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Localization of Cohomological Induction

by

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Abstract

We give a geometric realization of cohomologically induced $(\mathfrak{g}, K)$-modules. Let $(\mathfrak{h}, L)$ be a subpair of $(\mathfrak{g}, K)$. The cohomological induction is an algebraic construction of $(\mathfrak{g}, K)$-modules from an $(\mathfrak{h}, L)$-module V. For a real semisimple Lie group, the duality theorem by Hecht, Miličić, Schmid, and Wolf relates $(\mathfrak{g}, K)$-modules cohomologically induced from a Borel subalgebra to $\mathcal{D}$-modules on the flag variety of $\mathfrak{g}$. In this article we extend the theorem to more general pairs $(\mathfrak{g}, K)$ and $(\mathfrak{h}, L)$. We consider the tensor product of a $\mathcal{D}$-module and a certain module associated with V, and prove that its sheaf cohomology groups are isomorphic to cohomologically induced modules.

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§1. Introduction

The aim of this article is to realize cohomologically induced modules as sheaf cohomology groups of certain sheaves on homogeneous spaces.

Cohomological induction is defined as a functor between the categories of $(\mathfrak{g}, K)$-modules. Let $(\mathfrak{g}, K)$ be a pair (Definition 2.1) and let $\mathcal{C}(\mathfrak{g}, K)$ be the category of $(\mathfrak{g}, K)$-modules. Suppose that $(\mathfrak{h}, L)$ is a subpair of $(\mathfrak{g}, K)$ and that K and L are reductive. Following the book by Knapp and Vogan [KV95], we define the functors $P_{\mathfrak{h},L}^{\mathfrak{g},K}$ and $I_{\mathfrak{h},L}^{\mathfrak{g},K} : \mathcal{C}(\mathfrak{h}, L) \to \mathcal{C}(\mathfrak{g}, K)$ as $V \mapsto R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h},L)} V$ and $V \mapsto (\mathrm{Hom}_{R(\mathfrak{h},L)}(R(\mathfrak{g}, K), V))_K$, respectively. See Section 2 for the definition of the Hecke algebra $R(\mathfrak{g}, K)$. When $\mathfrak{g} = \mathfrak{h}$, the functor $I_{\mathfrak{h},L}^{\mathfrak{g},K} = I_{\mathfrak{g},L}^{\mathfrak{g},K}$ is called the Zuckerman functor. Let V be an $(\mathfrak{h}, L)$-module. We define the cohomologically induced module as the $(\mathfrak{g}, K)$-module $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_j(V)$ for $j \in \mathbb{N}$, where $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_j$ is the

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j-th left derived functor of $P^{\mathfrak{g},K}_{\mathfrak{h},L}$. Similarly, we define $(I^{\mathfrak{g},K}_{\mathfrak{h},L})^j(V)$, where $(I^{\mathfrak{g},K}_{\mathfrak{h},L})^j$ is the j-th right derived functor of $I^{\mathfrak{g},K}_{\mathfrak{h},L}$.

This construction produces a large family of representations of real reductive Lie groups. Let $G_\mathbb{R}$ be a real reductive Lie group with a Cartan involution θ so that the group of fixed points $K_\mathbb{R} := (G_\mathbb{R})^\theta$ is a maximal compact subgroup. Let $\mathfrak{g}$ be the complexified Lie algebra of $G_\mathbb{R}$ and K the complexification of $K_\mathbb{R}$. We give examples of cohomologically induced $(\mathfrak{g}, K)$-modules below. In the following three examples we suppose that $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$, and L is a maximal reductive subgroup of the normalizer $N_K(\mathfrak{h})$. We also suppose that V is a one-dimensional $(\mathfrak{h}, L)$-module.

- We assume the rank condition $\operatorname{rank}\mathfrak{g} = \operatorname{rank} K$ and that $\mathfrak{h}$ is a θ-stable Borel subalgebra. Then under a certain positivity condition on V, $(P^{\mathfrak{g},K}_{\mathfrak{h},L})_s(V)$ (or $(I^{\mathfrak{g},K}_{\mathfrak{h},L})^s(V)$) is the underlying $(\mathfrak{g}, K)$-module of a discrete series representation of $G_\mathbb{R}$. Here $s = \frac{1}{2}\dim K/L$.
- Suppose that $\mathfrak{h}$ is a θ-stable parabolic subalgebra. Then the $(\mathfrak{g}, K)$-module $(P^{\mathfrak{g},K}_{\mathfrak{h},L})_s(V)$ (or $(I^{\mathfrak{g},K}_{\mathfrak{h},L})^s(V)$) is called Zuckerman's derived functor module $A_\mathfrak{h}(\lambda)$. Here $s = \frac{1}{2}\dim K/L$.
- Let $P_\mathbb{R}$ be a parabolic subgroup of $G_\mathbb{R}$ and suppose that $\mathfrak{h}$ is its complexified Lie algebra. Then $(P^{\mathfrak{g},K}_{\mathfrak{h},L})_0(V)$ (or $(I^{\mathfrak{g},K}_{\mathfrak{h},L})^0(V)$) is the underlying $(\mathfrak{g}, K)$-module of a degenerate principal series representation realized on the real flag variety $G_\mathbb{R}/P_\mathbb{R}$.

The localization theory by Beĭlinson–Bernstein [BB81] provides another important construction of $(\mathfrak{g}, K)$-modules. It gives a realization of $(\mathfrak{g}, K)$-modules as K-equivariant twisted $\mathcal{D}$-modules on the full flag variety X of $\mathfrak{g}$.

These two constructions are related by a result of Hecht–Miličić–Schmid–Wolf [HMSW87]. We now recall their theorem. Let $G_\mathbb{R}$ be a connected real reductive Lie group and let $(\mathfrak{g}, K)$ be the pair defined in the above way. Suppose that $\mathfrak{h} = \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and L is a maximal reductive subgroup of the normalizer $N_K(\mathfrak{b})$. Let X be the full flag variety of $\mathfrak{g}$, Y the K-orbit through $\mathfrak{b} \in X$, and $i : Y \to X$ the inclusion map. Suppose that V is a $(\mathfrak{b}, L)$-module and $\mathfrak{b}$ acts as scalars given by $\lambda \in \mathfrak{b}^* := \operatorname{Hom}_\mathbb{C}(\mathfrak{b}, \mathbb{C})$. Write $\mathcal{V}_Y$ for the corresponding locally free $\mathcal{O}_Y$-module on Y and view it as a twisted $\mathcal{D}$-module. Let $\mathcal{D}_{X,\lambda}$ be the ring of twisted differential operators on X corresponding to λ and define the $\mathcal{D}_{X,\lambda}$-module direct image $i_+\mathcal{V}_Y$. Then the following is called the *duality theorem*:

Theorem 1.1 ([HMSW87]). *There is an isomorphism of* $(\mathfrak{g}, K)$*-modules*

$$\mathrm{H}^s(X, i_+\mathcal{V}_Y)^* \simeq (I^{\mathfrak{g},K}_{\mathfrak{b},L})^{u-s}\big(V^* \otimes {\textstyle\bigwedge^{\mathrm{top}}}(\mathfrak{g}/\mathfrak{b})^*\big)$$

for $s \in \mathbb{N}$ *and* $u = \dim K/L - \dim Y$. *Here the left side is the* K*-finite dual of the* $(\mathfrak{g}, K)$*-module* $\mathrm{H}^s(X, i_+\mathcal{V}_Y)$.

The proof in [HMSW87] is by describing the cohomology groups of both sides by using standard resolutions and giving an isomorphism between the two complexes. We note that by using the dual isomorphism ([KV95, Theorem 3.1]) $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_j(V)^* \simeq (I_{\mathfrak{h},L}^{\mathfrak{g},K})^j(V^*)$, Theorem 1.1 can be deduced from

$$\mathrm{H}^s(X, i_+\mathcal{V}_Y) \simeq (P_{\mathfrak{b},L}^{\mathfrak{g},K})_{u-s}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{b})\big). \tag{1.1}$$

The relation between cohomological induction and localization has been studied further (see [Bie90], [Cha93], [Kit10], [MP98], [Sch91]). Miličić–Pandžić [MP98] gave a more conceptual proof of Theorem 1.1 by using equivariant derived categories. In [Cha93] and [Kit10], Theorem 1.1 was extended to the case of partial flag varieties.

In this article we will realize geometrically the cohomologically induced modules in the following setting. Let $i : K \to G$ be a homomorphism between complex linear algebraic groups. Suppose that K is reductive and the kernel of i is finite so that the pair $(\mathfrak{g}, K)$ is defined. Let H be a closed subgroup of G. Put $M := i^{-1}(H)$ and take a Levi decomposition $M = L \ltimes U$. We write $i : Y = K/M \to G/H = X$ for the natural immersion. Let V be an $(\mathfrak{h}, M)$-module. We view V as an $(\mathfrak{h}, L)$-module by restriction and define the cohomologically induced module $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_j(V)$. In this generality, we can no longer realize it as a (twisted) $\mathcal{D}$-module on $X = G/H$. Instead we use the tensor product of an $i^{-1}\mathcal{D}_X$-module and an $i^{-1}\mathcal{O}_X$-module associated with V which is equipped with a $(\mathfrak{g}, K)$-action (see Definition 3.3). We now state the main theorem of this article.

Main Theorem (Theorem 4.1). *Suppose that* $\mathcal{V}$ *is an* $i^{-1}\widetilde{\mathfrak{g}}_X$*-module associated with* V (*see Definition* 3.3). *Then we have an isomorphism of* $(\mathfrak{g}, K)$*-modules*

$$\mathrm{H}^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \simeq (P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big)$$

for $s \in \mathbb{N}$ *and* $u = \dim U$.

Here $\mathcal{L}$ is the invertible sheaf on Y defined at the beginning of Section 4 and the direct image $i_+\mathcal{L}$ in the categories of $\mathcal{D}$-modules is defined as

$$i_*\big((\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X\big) \otimes_{\mathcal{O}_X} \Omega_X^\vee.$$

Hence its inverse image $i^{-1}i_+\mathcal{L}$ as a sheaf of abelian groups is given by

$$(\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee.$$

We note that if V comes from an algebraic H-module, then we can take $\mathcal{V}$ to be $i^{-1}\mathcal{V}_X$, where $\mathcal{V}_X$ is a G-equivariant locally free $\mathcal{O}_X$-module with typical fiber V (Example 3.5).

The work in this article was motivated by the study of branching laws of representations. In [Osh11] a special case of Theorem 4.1 was proved and it was used to get an estimate of the restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subalgebras.

This article is organized as follows. In Section 2 we recall the definition of cohomological induction following [KV95]. In Section 3 we give the definition of an $i^{-1}\mathcal{O}$-module associated with an $(\mathfrak{h}, M)$-module. We state and prove the main theorem (Theorem 4.1) in Section 4. Our proof basically follows the proof of the duality theorem in [HMSW87]. Section 5 is devoted to the construction of the $i^{-1}\mathcal{O}$-module associated with an $(\mathfrak{h}, M)$-module, which can be used for the geometric realization of cohomologically induced modules. In Section 6, we see that the module $i^{-1}i_+\mathcal{L}\otimes\mathcal{V}$ can be viewed as a twisted $\mathcal{D}$-module if $\mathfrak{h}$ acts as scalars on V. Therefore, Theorem 4.1 becomes the isomorphism (1.1) and hence Theorem 1.1 in the particular setting.

§2. Cohomological induction

In this section we recall the definition of cohomological induction following [KV95].

Let K be a complex reductive algebraic group and let $K_{\mathbb{R}}$ be a compact real form. Since any locally finite action of $K_{\mathbb{R}}$ uniquely extends to an algebraic action of K, the locally finite $K_{\mathbb{R}}$-modules are identified with the algebraic K-modules. Define the Hecke algebra $R(K_{\mathbb{R}})$ as the space of $K_{\mathbb{R}}$-finite distributions on $K_{\mathbb{R}}$. For $S \in R(K_{\mathbb{R}})$, the pairing with a smooth function f on $K_{\mathbb{R}}$ is written as

$$\int_{K_{\mathbb{R}}} f(k)\ dS(k).$$

The product of $S, T \in R(K_{\mathbb{R}})$ is given by

$$S * T : f \mapsto \int_{K_{\mathbb{R}}\times K_{\mathbb{R}}} f(kk')\, dS(k)\, dT(k').$$

The associative algebra $R(K_{\mathbb{R}})$ does not have the identity, but has an approximate identity (see [KV95, Chapter I]). The locally finite $K_{\mathbb{R}}$-modules are identified with the approximately unital left $R(K_{\mathbb{R}})$-modules. The action map $R(K_{\mathbb{R}}) \times V \to V$ is given by

$$(S, v) \mapsto \int_{K_{\mathbb{R}}} kv\, dS(k)$$

for a locally finite $K_{\mathbb{R}}$-module V. Here, kv is regarded as a smooth function on $K_{\mathbb{R}}$ that takes values in V. We have a natural isomorphism of $\mathbb{C}$-algebras

$$R(K_{\mathbb{R}}) \simeq \bigoplus_{\tau \in \widehat{K}} \mathrm{End}_{\mathbb{C}}(V_\tau),$$

where $\widehat{K}$ is the set of equivalence classes of irreducible K-modules, and V_τ is a representation space of $\tau \in \widehat{K}$. Hence $R(K_{\mathbb{R}})$ depends only on the complexification K up to natural isomorphisms, so in what follows, we also denote $R(K_{\mathbb{R}})$ by $R(K)$.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra and K a complex linear algebraic group such that the Lie algebra $\mathfrak{k}$ of K is a subalgebra of $\mathfrak{g}$. Suppose that a homomorphism $\phi : K \to \mathrm{Aut}(\mathfrak{g})$ of algebraic groups is given, where $\mathrm{Aut}(\mathfrak{g})$ is the automorphism group of $\mathfrak{g}$. We say $(\mathfrak{g}, K)$ is a *pair* if

- $\phi(\cdot)|_{\mathfrak{k}}$ is equal to the adjoint action $\mathrm{Ad}_{\mathfrak{k}}(K)$ of K, and
- the differential of ϕ is equal to the adjoint action $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$.

