



Partial Differential Equations — *Existence and stability of quasi-periodic solutions for derivative wave equations*, by MASSIMILIANO BERTI, LUCA BIASCO and MICHELA PROCESI, communicated on 14 December 2012.

ABSTRACT. — In this note we present the new KAM result in [3] which proves the existence of Cantor families of small amplitude, analytic, quasi-periodic solutions of derivative wave equations, with zero Lyapunov exponents and whose linearized equation is reducible to constant coefficients. In turn, this result is derived by an abstract KAM theorem for infinite dimensional reversible dynamical systems*.

KEY WORDS: Wave equation, KAM for PDEs, quasi-periodic solutions, small divisors, quasi-Töplitz property.

MATHEMATICS SUBJECT CLASSIFICATION: 37K55, 35L05.

1. INTRODUCTION

In the last years many progresses have been obtained concerning KAM theory for nonlinear PDEs, since the pioneering works of Kuksin [17] and Wayne [26] for 1- d semilinear wave (NLW) and Schrödinger (NLS) equations under Dirichlet boundary conditions, see [19] and [21] for further developments. The approach of these papers consists in generating iteratively a sequence of symplectic changes of variables which bring the Hamiltonian into a constant coefficients (= reducible) normal form with an elliptic (= linearly stable) invariant torus at the origin. Such a torus is filled by quasi-periodic solutions with zero Lyapunov exponents. This procedure requires to solve, at each step, constant-coefficients linear “homological equations” by imposing the “second order Melnikov” non-resonance conditions. Unfortunately these (infinitely many) conditions are violated already for periodic boundary conditions.

In this case, existence of quasi-periodic solutions for semilinear 1- d -NLW and NLS equations, was first proved by Bourgain [6] by extending the Lyapunov-Schmidt decomposition and the Newton approach introduced by Craig-Wayne [11] for periodic solutions. Its main advantage is to require only the “first order Melnikov” non-resonance conditions (the minimal assumptions) for solving the homological equations. It has allowed Bourgain to prove [7], [9] also the existence

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of quasi-periodic solutions for NLW and NLS (with Fourier multipliers) in any space dimension, see also the recent extensions in Berti-Bolle [4], [5]. The main drawback of this approach is that the homological equations are linear PDEs with non-constant coefficients. Translated in the KAM language this implies a non-reducible normal form around the torus and then a lack of information about the stability of the quasi-periodic solutions. Later on, existence of reducible elliptic tori was proved by Eliasson-Kuksin [13] for NLS (with Fourier multipliers) in any space dimension, see also Procesi-Xu [22].

A challenging frontier concerns PDEs with unbounded nonlinearities, i.e. containing derivatives. In this direction KAM theory has been extended for perturbed KdV equations by Kuksin [18], Kappeler-Pöschel [15], and, for the $1d$ -derivative NLS (DNLS) and Benjamin-Ono equations, by Liu-Yuan [14]. We remark that the KAM proof is more delicate for DNLS and Benjamin-Ono, because these equations are less “dispersive” than KdV, i.e. the eigenvalues of the principal part of the differential operator grow only quadratically at infinity, and not cubically as for KdV. This difficulty is reflected in the fact that the quasi-periodic solutions in [18], [15] are analytic, while those of [14] are only C^∞ . Actually, for the applicability of these KAM schemes, the more dispersive the equation is, the more derivatives in the nonlinearity can be supported. The limit case of the derivative nonlinear wave equation (DNLW)—which is not dispersive at all—is excluded by these approaches.

In this note we present the KAM theory developed in [3] which proves existence and stability of small amplitude analytic quasi-periodic solutions of the derivative wave equations. Such PDEs are not Hamiltonian, but may have a reversible structure, that we shall exploit.

All the previous results concern *Hamiltonian* PDEs. It was however remarked by Bourgain that the construction of periodic and quasi-periodic solutions, using the Newton iteration method of Craig-Wayne [11], is a-priori not restricted to Hamiltonian systems. This approach appears as a general implicit function type result, in large part independent of the Hamiltonian character of the equations. For example in [8] Bourgain proved the existence of periodic solutions for the non-Hamiltonian derivative wave equation

$$(1.1) \quad y_{tt} - y_{xx} + my + y_t^2 = 0, \quad m \neq 0, \quad x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}).$$

Actually also KAM theory is not only Hamiltonian in nature, but may be formulated for general vector fields, as realized in the seminal work of Moser [20]. This paper, in particular, started the analysis of reversible KAM theory for finite dimensional systems, later extended by Arnold [1] and Servyuk [24]. The reversibility property implies that the averages over the fast angles of some components of the vector field are zero, thus removing the “secular drifts” of the actions which are incompatible with a quasi-periodic behavior of the solutions.

Recently, Zhang-Gao-Yuan [25] have proved the existence of C^∞ -quasi periodic solutions for the derivative NLS equation

$$iu_t + u_{xx} + |u_x|^2 u = 0$$

with Dirichlet boundary conditions. Such equation is reversible, but not Hamiltonian. The result [25] is proved adapting the KAM scheme developed for the Hamiltonian DNLS in Liu-Yuan [14]. The derivative nonlinear wave equation (DNLW), which is not dispersive, is excluded.

