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On the strange duality conjecture for abelian surfaces

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Abstract. We study Le Potier's strange duality conjecture for moduli spaces of sheaves over generic abelian surfaces. We prove the isomorphism for abelian surfaces which are products of elliptic curves, when the moduli spaces consist of sheaves of equal ranks and fiber degree 1. The birational type of the moduli space of sheaves is also investigated. Generalizations to arbitrary product elliptic surfaces are given.

Keywords. Moduli spaces of sheaves, abelian surfaces, strange duality

1. Introduction

There are three versions of Le Potier's strange duality conjecture for abelian surfaces, formulated and supported numerically in [MO2]. In this article, we confirm two of them for product abelian surfaces. This establishes the conjecture over an open subset in the moduli space of polarized abelian surfaces.

1.1. The strange duality morphism

To set the stage, let (X, H) be a polarized complex abelian surface. For a coherent sheaf $V \to X$, denote by $v = \operatorname{ch} V \in H^\star(X, \mathbb{Z})$ its Mukai vector. Fix two Mukai vectors v and w such that the orthogonality condition $\chi(v \cdot w) = 0$ holds. We consider the two moduli spaces $\mathfrak{M}_v^+$ and $\mathfrak{M}_w^+$ of H-semistable sheaves of type v and w with fixed determinant. They carry two determinant line bundles $\Theta_w \to \mathfrak{M}_v^+$, $\Theta_v \to \mathfrak{M}_w^+$, whose global sections we seek to relate.

More precisely, according to Le Potier's strange duality conjecture [LP], [MO2], if the condition

$$c_1(v \cdot w) \cdot H > 0$$

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is satisfied, the jumping locus

$$\Theta_{vw}^{+} = \{(V, W) : h^1(V \otimes W) \neq 0\} \subset \mathfrak{M}_v^{+} \times \mathfrak{M}_w^{+}$$

is expected to be a divisor. Further, its defining equation is a section of the split bundle

$$\Theta_w \boxtimes \Theta_v \to \mathfrak{M}_v^{+} \times \mathfrak{M}_w^{+},$$

conjecturally inducing an isomorphism

$$\mathsf{D}^{+} : H^0(\mathfrak{M}_v^{+}, \Theta_w)^{\vee} \to H^0(\mathfrak{M}_w^{+}, \Theta_v).$$

This expectation is supported numerically: it was established in [MO2] that

$$\chi(\mathfrak{M}_v^{+}, \Theta_w) = \chi(\mathfrak{M}_w^{+}, \Theta_v) = \frac{\chi(X, L^{+})}{d_v + d_w}\binom{d_v + d_w}{d_v},$$

where L^{+} is a line bundle on X with

$$c_1(L^{+}) = c_1(v \cdot w) \quad \text{and} \quad d_v = \tfrac{1}{2}\dim \mathfrak{M}_v^{+}, \quad d_w = \tfrac{1}{2}\dim \mathfrak{M}_w^{+}.$$

Similarly, letting $\mathfrak{M}_v^{-}$ and $\mathfrak{M}_w^{-}$ denote the moduli spaces of sheaves with fixed determinant of their Fourier–Mukai transforms, we have the symmetry

$$\chi(\mathfrak{M}_v^{-}, \Theta_w) = \chi(\mathfrak{M}_w^{-}, \Theta_v) = \frac{\chi(X, L^{-})}{d_v + d_w}\binom{d_v + d_w}{d_v},$$

where L^{-} is a line bundle on X with $c_1(L^{-}) = c_1(\widehat{v} \cdot \widehat{w})$, the hats denoting the Fourier–Mukai transforms. The map

$$\mathsf{D}^{-} : H^0(\mathfrak{M}_v^{-}, \Theta_w)^{\vee} \to H^0(\mathfrak{M}_w^{-}, \Theta_v)$$

induced by the theta divisor $\Theta_{vw}^{-} \subset \mathfrak{M}_v^{-} \times \mathfrak{M}_w^{-}$ is expected to be an isomorphism.

1.2. Results

We will establish the isomorphisms for abelian surfaces which split as products of elliptic curves

$$X = B \times F.$$

We regard X as a trivial fibration $\pi_B : X \to B$, and write σ and f for the class of the zero section and of the fiber over zero. We assume that the polarization H is *suitable* in the sense of [F], i.e.

$$H = \sigma + Nf \quad \text{for } N \gg 0.$$

Over *simply connected* elliptic surfaces, for coprime rank and fiber degree, the moduli space of sheaves is birational to the Hilbert scheme of points, as shown by Bridgeland [B]. For sheaves with fixed determinant, the situation is subtler over elliptic *abelian* surfaces. A refinement of Bridgeland's argument, using a Fourier–Mukai transform with kernel given by a universal Atiyah bundle, allows us to prove the following result.

Theorem 1.1. *Let $X = B \times F$ be a product abelian surface, and let v be a Mukai vector such that the rank r and the fiber degree $d = c_1(v) \cdot f$ are coprime. Let*

$$a_B : X^{[d_v]} \to B$$

be the composition of the addition map $a : X^{[d_v]} \to X$ on the Hilbert scheme of points with the projection to the base elliptic curve. Then the moduli space $\mathfrak{M}_v^+$ is birational to

$$\mathfrak{X}_v^+ = \{(Z, b) : a_B(Z) = rb\} \subset X^{[d_v]} \times B.$$

A similar statement holds for the moduli space $\mathfrak{M}_v^-$. By contrast, the generalized Kummer variety K_v associated to the higher rank vectors is birational to the Kummer variety in rank 1, as noted in [Y]. This fact is recovered in two ways while establishing Theorem 1.1.

The following two theorems capture our main results concerning strange duality.

Theorem 1.2. *Let $X = B \times F$ be a product abelian surface. Let v and w be two orthogonal Mukai vectors of equal ranks $r \geq 3$, with*

$$c_1(v) \cdot f = c_1(w) \cdot f = 1.$$

Then $\mathsf{D}^+ : H^0(\mathfrak{M}_v^+, \Theta_w)^\vee \to H^0(\mathfrak{M}_w^+, \Theta_v)$ is an isomorphism.

Similarly, we show

Theorem 1.3. *Let $X = B \times F$ be a product abelian surface. Assume v and w are two orthogonal Mukai vectors of ranks $r, s \geq 3$ and equal Euler characteristics $\chi = \chi'$, with*

$$c_1(v) \cdot f = c_1(w) \cdot f = 1.$$

Then $\mathsf{D}^- : H^0(\mathfrak{M}_v^-, \Theta_w)^\vee \to H^0(\mathfrak{M}_w^-, \Theta_v)$ is an isomorphism.

In particular, the theorems imply that $\Theta^\pm$ are divisors in the products $\mathfrak{M}_v^\pm \times \mathfrak{M}_w^\pm$.

1.2.1. Higher genus. The requirement that B be elliptic can in fact be removed in Theorem 1.2. Indeed, keep F a smooth elliptic curve, and consider a smooth projective curve C of arbitrary genus $g \geq 1$. Let σ denote the zero section of the trivial fibration $X = C \times F \to C$. We show

Theorem 1.4. *Assume that $X = C \times F$ is a product surface as above, and let v, w be orthogonal Mukai vectors of equal ranks $r \geq g$ and $r \neq 2$, such that*

(i) *the determinants are fixed of the form*

$$\det v = \mathcal{O}(\sigma) \otimes \ell_v, \qquad \det w = \mathcal{O}(\sigma) \otimes \ell_w,$$

for generic line bundles ℓ_v and ℓ_w of fixed degree over the curve C;

(ii) $\dim \mathfrak{M}_v^+ + \dim \mathfrak{M}_w^+ \geq 8r(g-1)$.

Then $\mathsf{D}^+ : H^0(\mathfrak{M}_v^+, \Theta_w)^\vee \to H^0(\mathfrak{M}_w^+, \Theta_v)$ is an isomorphism.

1.3. Comparison

Theorems 1.2 and 1.4 parallel the strange duality results for *simply connected* elliptic surfaces. Let $\pi : Y \to \mathbb{P}^1$ be a simply connected elliptic fibration with a section and at worst irreducible nodal fibers. The dimension of the two complementary moduli spaces $\mathfrak{M}_v$ and $\mathfrak{M}_w$ will be taken large enough compared to the constant

$$\Delta = \chi(Y, \mathcal{O}_Y) \cdot ((r+s)^2 + (r+s) + 2) - 2(r+s).$$

The polarization is still assumed suitable. The following was proved by combining [MOY] and [BH]:

Theorem. *Let v and w be two orthogonal topological types of rank $r, s \geq 3$, such that*

(i) *the fiber degrees satisfy* $c_1(v) \cdot f = c_1(w) \cdot f = 1$,
(ii) $\dim \mathfrak{M}_v + \dim \mathfrak{M}_w \geq \Delta$.

Then $\mathsf{D} : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v)$ *is an isomorphism.*

We suspect that an analogous statement can be made for all (not necessarily simply connected, with possibly reducible fibers) elliptic surfaces $Y \to C$, going beyond the scope of Theorems 1.2 and 1.4.

The results for simply connected fibrations and abelian surfaces both rely on Fourier–Mukai techniques, but the geometry is more involved in the abelian case, as already illustrated by the birationality statement of Theorem 1.1. Via Fourier–Mukai, instead of a rather standard analysis of tautological line bundles over Hilbert scheme of points in the simply connected case, one is led here to studying sections of suitable theta bundles over the schemes $\mathfrak{X}_v^+$. This requires new ideas. We prove the duality for spaces of sheaves of equal ranks and fiber degree 1, but believe these assumptions may be relaxed. The case $r = s = 2$ is also left out of our theorems: while the Fourier–Mukai arguments do not cover it, we believe strange duality holds here as well.

1.4. Variation in moduli

Let $\mathcal{A}_d$ be the moduli space of pairs (X, H), where X is an abelian surface and H is an ample line bundle inducing a polarization of type $(1, d)$ on X. For a Mukai vector v with $c_1(v) = c_1(H)$, we consider the relative moduli space of sheaves $\pi : \mathfrak{M}[v]^+ \to \mathcal{A}_d$ whose fiber over a surface (X, H) is the moduli space of H-semistable sheaves with Mukai vector v and fixed determinant equal to H. Associated with two orthogonal Mukai vectors v and w, there is a universal canonical theta divisor

$$\Theta_{vw}^+ = \{(X, H, V, W) : h^1(X, V \otimes W) \neq 0\} \subset \mathfrak{M}[v]^+ \times_{\mathcal{A}_d} \mathfrak{M}[w]^+,$$

giving rise to a line bundle which splits as a product

$$\Theta_{vw}^+ = \Theta_w \boxtimes \Theta_v \quad \text{on } \mathfrak{M}[v]^+ \times_{\mathcal{A}_d} \mathfrak{M}[w]^+.$$

Pushforward via the morphisms

$$\mathfrak{M}[v]^{+} \to \mathcal{A}_d, \quad \mathfrak{M}[w]^{+} \to \mathcal{A}_d$$

yields two coherent sheaves of generalized theta functions over $\mathcal{A}_d$,

$$\mathbb{W} = R^0\pi_\star\Theta_w, \quad \mathbb{V} = R^0\pi_\star\Theta_v.$$

Let $\mathcal{H} \subset \mathcal{A}_d$ be the Humbert hypersurface parametrizing split abelian surfaces (X, H) with

$$X = B \times F \quad \text{and} \quad H = L_B \boxtimes L_F,$$

for line bundles L_B, L_F over B and F of degrees d and 1 respectively.

Theorem 1.2 can be rephrased as the following generic strange duality statement:

Theorem 1.5. *Assume v and w are orthogonal Mukai vectors of equal ranks $r \geq 3$ with*

(i) $c_1(v) = c_1(w) = H$;
(ii) $\langle v, v\rangle \geq 2(r^2 + r - 1)$, $\langle w, w\rangle \geq 2(r^2 + r - 1)$. [1]

The sheaves $\mathbb{V}$ and $\mathbb{W}$ are then locally free when restricted to the Humbert hypersurface $\mathcal{H}$, and Θ^+_{vw} induces an isomorphism

$$\mathsf{D} : \mathbb{W}^\vee \to \mathbb{V}$$

along $\mathcal{H}$.

2. Moduli spaces of sheaves of fiber degree 1

2.1. Setting

We consider a complex abelian surface X which is a product of two elliptic curves, $X \simeq B \times F$. Letting o_B, o_F denote the origins on B and F, we write σ and f for the divisors $B \times o_F$ respectively $o_B \times F$ in the product $B \times F$. Note that

$$\sigma^2 = f^2 = 0, \quad \sigma \cdot f = 1.$$

2.1.1. Line bundles over X. For any positive integers a and b, the line bundle $\mathcal{O}(a\sigma + bf)$ on X is ample. Its higher cohomology vanishes, and we have

$$h^0(X, \mathcal{O}(a\sigma + bf)) = \chi(X, \mathcal{O}(a\sigma + bf)) = ab. \tag{2.1}$$

Letting $\pi_B : X \to B$ and $\pi_F : X \to F$ be the natural projections from X, we also note that for any $\ell > 0$,

$$\begin{aligned} h^0(X, \mathcal{O}(\ell\sigma)) &= h^0(X, \pi_F^\star\mathcal{O}(\ell o_F)) = h^0(F, \mathcal{O}(\ell o_F)) = \ell, \\ h^0(X, \mathcal{O}(\ell f)) &= h^0(X, \pi_B^\star\mathcal{O}(\ell o_B)) = h^0(B, \mathcal{O}(\ell o_B)) = \ell. \end{aligned} \tag{2.2}$$

[1] Assumption (ii) allows us to exchange H-stability with stability with respect to a suitable polarization, since in this case the ensuing moduli spaces agree in codimension 1. This was proved for $K3$ surfaces in the appendix of [MOY]. The case of abelian surfaces follows by the same argument.