Let $i : K \to G$ be a homomorphism of complex linear algebraic groups with finite kernel and let $\mathfrak{g}$ be the Lie algebra of G. Then $(\mathfrak{g}, K)$ with the homomorphism $\phi := \mathrm{Ad} \circ i$ is a pair in the above sense.

Definition 2.2. Let $(\mathfrak{g}, K)$ be a pair. Let V be a complex vector space with a Lie algebra action of $\mathfrak{g}$ and an algebraic action of K. We say that V is a $(\mathfrak{g}, K)$-*module* if

- the differential of the action of K coincides with the restriction of the action of $\mathfrak{g}$ to $\mathfrak{k}$, and
- $(\phi(k)\xi)v = k(\xi(k^{-1}(v)))$ for $k \in K$, $\xi \in \mathfrak{g}$, and $v \in V$.

For a pair $(\mathfrak{g}, K)$, we denote by $\mathcal{C}(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$-modules. Suppose moreover that K is reductive. We extend the representation $\phi : K \to \mathrm{Aut}(\mathfrak{g})$ to a representation $\phi : K \to \mathrm{Aut}(U(\mathfrak{g}))$ on the universal enveloping algebra. We define the Hecke algebra $R(\mathfrak{g}, K)$ as

$$R(\mathfrak{g}, K) := R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}).$$

The product is given by

$$(S \otimes \xi) \cdot (T \otimes \eta) = \sum_i \bigl(S * (\langle \xi_i^*, \phi(\cdot)^{-1}\xi \rangle T) \otimes \xi_i \eta\bigr)$$

for $S, T \in R(K)$ and $\xi, \eta \in U(\mathfrak{g})$. Here ξ_i is a basis of the linear span of $\phi(K)\xi$, and ξ_i^* is its dual basis. We regard $\langle \xi_i^*, \phi(\cdot)^{-1}\xi \rangle$ as a function on $K_{\mathbb{R}}$. As in the group case, the $(\mathfrak{g}, K)$-modules are identified with the approximately unital left $R(\mathfrak{g}, K)$-modules. The action map $R(\mathfrak{g}, K) \times V \to V$ is given by

$$(S \otimes \xi,\, v) \mapsto \int_{K_{\mathbb{R}}} k(\xi v)\, dS(k)$$

for a $(\mathfrak{g}, K)$-module V.

Let $(\mathfrak{g}, K)$ and $(\mathfrak{h}, L)$ be pairs in the sense of Definition 2.1. Suppose that K and L are reductive. Let $i : (\mathfrak{h}, L) \to (\mathfrak{g}, K)$ be a map between pairs, namely, a Lie algebra homomorphism $i_{\mathrm{alg}} : \mathfrak{h} \to \mathfrak{g}$ and an algebraic group homomorphism $i_{\mathrm{gp}} : L \to K$ satisfy the following two assumptions:

- The restriction of i_{alg} to the Lie algebra $\mathfrak{l}$ of L is equal to the differential of i_{gp}.
- $\phi_K(i_{\mathrm{gp}}(l)) \circ i_{\mathrm{alg}} = i_{\mathrm{alg}} \circ \phi_L(l)$ for $l \in L$, where ϕ_K denotes ϕ for $(\mathfrak{g}, K)$ in Definition 2.1 and ϕ_L denotes ϕ for $(\mathfrak{h}, L)$.

We define the functors $P^{\mathfrak{g},K}_{\mathfrak{h},L}, I^{\mathfrak{g},K}_{\mathfrak{h},L} : \mathcal{C}(\mathfrak{h}, L) \to \mathcal{C}(\mathfrak{g}, K)$ by

$$P^{\mathfrak{g},K}_{\mathfrak{h},L} : V \mapsto R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h},L)} V,$$
$$I^{\mathfrak{g},K}_{\mathfrak{h},L} : V \mapsto (\mathrm{Hom}_{R(\mathfrak{h},L)}(R(\mathfrak{g}, K),\, V))_K,$$

where $(\cdot)_K$ is the subspace of K-finite vectors. Then $P^{\mathfrak{g},K}_{\mathfrak{h},L}$ is right exact and $I^{\mathfrak{g},K}_{\mathfrak{h},L}$ is left exact. Write $(P^{\mathfrak{g},K}_{\mathfrak{h},L})_j$ for the j-th left derived functor of $P^{\mathfrak{g},K}_{\mathfrak{h},L}$ and write $(I^{\mathfrak{g},K}_{\mathfrak{h},L})^j$ for the j-th right derived functor of $I^{\mathfrak{g},K}_{\mathfrak{h},L}$. We can see that $I^{\mathfrak{g},K}_{\mathfrak{h},L}$ is the right adjoint functor of the forgetful functor

$$\mathrm{For}^{\mathfrak{h},L}_{\mathfrak{g},K} : \mathcal{C}(\mathfrak{g}, K) \to \mathcal{C}(\mathfrak{h}, L), \qquad V \mapsto R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g},K)} V \simeq V,$$

and $P^{\mathfrak{g},K}_{\mathfrak{h},L}$ is the left adjoint functor of the functor

$$\mathrm{For}^{\vee\,\mathfrak{h},L}_{\mathfrak{g},K} : \mathcal{C}(\mathfrak{g}, K) \to \mathcal{C}(\mathfrak{h}, L), \qquad V \mapsto (\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g}, K), V))_L.$$

For an $(\mathfrak{h}, L)$-module V, the $(\mathfrak{g}, K)$-modules $(P^{\mathfrak{g},K}_{\mathfrak{h},L})_j(V)$ and $(I^{\mathfrak{g},K}_{\mathfrak{h},L})^j(V)$ are called *cohomologically induced modules.*

§3. $\mathcal{O}$-modules associated with $(\mathfrak{g}, K)$-modules

Let G be a complex linear algebraic group acting on a variety (or more generally a scheme) X. Let $a : G \times X \to X$ be the action map and $p_2 : G \times X \to X$ the second projection. Write $\mathcal{O}_X$ for the structure sheaf of X and a^*, p_2^* for the inverse image functors as $\mathcal{O}$-modules. We say that an $\mathcal{O}_X$-module $\mathcal{M}$ is *G-equivariant* if there is an isomorphism $a^*\mathcal{M} \simeq p_2^*\mathcal{M}$ satisfying the cocycle condition. For a G-equivariant $\mathcal{O}_X$-module $\mathcal{M}$, the G-action on $\mathcal{M}$ differentiates to a $\mathfrak{g}$-action on $\mathcal{M}$.

Definition 3.1. Suppose that H is a closed algebraic subgroup of G, and $X = G/H$ is the quotient variety. For an algebraic H-module V, define $\mathcal{V}_X$ as the G-equivariant quasi-coherent $\mathcal{O}_X$-module that has typical fiber V.

The category of G-equivariant quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category of algebraic H-modules, and $\mathcal{V}_X$ is the $\mathcal{O}_X$-module which corresponds to V via this equivalence. It also corresponds to the associated bundle $G \times_H V \to G/H$. The local sections of $\mathcal{V}_X$ can be identified with the V-valued regular functions f on open subsets of G satisfying $f(gh) = h^{-1} \cdot f(g)$ for $h \in H$. We often use this identification in the following.

Note that $\mathcal{V}_X$ is locally free if V is finite-dimensional. Indeed, let $v_1, \dots, v_n$ be a basis of V and take local sections $\widetilde{v}_1, \dots, \widetilde{v}_n$ such that $\widetilde{v}_i(e) = v_i$ for the identity element $e \in G$. Then the map $\mathcal{O}_X^{\oplus n} \to \mathcal{V}_X$ given by $(f_i)_i \mapsto \sum_{i=1}^n f_i \widetilde{v}_i$ is defined near the base point $eH \in G/H$ and is an isomorphism on some open neighborhood of eH.

Suppose that X is a smooth G-variety. Then the infinitesimal action is defined as a Lie algebra homomorphism from the Lie algebra $\mathfrak{g}$ of G to the space of vector fields $\mathcal{T}(X)$ on X. Denote the image of $\xi \in \mathfrak{g}$ by $\xi_X \in \mathcal{T}(X)$. Then ξ_X gives a first-order differential operator on the structure sheaf $\mathcal{O}_X$. Let $\widetilde{\mathfrak{g}}_X := \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$. This module becomes a Lie algebroid in a natural way (see [BB93, §1.2]): the Lie bracket is defined by

$$[f \otimes \xi, g \otimes \eta] = fg \otimes [\xi, \eta] + f\xi_X(g) \otimes \eta - g\eta_X(f) \otimes \xi$$

for $f, g \in \mathcal{O}_X$ and $\xi, \eta \in \mathfrak{g}$. Here $f \in \mathcal{O}_X$ means that f is a local section of $\mathcal{O}_X$. Similar notation will be used for other sheaves. Write $U(\widetilde{\mathfrak{g}}_X)$ ($\simeq \mathcal{O}_X \otimes U(\mathfrak{g})$) for the universal enveloping algebra of $\widetilde{\mathfrak{g}}_X$. Then a $U(\widetilde{\mathfrak{g}}_X)$-module is identified with an $\mathcal{O}_X$-module $\mathcal{M}$ with a $\mathfrak{g}$-action satisfying $\xi(fm) = \xi_X(f)m + f(\xi m)$ for $\xi \in \mathfrak{g}$, $f \in \mathcal{O}_X$, and $m \in \mathcal{M}$.

Let $\mathcal{T}_X$ be the tangent sheaf of X and let $p : \widetilde{\mathfrak{g}}_X$ ($= \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$) $\to \mathcal{T}_X$ be the map given by $f \otimes \xi \mapsto f\xi_X$. Then the kernel $\mathcal{H} := \ker p$ is isomorphic to the G-equivariant locally free $\mathcal{O}_X$-module with typical fiber $\mathfrak{h}$. Let $\mathcal{D}_X$ be the ring of differential operators on X. The map p extends to $p : U(\widetilde{\mathfrak{g}}_X) \to \mathcal{D}_X$ and descends to an isomorphism of algebras

$$U(\widetilde{\mathfrak{g}}_X)/U(\widetilde{\mathfrak{g}}_X)\mathcal{H} \xrightarrow{\sim} \mathcal{D}_X. \tag{3.1}$$

We will work in the following setting.

Setting 3.2. Let $i : K \to G$ be a homomorphism of complex linear algebraic groups with finite kernel. Let H be a closed algebraic subgroup of G. Put $M :=$

$i^{-1}(H)$, which is an algebraic subgroup of K, and write $X := G/H$ and $Y := K/M$ for the quotient varieties. The map $i : K \to G$ induces an injective morphism $i : Y \to X$ between the quotient varieties and an injective homomorphism $di : \mathfrak{k} \to \mathfrak{g}$ between Lie algebras. We identify $\mathfrak{k}$ with its image $di(\mathfrak{k})$ and regard $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$.

In particular, $(\mathfrak{g}, K)$ and $(\mathfrak{h}, M)$ become pairs in the sense of Definition 2.1, where $\mathfrak{h}$ is the Lie algebra of H.

Let $e \in K$ be the identity element and let $o := eM \in Y$ be the base point of Y. Write

$$\mathcal{I}_Y := \{f \in \mathcal{O}_X : f(y) = 0 \text{ for } y \in Y\}, \quad \mathcal{I}_o := \{f \in \mathcal{O}_X : f(o) = 0\},$$

so $\mathcal{I}_Y$ is the defining ideal of the closure $\overline{Y}$ of Y. It follows that $i^{-1}\mathcal{O}_X/\mathcal{I}_Y \simeq \mathcal{O}_Y$. Here i^{-1} denotes the inverse image functor for the sheaves of abelian groups. For an $i^{-1}\mathcal{O}_X$-module $\mathcal{M}$, the support of the sheaf $\mathcal{M}/(i^{-1}\mathcal{I}_o)\mathcal{M}$ is contained in $\{o\}$ so it is regarded as a vector space.

Let Y_p be the scheme $(Y, i^{-1}\mathcal{O}_X/(\mathcal{I}_Y)^p)$ for $p \geq 1$. If locally we have $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} I$, and Y is closed in X, then $Y_p = \operatorname{Spec}(A/I^p)$. The scheme Y_1 is identified with the algebraic variety Y. If $\mathcal{M}$ is an $i^{-1}\mathcal{O}_X$-module, then the sheaf $\mathcal{M}/(i^{-1}\mathcal{I}_Y)^p\mathcal{M}$ can be viewed as an $\mathcal{O}_{Y_p}$-module.

The inverse image $i^{-1}U(\widetilde{\mathfrak{g}}_X)$ of $U(\widetilde{\mathfrak{g}}_X)$ is a sheaf of algebras on Y and an $i^{-1}\mathcal{O}_X$-bimodule. We will call $i^{-1}U(\widetilde{\mathfrak{g}}_X)$-modules simply $i^{-1}\widetilde{\mathfrak{g}}_X$-modules. The K-action on $i^{-1}\widetilde{\mathfrak{g}}_X$ is given by $f \otimes \xi \mapsto (k \cdot f) \otimes \operatorname{Ad}(i(k))(\xi)$ for $f \in i^{-1}\mathcal{O}_X$, $\xi \in \mathfrak{g}$, $k \in K$. Suppose that $\mathcal{M}$ is an $i^{-1}\widetilde{\mathfrak{g}}_X$-module and let $i^{-1}\widetilde{\mathfrak{g}}_X \otimes \mathcal{M} \to \mathcal{M}$ be the action map. Then the inclusion $\mathfrak{g} \cdot (\mathcal{I}_Y)^p \subset (\mathcal{I}_Y)^{p-1}$ induces a map $i^{-1}\widetilde{\mathfrak{g}}_X \otimes \mathcal{M}/(i^{-1}\mathcal{I}_Y)^p\mathcal{M} \to \mathcal{M}/(i^{-1}\mathcal{I}_Y)^{p-1}\mathcal{M}$. The K-actions on X and Y induce a K-action on Y_p. Since Y is K-stable in X, we have $\mathfrak{k} \cdot (\mathcal{I}_Y)^p \subset (\mathcal{I}_Y)^p$. Therefore, we can define a $\mathfrak{k}$-action on $\mathcal{M}/(i^{-1}\mathcal{I}_Y)^p\mathcal{M}$. Similarly, we have $\mathfrak{h} \cdot \mathcal{I}_o \subset \mathcal{I}_o$ and we can equip $\mathcal{M}/(i^{-1}\mathcal{I}_o)\mathcal{M}$ with an $\mathfrak{h}$-module structure.