In the recent paper [2] we have extended KAM theory to deal with Hamiltonian derivative wave equations like

$$y_{tt} - y_{xx} + my + f(Dy) = 0, \quad m > 0, \quad D := \sqrt{-\partial_{xx} + m}, \quad x \in \mathbb{T}.$$

This kind of Hamiltonian pseudo-differential equations has been introduced by Bourgain [6] and Craig [10] as models to study the effect of derivatives versus dispersive phenomena. The key of [2] is the proof of the first order asymptotic expansion of the perturbed normal frequencies, obtained using the notion of quasi-Töplitz function. This concept was introduced by Procesi-Xu [22] and is connected to the Töplitz-Lipschitz property in Eliasson-Kuksin [13]. Of course we could not deal in [2] with the derivative wave equation, which is not Hamiltonian.

In [3] we develop KAM theory for a class of reversible derivative wave equations

$$(1.2) \quad y_{tt} - y_{xx} + my = g(x, y, y_x, y_t), \quad x \in \mathbb{T},$$

implying the *existence* and the *stability* of *analytic* quasi-periodic solutions, see Theorem 1.1. Note that the nonlinearity in (1.2) has an explicit x -dependence (unlike [2]). The search for periodic/quasi-periodic solutions for derivative wave equations is a natural question, which was pointed out, for instance, by Craig [10] as an important open problem (see section 7.3 of [10]).

Clearly we can not expect an existence result for any nonlinearity. For example, (1.2) with the nonlinear friction term $g = y_t^3$ has no smooth periodic/quasi-periodic solutions except the constants, see Proposition 1.2. This case is ruled out by assuming the condition

$$(1.3) \quad g(x, y, y_x, -v) = g(x, y, y_x, v),$$

satisfied, for example, by (1.1). Under condition (1.3) the equation (1.2) is *reversible*, namely the associated first order system

$$(1.4) \quad y_t = v, \quad v_t = y_{xx} - my + g(x, y, y_x, v),$$

is reversible with respect to the involution

$$(1.5) \quad S(y, v) := (y, -v), \quad S^2 = I.$$

Reversibility is an important property in order to allow the existence of periodic/quasi-periodic solutions, albeit not sufficient. For example, the reversible equation

$$y_{tt} - y_{xx} = y_x^3, \quad x \in \mathbb{T},$$

(proposed in [10], page 89) has no smooth periodic/quasi-periodic solutions except the constants, see Proposition 1.1. In order to find quasi-periodic solutions we also require the parity assumption

$$(1.6) \quad g(-x, y, -y_x, v) = g(x, y, y_x, v),$$

which rules out nonlinearities like y_x^3 . Actually, for the wave equation (1.2) the role of the time and space variables (t, x) is highly symmetric. Then, considering x “as time” (spatial dynamics idea) the term y_x^3 is a friction and condition (1.6) is the corresponding reversibility condition.

After Theorem 1.1 we shall further comment on the assumptions.

Before stating our main results, we mention the classical bifurcation theorems of Rabinowitz [23] about periodic solutions (with rational periods) of dissipative forced derivative wave equations

$$y_{tt} - y_{xx} + \alpha y_t + \varepsilon F(x, t, y_x, y_t) = 0, \quad x \in [0, \pi]$$

with Dirichlet boundary conditions, and for fully-non-linear forced wave equations

$$y_{tt} - y_{xx} + \alpha y_t + \varepsilon F(x, t, y_x, y_t, y_{tt}, y_{tx}, y_{xx}) = 0, \quad x \in [0, \pi].$$

This latter result is quite subtle because, from the point of view of the initial value problem, it is uncertain whether a solution can exist for more than a finite time due to the formation of shocks. Here the presence of the dissipation $\alpha \neq 0$ allows the existence of a periodic solution.

Finally, concerning quasi-linear wave equations we mention the Birkhoff normal form results of Delort [12] (and references therein), which imply long time existence for solutions with small initial data. To our knowledge, these are the only results of this type on compact manifolds. For quasi-linear wave equations in \mathbb{R}^d there is a huge literature, since the nonlinear effects of derivatives may be controlled by dispersion.

1.1. Main results

We consider derivative wave equations (1.2) where $m > 0$, the nonlinearity

$$g : \mathbb{T} \times \mathcal{U} \rightarrow \mathbb{R}, \quad \mathcal{U} \subset \mathbb{R}^3 \text{ open neighborhood of } 0,$$

is real analytic and satisfies the “reversibility” and “parity” assumptions (1.3), (1.6). Moreover g vanishes at least quadratically at $(y, y_x, v) = (0, 0, 0)$, namely

$$(1.7) \quad g(x, 0, 0, 0) = (\partial_y g)(x, 0, 0, 0) = (\partial_{y_x} g)(x, 0, 0, 0) = (\partial_v g)(x, 0, 0, 0) = 0.$$