Thus every section of $\mathcal{O}_X(\ell f)$ corresponds to a divisor of the form

$$\bigsqcup_{i=1}^{\ell} z_i \times F \quad \text{for } z_1 + \cdots + z_\ell = o_B.$$

Furthermore, (2.1) and (2.2) imply that every section of the line bundle $\mathcal{O}_X(\sigma + \ell f)$ vanishes along a divisor of the form

$$\sigma \cup \bigsqcup_{i=1}^{\ell} z_i \times F \quad \text{for } z_1 + \cdots + z_\ell = o_B.$$

2.1.2. Fourier–Mukai functors. Two different Fourier–Mukai transforms will be considered on X. First, there is the Fourier–Mukai transform with respect to the standardly normalized Poincaré bundle $\mathcal{P} \to X \times \widehat{X}$. We identify $\widehat{X} \cong X$ using the polarization $o_B \times F + B \times o_F$. Specifically, our (perhaps nonstandard) sign choice is so that $y = (y_B, y_F) \in X$ is viewed as a degree 0 line bundle $y \to X$ via the association

$$y \mapsto \mathcal{O}_B(y_B - o_B) \boxtimes \mathcal{O}_F(y_F - o_F).$$

The Poincaré bundle is then given by

$$\mathcal{P} \to X \times X, \quad \mathcal{P} = \mathcal{P}_B \boxtimes \mathcal{P}_F,$$

where

$$\mathcal{P}_B \to B \times B, \quad \mathcal{P}_B = \mathcal{O}_{B\times B}(\Delta_B) \otimes p^\star \mathcal{O}_B(-o_B) \otimes q^\star \mathcal{O}_B(-o_B), \quad \text{and}$$
$$\mathcal{P}_F \to F \times F, \quad \mathcal{P}_F = \mathcal{O}_{F\times F}(\Delta_F) \otimes p^\star \mathcal{O}_F(-o_F) \otimes q^\star \mathcal{O}_F(-o_F).$$

The Fourier–Mukai transform with kernel $\mathcal{P}$ is denoted

$$\mathbf{R}\mathcal{S} : \mathbf{D}(X) \to \mathbf{D}(X).$$

A second Fourier–Mukai transform is defined by considering the relative Picard variety of $\pi_B : X \to B$. We identify $F \cong \widehat{F}$ so that $y_F \in F \mapsto \mathcal{O}_F(y_F - o_F)$, and we let

$$\mathbf{R}\mathcal{S}^\dagger : \mathbf{D}(X) \to \mathbf{D}(X)$$

be the Fourier–Mukai transform whose kernel is the pullback of the Poincaré sheaf $\mathcal{P}_F \to F \times F$ to the product $X \times_B X \cong F \times F \times B$. Both Fourier–Mukai transforms are known to be equivalences of derived categories.

Several properties of the Fourier–Mukai will be used below. First, for integers a, b we have

$$\det \mathbf{R}\mathcal{S}(\mathcal{O}(a\sigma + bf)) = \mathcal{O}(-b\sigma - af), \quad \det \mathbf{R}\mathcal{S}^\dagger(\mathcal{O}(a\sigma + bf)) = \mathcal{O}(-\sigma + abf).$$

Next, we have the following standard result, which will in fact be proved in greater generality in Lemma 3.4.

Lemma 2.1. *For $x = (x_B, x_F) \in X$ and $y = (y_B, y_F) \in \widehat{X}$, we have*

$$\mathbf{R}\mathcal{S}^\dagger(t_x^\star E) = t_{x_B}^\star \mathbf{R}\mathcal{S}^\dagger(E) \otimes x_F^\vee, \quad \mathbf{R}\mathcal{S}^\dagger(E \otimes y) = t_{y_F}^\star \mathbf{R}\mathcal{S}^\dagger(E) \otimes y_B.$$

2.2. Moduli spaces of sheaves

We consider sheaves over X of Mukai vector v such that

$$\operatorname{rank} v = r, \quad \chi(v) = \chi, \quad c_1(v) \cdot f = 1.$$

We set

$$d_v = \tfrac{1}{2}\langle v, v\rangle = \frac{c_1(v)^2}{2} - r\chi,$$

which is half the dimension of the moduli space $\mathfrak{M}_v^+$ below. Recall from the introduction that the polarization H is suitable, i.e.,

$$H = \sigma + Nf \quad \text{for } N \gg 0.$$

As in [MO2], we are concerned with three moduli spaces of sheaves:

(i) The moduli space K_v of H-semistable sheaves V on X with

$$\det V = \mathcal{O}(\sigma + mf), \quad \det \mathbf{R}\mathcal{S}(V) = \mathcal{O}(-m\sigma - f).$$

Thus the determinants of the sheaves and of the Fourier–Mukai transforms of sheaves in K_v are fixed. Here, we wrote

$$c_1(v) = \sigma + mf \quad \text{where} \quad m = d_v + r\chi.$$

(ii) The moduli space $\mathfrak{M}_v^+$ of H-semistable sheaves V on X with

$$\det V = \mathcal{O}(\sigma + mf).$$

The two spaces (i) and (ii) are related via the degree d_v^4 étale morphism

$$\Phi_v^+ : K_v \times X \to \mathfrak{M}_v^+, \quad \Phi_v^+(V, x) = t_{rx}^\star V \otimes t_x^\star \det V^{-1} \otimes \det V.$$

In explicit form, we write equivalently

$$\Phi_v^+(V, x) = t_{rx}^\star V \otimes (x_F \boxtimes x_B^m). \tag{2.3}$$

(iii) Finally, there is a moduli space $\mathfrak{M}_v^-$ of semistable sheaves whose Fourier–Mukai transform has fixed determinant

$$\det \mathbf{R}\mathcal{S}(V) = \mathcal{O}(-f - m\sigma).$$

In this case, we shall make use of the étale morphism

$$\Phi_v^- : K_v \otimes \widehat{X} \to \mathfrak{M}_v^-, \quad \Phi_v^-(V, y) = t_{(y_B, my_F)}^\star V \otimes y^\chi.$$

Note that we have indeed

$$\begin{aligned} \det \mathbf{R}\mathcal{S}\Phi_v^-(V, y) &= \det \mathbf{R}\mathcal{S}(t_{(y_B, my_F)}^\star V \otimes y^\chi) = t_{\chi y}^\star \det \mathbf{R}\mathcal{S}(t_{(y_B, my_F)}^\star V) \\ &= t_{\chi y}^\star \det(\mathbf{R}\mathcal{S}(V) \otimes (y_B, my_F)^{-1}) \\ &= t_{\chi y}^\star \mathcal{O}(-f - m\sigma) \otimes (y_B, my_F)^{-\chi} = \mathcal{O}(-f - m\sigma). \end{aligned}$$

For further details regarding the morphisms Φ_v^+, Φ_v^-, we refer the reader to [MO2, Sections 4 and 5].

2.3. Birationality of K_v with the generalized Kummer variety $K^{[d_v]}$

We now establish in two different ways an explicit birational map

$$\Psi_r : K^{[d_v]} \dashrightarrow K_v,$$

where $K^{[d_v]}$ is the generalized Kummer variety of dimension $2d_v - 2$,

$$K^{[d_v]} = \{Z \in X^{[d_v]} : a(Z) = 0\}.$$

As usual, $a : X^{[d_v]} \to X$ denotes the addition map on the Hilbert scheme.

2.3.1. O'Grady's construction. We first obtain the map Ψ_r by induction on the rank r of the sheaves, following O'Grady's work [OG] for elliptically fibered $K3$ surfaces. To start the induction, we let $Z \subset X$ be a zero dimensional subscheme of length $\ell = \ell(Z) = d_v$, such that $a(Z) = 0$, and further satisfying the following conditions:

(i) Z consists of distinct points contained in different fibers of π_B,
(ii) Z does not intersect the section σ.

Such a Z is a generic point in the generalized Kummer variety $K^{[d_v]}$. We let

$$m_1 = \chi + d_v,$$

which is the correct number of fibers needed when $r = 1$, and set

$$V_1 = I_Z \otimes \mathcal{O}(\sigma + m_1 f).$$

Note that $\chi(V_1) = \chi$, and that V_1 thus constructed belongs to K_v. Indeed, the determinant of the Fourier–Mukai of V_1 is fixed since

$$\begin{aligned}\det \mathbf{R}\mathcal{S}(V_1) &= \det \mathbf{R}\mathcal{S}(\mathcal{O}(\sigma + m_1 f)) \otimes \det \mathbf{R}\mathcal{S}(\mathcal{O}_Z)^\vee = \mathcal{O}(-m_1\sigma - f) \otimes \mathcal{P}_{-a(Z)} \\ &= \mathcal{O}(-m_1\sigma - f).\end{aligned}$$

For the last equality, we made use of the fact that $a(Z) = 0$.

We claim that for Z as above we have

$$h^1(V_1(-\chi f)) = h^1(I_Z \otimes \mathcal{O}(\sigma + \ell f)) = h^0(I_Z(\sigma + \ell f)) = 1. \tag{2.4}$$

Indeed, as explained in Section 2.1, every section of $\mathcal{O}_X(\sigma + \ell f)$ vanishes along a divisor of the form

$$\sigma + \pi_B^\star(z_1 + \cdots + z_\ell)$$

where $z_1, \ldots, z_\ell$ are points of B with $z_1 + \cdots + z_\ell = o_B$. Conditions (i) and (ii) ensure that there is a unique such divisor passing through Z: $z_1, \ldots, z_\ell$ are the B-coordinates of the distinct points of Z.

We inductively construct extensions

$$0 \to \mathcal{O}(\chi f) \to V_{r+1} \to V_r \to 0, \tag{2.5}$$

with stable middle term. The sheaves obtained will satisfy $\chi(V_r(-\chi f)) = 0$. In order to get (2.5), we show

$$\operatorname{ext}^1(V_r, \mathcal{O}(\chi f)) = \operatorname{ext}^1(\mathcal{O}(\chi f), V_r) = h^1(V_r(-\chi f)) = h^0(V_r(-\chi f)) = 1.$$

In the above calculation, we used $H^2(V_r(-\chi f)) = 0$, which follows from the stability of V_r. We will inductively prove the following four statements at once:

(a) $h^0(V_r(-\chi f)) = 1$;
(b) for all fibers f, we have $h^0(V_r(-\chi f - \mathsf{f})) = 0$;
(c) for fibers f_η avoiding Z, we have

$$V_r|_{f_\eta} \cong \mathsf{E}_{r,o}, \tag{2.6}$$

where the Atiyah bundle $\mathsf{E}_{r,o}$ is the the unique stable bundle over f_η of rank r and determinant $\mathcal{O}_{f_\eta}(o)$;
(d) for fibers f_z containing points $z \in Z$ we have

$$V_r|_{f_z} = \mathsf{E}_{r-1,z} \oplus \mathcal{O}_{f_z}(o - z). \tag{2.7}$$

Here, $\mathsf{E}_{r-1,z}$ is the Atiyah bundle of rank $r - 1$ and determinant $\mathcal{O}_{f_\eta}(z)$.

Equations (2.6) and (2.7) are analogous to Lemma 1 in [MOY].

The base case $r = 1$ of the induction is easily verified. Indeed, the ext-dimension in (a) follows by (2.4), and (b) is checked in the same way. Items (c) and (d) are shown by direct calculation (cf. [MOY]). Assuming these four statements hold for V_r, we construct the unique nontrivial extension (2.5). We now argue for stability of the middle term V_{r+1}, and we explain that statements (a)–(d) hold for V_{r+1}.

First, Proposition I.4.7 of [OG] gives stability of the middle term V_{r+1} if a suitable vanishing hypothesis holds. Precisely, we require that for all fibers f, we have

$$\mathrm{Hom}(V_r|_{\mathsf{f}}, \mathcal{O}_{\mathsf{f}}) = 0. \tag{2.8}$$

This vanishing is indeed satisfied, as the restriction of V_r to fibers is given by (c) and (d). Proposition I.4.7 also asserts that the restriction of (2.5) to any fiber f does not split. Thus, we obtain nontrivial extensions

$$0 \to \mathcal{O}_{\mathsf{f}} \to V_{r+1}|_{\mathsf{f}} \to V_r|_{\mathsf{f}} \to 0.$$

From the inductive hypothesis (c) and (d) we now deduce

$$V_{r+1}|_{f_\eta} \cong \mathsf{E}_{r+1,o}$$

for fibers f_η avoiding Z, while for fibers f_z containing points $z \in Z$ we have

$$V_{r+1}|_{f_z} = \mathsf{E}_{r,z} \oplus \mathcal{O}_{f_z}(o - z).$$

We have therefore established statements (c) and (d) for V_{r+1}. Next we prove (b):

$$h^0(V_{r+1}(-\chi f - \mathsf{f})) = 0.$$

This is immediate from the short exact sequence (2.5). In fact, the same exact sequence shows that $h^0(V_{r+1}(-\chi f)) \geq 1$.