Definition 3.3. Let V be an $(\mathfrak{h}, M)$-module. We say an $i^{-1}\widetilde{\mathfrak{g}}_X$-module $\mathcal{V}$ is *associated with* V if $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ is a K-equivariant quasi-coherent $\mathcal{O}_{Y_p}$-module for all $p \geq 1$ and the following five assumptions hold.

(1) The canonical map

$$\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V} \to \mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1}\mathcal{V}$$

commutes with K-actions for $p \geq 2$.

(2) $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ is a flat $\mathcal{O}_{Y_p}$-module for $p \geq 1$.

(3) The action map $i^{-1}\widetilde{\mathfrak{g}}_X \otimes \mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V} \to \mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1}\mathcal{V}$ commutes with K-actions for $p \geq 2$. Here K acts on $i^{-1}\widetilde{\mathfrak{g}}_X \otimes \mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ diagonally.

(4) The $\mathfrak{k}$-action on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ induced from the $\mathfrak{g}$-action on $\mathcal{V}$ coincides with the differential of the K-action on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ for $p \geq 1$.

(5) There is an isomorphism $\iota : \mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V} \xrightarrow{\sim} V$ which commutes with $\mathfrak{h}$-actions and M-actions.

Remark 3.4. The $\mathfrak{g}$-action and the K-action on $\mathcal{V}$ induce an $\mathfrak{h}$-action and an M-action on $\mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V}$. The conditions (3) and (4) imply that $\mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V}$ becomes an $(\mathfrak{h}, M)$-module.

Example 3.5. Suppose that V is an H-module and define the G-equivariant quasi-coherent $\mathcal{O}_X$-module $\mathcal{V}_X$ as in Definition 3.1. The G-action on $\mathcal{V}_X$ induces a $\mathfrak{g}$-action and a K-action on $\mathcal{V}_X$. Then by regarding V as an $(\mathfrak{h}, M)$-module, $i^{-1}\mathcal{V}_X$ is associated with V.

We will construct an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with an arbitrary $(\mathfrak{h}, M)$-module in Section 5.

Example 3.6. Let $\mathcal{V}$ and $\mathcal{W}$ be $i^{-1}\widetilde{\mathfrak{g}}_X$-modules associated with $(\mathfrak{h}, M)$-modules V and W, respectively. Then the tensor product $\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{W}$ is associated with the $(\mathfrak{h}, M)$-module $V \otimes W$.

We can define the pull-back of $i^{-1}\widetilde{\mathfrak{g}}_X$-modules associated with V in the following way. Let K', G', H' be another triple of algebraic groups satisfying the assumptions in Setting 3.2. In particular, the map $i' : K' \to G'$ induces a morphism of the quotient varieties $i' : K'/M' \to G'/H'$, where $M' := (i')^{-1}(H')$. Suppose that $\varphi_K : K' \to K$ and $\varphi : G' \to G$ are homomorphisms such that the diagram

$$\begin{array}{ccc} K' & \xrightarrow{i'} & G' \\ {\scriptstyle \varphi_K}\downarrow & & \downarrow{\scriptstyle \varphi} \\ K & \xrightarrow{i} & G \end{array}$$

commutes and that $\varphi(H') \subset H$. Then $\varphi_K(M') \subset M$. The maps φ, φ_K induce morphisms $\varphi : X' := G'/H' \to X$, $\varphi_K : Y' := K'/M' \to Y$ and $\varphi_p : Y'_p := (Y', (i')^{-1}\mathcal{O}_{X'}/(\mathcal{I}_{Y'})^p) \to Y_p$. We get the commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ {\scriptstyle \varphi_K}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{i} & X \end{array}$$

Suppose that $\mathcal{V}$ is an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with an $(\mathfrak{h}, M)$-module V. Let $\mathcal{V}' := (i')^{-1}\mathcal{O}_{X'} \otimes_{(\varphi\circ i')^{-1}\mathcal{O}_X} \varphi_K^{-1}\mathcal{V}$. We define a $\mathfrak{g}'$-action on $\mathcal{V}'$ by $\xi(f \otimes v) = \xi_{X'}(f) \otimes v + f \otimes \varphi(\xi)v$ for $\xi \in \mathfrak{g}'$, $f \in (i')^{-1}\mathcal{O}_{X'}$, and $v \in \varphi_K^{-1}\mathcal{V}$ so that $\mathcal{V}'$ becomes an $(i')^{-1}\widetilde{\mathfrak{g}'}_{X'}$-module. Since

$$\mathcal{V}'/((i')^{-1}\mathcal{I}_{Y'})^p\mathcal{V}' \simeq (i')^{-1}\mathcal{O}_{X'}/(\mathcal{I}_{Y'})^p \otimes_{(\varphi\circ i')^{-1}\mathcal{O}_X} \varphi_K^{-1}\mathcal{V} \simeq \varphi_p^*(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}),$$

the sheaf $\mathcal{V}'/((i')^{-1}\mathcal{I}_{Y'})^p\mathcal{V}'$ is a K'-equivariant quasi-coherent $\mathcal{O}_{Y'_p}$-module. We can easily show the following proposition by checking the five assumptions in Definition 3.3.

Proposition 3.7. *Let V be an $(\mathfrak{h}, M)$-module and $\mathcal{V}$ an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with V. Then the $(i')^{-1}\widetilde{\mathfrak{g}'}_{X'}$-module $(i')^{-1}\mathcal{O}_{X'} \otimes_{(\varphi\circ i')^{-1}\mathcal{O}_X} \varphi_K^{-1}\mathcal{V}$ is associated with the $(\mathfrak{h}', M')$-module $\mathrm{For}_{\mathfrak{h},M}^{\mathfrak{h}',M'}(V)$.*

§4. Localization of cohomological induction

We retain Setting 3.2. In this section, we assume moreover that K is reductive. Let $M = L \ltimes U$ be a Levi decomposition of M, where L is a maximal reductive subgroup of M, and U is the unipotent radical of M. The corresponding decomposition of the Lie algebra is $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{u}$.

Let V be an $(\mathfrak{h}, M)$-module. We can view V as an $(\mathfrak{h}, L)$-module by restriction and then define the cohomologically induced module $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_j(V)$ as in Section 2.

In order to state the main theorem, we need a shift of modules by a character (or an invertible sheaf) that we will define in the following. Write $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})$ for the top exterior product of $\mathfrak{k}/\mathfrak{l}$ and view it as a one-dimensional L-module by the adjoint action. Since K and L are reductive, the identity component of L acts trivially on $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})$. We extend the L-action on $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})$ to an M-action by letting U act trivially. Define $\mathcal{L}$ as the K-equivariant locally free $\mathcal{O}_Y$-module on $Y := K/M$ whose typical fiber is isomorphic to the M-module $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})$. The K-action on $\mathcal{L}$ differentiates to a $\mathfrak{k}$-action. Then $\mathcal{L}$ becomes a $U(\widetilde{\mathfrak{k}}_Y)$-module and the kernel of the map $\widetilde{\mathfrak{k}}_Y \to \mathcal{T}_Y$ acts by zero because the identity component of M acts trivially on $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})$. Therefore, $\mathcal{L}$ has a structure of left $\mathcal{D}_Y$-module via the isomorphism (3.1) for Y.

Let $\mathcal{M}$ be a left $\mathcal{D}_Y$-module. Recall that the direct image of $\mathcal{M}$ by i in the category of left $\mathcal{D}$-modules is defined as

$$i_+\mathcal{M} := i_*((\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^\vee, \tag{4.1}$$

where i_* is the direct image functor for sheaves of abelian groups, Ω_Y is the canonical sheaf of Y, and $\Omega_X^\vee$ is the dual of the canonical sheaf of X. Via the map

$p : U(\widetilde{\mathfrak{g}}_X) \to \mathcal{D}_X$, we can view $i_+\mathcal{M}$ as a $\widetilde{\mathfrak{g}}_X$-module. The inverse image $i^{-1}i_+\mathcal{M}$ as a sheaf of abelian groups is

$$i^{-1}i_+\mathcal{M} = (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee,$$

which has an $i^{-1}\widetilde{\mathfrak{g}}_X$-module structure. We note that the functor $i^{-1}i_+$ is exact.

Define subsheaves of $\mathcal{D}_X$ by

$$F_p\mathcal{D}_X := \{D \in \mathcal{D}_X : D(\mathcal{I}_Y)^{p+1} \subset \mathcal{I}_Y\}$$

for $p \geq 0$. They are $\mathcal{O}_X$-bi-submodules of $\mathcal{D}_X$ and form a filtration of $\mathcal{D}_X$. It induces a filtration of $i^{-1}i_+\mathcal{L}$:

$$F_p i^{-1}i_+\mathcal{L} := (\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^* F_p\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee.$$

It follows from the definition of $F_p\mathcal{D}_X$ that $F_p i^{-1}i_+\mathcal{L}$ is annihilated by $(i^{-1}\mathcal{I}_Y)^{p+1}$ and hence is regarded as a quasi-coherent $\mathcal{O}_{Y_{p+1}}$-module.

Here is the main theorem of this article:

Theorem 4.1. *In Setting 3.2, assume that K is reductive. Let $M = L \ltimes U$ be a Levi decomposition. Suppose that V is an $(\mathfrak{h}, M)$-module and that $\mathcal{V}$ is an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with V (Definition 3.3). Then we have an isomorphism of $(\mathfrak{g}, K)$-modules*

$$\mathrm{H}^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \simeq (P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big)$$

for $s \in \mathbb{N}$ and $u = \dim U$. (See the remark below for the definition of the $(\mathfrak{g}, K)$-action on the left side.)

Remark 4.2. Since $i^{-1}i_+\mathcal{L}$ and $\mathcal{V}$ have $i^{-1}\widetilde{\mathfrak{g}}_X$-module structures, the tensor product $i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ becomes an $i^{-1}\widetilde{\mathfrak{g}}_X$-module. This gives a $\mathfrak{g}$-action on the cohomology group $\mathrm{H}^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$. In order to define a K-action, we use the filtration $F_p i^{-1}i_+\mathcal{L}$ defined above. By definition, $(i^{-1}\mathcal{I}_Y)^{p+1}$ annihilates $F_p i^{-1}i_+\mathcal{L}$ and hence

$$F_p i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq F_p i^{-1}i_+\mathcal{L} \otimes_{\mathcal{O}_{Y_q}} \mathcal{V}/(i^{-1}\mathcal{I}_Y)^q\mathcal{V} \tag{4.2}$$

for $p < q$. Since $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^q\mathcal{V}$ is a flat $\mathcal{O}_{Y_q}$-module by Definition 3.3(2), the map

$$F_{p-1} i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \to F_p i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$$

is injective. We let K act diagonally on the right side of (4.2). This gives a K-action on $\mathrm{H}^s(Y, F_p i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$. Using the isomorphisms

$$\mathrm{H}^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \simeq \mathrm{H}^s\big(Y, (\varinjlim_p F_p i^{-1}i_+\mathcal{L}) \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}\big)$$
$$\simeq \mathrm{H}^s\big(Y, \varinjlim_p (F_p i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})\big) \simeq \varinjlim_p \mathrm{H}^s(Y, F_p i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}),$$

we define a K-action on $\mathrm{H}^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$. With these actions, $\mathrm{H}^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$ becomes a $(\mathfrak{g}, K)$-module because of Definition 3.3(3), (4).

Proof of Theorem 4.1. Let $\widetilde{X} := G/L$ and $\widetilde{Y} := K/L$ be the quotient varieties. We have the commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\ \tilde{\imath}\ } & \widetilde{X} \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\ i\ } & X \end{array}$$

where the maps are defined canonically.

The direct image functor i_+ defined as in (4.1) induces the direct image functor between the bounded derived categories of left $\mathcal{D}$-modules, which we denote by $i_+ : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$. Similarly for $\tilde{\imath}_+$, π_+, and $(\pi_K)_+$. We have $\pi_+ \circ \tilde{\imath}_+ \simeq i_+ \circ (\pi_K)_+$. Since π_K is a smooth morphism and the fiber is isomorphic to the affine space $\mathbb{C}^u$, it follows that $(\pi_K)_+\Omega^\vee_{\widetilde{Y}} \simeq \mathcal{L}[u]$ (see [HMSW87]). Here $\mathcal{L}[u] \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$ is the complex $(\cdots \to 0 \to \mathcal{L} \to 0 \to \cdots)$, concentrated in degree $-u$. Therefore, $i_+(\pi_K)_+\Omega^\vee_{\widetilde{Y}} \simeq i_+\mathcal{L}[u]$ in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$.

Since L is reductive, the varieties $\widetilde{X}$ and $\widetilde{Y}$ are affine by Matsushima's criterion. Hence the functor $\tilde{\imath}_+$ is exact for quasi-coherent $\mathcal{D}$-modules and π_* is exact for quasi-coherent $\mathcal{O}$-modules.

