. This proof is much like the proof of Proposition 4.1, except that it uses Propositions 6.2 and 6.5 in the appropriate places.

Proof of Proposition 6.1. For any non-negative random variable Z and any $0 < \alpha < 2$, $0 < r < 1$, Hölder's inequality applied with $p = 2r/\gamma$ implies that

$$\mathbb{E}Z^2 = \mathbb{E}Z^\gamma Z^{2-\gamma} \le (\mathbb{E}Z^{2r})^{\gamma/(2r)} (\mathbb{E}Z^{2r(2-\gamma)/(2r-\gamma)})^{(2r-\gamma)/(2r)}.$$

In particular, if we set $q = 2r(2-\gamma)/(2r-\gamma)$ then we obtain

$$(\mathbb{E}Z^{2r})^{1/r} \ge \left(\frac{\mathbb{E}Z^2}{(\mathbb{E}Z^q)^{(2-\gamma)/q}}\right)^{2/\gamma}. \tag{6.2}$$

Now, set $Z = |\nabla v_t(X) - \nabla v_t(Y)|$, $a = \mathbb{E}\nabla v_t$ and $\epsilon(v_t) = \mathbb{E}(v_t(X) - \langle X, a\rangle - \mathbb{E}v_t)^2$. Lemma 4.2 and Proposition 6.5 then imply that the right hand side of (6.2) is at least

$$\left(\frac{2(1-\rho)\epsilon(v_t)}{\big(ck_t^2\sqrt{q(1-\rho)}\big((1+k_t)\sqrt{\log(1/m(f))} + \sqrt{q}\, k_t\big)\big)^{2-\gamma}}\right)^{2/\gamma}$$
$$= c(1-\rho)\left(\frac{\epsilon(v_t)}{\big(k_t^2\sqrt{q}\big((1+k_t)\sqrt{\log(1/m(f))} + \sqrt{q}\, k_t\big)\big)^{2-\gamma}}\right)^{2/\gamma}.$$

Now define $\eta = 8k_t^2/(1+8k_t^2)$ and choose $r = 1-\eta$ (so as to satisfy the hypothesis of Proposition 6.2). If we then define $\gamma = 2r - \alpha\eta = 2-(2+\alpha)\eta$ for some $0 < \alpha < 1$, we will find that $q = 2r(2+\alpha)/\alpha \le 6/\alpha$. In particular, the last displayed quantity is at least

$$(1-\rho)(c\alpha)^{(2-\gamma)/\gamma} \frac{\epsilon(v_t)^{2/\gamma}}{\big((k_t^3+1)\sqrt{\log(1/m(f))}\big)^{(2-\gamma)/\gamma}}.$$

Since $(k_t^3+1)^{(2-\gamma)/\gamma}$ depends only on t, we can put this all together (going back to (6.2)) to obtain

$$\begin{aligned}\big(\mathbb{E}|\nabla v_t(X) - \nabla v_t(Y)|^{2r}\big)^{1/r} &\ge c(t,\alpha)(1-\rho)\frac{\epsilon(v_t)^{2/\gamma}}{\log^{C(t)}(1/m(f))} \\ &= c(t,\alpha)(1-\rho)\frac{\epsilon(v_t)^{\frac{1+8k_t^2}{1-4\alpha k_t^2}}}{\log^{C(t)}(1/m(f))}.\end{aligned}$$

Combined with Proposition 6.2, this implies

$$\begin{aligned}\frac{dR_t}{dt} &\ge c(t)\rho\sqrt{1-\rho}\,\frac{m(f)^{8k_t^2(1+k_t)^2}}{\log^{C(t)}(1/m(f))}\epsilon(v_t)^{\frac{1+8k_t^2}{1-4\alpha k_t^2}} \\ &\ge c(t,\alpha)\rho\sqrt{1-\rho}\,m(f)^{8k_t^2(1+k_t)^2+\alpha}\epsilon(v_t)^{\frac{1+8k_t^2}{1-4\alpha k_t^2}},\end{aligned} \tag{6.3}$$

where the last line follows because for every $\alpha > 0$ and every C, there is a $C'(\alpha)$ such that for every $x \le 1/4$, $\log^C(1/x) \le C'(\alpha)x^{-\alpha}$. Now, with (6.3) as an analogue of (4.11), we complete the proof by following that of Proposition 6.1. Let us reiterate the main steps: recalling that $\delta = \int_0^\infty \frac{dR_s}{ds}\,ds$, we see that for any $\alpha, t > 0$, there is some $s \in [t, t(1+\alpha)]$ so that $\frac{dR_t}{dt}\big|_s \le \frac{\delta}{\alpha t}$. By (6.3) applied with $t = s$, we have

$$\epsilon(v_s) \le C(t,\alpha)m^{\frac{8k_t^2(1+k_t)^2(1-4\alpha k_t^2)}{1+8k_t^2}-\alpha}\left(\frac{\delta}{\rho\sqrt{1-\rho}}\right)^{\frac{1-4\alpha k_t^2}{1+8k_t^2}}.$$

Now, note that Φ is a contraction, and so Lemma 4.7 implies that

$$\begin{aligned}&\mathbb{E}\big(f_t(X) - P_{s-t}^{-1}\Phi(\langle X, \mathbb{E}\nabla v_s\rangle - \mathbb{E}v_s)\big)^2 \\ &\qquad\le C(t,\alpha)m^{\frac{8k_t^2(1+k_t)^2(1-4\alpha k_t^2)}{1+8k_t^2}-\alpha}\left(\frac{\delta}{\rho\sqrt{1-\rho}}\right)^{\frac{1-4\alpha k_t^2}{1+8k_t^2}-\alpha}.\end{aligned}$$

By changing α and adjusting $C(t,\alpha)$ accordingly, we can put this inequality into the form that was claimed in the proposition.

Finally, recall that $|\mathbb{E}\nabla v_s| \le k_s$ by Lemma 3.2, and so $P_{s-t}^{-1}\Phi(\langle X, \mathbb{E}\nabla v_s\rangle - \mathbb{E}v_s)$ can be written in the form $\Phi(\langle X, a\rangle - b)$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ with $|a| \le k_t$. □

6.3. On the monotonicity of δ with respect to ρ

The final step in the proof of Theorem 1.5 is to improve the application of Lemma 5.4. Assuming, for now, that f is the indicator function of a set A, the hypothesis of Theorem 1.5 tells us that if $e^{-t} = \rho$ then $\mathbb{E} 1_A P_t 1_A$ is almost as large as possible; that is, it is almost as large as $\mathbb{E} 1_B P_t 1_B$ where B is a half-space of probability $\Pr(A)$. This assumption allows us to apply Lemma 5.4, but only with $t = \log(1/\rho)$. In particular, this means that we will need to use this value of t in Proposition 6.1, which implies a poor dependence on ρ in our final answer.

To avoid all these difficulties, we will follow Kane [22] and Kindler and O'Donnell [28] to show that if $\mathbb{E} 1_A P_t 1_A$ is almost as large as possible for $t = \log(1/\rho)$, then it is also large for certain values of t that are larger.

Proposition 6.8. *Suppose $A \subset \mathbb{R}^n$ has $\Pr(A) = 1/2$. If $\theta = \cos(k \arccos \rho)$ for some $k \in \mathbb{N}$, and*

$$J(1/2, 1/2; \rho) - \mathbb{E}_\rho J(1_A(X), 1_A(Y); \rho) \le \delta,$$

then

$$J(1/2, 1/2; \theta) - \mathbb{E}_\theta J(1_A(X), 1_A(Y); \theta) \le k\delta.$$

Proof. Let Z_1 and Z_2 be independent standard Gaussians on $\mathbb{R}^n$ and define $Z(\gamma) = Z_1 \cos\gamma + Z_2 \sin\gamma$. Note that for any γ and any $j \in \mathbb{N}$, $Z((j+1)\gamma)$ and $Z(j\gamma)$ have correlation $\cos\gamma$. In particular, if $\gamma = \arccos\rho$, then the union bound implies that

$$\begin{aligned}
\Pr_\theta(X \in A,\ Y \notin A) &= \Pr(Z(0) \in A,\ Z(k\gamma) \notin A) \\
&\le \sum_{j=0}^{k-1} \Pr\big(Z(j\gamma) \in A,\ Z((j+1)\gamma) \notin A\big) \\
&= k \Pr_\rho(X \in A,\ Y \notin A).
\end{aligned} \tag{6.4}$$

The remarkable thing about this inequality is that it becomes equality when A is a half-space of measure $1/2$, because in this case, $\Pr_\rho(X \in A, Y \notin A) = \frac{1}{2\pi}\arccos\rho$.