To complete the argument, we establish now that (a) holds for V_{r+1}, i.e., we show that $h^0(V_{r+1}(-\chi f)) = 1$. Assume for a contradiction that $h^0(V_{r+1}(-\chi f)) \geq 2$. The cohomology of the exact sequence

$$0 \to V_{r+1}(-\chi f - \mathsf{f}) \to V_{r+1}(-\chi f) \to V_{r+1}|_{\mathsf{f}} \to 0$$

yields an injective map $H^0(V_{r+1}(-\chi f)) \to H^0(V_{r+1}|_{\mathsf{f}})$, so $h^0(V_{r+1}|_{\mathsf{f}}) \geq 2$. This contradicts (2.6) and (2.7) for V_{r+1}. We deduce $h^0(V_{r+1}(-\chi f)) = 1$, completing the inductive step.

Finally, (2.6) and (2.7) show that V_r uniquely determines the subscheme Z, proving that the map Ψ_r is injective. Therefore, we obtain a birational map

$$\Psi_r : K^{[d_v]} \dashrightarrow K_v,$$

by the equality of dimensions for the two irreducible spaces.

2.3.2. Fourier–Mukai construction. We next point out that Ψ_r can also be viewed as a fiberwise Fourier–Mukai transform. This is similar to the case of $K3$ surfaces analyzed in [MOY]. We show

Proposition 2.2. *For subschemes Z with $a(Z) = 0$, satisfying* (i) *and* (ii)*, we have*

$$\mathbf{R}\mathcal{S}^\dagger(V_r^\vee) = I_{\widetilde{Z}}(r\sigma - \chi f)[-1], \qquad \mathbf{R}\mathcal{S}^\dagger(V_r) = I_Z^\vee(-r\sigma + \chi f),$$

where $\widetilde{Z}$ is the reflection of Z along the zero section σ.

Proof. We prove the first equality. By the calculation of the restrictions of V_r to the fibers in (2.6) and (2.7), it follows that $V_r^\vee$ contains no subbundles of positive degree over each fiber. Therefore, $\mathbf{R}\mathcal{S}^\dagger(V_r^\vee)[1]$ is torsion free, by Proposition 3.7 of [BH]. The agreement of the two sheaves $\mathbf{R}\mathcal{S}^\dagger(V_r^\vee)[1]$ and $I_{\widetilde{Z}}(r\sigma - \chi f)$ holds fiberwise. This can be seen using (2.6) and (2.7), just as in [MOY, Proposition 1]. Hence, the two sheaves have rank 1, and furthermore, the ideal sheaves they involve must agree (recall that Z consists of distinct points). The proof is completed by proving equality of determinants. In turn, this follows inductively from the exact sequence (2.5):

$$\det \mathbf{R}\mathcal{S}^\dagger(V_{r+1}^\vee) = \det \mathbf{R}\mathcal{S}^\dagger(V_r^\vee) \otimes \det \mathbf{R}\mathcal{S}^\dagger(\mathcal{O}(-\chi f)) = \det \mathbf{R}\mathcal{S}^\dagger(V_r^\vee) \otimes \mathcal{O}(-\sigma).$$

The base case $r = 1$ is immediate, as

$$\begin{aligned}
\det \mathbf{R}\mathcal{S}^\dagger(I_Z^\vee(-\sigma - m_1 f)) &= \det \mathbf{R}\mathcal{S}^\dagger(\mathcal{O}(-\sigma - m_1 f)) \otimes \bigotimes_{z\in Z} \det(\mathbf{R}\mathcal{S}^\dagger(\mathcal{O}_z^\vee))^\vee \\
&= \mathcal{O}(-\sigma + m_1 f) \otimes \bigotimes_{z\in Z} \det(\mathbf{R}\mathcal{S}^\dagger(\mathcal{O}_z[-2]))^\vee \\
&= \mathcal{O}(-\sigma + m_1 f) \otimes \bigotimes_{z\in Z} \mathcal{O}(-f_z) \\
&= \mathcal{O}(-\sigma + m_1 f) \otimes \mathcal{O}(-\ell f) = \mathcal{O}(-\sigma + \chi f),
\end{aligned}$$

using that $a(Z) = 0$. Here f_z denotes the fiber through $z \in Z$. A similar calculation can be carried out for the Fourier–Mukai with kernel $\mathcal{P}^\vee$.

The second statement follows then by Grothendieck duality. We refer the reader to [MOY, Proposition 2] for an identical computation. □

Proposition 2.2 gives an explicit description of O'Grady's construction in terms of the Fourier–Mukai transform, at least along the loci (i) and (ii) whose complements have codimension 1. Explicitly, under the correspondence

$$K^{[d_v]} \ni Z \mapsto I_{\widetilde{Z}}(r\sigma - \chi f)[-1],$$

the birational map $\Psi_r : K^{[d_v]} \dashrightarrow K_v$ is given by the inverse of $\mathbf{R}\mathcal{S}^\dagger$, followed by dualizing,

$$\Psi_r \cong \mathbf{D}_X \circ (\mathbf{R}\mathcal{S}^\dagger)^{-1}.$$

However, viewed as a Fourier–Mukai transform, Ψ_r extends to an isomorphism in codimension 1, whenever $r \geq 3$. This is established in [BH, Sections 3 and 5] on general grounds, but can also be argued directly.

Indeed, the inverse of $\mathbf{R}\mathcal{S}^\dagger$ is, up to shifts, the Fourier–Mukai transform with kernel the dual $\pi^\star_{F\times F}\mathcal{P}_F{}^\vee$ of the fiberwise Poincaré line bundle on $X \times_B X \simeq F \times F \times B$. We claim that as long as $r \geq 3$ and Z contains no more than two points in the same fiber of $\pi_B : X \to B$, the Fourier–Mukai image of the sheaf $I_Z(r\sigma - \chi f)$ is a vector bundle. Stability is automatic, as the restrictions to all elliptic fibers which do not pass through Z are isomorphic to the Atiyah bundle of rank r and degree 1, therefore the restriction to the generic fiber is stable. The locus of $Z \in K^{[d_v]}$ with at least three points in the same elliptic fiber has codimension 2.

To see the claim above, note that the restriction of the ideal sheaf of a point to the elliptic fiber f through that point is

$$I_p|_{\mathsf{f}} = \mathcal{O}_{\mathsf{f}}(-p) + \mathcal{O}_p,$$

and similarly

$$I_{p,p'}|_{\mathsf{f}} = \mathcal{O}_{\mathsf{f}}(-p-p') + \mathcal{O}_p + \mathcal{O}_{p'},$$

for (possibly coincident) points p, p' in the fiber f. Thus, the restriction of $I_Z(r\sigma - \chi f)$ to fibers f can take the form

$$\mathcal{O}_{\mathsf{f}}(ro_F), \quad \mathcal{O}_{\mathsf{f}}(ro_F) \otimes (\mathcal{O}_{\mathsf{f}}(-p) + \mathcal{O}_p) \quad \text{or} \quad \mathcal{O}_{\mathsf{f}}(ro_F) \otimes (\mathcal{O}_{\mathsf{f}}(-p-p') + \mathcal{O}_p + \mathcal{O}_{p'})$$

which are all IT_0 with respect to $\mathcal{P}_F^\vee$, for $r \geq 3$. Thus the Fourier–Mukai transform $(\mathbf{R}\mathcal{S}^\dagger)^{-1}$ of $I_Z(r\sigma - \chi f)[-1]$ is a vector bundle by cohomology and base-change.

When $r = 2$, Ψ_2 is defined away from the divisor of subschemes $Z \in K^{[d_v]}$ with at least two points in the same elliptic fiber. As $K^{[d_v]}$ and K_v are irreducible holomorphic symplectic, Ψ_2 extends anyway to a birational map which is regular outside of codimension 2, but this extension is no longer identical to the Fourier–Mukai transform. In fact, semistable reduction is necessary to construct the extension, as in [OG, Section I.4]. We will not pursue it in this paper.

2.4. The moduli space $\mathfrak{M}_v^+$ via Fourier–Mukai

We investigate how sheaves of fixed determinant change under Fourier–Mukai. Recall the morphism (2.3), and let

$$V = \Phi_v^+(E, x),$$

for a pair $(E, x) \in K_v \times X$. Using Lemma 2.1 and Proposition 2.2, we calculate

$$\begin{aligned}\mathbf{R}\mathcal{S}^\dagger(V) &= \mathbf{R}\mathcal{S}^\dagger(t_{rx}^\star E \otimes (x_B^m \boxtimes x_F)) = t_{x_F}^\star \mathbf{R}\mathcal{S}^\dagger(t_{rx}^\star E) \otimes x_B^m \\ &= t_{x_F}^\star(t_{rx_B}^\star \mathbf{R}\mathcal{S}^\dagger(E) \otimes x_F^{-r}) \otimes x_B^m = t_{x_F+rx_B}^\star(I_Z^\vee(-r\sigma + \chi f)) \otimes (x_F^{-r} \boxtimes x_B^m) \\ &= I_{Z^+}^\vee(-r\sigma + \chi f) \otimes x_B^{d_v},\end{aligned}$$

where

$$Z^+ = t^\star_{x_F + r x_B} Z, \quad \text{so that} \quad a_B(Z^+) = -d_v r x_B.$$

In a similar fashion, we prove that

$$\mathbf{R}\mathcal{S}^\dagger(V^\vee) = I_{Z_+}(r\sigma - \chi f)[-1] \otimes x_B^{-d_v},$$

where now

$$Z_+ = t^\star_{-x_F + r x_B} \widetilde{Z} = \widetilde{Z^+}.$$

From the first equation, we obtain the rational map

$$\mathbf{R}\mathcal{S}^\dagger : K_v \times X \dashrightarrow \mathfrak{X}_v^+$$

where

$$\mathfrak{X}_v^+ = \{(Z^+, z_B) : a_B(Z^+) = r z_B\} \subset X^{[d_v]} \times B$$

via the assignment

$$(E, x) \mapsto (Z^+, -d_v x_B).$$

This map has degree d_v^4 and descends to $\mathfrak{M}_v^+$. Since Φ_v^+ is also of degree d_v^4 and $\mathfrak{X}_v^+$ is irreducible of the same dimension as $\mathfrak{M}_v^+$, we obtain a birational isomorphism $\mathfrak{M}_v^+ \dashrightarrow \mathfrak{X}_v^+$ in such a fashion that

$$\mathbf{R}\mathcal{S}^\dagger(V) = I^\vee_{Z^+}(-r\sigma + \chi f) \otimes z_B^{-1}, \quad \mathbf{R}\mathcal{S}^\dagger(V^\vee) = I_{\widetilde{Z^+}}(r\sigma - \chi f) \otimes z_B[-1].$$

The discussion of the previous subsection shows the birational isomorphism is given (explicitly as a Fourier–Mukai transform) away from codimension 2.

2.5. *The moduli space $\mathfrak{M}_v^-$ via Fourier–Mukai*

A similar argument applies to the moduli space $\mathfrak{M}_v^-$ of sheaves with fixed determinant of their Fourier–Mukai transform

$$\det \mathbf{R}\mathcal{S}(V) = \mathcal{O}(-f - m\sigma).$$

In this case, we have a morphism

$$\Phi_v^- : K_v \otimes \widehat{X} \to \mathfrak{M}_v^- \quad \text{given by} \quad \Phi_v^-(E, y) = t^\star_{(y_B, m y_F)} E \otimes y^\chi.$$

We calculate

$$\begin{aligned}\mathbf{R}\mathcal{S}^\dagger(\Phi_v^-(E, y)) &= \mathbf{R}\mathcal{S}^\dagger(t^\star_{(y_B, m y_F)} E \otimes y^\chi) = t^\star_{\chi y_F} \mathbf{R}\mathcal{S}^\dagger(t^\star_{(y_B, m y_F)} E) \otimes y_B^\chi \\ &= t^\star_{\chi y_F}(t^\star_{y_B} \mathbf{R}\mathcal{S}^\dagger(E) \otimes y_F^{-m}) \otimes y_B^\chi \\ &= t^\star_{y_B + \chi y_F}(I_Z^\vee(-r\sigma + \chi f)) \otimes (y_F^{-m} \boxtimes y_B^\chi) = I^\vee_{Z^-}(-r\sigma + \chi f) \otimes y_F^{-d_v}\end{aligned}$$

where $Z^- = t^\star_{y_B + \chi y_F} Z$, so that the addition in the fibers is $a_F(Z^-) = -\chi d_v y_F$. We therefore obtain the birational isomorphism

$$\mathfrak{M}_v^- \dashrightarrow \mathfrak{X}_v^-, \quad \Phi_v^-(E, y) \mapsto (Z^-, -d_v y_F),$$

where

$$\mathfrak{X}_v^- = \{(Z^-, z_F) : a_F(Z^-) = \chi z_F\} \hookrightarrow X^{[d_v]} \times F.$$

For further use, we record the identities

$$\mathbf{R}\mathcal{S}^\dagger(V) = I_{Z^-}^\vee(-r\sigma + \chi f) \otimes z_F, \quad \mathbf{R}\mathcal{S}^\dagger(V^\vee) = I_{\widetilde{Z^-}}(r\sigma - \chi f) \otimes z_F[-1].$$

3. Rank-coprime arbitrary fiber degree

Theorem 1.1 was proved in the previous section for fiber degree 1 in two ways: via an explicit analysis of O'Grady's description of the moduli space, and via Fourier–Mukai methods. In this section, we use Fourier–Mukai to prove Theorem 1.1 for arbitrary fiber degree. The argument builds on results of Bridgeland [B]. At the end of the section, we briefly consider the case of a surface $X = C \times F$ with F elliptic, but C of higher genus.