Denote by $\mathcal{T}_{\widetilde{X}/X}$ the sheaf of local vector fields on $\widetilde{X}$ tangent to the fiber of π, and denote by $\Omega_{\widetilde{X}/X}$ the top exterior product of its dual $\mathcal{T}^\vee_{\widetilde{X}/X}$. We note that there is a natural isomorphism $\Omega_{\widetilde{X}/X} \simeq \Omega_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*\Omega^\vee_X$. Recall that for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\widetilde{X}})$ the direct image $\pi_+\mathcal{M}$ is defined as

$$\pi_+\mathcal{M} = \pi_*\big((\mathcal{M} \otimes_{\mathcal{O}_{\widetilde{X}}} \Omega_{\widetilde{X}}) \otimes^{\mathbb{L}}_{\mathcal{D}_{\widetilde{X}}} \pi^*\mathcal{D}_X\big) \otimes_{\mathcal{O}_X} \Omega^\vee_X.$$

The left $\mathcal{D}_{\widetilde{X}}$-module $\pi^*\mathcal{D}_X$ has the resolution (see [HMSW87, Appendix A.3.3])

$$\mathcal{D}_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \bigwedge\nolimits^{\bullet} \mathcal{T}_{\widetilde{X}/X} \to \pi^*\mathcal{D}_X, \tag{4.3}$$

where the boundary map ∂ on $\mathcal{D}_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \bigwedge^{\bullet} \mathcal{T}_{\widetilde{X}/X}$ is given as

$$D \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widetilde{\xi}_d \mapsto \sum_{i=1}^{d} (-1)^{i+1} D\widetilde{\xi}_i \otimes \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widetilde{\xi}_d$$
$$+ \sum_{1 \le i < j \le d} (-1)^{i+j} D \otimes [\widetilde{\xi}_i, \widetilde{\xi}_j] \wedge \widetilde{\xi}_1 \wedge \cdots \wedge \widehat{\widetilde{\xi}_i} \wedge \cdots \wedge \widehat{\widetilde{\xi}_j} \wedge \cdots \wedge \widetilde{\xi}_d.$$

The right $\pi^{-1}\mathcal{D}_X$-module structure is not canonically defined on the complex, but the $\mathfrak{g}$-action can be described as

$$\xi(D\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widetilde{\xi_d})=-D\xi_{\widetilde{X}}\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widetilde{\xi_d}+D\otimes\xi(\widetilde{\xi_1}\wedge\cdots\wedge\widetilde{\xi_d})$$

for $\xi\in\mathfrak{g}$. Here we use the $\mathfrak{g}$-action on $\bigwedge\mathcal{T}_{\widetilde{X}/X}$ induced from the G-equivariant structure.

By using the resolution (4.3), the direct image $\pi_+\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee$ is given as the complex

$$\pi_*\big(\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}}\otimes_{\mathcal{O}_{\widetilde{X}}}\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\big)\otimes_{\mathcal{O}_X}\Omega_X^\vee.$$

As a result, we have

$$i_+\mathcal{L}[u]\simeq\pi_*\big(\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\mathcal{O}_{\widetilde{X}}}\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X}\big)$$

and hence

$$\begin{aligned}(4.4)\qquad i^{-1}i_+\mathcal{L}[u]&\simeq i^{-1}\pi_*\big(\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\mathcal{O}_{\widetilde{X}}}\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X}\big)\\&\simeq i^{-1}\pi_*\tilde{\imath}_*\big(\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}\Omega_{\widetilde{X}/X}\big)\\&\simeq(\pi_K)_*\big(\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\big).\end{aligned}$$

There is a natural morphism of complexes of $i^{-1}\mathcal{O}_X$-modules

$$\begin{aligned}(4.5)\qquad\psi:{}&(\pi_K)_*\big(\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\big)\otimes_{i^{-1}\mathcal{O}_X}\mathcal{V}\\&\to(\pi_K)_*\big(\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}\mathcal{V}\big).\end{aligned}$$

We claim that ψ is an isomorphism. Indeed, if $F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee$ denotes the filtration of $\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee$ defined in a way similar to $F_pi^{-1}i_+\mathcal{L}$, then we get a map

$$\begin{aligned}\psi_p:{}&(\pi_K)_*\big(F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\big)\otimes_{i^{-1}\mathcal{O}_X}\mathcal{V}\\&\to(\pi_K)_*\big(F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}\mathcal{V}\big).\end{aligned}$$

It is enough to show that ψ_p is an isomorphism for all $p\geq 0$ because $\varinjlim_p F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\simeq\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee$. Since the ideal $\pi_K^{-1}(i^{-1}\mathcal{I}_Y)^{p+1}$ of $\pi_K^{-1}i^{-1}\mathcal{O}_X$ annihilates $F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee$, we have

$$\begin{aligned}&(\pi_K)_*\big(F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\big)\otimes_{i^{-1}\mathcal{O}_X}\mathcal{V}\\&\simeq(\pi_K)_*\big(F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\big)\otimes_{\mathcal{O}_{Y_{p+1}}}(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p+1}\mathcal{V}).\end{aligned}$$

By Definition 3.3(2), $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p+1}\mathcal{V}$ is a flat $\mathcal{O}_{Y_{p+1}}$-module. Hence the projection formula shows that ψ_p is an isomorphism and the claim is verified.

The successive quotient of the filtration

$$F_p\mathcal{M}:=F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^d\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}\mathcal{V}$$

is

$$(F_p\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee/F_{p-1}\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee)\otimes_{\mathcal{O}_{\widetilde{Y}}}\tilde{\imath}^*(\textstyle\bigwedge^d\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\otimes_{\mathcal{O}_{\widetilde{Y}}}\pi_K^*(\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}),$$

which is a quasi-coherent $\mathcal{O}_{\widetilde{Y}}$-module. Since $\widetilde{Y}$ is affine, $\mathrm{H}^s(\widetilde{Y},F_p\mathcal{M}/F_{p-1}\mathcal{M})=0$ for $s>0$. Hence $\mathrm{H}^s(\widetilde{Y},F_p\mathcal{M})=0$ and

$$\mathrm{H}^s\big(\widetilde{Y},\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^d\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}\mathcal{V}\big)=0$$

for $s>0$. By (4.4) and (4.5), we conclude that

$$\begin{aligned}&\mathrm{H}^s(Y,i^{-1}i_+\mathcal{L}\otimes_{i^{-1}\mathcal{O}_X}\mathcal{V})\\&\quad\simeq\mathrm{H}^{s-u}\Gamma\big(\widetilde{Y},\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}\mathcal{V}\big).\end{aligned}$$

Since $\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}\Omega_{\widetilde{X}}\simeq\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{D}_{\widetilde{Y}}}\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}$, we have

$$\begin{aligned}&\tilde{\imath}^{-1}\tilde{\imath}_+\Omega_{\widetilde{Y}}^\vee\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}(\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\mathcal{O}_{\widetilde{X}}}\Omega_{\widetilde{X}/X})\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}\mathcal{V}\\&\quad\simeq\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{D}_{\widetilde{Y}}}\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\tilde{\imath}^{-1}\textstyle\bigwedge^\bullet\mathcal{T}_{\widetilde{X}/X}\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}(\mathcal{V}\otimes_{i^{-1}\mathcal{O}_X}i^{-1}\Omega_X^\vee).\end{aligned}$$

If we put

$$\mathcal{V}^{-d}:=\tilde{\imath}^{-1}\textstyle\bigwedge^d\mathcal{T}_{\widetilde{X}/X}\otimes_{\pi_K^{-1}i^{-1}\mathcal{O}_X}\pi_K^{-1}(\mathcal{V}\otimes_{i^{-1}\mathcal{O}_X}i^{-1}\Omega_X^\vee),$$

then we obtain

$$\mathrm{H}^s(Y,i^{-1}i_+\mathcal{L}\otimes_{i^{-1}\mathcal{O}_X}\mathcal{V})\simeq\mathrm{H}^{s-u}\Gamma\big(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{D}_{\widetilde{Y}}}\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\mathcal{V}^\bullet\big).\tag{4.6}$$

The boundary map

$$\partial:\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{D}_{\widetilde{Y}}}\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\mathcal{V}^{-d}\to\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{D}_{\widetilde{Y}}}\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\mathcal{V}^{-d+1}$$

is given by

$$\begin{aligned}&f\otimes D\otimes\widetilde{\xi}_1\wedge\cdots\wedge\widetilde{\xi}_d\otimes v\\&\quad\mapsto\sum_{i=1}^d(-1)^{i+1}f\otimes D\widetilde{\xi}_i\otimes\widetilde{\xi}_1\wedge\cdots\wedge\widehat{\widetilde{\xi}_i}\wedge\cdots\wedge\widetilde{\xi}_d\otimes v\\&\qquad+\sum_{1\le i<j\le d}(-1)^{i+j}f\otimes D\otimes[\widetilde{\xi}_i,\widetilde{\xi}_j]\wedge\widetilde{\xi}_1\wedge\cdots\wedge\widehat{\widetilde{\xi}_i}\wedge\cdots\wedge\widehat{\widetilde{\xi}_j}\wedge\cdots\wedge\widetilde{\xi}_d\otimes v,\end{aligned}$$

where $f\in\mathcal{O}_{\widetilde{Y}}$, $D\in\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}$, $\widetilde{\xi}_1,\ldots,\widetilde{\xi}_d\in\tilde{\imath}^{-1}\mathcal{T}_{\widetilde{X}/X}$, and $v\in\pi_K^{-1}(\mathcal{V}\otimes_{i^{-1}\mathcal{O}_X}i^{-1}\Omega_X^\vee)$.

The right side of (4.6) can be computed by using the following lemma.

Lemma 4.3. *Let V' be an L-module, or equivalently an $(\mathfrak{l},L)$-module. Let $\mathcal{V}'$ be an $\tilde{\imath}^{-1}\widetilde{\mathfrak{g}}_{\widetilde{X}}$-module associated with V'. Then*

$$\Gamma(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}\otimes_{\mathcal{D}_{\widetilde{Y}}}\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\mathcal{V}')\simeq R(\mathfrak{g},K)\otimes_{R(L)}V'.$$

Proof. The proof is similar to that of [Osh11, Lemma 3.4].

Using the right $\tilde{\imath}^{-1}\mathcal{D}_{\widetilde{X}}$-module structure of $\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}$, we can define a $\mathfrak{g}$-action ρ on the sheaf $\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}'$ by

$$\rho(\xi)(D \otimes v) := -D\xi_{\widetilde{X}} \otimes v + D \otimes \xi v$$

for $\xi \in \mathfrak{g}$, $D \in \tilde{\imath}^*\mathcal{D}_{\widetilde{X}}$, and $v \in \mathcal{V}'$. Moreover, the sheaf $\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}'$ is K-equivariant. We denote this K-action and also its infinitesimal $\mathfrak{k}$-action by ν. Definition 3.3(4) implies that the $\mathfrak{k}$-action ν is given by

$$\nu(\eta)(D \otimes v) = \eta_{\widetilde{Y}} D \otimes v - D\eta_{\widetilde{X}} \otimes v + D \otimes \eta v$$

for $\eta \in \mathfrak{k}$. Here, $\eta_{\widetilde{Y}} D$ and $D\eta_{\widetilde{X}}$ are defined by the $(\mathcal{D}_{\widetilde{Y}}, \tilde{\imath}^{-1}\mathcal{D}_{\widetilde{X}})$-bimodule structure on $\tilde{\imath}^*\mathcal{D}_{\widetilde{X}}$. Then it follows from Definition 3.3(3) that $\Gamma(\widetilde{Y}, \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}')$ is a weak $(\mathfrak{g}, K)$-module in the sense of [BL95], namely,

$$\nu(k)\rho(\xi)\nu(k^{-1}) = \rho(\mathrm{Ad}(i(k))\xi)$$

for $k \in K$ and $\xi \in \mathfrak{g}$. Put $\omega(\eta) := \nu(\eta) - \rho(\eta)$ for $\eta \in \mathfrak{k}$. Then $\omega(\eta)$ is given by

$$\omega(\eta)(D \otimes v) = \eta_{\widetilde{Y}} D \otimes v.$$

Since $\widetilde{Y}$ is an affine variety, $\Gamma(\widetilde{Y}, \mathcal{D}_{\widetilde{Y}})$ is generated by $U(\mathfrak{k})$ and $\mathcal{O}(\widetilde{Y})$ as an algebra. Therefore,

$$\begin{aligned}\Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}} \otimes_{\mathcal{D}_{\widetilde{Y}}} \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}') &\simeq \mathcal{O}(\widetilde{Y}) \otimes_{\Gamma(\widetilde{Y}, \mathcal{D}_{\widetilde{Y}})} \Gamma(\widetilde{Y}, \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}') \\ &\simeq \Gamma(\widetilde{Y}, \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}') \,/\, \omega(\mathfrak{k})\Gamma(\widetilde{Y}, \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}').\end{aligned}$$

Let $e \in K$ be the identity element. Write $o := eL \in \widetilde{Y}$ for the base point and $i_o : \{o\} \to \widetilde{Y}$ for the inclusion map. Let $\mathcal{I}_o$ be the maximal ideal of $\mathcal{O}_{\widetilde{Y}}$ corresponding to o. The fiber of $\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}'$ at o is given by

$$\begin{aligned}W &:= i_o^*(\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}') \\ &\simeq \Gamma(\widetilde{Y}, \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}') \,/\, \mathcal{I}_o(\widetilde{Y})\Gamma(\widetilde{Y}, \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}').\end{aligned}$$

The actions ρ and ν on $\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}'$ induce a $\mathfrak{g}$-action and an L-action on W. With these actions, W becomes a $(\mathfrak{g}, L)$-module and there is an isomorphism

$$\varphi : U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V' \xrightarrow{\sim} W. \tag{4.7}$$

This can be proved by using [Osh11, Lemma 3.3] and Definition 3.3 (see the proof of [Osh11, Lemma 3.4]). Hence we have

$$\Gamma(\widetilde{Y}, \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}') \simeq R(K) \otimes_{R(L)} (U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V').$$