Recall that $\mathbb{E}_\rho J(1_A(X), 1_A(Y); \rho) = \Pr_\rho(X \in A, Y \in A)$. Thus, the hypothesis of the proposition can be rewritten as

$$\left(\frac{1}{2} - \frac{1}{2\pi}\arccos\rho\right) - \big(\Pr(A) - \Pr_\rho(X \in A,\ Y \notin A)\big) \le \delta,$$

which rearranges to read

$$\Pr_\rho(X \in A,\ Y \notin A) \le \delta + \frac{1}{2\pi}\arccos\rho.$$

By (6.4), this implies that

$$\Pr_\theta(X \in A,\ Y \notin A) \le k\delta + \frac{1}{2\pi}\arccos\theta,$$

which can then be rearranged to yield the conclusion of the proposition. □

Let us point out two deficiencies in Proposition 6.8: the requirement that $\Pr(A) = 1/2$ and that k be an integer. The first of these deficiencies is responsible for the assumption $\mathbb{E}f = 1/2$ in Theorem 1.5, and the second one prevents us from obtaining a better constant in the exponent of δ. Both of these restrictions come from the subadditivity condition (6.4), which only makes sense for an integer k, and only achieves equality for a half-space of volume $1/2$. But beyond the fact that our proof fails, we have no reason not to believe that some version of Proposition 6.8 is true without these restrictions. In particular, we make the following conjecture:

Conjecture 6.9. *There is a function $k(\rho, a)$ such that*

- *for any fixed $a \in (0, 1)$, $k(\rho, a) \sim \sqrt{1-\rho}$ as $\rho \to 1$;*
- *for any fixed $a \in (0, 1)$, $k(\rho, a) \sim \rho$ as $\rho \to 0$; and*
- *for any $a \in (0, 1)$ and any $A \subset \mathbb{R}^n$ the quantity*

$$\frac{J(a, a; \rho) - \mathbb{E}_\rho J(1_A(X), 1_A(Y); \rho)}{k(\rho, a)}$$

is increasing in ρ.

If this conjecture were true, it would tell us that sets which are almost optimal for some ρ are also almost optimal for smaller ρ, where the function $k(\rho, a)$ quantifies the almost optimality.

In any case, let us move on to the proof of Theorem 1.5. If the conjecture is true, then the following proof will directly benefit from the improvement.

Proof of Theorem 1.5. We will prove the theorem when f is the indicator function of a set A. The extension to general f follows from the same argument that was made in the proof of Theorem 1.4.

Fix $\epsilon > 0$. If ρ_0 is close enough to 1 then for every $\rho_0 < \rho < 1$, there is a $k \in \mathbb{N}$ such that $k \arccos \rho \in [\pi/2 - \epsilon, \pi/2 - \epsilon/2]$. In particular, this means that $\cos(k \arccos \rho) \in [c_1(\epsilon), c_2(\epsilon)]$, where $c_1(\epsilon)$ and $c_2(\epsilon)$ converge to zero as $\epsilon \to 0$. Moreover, this k must satisfy

$$k \le \frac{C(\epsilon)}{\arccos \rho} \le \frac{C(\epsilon)}{\sqrt{1-\rho}}.$$

Now let $\theta = \cos(k \arccos \rho)$. By Proposition 6.8, A satisfies

$$J(1/2, 1/2; \theta) - \mathbb{E}_\theta J(1_A(X), 1_A(Y); \theta) \le C(\epsilon) \frac{\delta}{\sqrt{1-\rho}}.$$

Now we will apply Proposition 6.1 with ρ replaced by θ and $t = \log(1/\theta)$. Since $\theta \le c_2(\epsilon)$, it follows that $k_t = \theta/\sqrt{1-\theta^2} \le c_3(\epsilon)$ (where $c_3(\epsilon) \to 0$ with ϵ). Thus, the conclusion of Proposition 6.1 gives us $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\begin{aligned}\mathbb{E}\big((P_t 1_A)(X) - \Phi(\langle X, a\rangle - b)\big)^2 &\le C\left(\frac{\delta}{\theta\sqrt{(1-\theta)(1-\rho)}}\right)^{1-c_4(\epsilon)} \\ &\le C(\epsilon)\left(\frac{\delta}{\sqrt{1-\rho}}\right)^{1-c_4(\epsilon)}. \qquad (6.5)\end{aligned}$$

Now we apply the same time-reversal argument as in Theorem 1.4: Lemma 3.1 implies that there is some $s > 0$ and a half-space B such that

$$\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2 \le C(\epsilon)(\delta/\sqrt{1-\rho})^{1-c_4(\epsilon)}$$

and we can assume, at the cost of increasing $C(\epsilon)$, that $\Pr(B) = \Pr(A)$. Then Proposition 5.3 implies that

$$\mathbb{E}(P_t 1_A - P_t 1_B)^2 \le C(\epsilon)(\delta/\sqrt{1-\rho})^{1-c_4(\epsilon)},$$

and we apply Proposition 5.1 (recalling that t is bounded above and below by constants depending on ϵ) to conclude that

$$\Pr(A \,\triangle\, B) \le C(\epsilon)(\delta/\sqrt{1-\rho})^{1/4-c_4(\epsilon)/4}.$$

Recall that $c_4(\epsilon)$ is some quantity tending to zero with ϵ. Therefore, we can derive the claim of the theorem from the equation above by modifying $C(\epsilon)$. □

Finally, we will prove Corollary 1.9.

Proof of Corollary 1.9. Since $xy \le J(x, y)$, the hypothesis of Corollary 1.9 implies that

$$\mathbb{E}J(f(X), f(Y)) \ge \frac{1}{4} + \frac{1}{2\pi}\arcsin\rho - \delta.$$

Now, consider Theorem 1.5 with $\epsilon = 1/8$. If $\rho > \rho_0$ then apply it; if not, apply Theorem 1.4. In either case, the conclusion is that there is some $a \in \mathbb{R}^n$ such that

$$\mathbb{E}|f(X) - \Phi(\langle X, a\rangle)| \le C(\rho)\delta^c.$$

Setting $g(X) = \Phi(\langle X, a\rangle)$, Hölder's inequality implies that

$$\begin{aligned}|\mathbb{E}g(X)g(Y) - \mathbb{E}f(X)f(Y)| &= \big|\mathbb{E}(g(X) - f(X))g(Y) + \mathbb{E}f(X)(g(Y) - f(Y))\big| \\ &\le 2\mathbb{E}|f - g|.\end{aligned}$$

In particular,

$$\mathbb{E}g(X)g(Y) \ge \frac{1}{4} + \frac{1}{2\pi}\arcsin\rho - \delta - C(\rho)\delta^c. \tag{6.6}$$

But the left hand side can be computed exactly: if $|a| = (e^{2t} - 1)^{-1/2}$ and $A = \{x \in \mathbb{R}^n : x_1 \le 0\}$ then

$$\begin{aligned}\mathbb{E}g(X)g(Y) = \mathbb{E}P_t 1_A(X) P_t 1_A(Y) = \mathbb{E}1_A(X) P_{2t-\log(\rho)} 1_A(X) &= \frac{1}{4} + \frac{1}{2\pi}\arcsin(e^{-2t}\rho) \\ &\le \frac{1}{4} + \frac{1}{2\pi}\arcsin\rho - \frac{1}{2\pi}\rho(1 - e^{-2t}),\end{aligned}$$

where the last line used the fact that the derivative of arcsin is at least 1. Combining this with (6.6), we have

$$1 - e^{-2t} \le C(\rho)\delta^c. \tag{6.7}$$

On the other hand,

$$\mathbb{E}|g - 1_A| = 2(1/2 - \mathbb{E}g1_A) = \frac{1}{2} - \frac{1}{\pi}\arcsin(e^{-t}) \le \sqrt{1 - e^{-2t}},$$

which combines with (6.7) to prove that $\mathbb{E}|g - 1_A| \le C(\rho)\delta^c$. Applying the triangle inequality, we conclude that

$$\mathbb{E}|f - 1_A| \le \mathbb{E}|f - g| + \mathbb{E}|g - 1_A| \le C(\rho)\delta^c. \qquad \square$$