3.1. Fourier–Mukai transforms in the general coprime setting

We write

$$d = c_1(v) \cdot f$$

for the fiber degree, which we assume to be coprime to the rank r. Thus $c_1(v) = d\sigma + mf$. Pick integers a and b such that

$$ad + br = 1,$$

with $0 < a < r$. The following lemma gives the kernel of the Fourier–Mukai transform we will use:

Lemma 3.1. *There exists a vector bundle $\mathcal{U} \to F \times F$ with the following properties:*

(i) *the restriction of $\mathcal{U}$ to $F \times \{y\}$ is stable of rank a and degree b;*
(ii) *the restriction of $\mathcal{U}$ to $\{x\} \times F$ is stable of rank a and degree r.*

Furthermore,

$$c_1(\mathcal{U}) = b[o_F \times F] + r[F \times o_F] + c_1(\mathcal{P}_F).$$

Proof. This result is known (see for instance [B]), and can be explained in several ways. We consider the moduli space $M_F(a, b)$ of bundles of rank a and degree b over F. By the classical result of Atiyah [A], we have $M_F(a, b) \cong F$. We let $\mathcal{U} \to F \times F$ denote the universal bundle. Therefore, for all $y \in F$, $\mathcal{U}|_{E\times y}$ has rank a and degree b; in fact the determinant equals $\mathcal{O}_F((b-1) \cdot o_F + y)$. The bundle $\mathcal{U}$ is not unique and we can normalize it in several ways. Indeed, for any matrix

$$A = \begin{bmatrix} \lambda & a \\ \mu & b \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

we may assume that $\mathcal{U}|_{x\times F}$ has type (a, λ). In fact, we can regard the first factor F as the moduli space of bundles of rank a and degree λ over the second factor F (cf. [B]). We may pick the pair $\lambda = r$ and $\mu = -d$. What we showed above allows us to conclude

$c_1(\mathcal{U}) = b[o_F \times F] + r[F \times o_F] + c_1(\mathcal{P}_F)$. Replacing $\mathcal{U}$ by a suitable twist, we may in fact achieve

$$\det \mathcal{U} = \mathcal{O}(b[o_F \times F] + r[F \times o_F]) \otimes \mathcal{P}_F.$$

The lemma is proved. □

Lemma 3.2. *The sheaf* $\mathcal{U} \to F \times F$ *is semihomogeneous. More precisely,*

$$t_{(x,y)}^{\star}\mathcal{U} = \mathcal{U} \otimes \pi_1^{\star}(x^{-b/a}y^{1/a}) \otimes \pi_2^{\star}(x^{1/a}y^{-r/a}). \tag{3.1}$$

In particular,

$$\operatorname{ch}\mathcal{U} = a \exp\left(\frac{c_1(\mathcal{U})}{a}\right), \quad \textit{so} \quad \chi(\mathcal{U}) = -d.$$

Proof. By symmetry it suffices to argue that

$$t_{(x,0)}^{\star}\mathcal{U} = \mathcal{U} \otimes \pi_1^{\star}x^{-b/a} \otimes \pi_2^{\star}x^{1/a}.$$

We note that the choice of roots for the line bundles on the right hand side is not relevant. The restrictions of both sides to $F \times \{y\}$ agree: $\mathcal{U}_y = \mathcal{U}|_{F\times\{y\}}$ is stable on F, and thus semihomogeneous, satisfying

$$t_x^{\star}\mathcal{U}_y = \mathcal{U}_y \otimes x^{-b/a}.$$

We check agreement over $o_F \times F$. This is the statement that

$$\mathcal{U}|_{x\times F} = \mathcal{U}|_{o\times F} \otimes x^{1/a},$$

which holds by comparing ranks and determinants. Finally, agreement over $F \times F$ follows from the generalized see-saw Lemma 2.5 of Ramanan [R].

The second part of the lemma concerning the numerical invariants of $\mathcal{U}$ follows from general facts about semihomogeneous bundles [M]. □

Remark 3.3. A family of semihomogeneous bundles with fixed numerical invariants were constructed in arbitrary dimension in [O], and played a role in the decomposition of the Verlinde bundles. The dimension 1 case specializes to the bundle $\mathcal{U}$ considered here.

Letting $\pi_{F\times F} : F \times F \times B \to F \times F$ be the projection, we consider the relative Fourier–Mukai transform $\mathbf{R}\mathcal{S}^{\dagger} : \mathbf{D}(X) \to \mathbf{D}(X)$ with kernel

$$\pi_{F\times F}^{\star}\mathcal{U} \to F \times F \times B \cong X \times_B X.$$

This is a higher rank generalization of the relative Fourier–Mukai functor $\mathbf{R}\mathcal{S}^{\dagger}$ considered in the previous section. It follows from [B] that the kernel $\mathcal{U}$ is strongly simple over each factor, hence $\mathbf{R}\mathcal{S}^{\dagger}$ is an equivalence, with inverse having kernel $\pi_{F\times F}^{\star}\mathcal{U}^{\vee}[1]$. We further show:

Lemma 3.4. *Let* x, y *be points in* X*, suitably identified with points in* $\widehat{X}$ *as needed. Then*

$$\mathbf{R}\mathcal{S}^{\dagger}(E \otimes y) = t_{ay_F}^{\star}\mathbf{R}\mathcal{S}^{\dagger}(E) \otimes (y_B \boxtimes y_F^r), \quad \mathbf{R}\mathcal{S}^{\dagger}(t_x^{\star}E) = t_{x_B+bx_F}^{\star}\mathbf{R}\mathcal{S}^{\dagger}(E) \otimes x_F^{-d}.$$

Proof. Both formulas follow from (3.1). For the first, note that over $F \times F$ we have

$$\mathcal{U} \otimes \pi_1^\star y_F = t^\star_{(0,ay_F)} \mathcal{U} \otimes \pi_2^\star y_F^r,$$

by (3.1). Using the projections from $F \times F \times B$ onto its factors, we compute

$$\begin{aligned}
\mathbf{R}\mathcal{S}^\dagger(E \otimes y) &= \mathbf{R}\pi_{23\star}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star(E \otimes y))\\
&= \mathbf{R}\pi_{23\star}((\pi_{12}^\star \mathcal{U} \otimes \pi_1^\star y_F) \otimes \pi_{13}^\star E \otimes \pi_3^\star y_B)\\
&= \mathbf{R}\pi_{23\star}((t^\star_{(0,ay_F,0)} \pi_{12}^\star \mathcal{U} \otimes \pi_2^\star y_F^r) \otimes \pi_{13}^\star E) \otimes y_B\\
&= t^\star_{ay_F} \mathbf{R}\pi_{23\star}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star E) \otimes (y_F^r \boxtimes y_B) = t^\star_{ay_F} \mathbf{R}\mathcal{S}^\dagger(E) \otimes (y_F^r \boxtimes y_B).
\end{aligned}$$

We now explain the second formula. First, note that

$$\mathbf{R}\mathcal{S}^\dagger(t_x^\star E) = \mathbf{R}\pi_{23\star}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star t_x^\star E) = t^\star_{x_B} \mathbf{R}\pi_{23\star}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star t^\star_{x_F} E).$$

It suffices to prove that

$$\mathbf{R}\pi_{23\star}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star t^\star_{x_F} E) = t^\star_{bx_F} \mathbf{R}\mathcal{S}^\dagger(E) \otimes x_F^{-d}.$$

Now, (3.1) shows that over $F \times F$ we have

$$t^\star_{(x_F,0)} \mathcal{U} = \mathcal{U} \otimes \pi_1^\star x_F^{-b/a} \otimes \pi_2^\star x_F^{1/a},$$

so we calculate

$$\begin{aligned}
\mathbf{R}\pi_{23\star}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star t^\star_{x_F} E) &= \mathbf{R}\pi_{23\star}(\pi_{12}^\star t^\star_{(x_F,0)} \mathcal{U} \otimes \pi_1^\star x_F^{b/a} \otimes \pi_2^\star x_F^{-1/a} \otimes \pi_{13}^\star t^\star_{x_F} E)\\
&= \mathbf{R}\pi_{23\star}(t^\star_{(x_F,0,0)}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star E \otimes \pi_1^\star x_F^{b/a})) \otimes x_F^{-1/a}\\
&= \mathbf{R}\pi_{23\star}(\pi_{12}^\star \mathcal{U} \otimes \pi_{13}^\star E \otimes \pi_1^\star x_F^{b/a}) \otimes x_F^{-1/a} = \mathbf{R}\mathcal{S}^\dagger(E \otimes x_F^{b/a}) \otimes x_F^{-1/a}\\
&= t^\star_{bx_F} \mathbf{R}\mathcal{S}^\dagger(E) \otimes x_F^{br/a} x_F^{-1/a} \quad \text{(using the first half of the lemma)}\\
&= t^\star_{bx_F} \mathbf{R}\mathcal{S}^\dagger(E) \otimes x_F^{-d}.
\end{aligned}$$

The proof is complete. □

Proposition 3.5. *For a generic sheaf V of fixed determinant and determinant of Fourier–Mukai*

$$\det V = \mathcal{O}(d\sigma + mf), \qquad \det \mathbf{R}\mathcal{S}(V) = \mathcal{O}(-m\sigma - df)$$

we have

$$\mathbf{R}\mathcal{S}^\dagger(V) = I_Z^\vee \otimes \mathcal{O}_X((a\chi + bm)f),$$

for a subscheme Z of length d_v with $a(Z) = 0$.

Proof. In the proof, it will be important to distinguish between the two copies of X which are the source and the target of the Fourier–Mukai transform, because of the asymmetry present in the bundle $\mathcal{U}$. We will write X_1 and X_2 for these two copies.

By Grothendieck duality, to prove that

$$\mathbf{R}\mathcal{S}^{\dagger}(V) = I_Z^{\vee} \otimes \mathcal{O}_X((a\chi + bm)f),$$

it suffices to show that

$$\Psi^{\mathcal{U}^{\vee}}_{X_1 \to X_2}(V^{\vee}) = I_Z \otimes \mathcal{O}(-(a\chi + bm)f)[-1],$$

where Ψ is the Fourier–Mukai transform with kernel $\mathcal{U}^{\vee}$. Using [B, Lemma 6.4], we know that $V^{\vee}$ is WIT_1 with respect to Ψ, since the restriction to the general fiber is stable, of slope $-d/r < b/a$. Note that $\Psi(V^{\vee})[1]$ has rank

$$-\chi(V^{\vee}|_{F\times y} \otimes \mathcal{U}^{\vee}|_{F\times y}) = ad + br = 1.$$

Section 7 of [B], or Sections 3 and 5 in [BH], show that for generic V, the Ψ-transform is torsion-free, hence it must be of the form $I_Z \otimes L[-1]$ for some line bundle L. Bridgeland's argument moreover shows that the subscheme Z has length d_v.

The fiber degree of $\Psi(V^{\vee})$ equals

$$c_1(\mathbf{R}\pi_{2\star}(\pi_1^{\star} V|_{\mathsf{f}}^{\vee} \otimes \mathcal{U}^{\vee})) = \pi_{2\star}(\pi_1^{\star}(r - d\omega)(a - c_1(\mathcal{U}) + \mathrm{ch}_2(\mathcal{U})))_{(2)} = 0,$$

where Lemmas 3.1 and 3.2 are used to express the numerical invariants of $\mathcal{U}$. In fact more is true. Since the restriction of V to a generic fiber is stable, it must equal the Atiyah bundle $\mathsf{E}_{r,d}$. This implies that the restriction of L to a generic fiber must coincide with $\Psi(\mathsf{E}_{r,d})[1]$, which is trivial. Therefore, L must be a sheaf of the form $\pi_B^{\star} M^{\vee}$, where M is a degree $-\beta$ line bundle over B. We will establish that

$$\beta = -a\chi - bm.$$

To this end, we calculate the Euler characteristic of $\Psi(V^{\vee}(\sigma))$:

$$\begin{aligned}\chi(\Psi(V^{\vee})(\sigma)) &= \chi(F \times F \times B, \pi_{13}^{\star} V^{\vee} \otimes \pi_{12}^{\star}\mathcal{U}^{\vee} \otimes \pi_{23}^{\star}\mathcal{O}(\sigma)) \\ &= \int_{F\times F\times B} \pi_{13}^{\star}(r - (d\sigma + mf) + \chi\omega) \cdot \pi_{12}^{\star}(a - c_1(\mathcal{U}) + \mathrm{ch}_2(\mathcal{U})) \cdot \pi_{23}^{\star}(1+\sigma) \\ &= bm + a\chi - \chi r + md.\end{aligned}$$

On the other hand,

$$\chi(I_Z \otimes L(\sigma)) = \beta - d_v = \beta - (dm - r\chi),$$

hence $\beta = -a\chi - bm$.