The rest is the same as in [Osh11, Lemma 3.4]. ∎

Returning to the proof of Theorem 4.1, let us compute the cohomological induction $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_s(V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h}))$ by using the standard resolution ([KV95, §II.7]). The standard resolution is a projective resolution of the $(\mathfrak{h}, L)$-module $V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$ given by the complex

$$U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} \big(\textstyle\bigwedge^{\bullet}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big),$$

where the boundary map

$$\partial' : U(\mathfrak{h})\otimes_{U(\mathfrak{l})} \big(\textstyle\bigwedge^{d}(\mathfrak{h}/\mathfrak{l})\otimes V\otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \to U(\mathfrak{h})\otimes_{U(\mathfrak{l})} \big(\bigwedge^{d-1}(\mathfrak{h}/\mathfrak{l})\otimes V\otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big)$$

is

$$\begin{aligned} &D \otimes \overline{\xi_1} \wedge \cdots \wedge \overline{\xi_d} \otimes v \\ &\mapsto \sum_{i=1}^{d} (-1)^{i+1} (D\xi_i \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes v - D \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes \xi_i v) \\ &\quad + \sum_{1 \le i < j \le d} (-1)^{i+j} D \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \widehat{\overline{\xi_j}} \wedge \cdots \wedge \overline{\xi_d} \otimes v \end{aligned}$$

for $D \in U(\mathfrak{h})$, $\xi_1, \ldots, \xi_d \in \mathfrak{h}$, and $v \in V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$. Therefore,

$$\begin{aligned} (4.8) \qquad & (P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \\ &\simeq \mathrm{H}^{s-u} P_{\mathfrak{h},L}^{\mathfrak{g},K} \big(U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} \big(\textstyle\bigwedge^{\bullet}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big)\big) \\ &\simeq \mathrm{H}^{s-u} R(\mathfrak{g}, K) \otimes_{R(L)} \big(\textstyle\bigwedge^{\bullet}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big), \end{aligned}$$

where the boundary map

$$\begin{aligned} \partial' : R(\mathfrak{g}, K) \otimes_{R(L)} &\big(\textstyle\bigwedge^{d}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \\ &\to R(\mathfrak{g}, K) \otimes_{R(L)} \big(\textstyle\bigwedge^{d-1}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \end{aligned}$$

is given by

$$\begin{aligned} &D \otimes \overline{\xi_1} \wedge \cdots \wedge \overline{\xi_d} \otimes v \\ &\mapsto \sum_{i=1}^{d} (-1)^{i+1} (D\xi_i \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes v - D \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes \xi_i v) \\ &\quad + \sum_{1 \le i < j \le d} (-1)^{i+j} D \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \widehat{\overline{\xi_j}} \wedge \cdots \wedge \overline{\xi_d} \otimes v \end{aligned}$$

for $D \in R(\mathfrak{g}, K)$, $\xi_1, \ldots, \xi_d \in \mathfrak{h}$, and $v \in V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$.

Put

$$V^{-d} := \textstyle\bigwedge^{d}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$$

for simplicity. We identify the fiber of $\mathcal{T}_{\widetilde{X}/X}$ with $\mathfrak{h}/\mathfrak{l}$ in the following way: if a vector field $\widetilde{\xi} \in \mathcal{T}_{\widetilde{X}/X}$ equals $-\xi_{\widetilde{X}}$ at the base point $eL \in \widetilde{X}$ for $\xi \in \mathfrak{h}$, then $\widetilde{\xi}$ takes the value $\overline{\xi} \in \mathfrak{h}/\mathfrak{l}$ at $e \in G$. Similarly, the fiber of $\Omega^{\vee}_{\widetilde{X}/X}$ is identified with $\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$. Then $\mathcal{V}^{-d}$ is associated with V^{-d} by Examples 3.5 and 3.6. From (4.6) and (4.8) it is enough to show that the isomorphisms φ given in Lemma 4.3 for $V' = V^{-d}$, $0 \le d \le \dim(\mathfrak{h}/\mathfrak{l})$, commute with the boundary maps, that is, the diagram

$$\begin{array}{ccc}
R(\mathfrak{g},K) \otimes_{R(L)} V^{-d} & \xrightarrow{\partial'} & R(\mathfrak{g},K) \otimes_{R(L)} V^{-d+1} \\
\downarrow & & \downarrow \\
\Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}} \otimes_{\mathcal{D}_{\widetilde{Y}}} \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}^{-d}) & \xrightarrow{\partial} & \Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}} \otimes_{\mathcal{D}_{\widetilde{Y}}} \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}^{-d+1})
\end{array}$$

commutes. In view of the proof of Lemma 4.3, the above diagram is obtained by applying the functor $P^{\mathfrak{g},K}_{\mathfrak{g},L}$ to

$$\begin{array}{ccc}
U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d} & \xrightarrow{\partial'} & U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d+1} \\
\downarrow{\scriptstyle \varphi^d} & & \downarrow{\scriptstyle \varphi^{d-1}} \\
i_o^*(\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}^{-d}) & \xrightarrow{\partial} & i_o^*(\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}^{-d+1})
\end{array} \tag{4.9}$$

where φ^d is the map φ of (4.7) for $V' = V^{-d}$. Therefore, it suffices to show that the diagram (4.9) commutes.

To see this, we use the following notation. A section $f \in \tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}^{-d}$ defines a section of $i_o^*(\tilde{\imath}^*\mathcal{D}_{\widetilde{X}} \otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}} \mathcal{V}^{-d})$ and hence defines an element of $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d}$ via the isomorphism φ^d. We write $i_o^* f \in U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d}$ for this element. Put $Z := H/L$ and write $i_Z : Z \to \widetilde{X}$ for the inclusion map. Then $i_Z(Z) = \pi^{-1}(\{o\})$ and there is a canonical isomorphism $i_Z^*\mathcal{T}_{\widetilde{X}/X} \simeq \mathcal{T}_Z$. For $\xi_1, \dots, \xi_d \in \mathfrak{h}$ and $v \in V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$, put

$$m := \overline{\xi_1} \wedge \cdots \wedge \overline{\xi_d} \otimes v \ \in \ V^{-d}.$$

We will choose sections $\widetilde{\xi}_i \in \tilde{\imath}^{-1}\mathcal{T}_{\widetilde{X}/X}$ and $\widetilde{v} \in \pi_K^{-1}(\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^{\vee})$ on a neighborhood of the base point $o \in \widetilde{Y}$ in the following way. Take $\widetilde{\xi}_i \in \mathcal{T}_{\widetilde{X}/X}$ such that $\widetilde{\xi}_i|_Z \in i_Z^*\mathcal{T}_{\widetilde{X}/X}$ corresponds to $-(\xi_i)_Z$. It gives a section of $\tilde{\imath}^{-1}\mathcal{T}_{\widetilde{X}/X}$, which we denote by the same letter $\widetilde{\xi}_i$. We take a section $\widetilde{v} \in \pi_K^{-1}(\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^{\vee})$ on a neighborhood of o such that $i_o^*\widetilde{v}$ corresponds to v. Define a section $\widetilde{m} \in \mathcal{V}^{-d}$ in a neighborhood of o as

$$\widetilde{m} := \widetilde{\xi}_1 \wedge \cdots \wedge \widetilde{\xi}_d \otimes \widetilde{v} \in \mathcal{V}^{-d}.$$

Then the element $\varphi^d(1\otimes m)$ is represented by the section

$$1\otimes \widetilde{m}\in \tilde{\imath}^*\mathcal{D}_{\widetilde{X}}\otimes_{\tilde{\imath}^{-1}\mathcal{O}_{\widetilde{X}}}\mathcal{V}^{-d},$$

in other words, $i_o^*(1\otimes\widetilde{m})=1\otimes m$.

We have

$$\begin{aligned}\partial(1\otimes\widetilde{m}) = &\sum_{i=1}^{d}(-1)^{i+1}(\widetilde{\xi_i}\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widehat{\widetilde{\xi_i}}\wedge\cdots\wedge\widetilde{\xi_d}\otimes\widetilde{v})\\ &+\sum_{1\le i<j\le d}(-1)^{i+j}\,(1\otimes[\widetilde{\xi_i},\widetilde{\xi_j}]\wedge\widetilde{\xi_1}\wedge\cdots\wedge\widehat{\widetilde{\xi_i}}\wedge\cdots\wedge\widehat{\widetilde{\xi_j}}\wedge\cdots\wedge\widetilde{\xi_d}\otimes\widetilde{v})\end{aligned}$$

and

$$\begin{aligned}&\partial'(1\otimes m)\\ &=\sum_{i=1}^{d}(-1)^{i+1}(\xi_i\otimes\overline{\xi_1}\wedge\cdots\wedge\widehat{\overline{\xi_i}}\wedge\cdots\wedge\overline{\xi_d}\otimes v-1\otimes\overline{\xi_1}\wedge\cdots\wedge\widehat{\overline{\xi_i}}\wedge\cdots\wedge\overline{\xi_d}\otimes\xi_i v)\\ &\quad+\sum_{1\le i<j\le d}(-1)^{i+j}\,(1\otimes\overline{[\xi_i,\xi_j]}\wedge\overline{\xi_1}\wedge\cdots\wedge\widehat{\overline{\xi_i}}\wedge\cdots\wedge\widehat{\overline{\xi_j}}\wedge\cdots\wedge\overline{\xi_d}\otimes v).\end{aligned}$$

Since $\widetilde{\xi_i}|_Z$ corresponds to $-(\xi_i)_Z$, the vector fields $\widetilde{\xi_i}$ and $(\xi_i)_{\widetilde{X}}$ satisfy the relation $\widetilde{\xi_i}=-(\xi_i)_{\widetilde{X}}$ at o. Recall that the $\mathfrak{g}$-action on $\mathcal{T}_{\widetilde{X}/X}$ is defined as the differential of the G-equivariant structure on it. Hence our choice implies that $\xi_i\cdot\widetilde{\xi_j}|_Z=-([\xi_i,\xi_j])_Z$. As a result,

$$\begin{aligned}&i_o^*(\widetilde{\xi_i}\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widehat{\widetilde{\xi_i}}\wedge\cdots\wedge\widetilde{\xi_d}\otimes\widetilde{v})\\ &=i_o^*\big(\rho(\xi_i)(1\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widehat{\widetilde{\xi_i}}\wedge\cdots\wedge\widetilde{\xi_d}\otimes\widetilde{v})\big)-i_o^*(1\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widehat{\widetilde{\xi_i}}\wedge\cdots\wedge\widetilde{\xi_d}\otimes\xi_i\widetilde{v})\\ &\quad-\sum_{1\le i<j\le d}i_o^*(1\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widehat{\widetilde{\xi_i}}\wedge\cdots\wedge\widetilde{\xi_{j-1}}\wedge(\xi_i\cdot\widetilde{\xi_j})\wedge\widetilde{\xi_{j+1}}\wedge\cdots\wedge\widetilde{\xi_d}\otimes\widetilde{v})\\ &\quad-\sum_{1\le j<i\le d}i_o^*(1\otimes\widetilde{\xi_1}\wedge\cdots\wedge\widetilde{\xi_{j-1}}\wedge(\xi_i\cdot\widetilde{\xi_j})\wedge\widetilde{\xi_{j+1}}\wedge\cdots\wedge\widehat{\widetilde{\xi_i}}\wedge\cdots\wedge\widetilde{\xi_d}\otimes\widetilde{v})\\ &=\xi_i\otimes\overline{\xi_1}\wedge\cdots\wedge\widehat{\overline{\xi_i}}\wedge\cdots\wedge\overline{\xi_d}\otimes v-1\otimes\overline{\xi_1}\wedge\cdots\wedge\widehat{\overline{\xi_i}}\wedge\cdots\wedge\overline{\xi_d}\otimes\xi_i v\\ &\quad+\sum_{1\le i<j\le d}(-1)^{j+1}(1\otimes\overline{[\xi_i,\xi_j]}\wedge\overline{\xi_1}\wedge\cdots\wedge\widehat{\overline{\xi_i}}\wedge\cdots\wedge\widehat{\overline{\xi_j}}\wedge\cdots\wedge\overline{\xi_d}\otimes v)\\ &\quad+\sum_{1\le j<i\le d}(-1)^{j}(1\otimes\overline{[\xi_i,\xi_j]}\wedge\overline{\xi_1}\wedge\cdots\wedge\widehat{\overline{\xi_j}}\wedge\cdots\wedge\widehat{\overline{\xi_i}}\wedge\cdots\wedge\overline{\xi_d}\otimes v).\end{aligned}$$

Moreover, $[\widetilde{\xi_i}, \widetilde{\xi_j}]|_Z$ corresponds to $[-(\xi_i)_Z, -(\xi_j)_Z] = ([\xi_i, \xi_j])_Z$. Hence

$$i_o^*(1 \otimes [\widetilde{\xi_i}, \widetilde{\xi_j}] \wedge \widetilde{\xi_1} \wedge \cdots \wedge \widehat{\widetilde{\xi_i}} \wedge \cdots \wedge \widehat{\widetilde{\xi_j}} \wedge \cdots \wedge \widetilde{\xi_d} \otimes \widetilde{v}) \\ = -1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \widehat{\overline{\xi_j}} \wedge \cdots \wedge \overline{\xi_d} \otimes v.$$

We thus conclude that

$$\begin{aligned}
&(\varphi^{d-1})^{-1} \circ \partial \circ \varphi^d (1 \otimes m) = i_o^*(\partial(1 \otimes \widetilde{m})) \\
&= i_o^*\Big(\sum_{i=1}^d (-1)^{i+1} (\widetilde{\xi_i} \otimes \widetilde{\xi_1} \wedge \cdots \wedge \widehat{\widetilde{\xi_i}} \wedge \cdots \wedge \widetilde{\xi_d} \otimes \widetilde{v}) \\
&\quad + \sum_{1 \le i < j \le d} (-1)^{i+j} (1 \otimes [\widetilde{\xi_i}, \widetilde{\xi_j}] \wedge \widetilde{\xi_1} \wedge \cdots \wedge \widehat{\widetilde{\xi_i}} \wedge \cdots \wedge \widehat{\widetilde{\xi_j}} \wedge \cdots \wedge \widetilde{\xi_d} \otimes \widetilde{v})\Big) \\
&= \sum_{i=1}^d (-1)^{i+1} \Big(\xi_i \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes v - 1 \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes \xi_i v \\
&\quad + \sum_{1 \le i < j \le d} (-1)^{j+1} (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \widehat{\overline{\xi_j}} \wedge \cdots \wedge \overline{\xi_d} \otimes v) \\
&\quad + \sum_{1 \le j < i \le d} (-1)^{j} (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_j}} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes v)\Big) \\
&\quad + \sum_{1 \le i < j \le d} (-1)^{i+j+1} (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \widehat{\overline{\xi_j}} \wedge \cdots \wedge \overline{\xi_d} \otimes v) \\
&= \sum_{i=1}^d (-1)^{i+1} (\xi_i \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes v - 1 \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \overline{\xi_d} \otimes \xi_i v) \\
&\quad + \sum_{1 \le i < j \le d} (-1)^{i+j} (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\overline{\xi_i}} \wedge \cdots \wedge \widehat{\overline{\xi_j}} \wedge \cdots \wedge \overline{\xi_d} \otimes v) \\
&= \partial'(1 \otimes m).
\end{aligned}$$

Since ∂, ∂' and φ^d commute with $\mathfrak{g}$-actions,

$$\partial(\varphi^d(D \otimes m)) = D\partial(\varphi^d(1 \otimes m)) = D\varphi^{d-1}(\partial'(1 \otimes m)) = \varphi^{d-1}(\partial'(D \otimes m))$$

for $D \in U(\mathfrak{g})$. Consequently, the diagram (4.9) commutes and the proof of the theorem is complete. □

§5. Construction of modules

In this section, we will construct an $i^{-1}\widetilde{\mathfrak{g}}_X$-module $\mathcal{V}$ associated with an $(\mathfrak{h}, M)$-module V, which can be used in Section 4 for the realization of cohomologically induced modules.