7. The robust "majority is stablest" theorem

In this section, we prove Theorem 1.10. We begin by recalling some Fourier-theoretic properties of $\{-1, 1\}^n$. For more background on the Fourier analysis of Boolean functions, see the book by O'Donnell [38]. For $S \subset [n]$, define $\chi_S : \{-1, 1\}^n \to \{-1, 1\}$ by $\chi_S(x) = \prod_{i \in S} x_i$. Then $\{\chi_S : S \subset [n]\}$ is an orthonormal basis of $L_2(\{-1, 1\}^n)$. We will write $\hat{f}_S$ for the coefficients of f in this basis; that is,

$$f(x) = \sum_{S \subset [n]} \hat{f}_S \chi_S(x). \tag{7.1}$$

It may be easily checked that the coordinate influences of f can be expressed in terms of the Fourier expansion as

$$\mathrm{Inf}_i(f) = \sum_{S \ni i} |S| \hat{f}_S^2. \tag{7.2}$$

Recall that $\Pr_\rho$ denotes the distribution on $\{-1, 1\}^n \times \{-1, 1\}^n$ under which $(\xi_i, \sigma_i)_{i=1}^n$ are independent, $\mathbb{E}_\rho \xi_i = \mathbb{E}_\rho \sigma_i = 0$, and $\mathbb{E}_\rho \xi_i \sigma_i = \rho$. Define the Bonami–Beckner semigroup Q_t by

$$(Q_t f)(\xi) = \mathbb{E}_{e^{-t}}(f(\sigma) \mid \xi).$$

In terms of the Fourier expansion, one can check that

$$Q_t f = \sum_{S \subset [n]} e^{-t|S|} \hat{f}_S \chi_S. \tag{7.3}$$

Also, Q_t is a self-adjoint operator, and it satisfies

$$\mathbb{E}_\rho f(\xi) g(\sigma) = \mathbb{E} f(\xi)(Q_{\log(1/\rho)} g)(\xi) = \mathbb{E} g(\xi)(Q_{\log(1/\rho)} f)(\xi). \tag{7.4}$$

7.1. The invariance principle

Note that any function $f : \{-1, 1\}^n \to \mathbb{R}$ can be extended to a multilinear function on $\mathbb{R}^n$ through the Fourier expansion (7.1): since $\chi_S(x)$ is defined for all $x \in \mathbb{R}^n$, we may define $g(x)$ for $x \in \mathbb{R}^n$ by $g(x) = \sum_S \hat{f}_S \chi_S(x)$. We will say that g is the *multilinear extension of* f; note that g and f agree on $\{-1, 1\}^n$, thereby justifying the term "extension." A word of caution: we will sometimes define functions $f : \{-1, 1\}^n \to \mathbb{R}$ by formulas that make sense on all of $\mathbb{R}^n$ (for example, $f(x) = 1_{\{\langle a, x - b\rangle \ge 0\}}$). In such a case, the multilinear extension of f is not the same as the function $1_{\{\langle a, x - b\rangle \ge 0\}} : \mathbb{R}^n \to \mathbb{R}$.

Let us remark on some well-known and important properties of multilinear polynomials. First of all, since $\mathbb{E}\xi_i = \mathbb{E}X_i = 0$ and $\mathbb{E}\xi_i^2 = \mathbb{E}X_i^2 = 1$, it is trivial to check that for multilinear functions f and g,

$$\mathbb{E}f(\xi) = \mathbb{E}f(X), \quad \mathbb{E}f^2(\xi) = \mathbb{E}f^2(X), \quad \mathbb{E}_\rho f(\xi)g(\sigma) = \mathbb{E}_\rho f(X)g(Y). \tag{7.5}$$

It is also easy to check that if f is a multilinear polynomial then for any $t > 0$, $Q_t f$ and $P_t f$ are the same polynomial. In particular, there no ambiguity in using the notation f_t for both $P_t f$ and $Q_t f$.

Despite these similarities, $g(X)$ and $g(\xi)$ can have very different distributions in general (for example, if $g(x) = x_1$). The main technical result of [36] is that when f has low influence and $t > 0$, then $f_t(X)$ and $f_t(\xi)$ have similar distributions. We will quote a much less general statement than the one proved in [36], which will nevertheless be sufficient for our purposes. In particular, we will only need to know that if $g(\xi)$ takes values in [0, 1], then $g(X)$ mostly takes values in [0, 1]. Before stating the theorem from [36], let us introduce some notation: for a function f taking values in $\mathbb{R}$, let $\bar{f}$ be its truncation which takes values in [0, 1]:

$$\bar{f}(x) = \begin{cases} 0 & \text{if } f(x) < 0, \\ f(x) & \text{if } 0 \le f(x) \le 1, \\ 1 & \text{if } 1 < f(x). \end{cases}$$

Theorem 7.1. *Suppose f is a multilinear polynomial such that $f(\xi) \in [0, 1]$ for all $\xi \in \{-1, 1\}^n$. If $\max_i \mathrm{Inf}_i(f) \le \tau$ then for any $\eta > 0$,*

$$\mathbb{E}(f_\eta(X) - \overline{f_\eta}(X))^2 \le C\tau^{c\eta}. \tag{7.6}$$

We will now use Theorem 7.1 to prove Theorem 1.10. First, (7.6) and the triangle inequality imply that for any $0 < \rho' < 1$,

$$\mathbb{E}_{\rho'} f_\eta(X) f_\eta(Y) \le \mathbb{E}_{\rho'} \overline{f_\eta}(X)\overline{f_\eta}(Y) + C\tau^{c\eta}. \tag{7.7}$$

By (7.5) and (7.4),

$$\mathbb{E}_{\rho'} f_\eta(X) f_\eta(Y) = \mathbb{E}_{\rho'} f_\eta(\xi) f_\eta(\sigma) = \mathbb{E}_{e^{-2\eta}\rho'} f(\xi) f(\sigma). \tag{7.8}$$

Now set $\rho' = e^{2\eta}\rho$ (assuming that η is small enough so that $e^{2\eta}\rho < 1$). By (7.8) and (7.7),

$$\mathbb{E}_\rho f(\xi) f(\sigma) = \mathbb{E}_{\rho'} f_\eta(X) f_\eta(Y) \le \mathbb{E}_{\rho'} \overline{f_\eta}(X)\overline{f_\eta}(Y) + C\tau^{c\eta}. \tag{7.9}$$

Applying Theorem 1.2 to $\overline{f_\eta}$, we see that $\mathbb{E}_\rho f(\xi) f(\sigma) \le J_{\rho'}(\mathbb{E}\overline{f_\eta}, \mathbb{E}\overline{f_\eta}) + C\tau^{c\eta}$. Now, Theorem 7.1 implies that $|\mathbb{E}\overline{f_\eta} - \mathbb{E}f| \le C\tau^{c\eta}$, and the derivatives of $J_\rho(x, x)$ in both x and ρ can be bounded by a constant depending only on ρ; hence,

$$\begin{aligned} J_{\rho'}(\mathbb{E}\overline{f_\eta}, \mathbb{E}\overline{f_\eta}) &\le J_\rho(\mathbb{E}f, \mathbb{E}f) + C(\rho)(|\rho - \rho'| + |\mathbb{E}\overline{f_\eta} - \mathbb{E}f|) \\ &\le J_\rho(\mathbb{E}f, \mathbb{E}f) + C(\rho)(\eta + C\tau^{c\eta}). \end{aligned}$$

Plugging this into (7.9), we have $\mathbb{E}_\rho f(\xi) f(\sigma) \leq J_\rho(\mathbb{E} f, \mathbb{E} f) + C(\rho)(\eta + C\tau^{c\eta})$, which proves (1.5) once we take η sufficiently small (depending on τ).