When the determinant and the determinant of Fourier–Mukai of V are fixed, we show that $a(Z) = 0$. First, we analyze the requirement that the determinant be fixed. The inverse of $\mathbf{R}\mathcal{S}^\dagger$ is given by $\Phi^{\mathcal{U}^\vee[1]}_{X_2\to X_1}$, the Fourier–Mukai whose kernel is $\mathcal{U}^\vee[1]$, considered as a transform $X_2 \to X_1$. Hence,

$$V = \Phi^{\mathcal{U}^\vee[1]}_{X_2\to X_1}(L^\vee \otimes I_Z^\vee), \qquad L = \pi_B^\star M^\vee,$$

has fixed determinant $\mathcal{O}(d\sigma + mf)$. In order to make the computations more explicit, we write

$$M = \mathcal{O}_B(-(\beta+1)[o_B] + [\mu])$$

for some $\mu \in B$. We have

$$\begin{aligned}
\det V^\vee &= \det \Phi^{\mathcal{U}^\vee}(L^\vee) \otimes \bigotimes_{z\in Z} \det \mathbf{R}\pi_{13\star}(\pi_{12}^\star \mathcal{U}^\vee \otimes \pi_{23}^\star \mathcal{O}_z^\vee)^\vee \\
&= \det \Phi^{\mathcal{U}^\vee}(L^\vee) \otimes \bigotimes \det(\mathcal{U}^\vee|_{F\times z_F} \boxtimes \mathcal{O}_{z_B}[2])^\vee \\
&= \det \Phi^{\mathcal{U}^\vee}(L^\vee) \otimes \bigotimes_{z\in Z} (\mathcal{O}_F \boxtimes \mathcal{O}_B(-a[z_B])) \\
&= \det \Phi^{\mathcal{U}^\vee}(L^\vee) \otimes \big(\mathcal{O}_F \boxtimes \mathcal{O}_B(-(ad_v - 1)[o_B] - [a\cdot a_B(Z)])\big) \\
&= \mathcal{O}(-d\sigma - mf) \otimes \pi^\star \mathcal{O}_B(-[a\cdot a_B(Z) + r\mu] + [o_B]). \qquad (3.2)
\end{aligned}$$

This gives

$$a \cdot a_B(Z) + r\mu = o_B.$$

In (3.2), we used the calculation

$$\begin{aligned}
\det \Phi^{\mathcal{U}^\vee}(L^\vee) &= \det \mathbf{R}p_{13\star}(p_{12}^\star \mathcal{U}^\vee \otimes p_3^\star M) = \det(\mathbf{R}p_{1\star}\mathcal{U}^\vee \boxtimes M) \\
&= \det \mathbf{R}p_{1\star}(\mathcal{U}^\vee) \boxtimes M^{-r} = \mathcal{O}_F(-do_F) \boxtimes \mathcal{O}_B((r\beta+1)[o_B] - [r\mu]),
\end{aligned}$$

where by Lemma 3, the pushforward of $\mathcal{U}^\vee$ has rank $-r$ and degree $-d$. Equation (3.2) also makes use of the identity $r\beta - ad_v = -m$.

We now analyze the requirement that the determinant of the Fourier–Mukai be fixed. We know that

$$\mathbf{R}\mathcal{S}(V) = \mathbf{R}\mathcal{S} \circ \Phi^{\mathcal{U}^\vee[1]}_{X_2\to X_1}(L^\vee \otimes I_Z^\vee).$$

This composition can be re-expressed as a Fourier–Mukai whose kernel equals the convolution of the following two kernels: $\widetilde{\mathcal{U}}^\vee[1]$ for Φ, and $\mathcal{P} = \mathcal{P}_F \times \mathcal{P}_B$ for $\mathbf{R}\mathcal{S}$. The tilde indicates that the kernel $\mathcal{U}^\vee[1]$ is considered in the opposite direction for Φ than it is for Ψ. This is the same as applying to $\mathcal{U} \to F \times F$ the involution that exchanges the factors. The new kernel can be expressed as

$$\mathbf{R}p_{13\star}(p_{12}^\star \widetilde{\mathcal{U}}^\vee[1] \otimes p_{23}^\star \mathcal{P}) = \mathbf{R}\pi_{13\star}(\pi_{12}^\star \widetilde{\mathcal{U}}^\vee \otimes \pi_{23}^\star \mathcal{P}_F)[1] \boxtimes \mathcal{P}_B = \mathcal{V} \boxtimes \mathcal{P}_B$$

where

$$\mathcal{V} \to F \times F, \qquad \mathcal{V} = \mathbf{R}\pi_{13\star}(\pi_{12}^\star \widetilde{\mathcal{U}}^\vee \otimes \pi_{23}^\star \mathcal{P}_F)[1],$$

is the fiberwise Fourier–Mukai image of $\widetilde{\mathcal{U}}^\vee$ up to a shift. The complex $\mathcal{V}$ has rank b, and a Riemann-Roch calculation shows that

$$c_1(\mathcal{V}) = d[o_F \times F] + a[F \times o_F] + c_1(\mathcal{P}_F).$$

The pushforward $\mathbf{R}(p_2^F)_\star(\mathcal{V})$ has rank d and determinant $-r[o_F]$. This will be used below. Fiberwise, note that

$$\det \mathcal{V}^\vee|_{z_F\times F} = \det \mathbf{R}p_{2\star}(p_1^\star \widetilde{\mathcal{U}}|^\vee_{z_F\times F}\otimes \mathcal{P}_F) = -[z_F]-(a-1)[o_F].$$

Now, we calculate

$$\begin{aligned}\det \mathbf{R}\mathcal{S}(V) &= \det \mathbf{R}p_{2\star}(\mathcal{V}\boxtimes \mathcal{P}_B\otimes p_1^\star(L^\vee\otimes I_Z^\vee))\\ &= \det \mathbf{R}p_{2\star}(\mathcal{V}\boxtimes \mathcal{P}_B\otimes p_1^\star L^\vee)\otimes \det \mathbf{R}p_{2\star}(\mathcal{V}\boxtimes\mathcal{P}_B\otimes p_1^\star \mathcal{O}_Z[2])^\vee. \end{aligned}\tag{3.3}$$

The first determinant is constant:

$$\begin{aligned}\det \mathbf{R}p_{2\star}(\mathcal{V}\boxtimes \mathcal{P}_B\otimes p_1^\star L^\vee) &= \det(\mathbf{R}(p_2^F)_\star(\mathcal{V})\boxtimes \mathbf{R}(p_2^B)_\star(\mathcal{P}_B\otimes (p_1^B)^\star M))\\ &= (\det \mathbf{R}(p_2^F)_\star(\mathcal{V}))^{-\beta}\boxtimes \mathcal{O}_B(-[-\mu])^d = \mathcal{O}_F(r\beta[o_F])\boxtimes \mathcal{O}_B(-d[-\mu]).\end{aligned}$$

The second determinant appearing in (3.3) needs to be fixed, and it equals

$$\begin{aligned}&\bigotimes_{z\in Z}\det \mathcal{V}^\vee|_{z_F\times F}\boxtimes \mathcal{O}_B(-[z_B]+[o_B])^b\\ &\quad= \bigotimes_{z\in Z}\mathcal{O}_F(-[z_F]-(a-1)[o_F])\boxtimes \mathcal{O}_B(-[z_B]+[o_B])^b\\ &\quad= \mathcal{O}_F(-[a_F(Z)]-(ad_v-1)[o_F])\boxtimes \mathcal{O}_B(-[b\cdot a_B(Z)]+[o_B]).\end{aligned}$$

Therefore

$$\det \mathbf{R}\mathcal{S}(V) = \mathcal{O}(-mf-d\sigma)\otimes \mathcal{O}_F(-[a_F(Z)]+[o_F])\boxtimes \mathcal{O}_B(-[b\cdot a_B(Z)-d\mu]+[o_B]).$$

Since $\det \mathbf{R}\mathcal{S}(V)=\mathcal{O}(-mf-d\sigma)$, this immediately yields

$$a_F(Z)=o_F,\qquad b\cdot a_B(Z)-d\mu = o_B.$$

Combining these two equations with $a\cdot a_B(Z)+r\mu = o_B$ shown above, and the fact that $ad+br=1$, we obtain $a_B(Z)=0$ and $\mu=0$. Thus

$$a(Z)=0,\qquad L=\mathcal{O}((a\chi+bm)f).$$

This completes the proof. □

Proof of Theorem 1.1. In the course of the above proof, we showed that for a generic sheaf V of rank r and determinant

$$\det V = \mathcal{O}(d\sigma+mf),$$

the Fourier–Mukai transform takes the form

$$\mathbf{R}\mathcal{S}^\dagger(V) = I_Z^\vee((a\chi+bm)f)\otimes \pi_B^\star\mu$$

for some $\mu\in B\cong \widehat{B}$ such that

$$a\cdot a_B(Z)+r\mu = o_B.$$

The assignment $V\mapsto (Z, ba_B(Z)-d\mu)$ gives the birational isomorphism $\mathfrak{M}_v^+ \dashrightarrow \mathfrak{X}_v^+$, claimed by Theorem 1.1.

Remark 3.6. In a similar fashion we could find the birational type of the moduli space $\mathfrak{M}_v^-$. However, it is not necessary to repeat the argument above to deal with this new case. We could instead make use of the map

$$\Phi_v^- : K_v \times X \to \mathfrak{M}_v^-, \qquad (E, y) \mapsto t^\star_{(dy_B, my_F)} E \otimes y^\chi,$$

and apply Lemma 3.4 to calculate

$$\begin{aligned}
\mathbf{R}\mathcal{S}^\dagger(\Phi_v^-(E, y)) &= \mathbf{R}\mathcal{S}^\dagger(t^\star_{(dy_B, my_F)} E \otimes y^\chi) = t^\star_{a\chi y_F} \mathbf{R}\mathcal{S}^\dagger(t^\star_{(dy_B, my_F)} E) \otimes (y_B^\chi \boxtimes y_F^{r\chi}) \\
&= t^\star_{a\chi y_F}(t^\star_{dy_B + bmy_F} \mathbf{R}\mathcal{S}^\dagger(E) \otimes y_F^{-dm}) \otimes (y_B^\chi \boxtimes y_F^{r\chi}) \\
&= t^\star_{dy_B + (a\chi + bm)y_F}(I_Z^\vee((a\chi + bm)f)) \otimes (y_B^\chi \boxtimes y_F^{-d_v}) \\
&= I_{Z^-}^\vee((a\chi + bm)f) \otimes (y_B^{\chi - (a\chi + bm)d} \boxtimes y_F^{-d_v}) \\
&= I_{Z^-}^\vee((a\chi + bm)f) \otimes (y_B^{-bd_v} \boxtimes y_F^{-d_v})
\end{aligned}$$

where $Z^- = t^\star_{dy_B + (a\chi + bm)y_F} Z$. Thus, the assignment

$$V \mapsto (Z^-, -d_v y)$$

gives a birational isomorphism

$$\mathfrak{M}_v^- \dashrightarrow \mathfrak{X}_v^- = \{(Z^-, z) : a(Z^-) = f(z)\} \hookrightarrow X^{[d_v]} \times X$$

where the isogeny $f : X \to X$ is given by

$$f(z) = (dz_B, (a\chi + bm)z_F).$$

In the case of fiber degree 1, this specializes to the subvariety

$$\mathfrak{X}_v^- = \{(Z^-, z_F) : a_F(Z^-) = \chi z_F\} \hookrightarrow X^{[d_v]} \times F$$

of Section 2.

3.2. *Higher genus*

Assume now (C, o) is a pointed smooth curve of genus $g \geq 1$, and F is still an elliptic curve. We set $\bar{g} = g - 1$. Consider the product surface $X = C \times F \to C$. Let $\mathfrak{M}_v^+$ be the moduli space of sheaves over X of rank r and determinant $\mathcal{O}(\sigma + mf)$, where σ is the zero section and f denotes the fiber over o. We describe the birational type of $\mathfrak{M}_v^+$, in codimension 1 for $r \neq 2$, using the Fourier–Mukai transform with kernel the Poincaré bundle

$$\pi^\star_{F \times F} \mathcal{P}_F \to F \times F \times C.$$

The proof is entirely similar to that of Proposition 3.5, so we just record the result.

Let $a_C : X^{[d_v]} \to C^{[d_v]}$ be the map induced by the projection $X \to C$. Thus, each scheme Z of length d_v in X yields a divisor $a_C(Z)$ of degree d_v over the curve C. The line bundle

$$\mathcal{M}_Z = \mathcal{O}_C(a_C(Z) - d_v \cdot o)$$

has degree 0, and therefore admits roots of order r. We define

$$\mathfrak{X}_v^+ = \{(Z, c) : c^r = \mathcal{M}_Z\} \hookrightarrow X^{[d_v]} \times \mathrm{Pic}^0(C).$$

Then, for $V \in \mathfrak{M}_v^+$, we have

$$\mathbf{R}\mathcal{S}^\dagger(V) = I_Z^\vee(-r\sigma + (\chi + \bar{g})f) \otimes c^{-1},$$

establishing the birational isomorphism $\mathfrak{M}_v^+ \dashrightarrow \mathfrak{X}_v^+$. The same statement holds in any fiber degree coprime to the rank, but we will not detail this fact.