Let $\mathcal{V}_Y$ be the K-equivariant quasi-coherent $\mathcal{O}_Y$-module with typical fiber the M-module V. Let $p : \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g} \to \mathcal{T}_X$ be the map given by $f \otimes \xi \mapsto f\xi_X$ and put $\mathcal{H} := \ker p$. The $\mathcal{O}_X$-module $\mathcal{H}$ is G-equivariant with typical fiber $\mathfrak{h}$. Hence a section $\xi \in \mathcal{H}$ is identified with an $\mathfrak{h}$-valued regular function on a subset of G satisfying $\xi(gh) = \mathrm{Ad}(h^{-1})(\xi(g))$ for $h \in H$. Let $\xi, \xi' \in \mathcal{H}$. By regarding $\widetilde{\mathfrak{g}}_X = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$ as a submodule of $U(\widetilde{\mathfrak{g}}_X) = \mathcal{O}_X \otimes_{\mathbb{C}} U(\mathfrak{g})$, we have $[\xi, \xi'] := \xi\xi' - \xi'\xi \in \mathcal{H}$ and $[\xi, \xi'](g) = [\xi(g), \xi'(g)]$ with the identification above. If we write $\xi = \sum_i f_i \otimes \xi_i$ for $f_i \in \mathcal{O}_X$ and $\xi_i \in \mathfrak{g}$, then $\xi(g) = \sum_i f_i(g)\,\mathrm{Ad}(g^{-1})(\xi_i)$.

Let $\mathcal{A}$ be the subalgebra of $i^{-1}U(\widetilde{\mathfrak{g}}_X) = i^{-1}\mathcal{O}_X \otimes U(\mathfrak{g})$ generated by $i^{-1}\mathcal{H}$, $1 \otimes \mathfrak{k}$, and $i^{-1}\mathcal{O}_X \otimes 1$. We view $i^{-1}U(\widetilde{\mathfrak{g}}_X)$ as an $i^{-1}\mathcal{O}_X$-module and consider the inverse image $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)$ ($\simeq \mathcal{O}_Y \otimes U(\mathfrak{g})$) of $U(\widetilde{\mathfrak{g}}_X)$. Let $\overline{\mathcal{A}}$ be the image of the map $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{A} \to \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)$ so that $\overline{\mathcal{A}} \simeq \mathcal{A}/(\mathcal{A} \cap (i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})))$. Since $\mathcal{A} \cdot (i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})) \subset i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})$ in the algebra $i^{-1}U(\widetilde{\mathfrak{g}}_X)$, the algebra structure of $\mathcal{A}$ induces that of $\overline{\mathcal{A}}$, and $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)$ becomes a left $\overline{\mathcal{A}}$-module.

We give a left $\overline{\mathcal{A}}$-module structure on $\mathcal{V}_Y$ in the following way. We view a local section of $\mathcal{V}_Y$ as a V-valued regular function on a subset of K and define a $(1 \otimes i^{-1}\mathcal{H})$-action and an $(\mathcal{O}_Y \otimes 1)$-action by

$$((1 \otimes \xi)v)(k) = \xi(i(k))v(k), \qquad (f \otimes 1)v = fv$$

for $\xi \in i^{-1}\mathcal{H}$, $v \in \mathcal{V}_Y$, $f \in \mathcal{O}_Y$, and $k \in K$; define a $(1 \otimes \mathfrak{k})$-action on $\mathcal{V}_Y$ by differentiating the K-action on $\mathcal{V}_Y$. These actions are compatible in the following sense: if $f_i \in i^{-1}\mathcal{O}_X$, $\eta_i \in \mathfrak{k}$ and $\xi \in i^{-1}\mathcal{H}$ satisfy

$$\sum_i (f_i \otimes \eta_i) - \xi \in i^{-1}\mathcal{I}_Y \otimes \mathfrak{g},$$

then we have

$$\sum_i (f_i|_Y \otimes 1)((1 \otimes \eta_i)v) = (1 \otimes \xi)v \tag{5.1}$$

for $v \in \mathcal{V}_Y$. In the proposition below, we will see that these actions give a well-defined $\overline{\mathcal{A}}$-module structure.

Let $\mathcal{V} := \mathcal{H}om_{\overline{\mathcal{A}}}(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X), \mathcal{V}_Y)$, namely, $\mathcal{V}$ consists of the sections $v \in \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X), \mathcal{V}_Y)$ satisfying

$$\begin{aligned} &v((1 \otimes \xi)(f \otimes D)) = (1 \otimes \xi)(v(f \otimes D)), \\ &v((1 \otimes \eta)(f \otimes D)) = (1 \otimes \eta)(v(f \otimes D)), \text{ and} \\ &v(f'f \otimes D) = (f' \otimes 1)(v(f \otimes D)) \end{aligned}$$

for $f, f' \in \mathcal{O}_Y$, $D \in U(\mathfrak{g})$, $\eta \in \mathfrak{k}$, and $\xi \in i^{-1}\mathcal{H}$. We endow $\mathcal{V}$ with an $i^{-1}\widetilde{\mathfrak{g}}_X$-module structure by defining $(f \otimes D) \cdot v$ as

$$((f \otimes D) \cdot v)(f' \otimes D') = v(f' \otimes (1 \otimes D')(f \otimes D))$$

for $v \in \mathcal{V}$, $f \in i^{-1}\mathcal{O}_X$, $f' \in \mathcal{O}_Y$, and $D, D' \in U(\mathfrak{g})$.

Proposition 5.1. *Let V be an $(\mathfrak{h}, M)$-module. Then the left $\overline{\mathcal{A}}$-action on $\mathcal{V}_Y$ given above is well-defined, and the $i^{-1}\widetilde{\mathfrak{g}}_X$-module*

$$\mathcal{V} := \mathcal{H}om_{\overline{\mathcal{A}}}(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} U(\widetilde{\mathfrak{g}}_X),\, \mathcal{V}_Y)$$

is associated with V in the sense of Definition 3.3.

Proof. Let $k_0 \in K$ and $y_0 := k_0 M \in Y$. We fix a trivialization near y_0 in the following way. Take sections $\xi_1, \dots, \xi_n \in i^{-1}\mathcal{H}$ on a neighborhood U of y_0 in Y such that the map

$$(i^{-1}\mathcal{O}_X)^{\oplus n}|_U \to (i^{-1}\mathcal{H})|_U, \qquad (f_1, \dots, f_n) \mapsto \sum_{i=1}^n f_i \xi_i,$$

is an isomorphism. Take elements $\eta_1, \dots, \eta_m \in \mathfrak{k}$ that form a basis of the quotient space $\mathfrak{k}/\mathrm{Ad}(k_0)(\mathfrak{m})$ and take $\zeta_1, \dots, \zeta_l \in \mathfrak{g}$ such that $\eta_1, \dots, \eta_m, \zeta_1, \dots, \zeta_l$ form a basis of the quotient space $\mathfrak{g}/\mathrm{Ad}(i(k_0))\mathfrak{h}$. Modifying U if necessary, we get an isomorphism

$$(5.2) \qquad (i^{-1}\mathcal{O}_X)^{\oplus n+m+l}|_U \to (i^{-1}\mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g})|_U,$$
$$(f_1, \dots, f_n, g_1, \dots, g_m, h_1, \dots, h_l) \mapsto \sum_{i=1}^n f_i \xi_i + \sum_{i=1}^m (g_i \otimes \eta_i) + \sum_{i=1}^l (h_i \otimes \zeta_i).$$

For integers $s, t \geq 0$, let

$$I_{s,t} := \{\boldsymbol{i} = (i(1), \dots, i(s)) : 1 \leq i(1) \leq \dots \leq i(s) \leq t\}, \qquad I_t := \coprod_{s=0}^\infty I_{s,t}.$$

If $s = 0$, the set $I_{0,t}$ consists of one element $()$. For $\boldsymbol{i} = (i(1), \dots, i(s)) \in I_{s,l}$, we put $\zeta_{\boldsymbol{i}} := 1 \otimes \zeta_{i(1)} \cdots \zeta_{i(s)} \in i^{-1}\mathcal{O}_X \otimes U(\mathfrak{g})$. If $s = 0$ and $\boldsymbol{i} = ()$ then put $\zeta_{\boldsymbol{i}} := 1 \otimes 1$. In the same way, for $\boldsymbol{i'} = (i'(1), \dots, i'(s)) \in I_{s,n}$ and $\boldsymbol{i''} = (i''(1), \dots, i''(s)) \in I_{s,m}$, put $\xi_{\boldsymbol{i'}} := \xi_{i'(1)} \cdots \xi_{i'(s)}$ and $\eta_{\boldsymbol{i''}} := 1 \otimes \eta_{i''(1)} \cdots \eta_{i''(s)}$. From the isomorphism (5.2) and the Poincaré–Birkhoff–Witt theorem, we see that a section of $i^{-1}U(\widetilde{\mathfrak{g}}_X)|_U$ is uniquely written as

$$\sum_{\boldsymbol{i} \in I_l,\, \boldsymbol{i'} \in I_n,\, \boldsymbol{i''} \in I_m} f_{\boldsymbol{i},\boldsymbol{i'},\boldsymbol{i''}} \xi_{\boldsymbol{i'}} \eta_{\boldsymbol{i''}} \zeta_{\boldsymbol{i}},$$

where $f_{\boldsymbol{i},\boldsymbol{i}',\boldsymbol{i}''} \in i^{-1}\mathcal{O}_X$, and $f_{\boldsymbol{i},\boldsymbol{i}',\boldsymbol{i}''} = 0$ except for finitely many $(\boldsymbol{i},\boldsymbol{i}',\boldsymbol{i}'')$. Hence a section of $(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X))|_U$ is uniquely written as a finite sum $\sum_{\boldsymbol{i},\boldsymbol{i}',\boldsymbol{i}''} f_{\boldsymbol{i},\boldsymbol{i}',\boldsymbol{i}''}\xi_{\boldsymbol{i}'}\eta_{\boldsymbol{i}''}\zeta_{\boldsymbol{i}}$ for $f_{\boldsymbol{i},\boldsymbol{i}',\boldsymbol{i}''} \in \mathcal{O}_Y$.

Lemma 5.2. *The subsheaf* $\overline{\mathcal{A}}|_U$ *of* $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)$ *consists of the sections written as a finite sum*

$$\sum_{\boldsymbol{i}'\in I_n,\, \boldsymbol{i}''\in I_m} f_{\boldsymbol{i}',\boldsymbol{i}''} \otimes \xi_{\boldsymbol{i}'}\eta_{\boldsymbol{i}''}$$

for $f_{\boldsymbol{i}',\boldsymbol{i}''} \in \mathcal{O}_Y$.

Proof. It is enough to prove that for any section $a \in \mathcal{A}|_U$ there exist functions $f_{\boldsymbol{i}',\boldsymbol{i}''} \in i^{-1}\mathcal{O}_X$ such that

$$a - \sum_{\boldsymbol{i}',\boldsymbol{i}''} f_{\boldsymbol{i}',\boldsymbol{i}''}\xi_{\boldsymbol{i}'}\eta_{\boldsymbol{i}''} \in i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g}). \tag{5.3}$$

For this we consider relations in the algebra $i^{-1}U(\widetilde{\mathfrak{g}}_X)$. By our choice of $\xi_1,\dots,\xi_n$ and $\eta_1,\dots,\eta_m$, we can find $f_i, g_i \in i^{-1}\mathcal{O}_X$ for each $\eta \in \mathfrak{k}$ such that

$$(1\otimes\eta) - \Big(\sum_{i=1}^{n} f_i\xi_i + \sum_{i=1}^{m} g_i \otimes \eta_i\Big) \in i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g}).$$

We also have

$$[\xi_i, f\otimes 1] = 0, \qquad [1\otimes\eta, 1\otimes\eta'] = 1\otimes[\eta,\eta'], \qquad [1\otimes\eta, f\otimes 1] = (\eta_X(f))\otimes 1$$

for $f \in i^{-1}\mathcal{O}_X$, $\eta,\eta' \in \mathfrak{k}$. Further $[\xi_i,\xi_j], [1\otimes\eta_i,\xi_j] \in i^{-1}\mathcal{H}$ and hence there exist $f_{i,j,k}, g_{i,j,k} \in i^{-1}\mathcal{O}_X$ such that

$$[\xi_i,\xi_j] = \sum_{k=1}^{n} f_{i,j,k}\xi_k, \qquad [1\otimes\eta_i,\xi_j] = \sum_{k=1}^{n} g_{i,j,k}\xi_k.$$

Since $\mathcal{A}$ is generated by $i^{-1}\mathcal{H}$, $1\otimes\mathfrak{k}$ and $i^{-1}\mathcal{O}_X\otimes 1$, we can prove (5.3) by using these relations iteratively and using $\mathcal{A}(i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})) \subset i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})$. □

From the lemma above and its proof, we see that the algebra $\overline{\mathcal{A}}$ is generated by $\mathcal{O}_Y\otimes 1$, $1\otimes\xi_1,\dots,1\otimes\xi_n$, and $1\otimes\mathfrak{k}$ with the relations:

$$1\otimes\eta = \sum_{i=1}^{n} f_i\otimes\xi_i + \sum_{i=1}^{m} g_i\otimes\eta_i,$$

$$[1\otimes\xi_i, f\otimes 1] = 0, \qquad [1\otimes\eta, 1\otimes\eta'] = 1\otimes[\eta,\eta'], \qquad [1\otimes\eta, f\otimes 1] = (\eta_Y(f))\otimes 1,$$

$$[1\otimes\xi_i, 1\otimes\xi_j] = \sum_{k=1}^{n} f_{i,j,k}\otimes\xi_k, \qquad [1\otimes\eta_i, 1\otimes\xi_j] = \sum_{k=1}^{n} g_{i,j,k}\otimes\xi_k,$$

where $f_i, g_i, f_{i,j,k}, g_{i,j,k}$ are the restrictions to Y of the corresponding functions in the proof of Lemma 5.2 and $f \in \mathcal{O}_Y$, $\eta, \eta' \in \mathfrak{k}$. We can check that these relations are compatible with the action on $\mathcal{V}_Y$ (see (5.1)) and hence the $\overline{\mathcal{A}}$-action on $\mathcal{V}_Y$ is well-defined.