Next, we prove (1.6). Under our assumption that $\mathbb{E}_\rho f(\xi) f(\sigma) \geq J_\rho(\mathbb{E} f, \mathbb{E} f) - \delta$, (7.9) implies that

$$\begin{aligned}\mathbb{E}_{\rho'} \overline{f_\eta}(X)\overline{f_\eta}(Y) &\geq J_\rho(\mathbb{E} f, \mathbb{E} f) - C\tau^{c\eta} - \delta \geq J_\rho(\mathbb{E}\overline{f_\eta}, \mathbb{E}\overline{f_\eta}) - C\tau^{c\eta} - \delta \\ &\geq J_{\rho'}(\mathbb{E}\overline{f_\eta}, \mathbb{E}\overline{f_\eta}) - C(\rho)\eta - C\tau^{c\eta} - \delta,\end{aligned}$$

where the second inequality follows because $|\mathbb{E} f - \mathbb{E}\overline{f_\eta}| \leq C\tau^{c\eta}$ and $\partial J(x, y; \rho)/\partial x$ is bounded. Applying Theorem 1.4 (with ρ' in place of ρ) to $\overline{f_\eta}$, we see that there are $a, b \in \mathbb{R}^n$ such that

$$\mathbb{E}(\overline{f_\eta}(X) - 1_{\{\langle a, X-b\rangle \geq 0\}})^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}.$$

By (7.6) and the triangle inequality, we may replace $\overline{f_\eta}$ by f_η:

$$\mathbb{E}(f_\eta(X) - 1_{\{\langle a, X-b\rangle \geq 0\}})^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}. \tag{7.10}$$

The next step is to pull (7.10) back to the discrete cube by replacing X with ξ on the left hand side of (7.10). We will do this using Theorem 7.1. As a prerequisite, we need to show that $1_{\{\langle a, x-b\rangle \geq 0\}}$ has small influences; this is essentially the same as saying that a is well-spread:

Lemma 7.2. *There is an $a \in \mathbb{R}^n$ satisfying (7.10) with $\sum a_i^2 = 1$ and $\max_i |a_i| \leq C\tau^c$.*

Once we have shown that $1_{\{\langle a, x-b\rangle \geq 0\}}$ has small influences, we can use Theorem 7.1 to show that the multilinear extension of $1_{\{\langle a, x-b\rangle \geq 0\}}$ is close to $1_{\{\langle a, x-b\rangle \geq 0\}}$:

Lemma 7.3. *Let $g^{a,b}$ be the multilinear extension of the function $x \mapsto 1_{\{\langle a, x-b\rangle \geq 0\}}$. If $\sum_i a_i^2 = 1$ and $\max_i |a_i| \leq \tau$ then for any $\eta > 0$,*

$$\mathbb{E}(g_\eta^{a,b}(X) - 1_{\{\langle a, X-b\rangle \geq 0\}})^2 \leq C(\eta + \tau^{c\eta}).$$

From Lemma 7.3 and the triangle inequality, we conclude from (7.10) that

$$\mathbb{E}(f_\eta(X) - g_\eta^{a,b}(X))^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}.$$

Since $f_\eta - g_\eta^{a,b}$ is a multilinear polynomial, its second moment remains unchanged when X is replaced by ξ:

$$\mathbb{E}(f_\eta(\xi) - g_\eta^{a,b}(\xi))^2 = \mathbb{E}(f_\eta(X) - g_\eta^{a,b}(X))^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}.$$

Now, $g^{a,b}$ is the indicator of a half-space on the cube; thus, $\mathbb{E}(g_\eta^{a,b}(\xi) - g^{a,b}(\xi))^2 \leq C\eta^c$ (see, for example, [6]). Applying this and the triangle inequality, we have

$$\mathbb{E}(f_\eta(\xi) - g^{a,b}(\xi))^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}. \tag{7.11}$$

The last piece is to replace f_η by f. We do this with a simple lemma which shows that for any function f, if f_η is close to some indicator function then f is also close to that function.

Lemma 7.4. *For any functions* $f : \{-1, 1\}^n \to [0, 1]$ *and* $g : \{-1, 1\}^n \to \{0, 1\}$ *and any* $\eta > 0$,

$$\mathbb{E}(f(\xi) - g(\xi))^2 \leq C\sqrt{\mathbb{E}(f_\eta(\xi) - g(\xi))^2}.$$

Applying Lemma 7.4 to (7.11), we obtain

$$\mathbb{E}(f(\xi) - g^{a,b}(\xi))^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}.$$

By choosing τ and η small enough compared to δ, the proof of Theorem 1.10 is complete, modulo the proofs of Lemmas 7.2, 7.3 and 7.4. We will prove them in the coming section.

7.2. Gaussian and Boolean half-spaces

Here we will prove the lemmas of the previous section. Before doing so, let us observe that $\mathbb{E}X_i 1_{\{\langle a, X-b\rangle \geq 0\}}$ is proportional to a_i, a fact which has already been noted by Matulef et al. [34]:

Lemma 7.5. *If* $|a| = 1$ *then*

$$\mathbb{E}X_i 1_{\{\langle a, X-b\rangle \geq 0\}} = a_i \phi(\langle a, b\rangle).$$

Proof. Let $e_i \in \mathbb{R}^n$ be the vector with 1 in position i and 0 elsewhere. We may write $e_i = a_i a + a^\perp$, where $a^\perp$ is some element of $\mathbb{R}^n$ which is orthogonal to a. Note that $\langle X, a^\perp\rangle$ is independent of $\langle X, a\rangle$ and so $\mathbb{E}\langle X, a^\perp\rangle 1_{\{\langle a, X-b\rangle \geq 0\}} = 0$. Hence,

$$\begin{aligned}\mathbb{E}X_i 1_{\{\langle a, X-b\rangle \geq 0\}} &= \mathbb{E}\langle a_i a + a^\perp, X\rangle 1_{\{\langle a, X-b\rangle \geq 0\}} = a_i \mathbb{E}\langle a, X\rangle 1_{\{\langle a, X-b\rangle \geq 0\}} \\ &= a_i \mathbb{E}X_1 1_{\{X_1 \geq \langle a, b\rangle\}},\end{aligned}$$

where the last equality follows because, by the rotational invariance of the Gaussian measure, $\langle a, X\rangle$ has the same distribution as X_1. Finally, integration by parts shows that $\mathbb{E}X_1 1_{\{X_1 \geq \langle a, b\rangle\}} = \phi(\langle a, b\rangle)$. □

Next, we prove Lemma 7.2. The point is that if a half-space is close to a low-influence function f then that half-space must also have low influences. We can then perturb the half-space to have even lower influences without increasing its distance to f by much.

Proof of Lemma 7.2. Suppose that f has influences bounded by τ, and that

$$\mathbb{E}(f(X) - 1_{\{\langle a, X-b\rangle \geq 0\}})^2 \leq \gamma, \tag{7.12}$$

where $\gamma = C(\rho)(\eta + \tau^{c\eta} + \epsilon)^c$. We will show that there is some $\tilde{a}$ such that $\sum_i \tilde{a}_i^2 = 1$, $\max_i |\tilde{a}_i| \leq C\tau^c$, and

$$\mathbb{E}(f(X) - 1_{\{\langle \tilde{a}, X-b\rangle \geq 0\}})^2 \leq C\gamma^c. \tag{7.13}$$

When applied to the function f_η, this will imply the claim of Lemma 7.2.