4. The strange duality isomorphism

We now proceed to prove Theorems 1.2–1.4 stated in the introduction. Throughout this section, we place ourselves in the context when the fiber degree is 1.

4.1. Reformulation

Let $X = B \times F$ be a product abelian surface. As a consequence of Section 2, under the birational map

$$\mathfrak{M}_v^+ \times \mathfrak{M}_w^+ \dashrightarrow \mathfrak{X}_v^+ \times \mathfrak{X}_w^+$$

induced by the relative Fourier–Mukai transform, the standard theta divisor

$$\Theta_{vw}^+ = \{(V, W) : h^1(V \otimes W) \neq 0\} \subset \mathfrak{M}_v^+ \times \mathfrak{M}_w^+$$

is identified with a divisor

$$\Theta^+ \subset \mathfrak{X}_v^+ \times \mathfrak{X}_w^+.$$

Note that for sheaves $(V, W) \in \mathfrak{M}_v^+ \times \mathfrak{M}_w^+$ corresponding to pairs $(Z^+, z_B) \in \mathfrak{X}_v^+$ and $(T^+, t_B) \in \mathfrak{X}_w^+$, we have

$$\begin{aligned} H^1(V \otimes W) &= \mathrm{Ext}^1(W^\vee, V) = \mathrm{Ext}^1(\mathbf{R}\mathcal{S}^\dagger(W^\vee), \mathbf{R}\mathcal{S}^\dagger(V)) \\ &= \mathrm{Ext}^1(I_{\widetilde{T^+}}(s\sigma - \chi' f)[-1] \otimes t_B, I_{Z^+}^\vee \otimes \mathcal{O}(-r\sigma + \chi f) \otimes z_B^{-1}) \\ &= \mathrm{Ext}^1(I_{Z^+}^\vee \otimes \mathcal{O}(-r\sigma + \chi f) \otimes z_B^{-1}, I_{\widetilde{T^+}}(s\sigma - \chi' f)[-1] \otimes t_B)^\vee \\ &= H^0(I_{Z^+} \otimes I_{\widetilde{T^+}} \otimes z_B \otimes t_B \otimes \mathcal{O}((r+s)\sigma - (\chi + \chi')f))^\vee. \end{aligned}$$

(The notation above has the obvious meaning: r, s are the ranks of v and w, while χ, χ' are their Euler characteristics.) We set

$$L = \mathcal{O}((r+s)\sigma - (\chi + \chi')f) \quad \text{on } X. \tag{4.1}$$

For $r, s \neq 2$ the birational isomorphisms $\mathfrak{M}_v^+ \dashrightarrow \mathfrak{X}_v^+$ and $\mathfrak{M}_w^+ \dashrightarrow \mathfrak{X}_w^+$ are regular in codimension 1. We argued above that the theta divisor Θ_{vw}^+ in the product $\mathfrak{M}_v^+ \times \mathfrak{M}_w^+$ corresponds to the divisor

$$\Theta^+ = \{(Z^+, z_B, T^+, t_B) : h^0(I_{Z^+} \otimes I_{\widetilde{T^+}} \otimes z_B \otimes t_B \otimes L) \neq 0\}$$

in the product $\mathfrak{X}_v^+ \times \mathfrak{X}_w^+$. Furthermore, the theta bundles on $\mathfrak{M}_v^+$ and $\mathfrak{M}_w^+$ also yield line bundles $\Theta_w \to \mathfrak{X}_v^+$ and $\Theta_v \to \mathfrak{X}_w^+$ such that

$$\mathcal{O}(\Theta^+) = \Theta_w \boxtimes \Theta_v.$$

Consequently, strange duality is demonstrated if we show that the divisor Θ^+ induces an isomorphism

$$\mathsf{D}^+ : H^0(\mathfrak{X}_v^+, \Theta_w)^\vee \to H^0(\mathfrak{X}_w^+, \Theta_v).$$

4.2. Theta bundles

For $r \neq 2$, we are interested in an explicit description of the determinant line bundle

$$\Theta_w \to \mathfrak{X}_v^+.$$

For a line bundle L on X, we standardly let

$$L^{[d_v]} = \det \mathbf{R} p_\star(\mathcal{O}_{\mathcal{Z}} \otimes q^\star L) \quad \text{on } X^{[d_v]},$$

where $\mathcal{Z} \subset X^{[d_v]} \times X$ is the universal subscheme, and p, q are the two projections. We also note the natural projections

$$c_v : \mathfrak{X}_v^+ \to X^{[d_v]}, \quad (Z^+, z_B) \mapsto Z^+,$$
$$\pi_2 : \mathfrak{X}_v^+ \to B, \quad (Z^+, z_B) \mapsto z_B.$$

The theta bundle is calculated by the following result:

Proposition 4.1. *We have*

$$\Theta_w = c_v^\star L^{[d_v]} \otimes \pi_2^\star \mathcal{O}_B((s-r)o_B),$$

with L given by (4.1)*.*

Proof. We give one proof here; another one is essentially contained in Section 4.4. We begin by noting the degree d_v^4 étale morphism

$$q_v : K^{[d_v]} \times X = K^{[d_v]} \times B \times F \to \mathfrak{X}_v^+, \quad (Z, x) \mapsto (t^\star_{rx_B + x_F} Z, -d_v x_B).$$

It is related to the standard less twisted map

$$\mu_v : K^{[d_v]} \times X \to X^{[d_v]}, \quad \mu_v(Z, x) = t_x^\star Z,$$

via the commutative diagram

$$\begin{array}{ccc} K^{[d_v]} \times B \times F & \xrightarrow{q_v} & \mathfrak{X}_v^+ \\ \downarrow{\scriptstyle (1,r,1)} & & \downarrow{\scriptstyle c_v} \\ K^{[d_v]} \times B \times F & \xrightarrow{\mu_v} & X^{[d_v]} \end{array}$$

Using the diagram, we calculate

$$q_v^\star c_v^\star L^{[d_v]} = (1, r, 1)^\star \mu_v^\star L^{[d_v]} = (1, r, 1)^\star (L^{[d_v]} \boxtimes L^{d_v}) = L^{[d_v]} \boxtimes ((r, 1)^\star L)^{d_v},$$

and further,

$$(r, 1)^\star L = (r, 1)^\star \mathcal{O}((r+s)\sigma - (\chi + \chi')f) = \mathcal{O}((r+s)\sigma - r^2(\chi + \chi')f).$$

If $\pi_B : K^{[d_v]} \times B \times F \to B$ is the projection to B, from the definitions we also have $\pi_2 \circ q_v = -d_v$ hence

$$q_v^\star \pi_2^\star \mathcal{O}_B((s-r)o_B) = \pi_B^\star \mathcal{O}_B((s-r)d_v^2 o_B).$$

Putting the previous three equations together we find

$$q_v^\star(c_v^\star L^{[d_v]} \otimes \pi_2^\star \mathcal{O}_B((s-r)o_B)) \\ = L^{[d_v]} \boxtimes \mathcal{O}((r+s)\sigma + ((s-r)d_v - r^2(\chi+\chi'))f)^{\otimes d_v}. \quad (4.2)$$

On the other hand, Proposition 2 of [MO2] (written in holomorphic K-theory, as in [O]) gives

$$q_v^\star \Theta_w = L^{[d_v]} \boxtimes \mathcal{O}((r+s)\sigma + (rn+sm)f)^{\otimes d_v}. \quad (4.3)$$

The two pullbacks (4.2) and (4.3) are seen equal on $K^{[d_v]} \times X$, as one shows that

$$rn + sm = (s-r)d_v - r^2(\chi+\chi').$$

This uses the numerical identity $m+n = -r\chi' - s\chi$, which expresses the strange duality orthogonality $\chi(v \cdot w) = 0$, also remembering that $d_v = m - r\chi$.

Now, consider

$$Q = \Theta_w \otimes (c_v^\star L^{[d_v]})^\vee \otimes \pi_B^\star \mathcal{O}_B((r-s)o_B).$$

We showed that $q_v^\star Q$ is trivial. Note the morphism

$$p_v : \mathfrak{X}_v^+ \to X, \quad (Z, x_B) \mapsto (a_F(Z), x_B),$$

with fibers isomorphic to $K^{[d_v]}$. In fact, each fiber $\iota : p_v^{-1}(x) \hookrightarrow \mathfrak{X}_v^+$ factors through the morphism

$$q_v : K^{[d_v]} \times X \to \mathfrak{X}_v^+.$$

Indeed, $\iota = q_v \circ j$, where $j : p_v^{-1}(x) \to K^{[d_v]} \times X$ is the map

$$j(Z) = (t^\star_{-ry_B - y_F} Z, y),$$

for any choice of $y \in X$ such that $d_v y = -x$. Therefore, the above argument implies that the restriction of Q to each fiber $p_v^{-1}(x)$ is trivial. Hence, $Q = p_v^\star N$ for some line bundle N over X.

We now argue that N is trivial, by constructing a suitable test family. Consider a subscheme Z_0 of length $d_v - 1$ supported at 0, and define

$$\alpha : X \dashrightarrow \mathfrak{X}_v^+, \quad x \mapsto (Z_0 + (rx_B, x_F), x_B).$$

The map α is defined away from the r^2 points in $B[r] \times o_F$. It suffices to show N is trivial along this open set. Pulling back the equality

$$p_v^\star N = \Theta_w \otimes (c_v^\star L^{[d_v]})^\vee \otimes \pi_B^\star \mathcal{O}_B((r-s)o_B)$$

under α, and noting $\alpha \circ p_v = 1$, it suffices to prove that

$$\alpha^\star \Theta_w = \overline{c}_v^\star L^{[d_v]} \otimes \pi_B^\star \mathcal{O}_B((s-r)o_B) \quad (4.4)$$

where

$$\overline{c}_v = c_v \circ \alpha : X \dashrightarrow X^{[d_v]}, \quad \overline{c}_v(x) = Z_0 + (rx_B, x_F).$$

We calculate the right hand side of (4.4). The universal family $\mathcal{Z} \subset X^{[d_v]} \times X$ becomes under pullback by $\overline{c}_v$ the family $Z_0 + \Delta_r \cong \Delta_r$, where $\Delta_r \subset X \times X$ is the r-fold diagonal

$$\Delta_r = (rx_B, x_F, x_B, x_F).$$

Then

$$\overline{c}_v^\star L^{[d_v]} = \overline{c}_v^\star \det \mathbf{R} p_\star(\mathcal{O}_{\mathcal{Z}} \otimes q^\star L) = \det \mathbf{R} p_\star(\mathcal{O}_{\Delta_r} \otimes q^\star L) = (r_B, 1_F)^\star L,$$

hence

$$\overline{c}_v^\star L^{[d_v]} = \mathcal{O}((r+s)\sigma - r^2(\chi + \chi')f).$$

For the left hand side of (4.4), fix (T, c) in $\mathfrak{X}_w^+$, where $c \in B \cong \widehat{B}$, so that $a_B(T) = sc$. Writing $\widetilde{T}$ for the reflection of T in the fibers, we have

$$\Theta_w = \det \mathbf{R} p_\star(I_{\mathcal{Z}} \otimes \mathcal{P}_B \otimes q^\star(L \otimes c \otimes I_{\widetilde{T}}))^\vee,$$

where $\mathcal{P}_B$ is the Poincaré bundle over $B \times B$. Write $M = L \otimes c \otimes I_{\widetilde{T}}$, so that in the K-theory of X we have

$$M = M_B \boxtimes M_F - \sum_{t \in T} \mathcal{O}_{t_B \times -t_F}$$

for

$$M_B = c \otimes \mathcal{O}_B(-(\chi + \chi')o_B),\ M_F = \mathcal{O}((r+s)o_F).$$

Since the universal family pulls back to $\alpha^\star \mathcal{Z} \cong \Delta_r$, we find that K-theoretically we have $\alpha^\star I_{\mathcal{Z}} = \mathcal{O} - \mathcal{O}_{\Delta_r}$. Therefore,

$$\begin{aligned}\alpha^\star \Theta_w &= \det \mathbf{R} p_\star(\mathcal{P}_B \otimes q^\star(M_B \boxtimes M_F))^\vee \otimes \det \mathbf{R} p_\star(\mathcal{O}_{\Delta_r} \otimes \mathcal{P}_B \otimes q^\star(M_B \boxtimes M_F)) \\ &\quad \otimes \bigotimes_{t \in T} \Big(\det \mathbf{R} p_\star(\mathcal{P}_B \otimes q^\star \mathcal{O}_{t_B \times -t_F}) \otimes \det \mathbf{R} p_\star(\mathcal{O}_{\Delta_r} \otimes \mathcal{P}_B \otimes q^\star \mathcal{O}_{t_B \times -t_F})^\vee\Big).\end{aligned}$$