By Lemma 5.2, $(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X))|_U$ is a free $\overline{\mathcal{A}}|_U$-algebra with basis $1 \otimes \zeta_{\boldsymbol{i}}$. Therefore, the map

$$\phi : \mathcal{V}|_U \to \prod_{\boldsymbol{i} \in I_l} \mathcal{V}_Y|_U$$

given by $\phi(v) = (v(1 \otimes \zeta_{\boldsymbol{i}}))_{\boldsymbol{i}}$ is bijective.

Our choice of $\zeta_1, \ldots, \zeta_l$ implies that they form a basis of the normal tangent bundle of U in X. Since ϕ is bijective, we see that

$$\phi((i^{-1}\mathcal{I}_Y)^p \mathcal{V}|_U) = \prod_{s=p}^{\infty} \prod_{\boldsymbol{i} \in I_{s,l}} \mathcal{V}_Y|_U,$$

and hence

$$(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p \mathcal{V})|_U \simeq \prod_{s=0}^{p-1} \prod_{\boldsymbol{i} \in I_{s,l}} \mathcal{V}_Y|_U.$$

If we endow the right side of the last isomorphism with an $\mathcal{O}_{Y_p}$-module structure via this isomorphism, it is written as follows. Let $f \in i^{-1}\mathcal{O}_X$ and $v = (v_{\boldsymbol{i}})_{\boldsymbol{i}}$. For a subset $A \subset \{1, \ldots, s\}$ with $A = \{a(1), \ldots, a(t)\}$, $a(1) < \cdots < a(t)$, and for $\boldsymbol{i} = (i(1), \ldots, i(s)) \in I_{s,l}$, let $\{b(1), \ldots, b(s-t)\} = \{1, \ldots, s\} \setminus A$ with $b(1) < \cdots < b(s-t)$ and put $\boldsymbol{i'} := (i(b(1)), \ldots, i(b(s-t))) \in I_{s-t,l}$. Then the $\boldsymbol{i}$-term of $f \cdot v$ is

$$(f \cdot v)_{\boldsymbol{i}} = \sum_{A \subset \{1,\ldots,s\}} ((\zeta_{i(a(1))})_X \cdots (\zeta_{i(a(t))})_X f)|_U \cdot v_{\boldsymbol{i'}}. \tag{5.4}$$

On the right side here, we use the $\mathcal{O}_Y$-action on $\mathcal{V}_Y$. This $i^{-1}\mathcal{O}_X$-action on $\prod_{s=0}^{p-1} \prod_{\boldsymbol{i} \in I_{s,l}} \mathcal{V}_Y|_U$ induces an $\mathcal{O}_{Y_p}$-action.

We now show that $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ is a quasi-coherent and flat $\mathcal{O}_{Y_p}$-module. Suppose first that $\mathcal{V}_Y|_U$ is a free $\mathcal{O}_U$-module on U so there exist sections $v_j \in \Gamma(U, \mathcal{V}_Y)$, $j \in J$, such that the map $\mathcal{O}_U^{\oplus J} \to \mathcal{V}_Y|_U$, $(f_j)_{j \in J} \mapsto \sum_{j \in J} f_j v_j$, is bijective. We define the map

$$\psi : (\mathcal{O}_{Y_p}|_U)^{\oplus J} \to \prod_{s=0}^{p-1} \prod_{\boldsymbol{i} \in I_{s,l}} \mathcal{V}_Y|_U$$

by letting the $\boldsymbol{i}$-term of $\psi(f)$ for $\boldsymbol{i} = (i(1), \ldots, i(s)) \in I_{s,l}$ and $f = (f_j)_{j \in J}$ be

$$\psi(f)_{\boldsymbol{i}} = \sum_{j \in J} ((\zeta_{i(1)})_X \cdots (\zeta_{i(s)})_X f_j)|_U \cdot v_j.$$

Then ψ is an isomorphism of $\mathcal{O}_{Y_p}|_U$-modules and hence $(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V})|_U$ is a free $\mathcal{O}_{Y_p}|_U$-module.

For the general case, we write V as a union of finite-dimensional M-submodules: $V = \bigcup_\alpha V^\alpha$. Then the K-equivariant quasi-coherent $\mathcal{O}_Y$-module $\mathcal{V}_Y^\alpha$ with fiber V^α is locally free. If we define the $\mathcal{O}_{Y_p}$-module structure on $\prod_{s=0}^{p-1}\prod_{\boldsymbol{i}\in I_{s,l}}\mathcal{V}_Y^\alpha|_U$ as in (5.4), then the preceding argument proves that it is a locally free $\mathcal{O}_{Y_p}|_U$-module. Since $\mathcal{V}_Y$ is the union of $\mathcal{V}_Y^\alpha$, we see that $(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V})|_U$ is isomorphic to the union of $\prod_{s=0}^{p-1}\prod_{\boldsymbol{i}\in I_{s,l}}\mathcal{V}_Y^\alpha|_U$ as an $\mathcal{O}_{Y_p}|_U$-module. Hence $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ is a quasi-coherent and flat $\mathcal{O}_{Y_p}$-module.

We define a K-action on $\mathcal{V}$ by

$$(k\cdot v)(f\otimes D) = k\cdot\big(v((k^{-1}\cdot f)\otimes \mathrm{Ad}(i(k)^{-1})D)\big)$$

for $k\in K$, $v\in\mathcal{V}$, $f\in\mathcal{O}_Y$, and $D\in U(\mathfrak{g})$. This action descends to a K-action on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ and makes it a K-equivariant $\mathcal{O}_{Y_p}$-module. From this definition, it immediately follows that the maps $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}\to\mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1}\mathcal{V}$ and $i^{-1}\widetilde{\mathfrak{g}}_X\otimes\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}\to\mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1}\mathcal{V}$ commute with K-actions for all $p>0$.

We have checked conditions (1), (2) and (3) of Definition 3.3. We can verify condition (4) by computing the $\mathfrak{k}$-action as

$$\begin{aligned}(\eta\cdot v)(f\otimes D) &= v(f\otimes D\eta)\\ &= -v(f\otimes[\eta,D]) + v((1\otimes\eta)(f\otimes D)) - v((\eta_Y(f))\otimes D)\\ &= -v(f\otimes[\eta,D]) + (1\otimes\eta)(v(f\otimes D)) - v((\eta_Y(f))\otimes D)\end{aligned}$$

for $\eta\in\mathfrak{k}$, $v\in\mathcal{V}$, $f\in\mathcal{O}_Y$, and $D\in U(\mathfrak{g})$.

For condition (5), we get an isomorphism of vector spaces $\iota:\mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V}\simeq V$ by taking the fiber of the isomorphism $\phi:\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}\simeq\mathcal{V}_Y$ at o. The map ι is written as $\iota(v) = (v(1\otimes 1))(e)$ for $v\in\mathcal{V}$. For $\xi\in\mathfrak{h}$, there exists a section $\xi'\in i^{-1}\mathcal{H}$ near the base point o such that $1\otimes\xi-\xi'\in i^{-1}\mathcal{I}_o\otimes\mathfrak{g}$, or equivalently, $\xi'(e)=\xi$. Then

$$\iota(\xi v) = ((\xi v)(1\otimes 1))(e) = (v(1\otimes\xi))(e) = (v(\xi'))(e) = \xi(v(1\otimes 1)(e)) = \xi\iota(v).$$

Moreover, we have

$$\iota(mv) = ((mv)(1\otimes 1))(e) = (m(v(1\otimes 1)))(e) = m(v(1\otimes 1)(e)) = m\iota(v)$$

for $m\in M$ and hence ι commutes with the $(\mathfrak{h},M)$-actions. □

Remark 5.3. The $i^{-1}\widetilde{\mathfrak{g}}_X$-module $\mathcal{V}$ constructed above in this section has the following universal property. If $\mathcal{V}'$ is another $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with V, then there exists a canonical map $\mathcal{V}'\to\mathcal{V}$ such that the induced map

$$V\simeq\mathcal{V}'/(i^{-1}\mathcal{I}_o)\mathcal{V}'\to\mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V}\simeq V$$

is the identity map. Moreover, it also induces an isomorphism

$$\mathcal{V}'/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}' \to \mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$$

for any $p \in \mathbb{N}$. Therefore, the tensor product $i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}'$ does not depend on the choice of $\mathcal{V}'$ up to canonical isomorphism. We will give another description of the $i^{-1}\widetilde{\mathfrak{g}}_X$-module $i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ in Proposition 6.1.

§6. Twisted $\mathcal{D}$-modules

Retain the notation of the previous sections. Let V be an $(\mathfrak{h}, M)$-module and $\mathcal{V}$ an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with V. Since $\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}$ is a K-equivariant quasi-coherent $\mathcal{O}_Y$-module with typical fiber V, there is a canonical isomorphism $\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V} \simeq \mathcal{V}_Y$. We view $\mathcal{H} := \ker(p : \mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{T}_X)$ as a subsheaf of $U(\widetilde{\mathfrak{g}}_X)$. Since $\mathcal{H}(\mathcal{I}_Y \otimes U(\mathfrak{g})) \subset \mathcal{I}_Y \otimes U(\mathfrak{g})$, the $i^{-1}\mathcal{H}$-action on $\mathcal{V}$ induces one on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}$. By regarding local sections of these equivariant modules as vector-valued regular functions, this action is written as

$$(\xi v)(k) = \xi(i(k))v(k) \tag{6.1}$$

for $\xi \in i^{-1}\mathcal{H}$, $v \in \mathcal{V}$ and $k \in K$. Indeed, since the action map $i^{-1}\mathcal{H} \otimes \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V} \to \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}$ commutes with K-actions by Definition 3.3(3), it is enough to prove (6.1) for $k = e$. This follows from $\mathcal{H}(\mathcal{I}_o \otimes U(\mathfrak{g})) \subset \mathcal{I}_o \otimes U(\mathfrak{g})$ and Definition 3.3(5).

The $\mathcal{O}_Y$-modules $\mathcal{L}$, $\mathcal{V}_Y$, Ω_Y, and $i^*\Omega_X^\vee$ are K-equivariant with typical fiber $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})$, V, $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{m})^*$, and $\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$, respectively. Hence the tensor product $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee$ is also K-equivariant and has typical fiber $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{m})^* \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$. We define a right $i^{-1}\mathcal{H}$-module structure, a right $\mathfrak{k}$-module structure, and a right $\mathcal{O}_Y$-module structure on the sheaf $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee$ by

$$\begin{aligned}
((f \otimes v \otimes \omega \otimes \omega')\xi)(k) &= -f(k) \otimes (\xi(i(k))v(k)) \otimes \omega(k) \otimes \omega'(k) \\
&\quad - f(k) \otimes v(k) \otimes \omega(k) \otimes \mathrm{ad}(\xi(i(k)))\omega'(k), \\
(f \otimes v \otimes \omega \otimes \omega')\eta &= -(\eta f) \otimes v \otimes \omega \otimes \omega' - f \otimes (\eta v) \otimes \omega \otimes \omega' \\
&\quad - f \otimes v \otimes (\eta\omega) \otimes \omega' - f \otimes v \otimes \omega \otimes (\eta\omega'), \\
(f \otimes v \otimes \omega \otimes \omega')f' &= f'f \otimes v \otimes \omega \otimes \omega'
\end{aligned}$$

for $f \in \mathcal{L}$, $\xi \in i^{-1}\mathcal{H}$, $\eta \in \mathfrak{k}$, $v \in \mathcal{V}_Y$, $\omega \in \Omega_Y$, $\omega' \in i^*\Omega_X^\vee$, $f' \in \mathcal{O}_Y$, and $k \in K$. These actions are compatible: if $f_i \in i^{-1}\mathcal{O}_X$, $\eta_i \in \mathfrak{k}$ and $\xi \in i^{-1}\mathcal{H}$ satisfy

$$\sum_i (f_i \otimes \eta_i) - \xi \in i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g}),$$

then we have

$$\sum_i \big((f \otimes v \otimes \omega \otimes \omega')f_i|_Y\big)\eta_i = (f \otimes v \otimes \omega \otimes \omega')\xi.$$

Therefore, we can prove in the same way as in Section 5 that these actions define a right $\overline{\mathcal{A}}$-module structure on $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee$.