Since $X_1, \dots, X_n$ are orthonormal,

$$\mathbb{E}(f(X) - 1_{\{\langle a, X-b\rangle \geq 0\}})^2 \geq \sum_{i=1}^n \big(\mathbb{E}X_i f(X) - \mathbb{E}X_i 1_{\{\langle a, X-b\rangle \geq 0\}}\big)^2$$
$$= \sum_{i=1}^n \big(\hat{f}_{\{i\}} - a_i \phi(\langle a, b\rangle)\big)^2, \tag{7.14}$$

where the equality used Lemma 7.5. Define $\kappa_{a,b} = \phi(\langle a, b\rangle)$, and note from (7.2) that since the influences of f are bounded by τ, $|\hat{f}_{\{i\}}| \leq \sqrt{\tau}$ for every i. Hence for any i with $|a_i|\kappa_{a,b} \geq 2\sqrt{\tau}$, we have $(\hat{f}_{\{i\}} - a_i\kappa_{a,b})^2 \geq a_i^2 \kappa_{a,b}^2/4$. Combining this with (7.12) and (7.14),

$$\gamma \geq \mathbb{E}(f(X) - 1_{\{\langle a, X-b\rangle \geq 0\}})^2 \geq \frac{\kappa_{a,b}^2}{4} \sum_{\{i : |a_i|\kappa_{a,b} \geq 2\sqrt{\tau}\}} a_i^2. \tag{7.15}$$

We now consider two cases, depending on whether $\kappa_{a,b}$ is large or small. First, suppose that $\kappa_{a,b} \leq \gamma^{1/3}$; suppose also, without loss of generality, that $\langle a, b\rangle \leq 0$ (if not, replace f by $1 - f$). Then $\kappa_{a,b} = \phi(\langle a, b\rangle) \geq \Phi(\langle a, b\rangle) = \mathbb{E}1_{\{\langle a, X-b\rangle \geq 0\}}$; on the other hand, (7.12) implies that $(\mathbb{E}f - \mathbb{E}1_{\{\langle a, X-b\rangle \geq 0\}})^2 \leq \mathbb{E}(f - 1_{\{\langle a, X-b\rangle \geq 0\}})^2 \leq \gamma$ and so

$$\mathbb{E}f \leq \sqrt{\gamma} + \mathbb{E}1_{\{\langle a, X-b\rangle \geq 0\}} \leq \sqrt{\gamma} + \kappa_{a,b} \leq 2\gamma^{1/3}.$$

Since f takes values in $[0, 1]$, it follows that $\mathbb{E}f^2 \leq C\gamma^c$; in particular, *any* half-space with small enough measure will satisfy (7.13).

Now suppose that $\kappa_{a,b} \geq \gamma^{1/3}$ (which is in turn larger than $\tau^{1/3}$ by definition); then (7.15) implies that

$$\sum_{\{i : |a_i| \geq 2\tau^{1/6}\}} a_i^2 \leq \sum_{\{i : |a_i|\kappa_{a,b} \geq 2\sqrt{\tau}\}} a_i^2 \leq 4\gamma^{1/3}.$$

If we define $\bar{a}$ to be the truncated version of a (i.e. $\bar{a}_i = a_i$ if $|a_i| < 2\tau^{1/6}$ and $\bar{a}_i = 0$ otherwise), then this implies that $|a - \bar{a}|^2 \leq 4\gamma^{1/3}$. Since $|a| = 1$, it then follows from the triangle inequality that $|\bar{a}| \geq 1 - 2\gamma^{1/6}$. Set $\tilde{a} = \bar{a}/|\bar{a}|$. If γ is small enough so that $1 - 2\gamma^{1/6} \leq 1/2$ then

$$\max_i |\tilde{a}_i| = \frac{1}{|\bar{a}|} \max_i |\bar{a}_i| \leq \frac{2\tau^{1/6}}{1 - 2\gamma^{1/6}} \leq 4\tau^{1/6}$$

and

$$|a - \tilde{a}| \leq |a - \bar{a}| + |\bar{a} - \tilde{a}| \leq 2\gamma^{1/6} + \frac{1 - |\bar{a}|}{|\bar{a}|} \leq 8\gamma^{1/6}.$$

By the triangle inequality, $\tilde{a}$ satisfies (7.13). □

Next, we will prove Lemma 7.3: if $g^{a,b}$ is the multilinear extension of a low-influence half-space, then $g^{a,b}$ is close to a half-space. Observe that this is very much not the case for general half-spaces: the multilinear extension of $1_{\{x_1 \geq 0\}}$ is x_1, which is not close, in $L_2(\mathbb{R}^n, \gamma_n)$, to any half-space.

The main idea of the proof is to study the quantity $\mathbb{E}\overline{g^{a,b}}(X)\langle a, X-b\rangle$. By showing that this is almost as large as $\mathbb{E}1_{\{\langle a,X-b\rangle\geq 0\}}\langle a, X-b\rangle$, we show that $\overline{g^{a,b}}(X)$ is close to $1_{\{\langle a,X-b\rangle\geq 0\}}$.

Proof of Lemma 7.3. Suppose without loss of generality that $\langle a,b\rangle \geq 0$. Let $h(x) = \langle a, x-b\rangle$ and let g be the multilinear extension of $1_{\{h\geq 0\}}$. First of all, the Berry–Esséen theorem implies that for any $t\in\mathbb{R}$, $|\Pr(\langle a,\xi\rangle \geq t) - \Pr(\langle a, X\rangle \geq t)| \leq C\tau$. By the formula $\mathbb{E}Z = \int_0^\infty \Pr(Z\geq t)\,dt$ for a non-negative random variable Z, we have

$$\begin{aligned}
\mathbb{E}g(\xi)h(\xi) = \mathbb{E}h(\xi)1_{\{h(\xi)\geq 0\}} &= \int_0^\infty \Pr(\langle a, \xi - b\rangle \geq t)\,dt = \int_{\langle a,b\rangle}^\infty \Pr(\langle a,\xi\rangle \geq t)\,dt \\
&\geq \int_{\langle a,b\rangle}^M \Pr(\langle a,\xi\rangle \geq t)\,dt \geq \int_{\langle a,b\rangle}^M \Pr(X_1\geq t)\,dt - CM\tau \\
&\geq \int_{\langle a,b\rangle}^\infty \Pr(X_1\geq t)\,dt - CM\tau - Ce^{-M^2/2} \\
&= \mathbb{E}h(X)1_{\{h(X)\geq 0\}} - CM\tau - Ce^{-M^2/2}.
\end{aligned}$$

Choosing $M = \sqrt{\log(1/\tau)}$, we have

$$\mathbb{E}g(\xi)h(\xi) \geq \mathbb{E}h(X)1_{\{h(X)\geq 0\}} - C\tau^c. \tag{7.16}$$

Now, h is linear and so $h_t = e^{-t}h$; since Q_η is self-adjoint, Theorem 7.1 implies that

$$\begin{aligned}
\mathbb{E}g(\xi)h(\xi) &= e^\eta \mathbb{E}g(\xi)h_\eta(\xi) = e^\eta\mathbb{E}g_\eta(\xi)h(\xi) = e^\eta\mathbb{E}g_\eta(X)h(X) \\
&\leq e^\eta\mathbb{E}\overline{g_\eta}(X)h(X) + Ce^\eta(\eta+\tau^{c\eta}) \leq \mathbb{E}\overline{g_\eta}(X)h(X) + C(\eta+\tau^{c\eta}),
\end{aligned}$$

where the last inequality assumes that $\eta < 1$ (if not then the lemma is trivial anyway), and uses the fact that $\mathbb{E}\overline{g_\eta}(X)h(X)$ is bounded by a universal constant. Combining this with (7.16) yields

$$\mathbb{E}h(X)1_{\{h(X)\geq 0\}} \leq \mathbb{E}\overline{g_\eta}(X)h(X) + C(\eta+\tau^{c\eta}). \tag{7.17}$$