We calculate the first term:

$$\begin{aligned}\det \mathbf{R} p_\star(\mathcal{P}_B \otimes q^\star(M_B \boxtimes M_F))^\vee &= \det(\mathbf{R}(p_B)_\star(\mathcal{P}_B \otimes q_B^\star M_B) \boxtimes H^\bullet(M_F) \otimes \mathcal{O}_F)^\vee \\ &= \det(\mathbf{R}(p_B)_\star(\mathcal{P}_B \otimes q_B^\star M_B))^{-(r+s)} \boxtimes \mathcal{O}_F = (c \otimes \mathcal{O}_B(-o_B))^{-(r+s)} \boxtimes \mathcal{O}_F.\end{aligned}$$

The second term becomes

$$\det \mathbf{R} p_\star(\mathcal{O}_{\Delta_r} \otimes \mathcal{P}_B \otimes q^\star(M_B \boxtimes M_F)) = \det(\mathbf{R}(p_B)_\star(\mathcal{O}_{\Delta_r^B} \otimes \mathcal{P}_B \otimes q_B^\star M_B) \boxtimes M_F)$$

where Δ_r^B is the image of

$$j : B \to \Delta_r^B, \quad x_B \mapsto (rx_B, x_B).$$

We calculate

$$\mathbf{R}(p_B)_\star(\mathcal{O}_{\Delta_r^B} \otimes (\mathcal{P}_B \otimes q_B^\star M_B)) = j^\star \mathcal{P}_B \otimes r^\star M_B = \mathcal{O}_B(-2ro_B) \otimes r^\star M_B.$$

Therefore, the second term equals

$$c^r \otimes \mathcal{O}_B(-2ro_B - r^2(\chi + \chi')o_B) \boxtimes \mathcal{O}_F((r+s)o_F).$$

The third term now equals

$$\det \mathbf{R}p_{\star}(\mathcal{P}_B \otimes q^{\star}\mathcal{O}_{t_B \times -t_F}) = \mathcal{P}_{t_B} \boxtimes \mathcal{O}_F,$$

hence the tensor product over all $t \in T$ yields

$$\mathcal{P}_{a_B(W)} \boxtimes \mathcal{O}_F = c^s \boxtimes \mathcal{O}_F.$$

Finally, the fourth term is easily seen to be trivial,

$$\det \mathbf{R}p_{\star}(\mathcal{O}_{\Delta_r} \otimes \mathcal{P}_B \otimes q^{\star}\mathcal{O}_{t_B \times -t_F}) = \mathcal{O}_X.$$

Equation (4.4) follows by putting all the terms together. This concludes the proof of Proposition 4.1. □

4.2.1. Fixed determinant of Fourier–Mukai. The discussion for the moduli space of sheaves with fixed determinant of their Fourier–Mukai is entirely parallel. We identify the theta divisor $\Theta^-_{vw} \subset \mathfrak{M}^-_v \times \mathfrak{M}^-_w$ with the divisor

$$\Theta^- = \{(Z^-, z_F, T^-, t_F) : h^0(I_{Z^-} \otimes I_{\widetilde{T}^-} \otimes z_F^{-1} \otimes t_F \otimes L) \neq 0\} \hookrightarrow \mathfrak{X}^-_v \times \mathfrak{X}^-_w,$$

where as before

$$L = \mathcal{O}(-(\chi + \chi')f + (r+s)\sigma).$$

For $r, s \geq 3$, the birational isomorphisms are defined in codimension 1, and we have

$$\Theta_w = (c_v^-)^{\star} L^{[d_v]} \otimes \pi_F^{\star}\mathcal{O}_F((\chi - \chi')o_F),$$

where $c_v^- : \mathfrak{X}^-_v \to X^{[d_v]}$ is the forgetful morphism.

4.3. Equal ranks and the proof of strange duality

We now consider the case $r = s \geq 3$, when we simply have

$$\Theta_w = c_v^{\star} L^{[d_v]}, \quad \Theta_v = c_w^{\star} L^{[d_w]}.$$

Furthermore, tensor product gives a rational map defined away from codimension 2,

$$\tau^+ : \mathfrak{X}^+_v \times \mathfrak{X}^+_w \dashrightarrow \mathfrak{X}^+, \quad (I_Z, z_B, I_T, t_B) \mapsto (I_Z \otimes I_{\widetilde{T}}, z_B \otimes t_B).$$

Here

$$\mathfrak{X}^+ = \{(U, u_B) : a_B(U) = r u_B\} \subset X^{[d_v+d_w]} \times B.$$

In other words, $\mathfrak{X}^+$ is the fiber product

$$\begin{array}{ccc} \mathfrak{X}^+ & \xrightarrow{c} & X^{[d_v+d_w]} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle a_B} \\ B & \xrightarrow{r} & B \end{array}$$

where c and π_2 are the natural projection maps,

$$c(U, u_B) = U, \quad \pi_2(U, u_B) = u_B.$$

The divisor $\Theta^+ \hookrightarrow \mathfrak{X}_v^+ \times X_w^+$ is the pullback under τ^+ of the divisor

$$\theta^+ = \{(U, u_B) : h^0(I_U \otimes u_B \otimes L) \neq 0\},$$

with corresponding line bundle

$$\mathcal{O}(\theta^+) \simeq c^\star L^{[d_v+d_w]}$$

on $\mathfrak{X}^+$.

To identify the space of sections $H^0(\mathfrak{X}^+, \mathcal{O}(\theta^+))$, we let $B[r]$ denote the group of r-torsion points on B, and fix an isomorphism

$$r_\star \mathcal{O} \simeq \bigoplus_{\tau \in B[r]} \tau.$$

Then

$$\begin{aligned} H^0(\mathfrak{X}^+, \mathcal{O}(\theta^+)) &= H^0(\mathfrak{X}^+, c^\star L^{[d_v+d_w]}) = H^0(X^{[d_v+d_w]}, L^{[d_v+d_w]} \otimes (c)_\star \mathcal{O}) \\ &= \bigoplus_{\tau \in B[r]} H^0(X^{[d_v+d_w]}, L^{[d_v+d_w]} \otimes a_B^\star \tau) = \bigoplus_{\tau \in B[r]} H^0(X^{[d_v+d_w]}, (L \otimes \tau)^{[d_v+d_w]}). \end{aligned}$$

Under the isomorphism above, the divisor θ^+ corresponds up to a $\mathbb{C}^\star$-scaling ambiguity to a tuple of sections,

$$\theta^+ \leftrightarrow (s_\tau)_{\tau \in B[r]}, \quad s_\tau \in H^0(X^{[d_v+d_w]}, (L \otimes \tau)^{[d_v+d_w]}).$$

The space of sections of $(L \otimes \tau)^{[d_v+d_w]} \to X^{[d_v+d_w]}$ can be identified with

$$\Lambda^{d_v+d_w} H^0(L \otimes \tau),$$

which is one-dimensional since $h^0(L \otimes \tau) = \chi(L \otimes \tau) = d_v + d_w$. Furthermore, any nonzero section vanishes along the divisor

$$\Theta_{L \otimes \tau} = \{I_U : h^0(I_U \otimes L \otimes \tau) \neq 0\}.$$

We have the important

Proposition 4.2. *For each τ, s_τ is not the trivial section, hence it vanishes along $\Theta_{L\otimes\tau}$.*

The proposition completes the proof of Theorem 1.2. Indeed, as explained in [MO1], each of the sections s_τ induces an isomorphism between spaces of sections

$$\bigoplus_{\tau \in B[r]} s_\tau : \bigoplus_{\tau \in B[r]} H^0(X^{[d_v}, (L \otimes \tau)^{[d_v]})^\vee \to \bigoplus_{\tau \in B[r]} H^0(X^{[d_w]}, (L \otimes \tau)^{[d_w]}).$$

Since we argued that the strange duality map coincides with $\bigoplus_{\tau \in B[r]} s_\tau$, Theorem 1.2 follows. [2]

The proof of Theorem 1.3 is identical. □

[2] Here, we assumed the determinant is $\mathcal{O}(\sigma + mf)$ where f denotes the fiber over zero. If the determinant involves the fiber over a different point, the same argument applies with the obvious changes in the definition of L.

Proof of Proposition 4.2. This is easily seen by restriction to $X_F^{[d_v+d_w]}$, the fiber over zero of the addition map $a_B : X^{[d_v+d_w]} \to B$. Letting $\mathfrak{X}_F^+$ be the fiber product

$$\begin{array}{ccc} \mathfrak{X}_F^+ & \xrightarrow{c} & X_F^{[d_v+d_w]} \\ \downarrow \iota & & \downarrow \iota \\ \mathfrak{X}^+ & \xrightarrow{c} & X^{[d_v+d_w]} \end{array}$$

we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathfrak{X}^+, \theta^+) & \longrightarrow & \bigoplus_{\tau \in B[r]} H^0(X^{[d_v+d_w]}, (L \otimes \tau)^{[d_v+d_w]}) \\ \downarrow & & \downarrow \\ H^0(\mathfrak{X}_F^+, \theta^+) & \longrightarrow & \bigoplus_{\tau \in B[r]} H^0(X_F^{[d_v+d_w]}, (L \otimes \tau)^{[d_v+d_w]}) \end{array}$$

where the horizontal maps are isomorphisms and the vertical maps are restrictions of sections. The bottom isomorphism is particularly easy to understand since

$$\mathfrak{X}_F^+ \simeq X_F^{[d_v+d_w]} \times B[r].$$

Let θ_F^+ be the restriction of θ^+ to $\mathfrak{X}_F^+$. Further restricting θ_F^+ to $X_F^{[d_v+d_w]} \times \{\tau\}$ we obtain the divisor $\Theta_{L\otimes\tau}$. We claim that each of the divisors $\Theta_{L\otimes\tau}$ restricts nontrivially to $X_F^{[d_v+d_w]}$. This implies in turn that s_τ is not the trivial section for any τ. The following lemma proves the claim above when $\tau = \mathcal{O}$; the arguments are identical for $\tau \neq \mathcal{O}$. □

Lemma 4.3. *Consider the line bundle $L_{k,n} = \mathcal{O}(k\sigma + nf)$ on $X = B \times F$, and assume that $k \geq 2$ and $n \geq 1$. Then for a generic $I_Z \in X_F^{[kn]}$, we have*

$$H^0(I_Z \otimes L_{k,n}) = 0.$$

Proof. We argue by induction on n. For $n = 1$, as established before, all sections of L vanish along divisors of the form

$$\bigsqcup_{i=1}^{k} (B \times y_i) \cup f \quad \text{for } y_1 + \cdots + y_k = o_F.$$

A zero-dimensional subscheme I_Z in $X_F^{[k]}$ consisting of distinct points $z_i = (x_i, y_i) \in B \times F$, $1 \leq i \leq k$, must satisfy

$$x_1 + \cdots + x_k = o_B.$$

Choose Z so that $y_1 + \cdots + y_k \neq o_F$, with $y_i \neq y_j$ for $i \neq j$, and so that $x_i \neq o_B$ for all $1 \leq i \leq k$. Then no section of L vanishes at Z.

To carry out the induction, note the exact sequence of X,

$$0 \to L_{k,n} \to L_{k,n+1} \to \mathcal{O}_f(ko_F) \to 0,$$

and the associated exact sequence on global sections,

$$0 \to H^0(X, L_{k,n}) \to H^0(X, L_{k,n+1}) \to H^0(X, \mathcal{O}_f(ko_F)) \to 0.$$

Let $I_Z \in X_F^{[kn]}$ be such that

$$H^0(I_Z \otimes L_{k,n}) = 0 \quad \text{and} \quad Z \cap (o_B \times F) = \emptyset.$$

Then $H^1(I_Z \otimes L_{k,n}) = 0$. Choose k additional points

$$z_i = (o_B, y_i), \quad 1 \leq i \leq k, \quad \text{so that} \quad y_1 + \cdots + y_k \neq o_F.$$

From the exact sequence of global sections, it follows that no section of $L_{k,n+1}$ vanishes at

$$Z \cup \{z_1, \ldots, z_k\} \in X_F^{[k(n+1)]}.$$

Indeed, the exact sequence and the induction hypothesis on Z show that

$$H^0(X, L_{k,n+1} \otimes I_Z) \cong H^0(F, \mathcal{O}_F(ko_F) \otimes I_Z) \cong H^0(F, \mathcal{O}_F(ko_F))$$

via restriction. But no section of $\mathcal{O}_F(ko_F)$ on the central fiber vanishes at the points $z_1, \ldots, z_k$, so no section of $L_{k,n+1}$ vanishes at $Z \cup \{z_1, \ldots, z_k\}$. We conclude that for a general $I_{Z'} \in X_F^{[k(n+1)]}$, we have

$$H^0(I_{Z'} \otimes L_{k,n+1}) = 0.$$

This ends the proof of the lemma. □

4.4. Proof of Theorem 1.4

The strategy of proof is similar to that of Theorem 1.2. We indicate the necessary changes. We write

$$\det v = \mathcal{O}(\sigma + m_v f) \otimes Q_v^{-r}, \qquad \det w = \mathcal{O}(\sigma + m_w f) \otimes Q_w^{-s},$$

for line bundles Q_v, Q_w of degree 0 over C. We recall the birational isomorphism in Subsection 3.2:

$$\mathfrak{M}_v^+ \dashrightarrow \mathfrak{X}_v^+ = \{(Z, z_C) : z_C^r = \mathcal{M}_Z\} \subset X^{[d_v]} \times \mathrm{Pic}^0(C)$$

where we set

$$\mathcal{M}_Z = \mathcal{O}_C(a_C(Z) - d_v \cdot o).$$

This was noted when Q_v is trivial in Section 3.2, but the twist by Q_v is an isomorphism of moduli spaces, yielding only a modified formula

$$\mathbf{R}\mathcal{S}^\dagger(V) = I_Z^\vee(-r\sigma + (\chi + \bar{g})f) \otimes z_C^{-1} \otimes Q_v^{-1}.$$

There is a similar birational isomorphism $\mathfrak{M}_w^+ \dashrightarrow \mathfrak{X}_w^+$.