By using this right $\overline{\mathcal{A}}$-module structure, we consider the sheaf

$$(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee) \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)),$$

which has a right $i^{-1}\widetilde{\mathfrak{g}}_X$-module structure. We view it as a left $i^{-1}\widetilde{\mathfrak{g}}_X$-module via the anti-isomorphism

$$S : U(\widetilde{\mathfrak{g}}_X) \to U(\widetilde{\mathfrak{g}}_X), \quad f \otimes 1 \mapsto f \otimes 1, \quad 1 \otimes \xi \mapsto -1 \otimes \xi,$$

for $f \in \mathcal{O}_X$, $\xi \in \mathfrak{g}$.

Proposition 6.1. *Let $\mathcal{L}$ be as in Section 4. Let $\mathcal{V}$ be an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with an $(\mathfrak{h}, M)$-module V. Then there exists a K-equivariant isomorphism of $i^{-1}\widetilde{\mathfrak{g}}_X$-modules*

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq (\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee) \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)).$$

Proof. Let $F_p i^{-1}i_+\mathcal{L}$ be the filtration of $i^{-1}i_+\mathcal{L}$ as in Section 4. Then $F_0 i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ is regarded as a subsheaf of $i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ (see Remark 4.2). We have

$$\begin{aligned} F_0 i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} &\simeq F_0 i^{-1}i_+\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V} \\ &\simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee \otimes_{\mathcal{O}_Y} \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}. \end{aligned}$$

Therefore, we get an isomorphism of K-equivariant $\mathcal{O}_Y$-modules

$$\begin{aligned} \psi_0 : \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee &\xrightarrow{\sim} F_0 i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}, \\ f \otimes v \otimes \omega \otimes \omega' &\mapsto (f \otimes \omega \otimes \omega') \otimes v. \end{aligned} \tag{6.2}$$

Here $v \in \mathcal{V}_Y$ and we choose a section of $\mathcal{V}$ that is sent to $v \in \mathcal{V}_Y \simeq \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}$ by the quotient map, which we denote by the same letter $v \in \mathcal{V}$. Write $\mathcal{V}'_Y := \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee$ for simplicity. The isomorphism (6.2) extends to the homomorphism of $i^{-1}\widetilde{\mathfrak{g}}_X$-modules

$$\begin{aligned} \psi : \mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) &\to i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}, \\ v \otimes (1 \otimes (f \otimes D)) &\mapsto S(f \otimes D) \cdot \psi_0(v). \end{aligned}$$

We can check that the map ψ descends to

$$\overline{\psi} : \mathcal{V}'_Y \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) \to i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}.$$

Let

$$\pi : \mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) \to \mathcal{V}'_Y \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X))$$

be the quotient map and put

$$\mathcal{V}_p := \pi\left(\mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{\mathbb{C}} U_p(\mathfrak{g}))\right),$$

where $\{U_p(\mathfrak{g})\}_{p\in\mathbb{N}}$ is the standard filtration of $U(\mathfrak{g})$. We have

$$\overline{\psi}(\mathcal{V}_p) = \psi(\mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{\mathbb{C}} U_p(\mathfrak{g})) \subset F_p i^{-1} i_+ \mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}.$$

Let us take an open set $U \subset Y$ and elements $\zeta_1, \dots, \zeta_l \in \mathfrak{g}$ as in the proof of Proposition 5.1 and use the same notation. Then by an argument similar to the proof of Proposition 5.1, we obtain a bijective map of sheaves

$$\prod_{s=0}^{p} \prod_{\boldsymbol{i} \in I_{s,l}} \mathcal{V}'_Y|_U \simeq \mathcal{V}_p|_U, \qquad (v_{\boldsymbol{i}})_{\boldsymbol{i}} \mapsto \sum_{\boldsymbol{i}} \pi(v_{\boldsymbol{i}} \otimes (1 \otimes \zeta_{\boldsymbol{i}})),$$

and hence

$$\prod_{\boldsymbol{i} \in I_{p,l}} \mathcal{V}'_Y|_U \simeq \mathcal{V}_p/\mathcal{V}_{p-1}|_U.$$

We also see that

$$(F_p i^{-1} i_+ \mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})/(F_{p-1} i^{-1} i_+ \mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \simeq (F_p i^{-1} i_+ \mathcal{L}/F_{p-1} i^{-1} i_+ \mathcal{L}) \otimes_{\mathcal{O}_Y} \mathcal{V}_Y$$

and

$$F_p i^{-1} i_+ \mathcal{L}/F_{p-1} i^{-1} i_+ \mathcal{L} \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^{\vee} \otimes_{\mathcal{O}_Y} i^{-1}((\mathcal{I}_Y)^p/(\mathcal{I}_Y)^{p+1})$$

by [Osh11, Lemma 3.3]. Since $\zeta_{\boldsymbol{i}}$ for $\boldsymbol{i} \in I_{p,l}$ give a trivialization of $i^{-1}((\mathcal{I}_Y)^p/(\mathcal{I}_Y)^{p+1})$, we conclude that the map

$$\mathcal{V}_p/\mathcal{V}_{p-1} \to (F_p i^{-1} i_+ \mathcal{L} \otimes \mathcal{V})/(F_{p-1} i^{-1} i_+ \mathcal{L} \otimes \mathcal{V})$$

induced by $\overline{\psi}$ on the successive quotient is an isomorphism. Therefore the map $\overline{\psi}$ is also an isomorphism. We can also see that $\overline{\psi}$ commutes with the K-action. Hence the proposition follows. □

Let $\lambda \in \mathfrak{h}^*$ be such that $\mathrm{Ad}^*(h)\lambda = \lambda$ for $h \in H$. For a section $\xi \in \mathcal{H}$, we define a function $f_{\xi,\lambda} \in \mathcal{O}_X$ as

$$f_{\xi,\lambda}(gH) = \lambda(\xi(g)).$$

Let $\mathcal{I}_\lambda$ be the two-sided ideal of the sheaf $U(\widetilde{\mathfrak{g}}_X) = \mathcal{O}_X \otimes U(\mathfrak{g})$ generated by $\xi - (f_{\xi,\lambda} \otimes 1)$ for all $\xi \in \mathcal{H}$. We define the ring of twisted differential operators as

$$\mathcal{D}_{X,\lambda} := U(\widetilde{\mathfrak{g}}_X)/\mathcal{I}_\lambda.$$

Let $\mu := \lambda|_{\mathfrak{m}}$ and define $\mathcal{D}_{Y,\mu}$ similarly. Then we can define the direct image of a left $\mathcal{D}_{Y,\mu}$-module $\mathcal{M}$ by

$$i_+\mathcal{M} := i_*\big((\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_{Y,-\mu}} i^*\mathcal{D}_{X,-\lambda}\big) \otimes_{\mathcal{O}_X} \Omega_X^{\vee}.$$

Suppose that V is an $(\mathfrak{h}, M)$-module and $\mathfrak{h}$ acts on V by $\lambda \in \mathfrak{h}^*$. The K-equivariant $\mathcal{O}_Y$-module $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y$ has a natural structure of left $\mathcal{D}_{Y,\mu}$-module. Therefore, we can define the direct image $i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$ as a left $\mathcal{D}_{X,\lambda}$-module.

Proposition 6.2. *Suppose that V is an $(\mathfrak{h}, M)$-module and $\mathfrak{h}$ acts on V by $\lambda \in \mathfrak{h}^*$ such that $\mathrm{Ad}^*(h)\lambda = \lambda$ for $h \in H$. Let $\mathcal{V}$ be an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with V. Then we have a K-equivariant isomorphism of $i^{-1}\widetilde{\mathfrak{g}}_X$-modules*

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y).$$

Proof. We define a filtration $F_p i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$ of $i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$ in the same way as $F_p i^{-1}i_+\mathcal{L}$. Then

$$F_0 i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y) \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^{\vee}.$$

By using the same argument as in Proposition 6.1, we define a map of $i^{-1}\widetilde{\mathfrak{g}}_X$-modules

$$\mathcal{V}_Y' \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) \to i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$$

and we see that it induces an isomorphism

$$\mathcal{V}_Y' \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) \simeq i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y).$$

Hence

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$$

by Proposition 6.1. □

Recall that $\mathcal{L}$ is the K-equivariant invertible sheaf on $Y = K/M$ with typical fiber $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})$. We view a one-dimensional vector space $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^*$ as an $(\mathfrak{h}, M)$-module in the following way: $\mathfrak{h}$ acts as zero; the Levi component L of M acts as the coadjoint action $\bigwedge \mathrm{Ad}^*$; the unipotent radical U of M acts trivially. Let $\mathcal{L}'$ be an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^*$. Then $\mathcal{L}'/(i^{-1}\mathcal{I}_Y)\mathcal{L}'$ is isomorphic to the dual of $\mathcal{L}$. Therefore, by Proposition 6.2 we have

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{L}' \simeq i^{-1}i_+\mathcal{V}_Y.$$

Example 3.6 shows that the $i^{-1}\widetilde{\mathfrak{g}}_X$-module $\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{L}'$ is associated with $V \otimes \bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^*$.

Theorem 6.3. *In Setting 3.2, assume that K is reductive. Suppose that V is an $(\mathfrak{h}, M)$-module and $\mathfrak{h}$ acts on V by $\lambda \in \mathfrak{h}^*$ such that $\mathrm{Ad}^*(h)\lambda = \lambda$ for $h \in H$. Let $M = L \ltimes U$ be a Levi decomposition. Then*

$$\begin{aligned} \mathrm{H}^s(Y, i^{-1}i_+\mathcal{V}_Y) &\simeq (P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^* \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \\ &\simeq (I_{\mathfrak{g},L}^{\mathfrak{g},K})^{y+s} P_{\mathfrak{h},L}^{\mathfrak{g},L}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \end{aligned}$$

for $s \in \mathbb{N}$, $u = \dim U$, and $y = \dim Y$.

Proof. The first isomorphism follows from Theorem 4.1 and the argument above. Since the functor $P_{\mathfrak{h},L}^{\mathfrak{g},L}$ is exact, $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s} \simeq (P_{\mathfrak{g},L}^{\mathfrak{g},K})_{u-s} \circ P_{\mathfrak{h},L}^{\mathfrak{g},L}$. Hence the duality ([KV95, Theorem 3.5])

$$(P_{\mathfrak{g},L}^{\mathfrak{g},K})_{\dim(K/L)-s}\big(\,\cdot \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^*\big) \simeq (I_{\mathfrak{g},L}^{\mathfrak{g},K})^s(\cdot)$$

and $\dim K/L = \dim U + \dim Y$ give the second isomorphism. □

By Theorem 6.3 we obtain the convergence of the spectral sequence

(6.3) $$\begin{aligned} \mathrm{H}^s(X, R^t i_+\mathcal{V}_Y) &\Rightarrow (P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s-t}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^* \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \\ &\simeq (I_{\mathfrak{g},L}^{\mathfrak{g},K})^{y+s+t} P_{\mathfrak{h},L}^{\mathfrak{g},L}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big). \end{aligned}$$

Here $R^t i_+$ is the higher direct image functor for a twisted left $\mathcal{D}$-module.

We will now see that this spectral sequence implies results of [HMSW87] and [Kit10].

Example 6.4. Let $G_\mathbb{R}$ be a connected real semisimple Lie group with a maximal compact subgroup $K_\mathbb{R}$ and the complexified Lie algebra $\mathfrak{g}$. Let K be the complexification of $K_\mathbb{R}$ and G the inner automorphism group of $\mathfrak{g}$. There is a canonical homomorphism $i : K \to G$, which has finite kernel. Suppose that H is a Borel subgroup of G. Let us apply Setting 3.2. Then $X = G/H$ is the full flag variety of $\mathfrak{g}$. Since L is abelian and K is connected, L acts trivially on $\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^*$. Moreover in this case it is known that Y is affinely embedded in X. Therefore, $R^t i_+ \simeq 0$ for $t > 0$ and the spectral sequence (6.3) collapses. We thus get (1.1) and hence the duality theorem (Theorem 1.1).

Example 6.5. Let $G_\mathbb{R}$ be a connected real semisimple Lie group with a maximal compact subgroup $K_\mathbb{R}$. We define K, G, and $i : K \to G$ as in the previous example. Suppose that H is a parabolic subgroup of G and apply Setting 3.2. Then $X = G/H$ is a partial flag variety of $\mathfrak{g}$. In this case Y is not necessarily affinely embedded in X. Let $\widetilde{X}$ be the full flag variety of $\mathfrak{g}$ and let $p : \widetilde{X} \to X$ be the natural surjective map. Then we have an isomorphism $\mathrm{H}^s(\widetilde{X}, p^*\mathcal{M}) \simeq \mathrm{H}^s(X, \mathcal{M})$ for any

$\mathcal{O}_X$-module $\mathcal{M}$. Hence (6.3) becomes

$$\mathrm{H}^s(\widetilde{X}, p^* R^t i_+ \mathcal{V}_Y) \Rightarrow (I^{\mathfrak{g},K}_{\mathfrak{g},L})^{y+s+t} P^{\mathfrak{g},L}_{\mathfrak{h},L}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big),$$

which is [Kit10, Theorem 25 (6.6)].

Let V be any $(\mathfrak{h}, M)$-module and $\mathcal{V}$ an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with V. Since $i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{L}' \simeq i^{-1}i_+\mathcal{O}_Y$, we have

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{L}' \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq i^{-1}i_+\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}.$$

We can thus rewrite Theorem 4.1 as

Theorem 6.6. *In Setting 3.2, assume that K is reductive. Let $M = L \ltimes U$ be a Levi decomposition. Suppose that V is an $(\mathfrak{h}, M)$-module and $\mathcal{V}$ is an $i^{-1}\widetilde{\mathfrak{g}}_X$-module associated with V (Definition 3.3). Then*

$$\begin{aligned}\mathrm{H}^s(Y, i^{-1}i_+\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) &\simeq (P^{\mathfrak{g},K}_{\mathfrak{h},L})_{u-s}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{l})^* \otimes \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big) \\ &\simeq (I^{\mathfrak{g},K}_{\mathfrak{g},L})^{y+s} P^{\mathfrak{g},L}_{\mathfrak{h},L}\big(V \otimes \textstyle\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\big)\end{aligned}$$

for $s \in \mathbb{N}$, $u = \dim U$, $y = \dim Y$.

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