Now, let $m(X) = 1_{\{\langle a,X-b\rangle\geq 0\}} - \overline{g_\eta}(X)$ and take $\epsilon = \mathbb{E}|m|$. Note that because $\overline{g_\eta}\in[0,1]$, when $m\neq 0$ then m and h have the same sign; in particular, $m(x)h(x)\geq 0$. Let $A = \{x : \langle a, x-b\rangle \in [-\epsilon/2, \epsilon/2]\}$. Then $\Pr(A)\leq \epsilon/2$, and since $|m|\leq 1$ pointwise we must have $\mathbb{E}|m|1_{A^c} \geq \mathbb{E}|m| - \Pr(A) \geq \epsilon/2$. But on A^c we have $|h(x)|\geq \epsilon/2$; since $m(x)h(x)\geq 0$,

$$\mathbb{E}m(X)h(X) \geq \mathbb{E}m(X)h(X)1_{\{X\in A^c\}} \geq \frac{\epsilon}{2}\mathbb{E}|m|1_{A^c} \geq \frac{\epsilon^2}{4}.$$

Applying this to (7.17) yields $\epsilon \leq C(\eta+\tau^{c\eta})^c$. So if we recall the definition of ϵ, then we see that

$$\mathbb{E}|1_{\{\langle a,X-b\rangle\geq 0\}} - \overline{g_\eta}(X)| \leq C(\eta+\tau^{c\eta})^c.$$

By changing the constant c, we may replace $\mathbb{E}|\cdot|$ with $\mathbb{E}(\cdot)^2$; by (7.6) and the triangle inequality, we may replace $\overline{g_\eta}$ by g_η. This completes the proof of the lemma. Note that the only reason for proving this lemma with g_η instead of g was for extra convenience when applying it; the statement of the lemma is also true with g instead of g_η. □

The only remaining piece is Lemma 7.4.

Proof of Lemma 7.4. Suppose $f : \{-1,1\}^n \to [-1,1]$ and $g : \{-1,1\}^n \to \{-1,1\}$. This does not exactly correspond to the statement of the lemma, but it will be more convenient for the proof; we can recover the statement of the lemma by replacing f by $(1+f)/2$ and g by $(1+g)/2$.

Let $\epsilon = \mathbb{E}(f_\eta(\xi) - g(\xi))^2$. Since g takes values in $\{-1,1\}$, we have $\mathbb{E}g^2 = 1$. Then the triangle inequality implies that $(\mathbb{E}g^2)^{1/2} \le (\mathbb{E}f_\eta^2)^{1/2} + \sqrt{\epsilon}$; squaring yields

$$\mathbb{E}f_\eta^2 \ge \mathbb{E}g^2 - 2\mathbb{E}(f_\eta^2)^{1/2}\sqrt{\epsilon} - \epsilon \ge 1 - 3\sqrt{\epsilon}.$$

Since $\mathbb{E}f^2 \le 1$, we have

$$\mathbb{E}(f - f_\eta)^2 = \sum_{S \subset [n]} \hat{f}_S^2 (1 - e^{-\eta|S|})^2 \le \sum_{S \subset [n]} \hat{f}_S^2 (1 - e^{-\eta|S|}) = \mathbb{E}f^2 - \mathbb{E}f_\eta^2 \le 3\sqrt{\epsilon}.$$

It then follows by the triangle inequality that $\mathbb{E}(f-g)^2 \le C\sqrt{\epsilon}$. □

8. Spherical noise stability

We now use Theorem 1.4 to prove Theorem 1.11. For a subset $A \subset S^{n-1}$, we define $\bar{A} \subset \mathbb{R}^n$ to be the *radial extension* of A:

$$\bar{A} = \{x \in \mathbb{R}^n : x \ne 0 \text{ and } x/|x| \in A\}.$$

From the spherical symmetry of the Gaussian distribution it immediately follows that $\Pr(\bar{A}) = Q(A)$. The proof of Theorem 1.11 crucially relies on the fact that $Q_\rho(A_1, A_2)$ is close to $\Pr_\rho(\bar{A}_1, \bar{A}_2)$ in high dimensions. More explicitly, it uses the following lemmas:

Lemma 8.1. *For any half-space $H = \{x \in \mathbb{R}^n : \langle a, x\rangle \le b\}$ there is a spherical cap $B = \{x \in S^{n-1} : \langle a, x\rangle \le b'\}$ such that $\Pr(\bar{B}) = \Pr(H)$ and*

$$\Pr(\bar{B} \mathbin{\triangle} H) \le Cn^{-1/2}\log n.$$

Lemma 8.2. *For any two sets $A_1, A_2 \subset S^{n-1}$ and any $\rho \in [-1+\epsilon, 1-\epsilon]$,*

$$|Q_\rho(A_1, A_2) - \Pr_\rho(\bar{A}_1, \bar{A}_2)| \le C(\epsilon) n^{-1/2}\log n.$$

Given Lemmas 8.2 and 8.1, Theorem 1.11 is an easy corollary of Theorem 1.4:

Proof of Theorem 1.11. Define $\delta_* = \delta(\bar{A}_1, \bar{A}_2)$. Let H_1, H_2 be parallel half-spaces with $\Pr(H_i) = \Pr(\bar{A}_i)$, and let B_1, B_2 be the corresponding caps whose existence is guaranteed by Lemma 8.1. Then

$$\begin{aligned}
\delta_* = \delta(\bar{A}_1, \bar{A}_2) &= \Pr_\rho(X \in H_1,\ Y \in H_2) - \Pr_\rho(X \in \bar{A}_1,\ Y \in \bar{A}_2) \\
&\le \Pr_\rho(X \in \bar{B}_1,\ Y \in \bar{B}_2) - \Pr_\rho(X \in \bar{A}_1,\ Y \in \bar{A}_2) + O(n^{-1/2}\log n) \\
&\le Q_\rho(X \in B_1,\ Y \in B_2) - Q_\rho(X \in A_1,\ Y \in A_2) + O(n^{-1/2}\log n) \\
&= \delta(A_1, A_2) + O(n^{-1/2}\log n),
\end{aligned}$$

where the first inequality follows from Lemma 8.1 and the second from Lemma 8.2.

From Theorem 1.4 it follows that there are parallel half-spaces H_1 and H_2 with $\Pr(H_i) = \Pr(\bar{A}_i)$ satisfying

$$\Pr(\bar{A}_i \bigtriangleup H_i) \le C(\rho) m^{-C(\rho)} \delta_*^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}}.$$

By Lemma 8.1, there are parallel caps B_1 and B_2 such that

$$Q(A_i \bigtriangleup B_i) = \Pr(\bar{A}_i \bigtriangleup \bar{B}_i) \le C(\rho) m^{-C(\rho)} \delta_*^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}}. \qquad \square$$

The proof of Lemma 8.1 is quite simple, so we present it first:

Proof of Lemma 8.1. Let $H = \{x \in \mathbb{R}^n : \langle a, x \rangle \le b\}$, and suppose without loss of generality that $|a| = 1$ and $b \ge 0$. For any $\epsilon > 0$, define

$$H_\epsilon^+ = \{x \in \mathbb{R}^n : \langle a, x \rangle \le b(1+\epsilon)\},$$
$$H_\epsilon^- = \{x \in \mathbb{R}^n : \langle a, x \rangle \le b(1-\epsilon)\}.$$

Note that $\Pr(H_\epsilon^+ \setminus H_\epsilon^-) \le C\epsilon$.