The divisor $\Theta^+_{vw} \subset \mathfrak{M}^+_v \times \mathfrak{M}^+_w$ is identified with $\Theta^+ \subset \mathfrak{X}^+_v \times \mathfrak{X}^+_w$ where

$$\Theta^+ = \{(Z, T, z_C, t_C) : h^0(I_Z \otimes I_T \otimes z_C \otimes t_C \otimes L) \neq 0\},$$

for the line bundle

$$L = \mathcal{O}((r+s)\sigma - (\chi_v + \chi_w + 2\bar{g}) f) \otimes Q \quad \text{with} \quad \chi(L) = d_v + d_w.$$

Here, we wrote $Q = Q_v \otimes Q_w$, which by assumption is a generic line bundle of degree 0 over the curve C. It can be seen that L has no higher cohomology if $\chi_v + \chi_w \leq -3\bar{g}$. In turn, when $r = s$, this is equivalent to the requirement

$$d_v + d_w = -2r(\chi_v + \chi_w + \bar{g}) \geq 4r\bar{g}$$

of the theorem.

There are a few steps in the proof of Theorem 1.4 that need modifications from the genus 1 case. They are:

(i) The identification of the theta bundles. We carry this out for $r = s$ only, making use of the natural addition map $\tau^+ : \mathfrak{X}^+_v \times \mathfrak{X}^+_w \dashrightarrow \mathfrak{X}^+$, where

$$\mathfrak{X}^+ = \{(U, u_C) : \mathcal{M}_U = u_C^r\} \subset X^{[d_v+d_w]} \times \mathrm{Pic}^0(C).$$

We have $\Theta^+ = (\tau^+)^\star \theta^+$ for

$$\theta^+ = \{(U, u_C) : h^0(I_U \otimes u_C \otimes L) \neq 0\}.$$

As before, we note the natural projections $c : \mathfrak{X}^+ \to X^{[d_v+d_w]}$ and $\mathrm{pr} : \mathfrak{X}^+ \to \mathrm{Pic}^0(C)$. We claim that

$$\mathcal{O}(\theta^+) = c^\star L^{[d_v+d_w]} \otimes \mathrm{pr}^\star \mathcal{P}^r_\alpha, \tag{4.5}$$

where $\mathcal{P}_\alpha$ is the line bundle over $\mathrm{Pic}^0(C)$ associated to the point

$$\alpha = K_C(-2\bar{g} \cdot o) \otimes Q^{-2} \in \mathrm{Pic}^0(C). \tag{4.6}$$

Pulling back under τ^+, it follows that

$$\Theta_w = c_v^\star L^{[d_v]} \otimes \mathrm{pr}^\star \mathcal{P}^r_\alpha, \qquad \Theta_v = c_w^\star L^{[d_w]} \otimes \mathrm{pr}^\star \mathcal{P}^r_\alpha.$$

Before proving (4.5), we simplify notations. We write $\ell = \chi(L) = d_v + d_w$. Over the Jacobian $A = \mathrm{Pic}^0(C)$, we fix a principal polarization, for instance

$$\Theta = \{y \in A : h^0(y \otimes \mathcal{O}_C(\bar{g} \cdot o)) \neq 0\}.$$

The standardly normalized Poincaré bundle $\mathcal{P} \to A \times A$ takes the form

$$\mathcal{P} = m^\star \Theta^{-1} \otimes (\Theta \boxtimes \Theta).$$

This differs in sign from the usual conventions, but it is compatible with our conventions on the Abel–Jacobi embedding

$$C \to A, \quad x \mapsto \mathcal{O}_C(x - o),$$

in the sense that $\mathcal{P}|_{A\times\alpha}$ restricts to α over C. For degree ℓ, we consider the Abel–Jacobi map

$$\pi : C^{(\ell)} \to A, \quad D \mapsto \mathcal{O}(D - \ell \cdot o),$$

and set

$$\mathcal{P}_C^{(\ell)} = (\pi, 1)^\star \mathcal{P} \to C^{(\ell)} \times A.$$

The bundle $\mathcal{P}^{(\ell)}$ satisfies

$$\mathcal{P}^{(\ell)}|_{C^{(\ell)}\times\{y\}} \cong y^{(\ell)}, \quad \mathcal{P}^{(\ell)}|_{\ell[o]\times A} \text{ is trivial over } A.$$

When $\ell = 1$, $\mathcal{P}^{(1)}$ is the Poincaré bundle $\mathcal{P}_C \to C \times A$ normalized over o.

With these notations out of the way, we first consider $\mathcal{O}(\theta^+)$ over the product $X^{[\ell]} \times A$. We claim that

$$\mathcal{O}(\theta^+) = c^\star L^{[\ell]} \otimes (a_C, 1)^\star \mathcal{P}^{(\ell)} \otimes \mathrm{pr}_A^\star \mathcal{M} \tag{4.7}$$

for some line bundle $\mathcal{M} \to A$. This follows from the see-saw theorem. The restriction of $\mathcal{O}(\theta^+)$ to $X^{[\ell]} \times \{u_C\}$ is

$$(L \otimes u_C)^{[\ell]} = L^{[\ell]} \otimes a_C^\star u_C^{(\ell)}.$$

This agrees with the restriction of $c^\star L^{[\ell]} \otimes (a_C, 1)^\star \mathcal{P}^{(\ell)}$ and proves (4.7). To identify $\mathcal{M}$, we restrict to $\{U\} \times A$, where U is a length ℓ subscheme of X supported over $o \times o_F$. We obtain

$$\mathcal{M} = \det \mathbf{R} p_\star (\mathcal{P}_C \otimes q^\star (L \otimes I_U))^{-1}.$$

We can rewrite $\mathcal{M}$ expressing in K-theory

$$I_U = \mathcal{O} - \ell \cdot \mathcal{O}_{o\times o_F}.$$

Recalling the normalization of $\mathcal{P}_C$ over o, and that

$$L = (\mathcal{O}_C(t \cdot o) \otimes Q) \boxtimes \mathcal{O}_F(2r \cdot o_F)$$

for $t = -(\chi_v + \chi_w + 2\bar{g})$, we obtain

$$\begin{aligned}\mathcal{M} &= \det \mathbf{R} p_\star (\mathcal{P}_C \otimes q^\star L)^{-1} = \det \mathbf{R} p_\star (\mathcal{P}_C \otimes q^\star (\mathcal{O}_C(t \cdot o) \otimes Q))^{-2r} \\ &= \det \mathbf{R} p_\star (\mathcal{P}_C \otimes q^\star (\mathcal{O}_C(\bar{g} \cdot o) \otimes Q))^{-2r} = \Theta^{2r} \otimes \mathcal{P}_{-Q}^{2r}.\end{aligned}$$

Therefore

$$\mathcal{O}(\theta^+) = c^\star L^{[\ell]} \otimes (a_C, 1)^\star \circ (\pi, 1)^\star \mathcal{P} \otimes \mathrm{pr}_A^\star (\Theta \otimes \mathcal{P}_{-Q})^{2r}.$$

Over $\mathfrak{X}^+ \hookrightarrow X^{[\ell]} \times A$, we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{X}^+ & \xrightarrow{(a_C,1)} & C^{(\ell)} \times A \\ \downarrow{\scriptstyle \mathrm{pr}_A} & & \downarrow{\scriptstyle (\pi,1)} \\ A & \xrightarrow{(r,1)} & A \times A \end{array}$$

so that $(\pi, 1) \circ (a_C, 1) = (r, 1) \circ \mathrm{pr}_A$. Hence, we obtain

$$\begin{aligned}\mathcal{O}(\theta^+) &= c^\star L^{[\ell]} \otimes \mathrm{pr}_A^\star((r,1)^\star \mathcal{P} \otimes \Theta^{2r} \otimes \mathcal{P}_{-Q}^{2r})\\ &= c^\star L^{[\ell]} \otimes \mathrm{pr}_A^\star(r^\star\Theta \otimes \Theta \otimes (r+1)^\star \Theta^{-1} \otimes \Theta^{2r} \otimes \mathcal{P}_{-Q}^{2r})\\ &= c^\star L^{[\ell]} \otimes \mathrm{pr}_A^\star(\Theta \otimes (-1)^\star \Theta^{-1} \otimes \mathcal{P}_{-Q}^{2})^r = c^\star L^{[\ell]} \otimes \mathrm{pr}_A^\star \mathcal{P}_\alpha^r,\end{aligned}$$

as claimed.

(ii) Next, we identify the space of sections $H^0(\mathfrak{X}^+, \mathcal{O}(\theta^+))$ using the cartesian diagram

$$\begin{array}{ccc} \mathfrak{X}^+ & \xrightarrow{\ c\ } & X^{[\ell]} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_A} & & \Big\downarrow{\scriptstyle f} \\ A & \xrightarrow{\ r\ } & A \end{array}$$

where $f = \pi \circ a_C$. We have

$$\begin{aligned}H^0(\mathfrak{X}^+, \mathcal{O}(\theta^+)) &= H^0(\mathfrak{X}^+, c^\star L^{[\ell]} \otimes \mathrm{pr}_A^\star \mathcal{P}_\alpha^r) = H^0(X^{[\ell]}, L^{[\ell]} \otimes c_\star \mathrm{pr}_A^\star \mathcal{P}_\alpha^r)\\ &= H^0(X^{[\ell]}, L^{[\ell]} \otimes f^\star r_\star \mathcal{P}_\alpha^r) = H^0(X^{[\ell]}, L^{[\ell]} \otimes f^\star r_\star r^\star \mathcal{P}_\alpha)\\ &= \bigoplus_{\tau \in A[r]} H^0(X^{[\ell]}, L^{[\ell]} \otimes f^\star(\mathcal{P}_\alpha \otimes \tau))\\ &= \bigoplus_{\tau \in A[r]} H^0(X^{[\ell]}, L^{[\ell]} \otimes a_C^\star(\alpha \otimes \tau))\\ &= \bigoplus_{\tau \in A[r]} H^0(X^{[\ell]}, (L \otimes \alpha \otimes \tau)^{[\ell]}).\end{aligned}$$

Similar expressions hold over each of the factors $\mathfrak{X}_v^+$ and $\mathfrak{X}_w^+$.

(iii) Finally, we need a suitable analogue of Lemma 4.3. This concerns subschemes Z in X, belonging to

$$X_F^{[\ell]} = \{Z : a_C(Z) \text{ is rationally equivalent to } \ell \cdot o\}.$$

We show that if $L_{k,n} = \mathcal{O}(k\sigma + nf)$ for $k \geq 2g$, $n \geq g$, then for all r-torsion line bundles τ over C, we have

$$H^0(L_{k,n} \otimes \alpha \otimes \tau \otimes I_Z) = 0$$

provided Z is generic in $X_F^{[k(n-\bar{g})]}$, and Q, which appears in the definition of α in (4.6), is generic over C. This is then applied to the line bundle L appearing in the expression of the theta bundles above.

We induct on n, starting with the base case $n = g$. Just as in genus 1, for generic Q, the sections of $L_{k,g} \otimes \alpha \otimes \tau$ vanish along divisors of the form

$$\bigsqcup_{i=1}^{k} (C \times y_i) \cup f_{p_1} + \cdots + f_{p_g} \quad \text{for } y_1 + \cdots + y_k = o_F,$$

and some $p_1, \dots, p_g \in C$. Indeed, it suffices to explain that

$$h^0(\mathcal{O}_C(g \cdot o) \otimes \alpha \otimes \tau) = 1,$$

the unique section vanishing at g points $p_1(o), \dots, p_g(o)$, which depend on o. Equivalently, recalling the definition of α in (4.6), and using Riemann–Roch and Serre duality, we prove that

$$h^0(\tau^{-1} \otimes \mathcal{O}_C((g-2) \cdot o) \otimes Q^2) = 0.$$

As the Euler characteristic of the above line bundle is -1, the space of sections vanishes for generic Q of degree 0.

To complete the proof of the base case, fix a generic Q as above. It suffices to explain that for all $k \geq 2g$, we can find $x_1, \dots, x_k \in C$ such that

$$x_1 + \cdots + x_k \equiv k \cdot o, \qquad x_i \neq p_j \qquad \text{for all } i, j.$$

This amounts to choosing a section from the nonempty set

$$H^0(\mathcal{O}_C(k \cdot o)) - \bigcup_{j=1}^{g} H^0(\mathcal{O}_C(k \cdot o - p_j)) - H^0(\mathcal{O}_C((k-1) \cdot o)),$$

and letting $x_1, \dots, x_k$ be its zeros. This ends the argument for the base case.

The inductive step does not require any changes from the original Lemma 4.3.

The proof of Theorem 1.4 is now completed. □

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