Now define $B = \{x \in S^{n-1} : \langle x, a \rangle \le b/\sqrt{n}\}$. Then $\bar{B} = \{x \in \mathbb{R}^n : \langle x, a \rangle \le b|x|/\sqrt{n}\}$, and so

$$\Pr(\bar{B} \setminus H_\epsilon^+) = \Pr\big((1+\epsilon)b \le \langle X, a \rangle \le b|X|/\sqrt{n}\big) \le \Pr(|X| \ge (1+\epsilon)\sqrt{n}) \le Ce^{-c\epsilon^2 n},$$

where the last line follows from standard concentration inequalities (Bernstein's inequalities, for example). Similarly,

$$\Pr(H_\epsilon^- \setminus \bar{B}) \le \Pr(|X| \le (1-\epsilon)\sqrt{n}) \le Ce^{-c\epsilon^2 n}.$$

Since $H_\epsilon^- \subset H \subset H_\epsilon^+$ and $\Pr(H_\epsilon^+ \setminus H_\epsilon^-) \le C\epsilon$, it follows that

$$\Pr(H \bigtriangleup \bar{B}) \le C\epsilon + Ce^{-c\epsilon^2 n}.$$

By choosing $\epsilon = Cn^{-1/2} \log n$, we have

$$\Pr(H \bigtriangleup \bar{B}) \le Cn^{-1/2} \log n. \tag{8.1}$$

Now, the lemma claimed that we could ensure $\Pr(\bar{B}) = \Pr(H)$. Since the volume of the cap $B' := \{\langle a, x \rangle \le b'|x|\}$ is continuous and strictly increasing in b', we may define b' to be the unique real number such that $\Pr(\bar{B}') = \Pr(H)$. Now, either $B \subset B'$ or $B' \subset B$; hence $\Pr(\bar{B} \bigtriangleup \bar{B}') = |\Pr(\bar{B}) - \Pr(\bar{B}')|$. On the other hand, (8.1) implies that

$$|\Pr(\bar{B}) - \Pr(\bar{B}')| = |\Pr(\bar{B}) - \Pr(H)| \le Cn^{-1/2} \log n,$$

and so the triangle inequality leaves us with

$$\Pr(H \bigtriangleup \bar{B}') \le \Pr(H \bigtriangleup \bar{B}) + \Pr(B \bigtriangleup \bar{B}') \le Cn^{-1/2} \log n. \qquad \square$$

We defer the proof of Lemma 8.2 until the next section, since this proof requires an introduction to spherical harmonics.

8.1. Spherical harmonics and Lemma 8.2

We will try to give an introduction to spherical harmonics which is as brief as possible, while still containing enough material for us to explain the proof of Lemma 8.2 adequately. A slightly less brief introduction is contained in [29]; for a full treatment, see [37].

Let $\mathcal{S}_k$ be the linear space consisting of harmonic, homogeneous, degree-k polynomials. We will think of $\mathcal{S}_k$ as a subspace of $L_2(S^{n-1}, Q)$; then $\{\mathcal{S}_k : k \geq 0\}$ spans $L_2(S^{n-1}, Q)$. One can easily check that $\mathcal{S}_k$ is invariant under rotations. Hence it is a representation of $SO(n)$. It turns out, moreover, that $\mathcal{S}_k$ is an irreducible representation of $SO(n)$; combined with Schur's lemma, this leads to the following important property:

Lemma 8.3. *If $T : L_2(S^{n-1}) \to L_2(S^{n-1})$ commutes with rotations then $\{\mathcal{S}_k : k \geq 0\}$ are the eigenspaces of T.*

In particular, we will apply Lemma 8.3 to the operators T_ρ defined by $(T_\rho f)(X) = \mathbb{E}(f(Y) \mid X)$, where $(X, Y) \sim Q_\rho$. In other words, $(T_\rho f)(x)$ is the average of f over the set $\{y \in S^{n-1} : \langle x, y\rangle = \rho\}$. Clearly, T_ρ commutes with rotations; hence Lemma 8.3 implies that $\{\mathcal{S}_k : k \geq 0\}$ are the eigenspaces of T_ρ. In particular, there exist $\{\mu_k(\rho) : k \geq 0\}$ such that $T_\rho f = \mu_k(\rho) f$ for all $f \in \mathcal{S}_k$. Moreover, to compute $\mu_k(\rho)$, it is enough to compute $T_\rho f$ for a single $f \in \mathcal{S}_k$. For this task, the Gegenbauer polynomials provide good candidates: define

$$G_k(t) = \mathbb{E}\big(t + iW_1\sqrt{1-t^2}\big)^k,$$

where the expectation is over $W = (W_1, \dots, W_{n-1})$ distributed uniformly on the sphere S^{n-2}. Define $f_k(x) = G_k(x_1)$; it turns out that $f_k \in \mathcal{S}_k$; on the other hand, one can easily check that $f_k(e_1) = 1$, while $(T_\rho f_k)(e_1) = G_k(\rho)$. From the discussion above, it then follows that

$$\mu_k(\rho) = \mathbb{E}\big(\rho + iW_1\sqrt{1-\rho^2}\big)^k.$$

With this explicit formula, we can show that $\mu_k(\rho)$ is continuous in ρ:

Lemma 8.4. *For any $\epsilon > 0$ there exists $C(\epsilon)$ such that if $\rho, \eta \in [-1+\epsilon, 1-\epsilon]$ then*

$$|\mu_k(\rho) - \mu_k(\eta)| \leq C(\epsilon)(|\rho - \eta| + n^{-1/2}).$$

We will leave the proof of Lemma 8.4 to the end. First, let us show how it can be used to prove that $Q_\rho(X \in A_1,\ Y \in A_2)$ is continuous in ρ.

Lemma 8.5. *For any $\epsilon > 0$ there exists $C(\epsilon)$ such that if $\rho, \eta \in [-1+\epsilon, 1-\epsilon]$ then*

$$\begin{aligned}|Q_\rho(X \in A_1,\ Y \in A_2) &- Q_\eta(X \in A_1,\ Y \in A_2)|\\ &\leq C(\epsilon) Q^{1/2}(A_1) Q^{1/2}(A_2)(|\rho - \eta| + n^{-1/2}).\end{aligned}$$

Proof. Take $f, g \in L_2(S^{n-1}, Q)$ and consider the decomposition $f = \sum_{k=0}^\infty f_k$ where $f_k \in \mathcal{S}_k$. Then

$$|\mathbb{E} g T_\rho f - \mathbb{E} g T_\eta f| \leq \|T_\rho f - T_\eta f\|_2 \|g\|_2$$

(where $\|f\|_2$ denotes $\sqrt{\mathbb{E}f^2}$) and

$$\|T_\rho f - T_\eta f\|_2^2 = \sum_{k=0}^{\infty} (\mu_k(\rho) - \mu_k(\eta))^2 \|f_k\|_2^2.$$

By Lemma 8.4, we have

$$\|T_\rho f - T_\eta f\|_2 \le C(\epsilon)(|\rho - \eta| + n^{-1/2})\|f\|_2,$$

and therefore

$$|\mathbb{E}gT_\rho f - \mathbb{E}gT_\eta f| \le C(\epsilon)\|f\|_2\|g\|_2(|\rho - \eta| + n^{-1/2}).$$

Note that if $f = 1_{A_1}$ and $g = 1_{A_2}$ then $\mathbb{E}gT_\rho f = Q_\rho(X \in A_1, Y \in A_2)$, while $\|f\|_2 = Q(A_1)^{1/2}$. □

The proof of Lemma 8.2 is straightforward once we know Lemma 8.5. As we have already mentioned, normalized Gaussian vectors from $\Pr_\rho$ have a joint distribution that is similar to Q_ρ, except that their inner products are close to ρ instead of being exactly ρ. But Lemma 8.5 implies that a small difference in ρ does not affect the noise sensitivity by much.

Proof of Lemma 8.2. Let X, Y be distributed according to $\Pr_\rho$. Then

$$\Pr_\rho(X \in \bar{A}_1, Y \in \bar{A}_2) = \Pr_\rho(X/|X| \in A_1,\ Y/|Y| \in A_2).$$

Note that conditioned on $|X|$, $|Y|$ and $\langle X, Y\rangle$, the variables $X/|X|$, $Y/|Y|$ are distributed according to Q_r, where $r = \langle X, Y\rangle/(|X|\,|Y|)$. Now with probability $1 - 1/n^2$,

$$|X|^2, |Y|^2 \in n \pm Cn^{1/2}\log n, \qquad \langle X, Y\rangle \in \rho n \pm Cn^{1/2}\log n.$$

On this event, we have

$$r = \langle X/|X|, Y/|Y|\rangle \in \rho \pm Cn^{-1/2}\log n.$$

Using Lemma 8.5 we get

$$\Pr_\rho(X \in \bar{A}_1,\ Y \in \bar{A}_2) \le Q_\rho(X \in A_1,\ Y \in A_2) + C(\epsilon)n^{-1/2}\log n.$$

A similar argument yields a bound in the other direction and concludes the proof. □

Our final task is the proof of Lemma 8.4:

Proof of Lemma 8.4. Define $Z_\rho = \rho + iW_1\sqrt{1-\rho^2}$ (recalling that $W = (W_1, \dots, W_{n-1})$ is uniformly distributed on S^{n-2}) so that $\mu_k(\rho) = \mathbb{E}Z_\rho^k$. Note that if $|W_1| \le 1/2$ (which happens with probability at least $1 - \exp(-cn)$) then

$$|Z_\rho| = \rho^2 + W_1(1-\rho^2) \le \frac{1+\rho^2}{2} \le 1 - \frac{\epsilon}{2} \le \exp(-c\epsilon).$$

Now,

$$\mu_k(\rho) - \mu_k(\eta) = \mathbb{E}(Z_\rho^k - Z_\eta^k) = \mathbb{E}(Z_\rho - Z_\eta) \sum_{j=1}^{k-1} Z_\rho^j Z_\eta^{k-1-j}. \tag{8.2}$$

If $|W_1| \le \frac{1}{2}$ then $|Z_\rho^j Z_\eta^{k-1-j}| \le \exp(-c\epsilon k)$ and so

$$\Big|\sum_j Z_\rho^j Z_\eta^{k-1-j}\Big| \le k \exp(-c\epsilon k) \le C(\epsilon).$$

Applying this to (8.2), we have

$$\begin{aligned}|\mu_k(\rho) - \mu_k(\eta)| &= \mathbb{E}(Z_\rho^k - Z_\eta^k) 1_{\{|W_1| \ge 1/2\}} + \mathbb{E} 1_{\{|W_1|<1/2\}} (Z_\rho - Z_\eta) \sum_{j=1}^{k-1} Z_\rho^j Z_\eta^{k-1-j} \\ &\le 2 \Pr(|W_1| \ge 1/2) + C(\epsilon) \mathbb{E}|Z_\rho - Z_\eta| \le \exp(-cn) + C(\epsilon)|\rho - \eta|,\end{aligned}$$

where $\mathbb{E}|Z_\rho - Z_\eta| \le C(\epsilon)|\rho - \eta|$ because $|\sqrt{1-\rho^2} - \sqrt{1-\eta^2}| \le C(\epsilon)|\rho - \eta|$. □

8.2. Spherical noise and Max-Cut

In this section, we will show how robust noise sensitivity on the sphere (Theorem 1.11) implies that half-space rounding for the Goemans–Williamson algorithm is robustly optimal (Theorem 1.12). The key for making this connection is Karloff's family of graphs [23]: for any $n, d \in \mathbb{N}$, let $G_{n,d} = (V_{n,d}, E_{n,d})$ be the graph whose vertices are the $\binom{n}{n/2}$ balanced elements of $\{-n^{-1/2}, n^{-1/2}\}^n$, and with an edge between u and v if $\langle u, v\rangle = d/n - 1$. Karloff showed that if $d \le n/3$ then the optimal cut of $G_{n,d}$ has value $|E_{n,d}|(1 - d/(2n))$. Moreover, the identity embedding (and any rotation of it) is an optimal embedding of $G_{n,d}$ into S^{n-1}. In these embeddings, every angle between two connected vertices is d/n; hence, it is easy to calculate the expected value of a rounding scheme:

Lemma 8.6. *Let (X, Y) be distributed according to $Q_{d/n}$. For any rounding scheme R,*

$$\mathrm{Cut}(G_{n,d}, R) \le \frac{|E_{n,d}|}{2} \mathbb{E}|R(X) - R(Y)|,$$

where the expectation is with respect to X, Y and R.

Proof. Recall that

$$\begin{aligned}\mathrm{Cut}(G, R) &= \frac{1}{2} \min_f \mathbb{E}_R \sum_{(u,v)\in E} |R(f(u)) - R(f(v))| \\ &\le \frac{1}{2} \mathbb{E}_R \mathbb{E}_f \sum_{(u,v)\in E} |R(f(u)) - R(f(v))|,\end{aligned}$$

where the expectation is taken over all rotations f. But if f is a uniformly random rotation then for every $(u, v) \in E_{n,d}$, the pair $(f(u), f(v))$ is equal in distribution to the pair (X, Y) (and both pairs are independent of R). □

Theorem 1.12 follows fairly easily from Lemma 8.6, Theorem 1.11, and the fact that $\text{MaxCut}(G_{n,d}) = |E_{n,d}|(1 - d/n)$. Indeed, choose integers n and d such that $|d/n - 1 - \cos\theta^*| \le n^{-1}$, where $\theta^* \approx 2.33$ minimizes α_θ, and suppose there is a rounding scheme R such that

$$\text{Cut}(G_{n,d}, R) \ge \text{MaxCut}(G_{n,d})(\alpha_{\theta^*} - \epsilon).$$

Let $\theta = \arccos(d/n - 1)$; since α_θ is continuous in θ, it follows that $|\alpha_\theta - \alpha_{\theta^*}| \le C/n$. Taking $\epsilon_\star = \max\{\epsilon, n^{-1/2}\log n\}$, we have $|\alpha_\theta - \alpha_{\theta^*}| \le C\epsilon_\star$ and so

$$\begin{aligned}\text{Cut}(G_{n,d}, R) &\ge \text{MaxCut}(G_{n,d})(\alpha_\theta - C\epsilon_\star) = \frac{1}{2}|E_{n,d}|(1-\cos\theta)(\alpha_\theta - C\epsilon_\star)\\ &= \frac{1}{\pi}\theta|E_{n,d}|(1 - C\epsilon_\star).\end{aligned}$$

By Lemma 8.6, $\frac{1}{2}\mathbb{E}|R(X) - R(Y)| \ge \frac{2}{\pi}\theta(1 - C\epsilon_\star)$. If we define the (random) subset $A_R \subset S^{n-1}$ by $A_R = \{x : R(x) = 1\}$, and set $\rho = \cos\theta$, then

$$\Pr(A_R) - \mathbb{S}_\rho(A_R) = \frac{1}{2}\mathbb{E}(|R(X) - R(Y)| \mid R).$$

Taking expectations gives

$$\mathbb{E}(\Pr(A_R) - \mathbb{S}_\rho(A_R)) = \frac{1}{2}\mathbb{E}|R(X) - R(Y)| \ge \frac{1}{\pi}\arccos\rho - C\epsilon_\star. \tag{8.3}$$

Let δ_R be the random deficit $\delta_R = (2/\pi)\arccos\rho - (\Pr(A_R) - \mathbb{S}_\rho(A_R))$, so that (8.3) implies $\mathbb{E}\delta_R \le C\epsilon_\star$. Take η_R to be the distance from A_R to the nearest hemisphere: $\eta_R = \min\{\Pr(A_R \triangle B) : B \text{ is a hemisphere}\}$ and let B_R be a hemisphere that achieves the minimum (which is attained because the set of hemispheres is compact with respect to the distance $d(A, B) = \Pr(A \triangle B)$). Recall that $\theta \approx \theta^* \approx 2.33$ and so $\rho = \cos\theta < 0$; by the same symmetries discussed following Theorem 1.4, Theorem 1.11 applies for $\rho < 0$, but with the deficit inequality reversed. Hence, $\eta_R \le C\max\{\delta_R, n^{-1/2}\log n\}^c$. Taking expectations yields

$$\mathbb{E}\eta_R \le C\mathbb{E}\max\{\delta_R, n^{-1/2}\log n\}^c \le C\max\{\mathbb{E}\delta_R, n^{-1/2}\log n\}^c = C'\epsilon_\star^c.$$

Consider the rounding scheme $\tilde{R}(y)$ which is 1 when $y \in B_R$ and -1 otherwise. Then $\mathbb{E}(|R(Y) - \tilde{R}(Y)| \mid R) = 2\eta_R$, and so the displayed equation above implies that

$$\mathbb{E}|R(Y) - \tilde{R}(Y)| \le C\epsilon_\star^c.$$

Since $\tilde{R}$ is a hyperplane rounding scheme, this completes the proof of Theorem 1.12.

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