In addition we assume a “non-degeneracy” condition on the leading order term of the nonlinearity (in order to verify the usual “twist” hypotheses required in KAM theory). For definiteness, we have developed all the calculations for

$$(1.8) \quad g = yy_x^2 + \text{h.o.t.}$$

Because of (1.3), it is natural to look for “reversible” quasi-periodic solutions such that

$$S(y, v)(t) = (y, v)(-t),$$

namely such that $y(t, x)$ is even and $v(t, x)$ is odd in time. Moreover, because of (1.6) the phase space of functions even in x ,

$$(1.9) \quad (y, v)(-x) = (y, v)(x), \quad \forall x \in \mathbb{T},$$

is invariant under the flow evolution of (1.4) and it is natural to study the dynamics on this subspace (standing waves). Note, in particular, that y satisfies the *Neumann* boundary conditions $y_x(t, 0) = y_x(t, \pi) = 0$.

Summarizing we look for reversible quasi-periodic *standing wave* solutions of (1.2), namely satisfying

$$(1.10) \quad y(t, x) = y(t, -x), \quad \forall t, \quad y(-t, x) = y(t, x), \quad \forall x \in \mathbb{T}.$$

For every choice of the *tangential* sites $\mathcal{J}^+ \subset \mathbb{N} \setminus \{0\}$, the linear Klein-Gordon equation

$$y_{tt} - y_{xx} + my = 0, \quad x \in \mathbb{T},$$

possesses the family of quasi-periodic standing wave solutions

$$(1.11) \quad y = \sum_{j \in \mathcal{J}^+} \frac{\sqrt{8\xi_j}}{\lambda_j} \cos(\lambda_j t) \cos(jx), \quad \lambda_j := \sqrt{j^2 + m},$$

parametrized by the amplitudes $\xi_j \in \mathbb{R}_+$.

THEOREM 1.1 [3]. *For every choice of finitely many tangential sites $\mathcal{J}^+ \subset \mathbb{N} \setminus \{0\}$, for all $m > 0$ except finitely many (depending on \mathcal{J}^+), the DNLW equation (1.2) with a real analytic nonlinearity satisfying (1.3), (1.6), (1.7), (1.8) admits small-amplitude, analytic (both in t and x), quasi-periodic solutions*

$$(1.12) \quad y = \sum_{j \in \mathcal{J}^+} \frac{\sqrt{8\xi_j}}{\lambda_j} \cos(\omega_j^\infty(\xi)t) \cos(jx) + o(\sqrt{\xi}), \quad \omega_j^\infty(\xi) \stackrel{\xi \rightarrow 0}{\approx} \sqrt{j^2 + m}$$

satisfying (1.10), for all values of the parameters ξ in a Cantor-like set with asymptotic density 1 at $\xi = 0$. Moreover the solutions have zero Lyapunov exponents and the linearized equations can be reduced to constant coefficients. The term $o(\sqrt{\xi})$ in (1.12) is small in some analytic norm.

This theorem completely answers the question posed by Craig in [10] of developing a general theory for quasi-periodic solutions for reversible derivative wave equations. With respect to Bourgain [8], we prove the existence of quasi-periodic solutions (not only periodic) as well as a stable KAM normal form nearby.

Let us comment on the hypothesis of Theorem 1.1.

1. **Reversibility and Parity.** As already said, the “reversibility” and “parity” assumptions (1.3), (1.6), rule out nonlinearities like y_t^3 and y_x^3 for which periodic/quasi-periodic solutions of (1.2) do not exist. We generalize these non-existence results in Propositions 1.1, 1.2.
2. **Mass $m > 0$.** The assumption on the mass $m \neq 0$ is, in general, necessary. When $m = 0$, a well known example of Fritz John (see (1.16)) shows that (1.1) has no smooth solutions for all times except the constants. In Proposition 1.3 we give other non-existence results of periodic/quasi-periodic solutions for DNLW equations satisfying both (1.3), (1.6), but with mass $m = 0$. For the KAM construction, the mass $m > 0$ is used in the Birkhoff normal form step. If the mass $m < 0$ then the Sturm-Liouville operator $-\partial_{xx} + m$ may possess finitely many negative eigenvalues and one should expect the existence of partially hyperbolic tori.
3. **x -dependence.** The nonlinearity g in (1.2) may explicitly depend on the space variable x , i.e. this equations are NOT invariant under x translations. This is an important novelty with respect to the KAM theorem in [2] which used the conservation of momentum. The key idea is the introduction of the a -weighted majorant norm for vector fields (see (2.37)) which penalizes the “high-momentum monomials”, see (2.38).
4. **Twist.** We have developed all the calculations for the cubic leading term $g = yy_x^2 + \text{h.o.t.}$. In this case the third order Birkhoff normal form of the PDE (1.2) turns out to be (partially) integrable and the frequency-to-action map is invertible. This is the so called “twist-condition” in KAM theory. It could be interesting to classify the allowed nonlinearities. For example, among the cubic nonlinearities, we already know that for y_x^3 , y^2y_x (and v^3) there are no non-trivial periodic/quasi-periodic solutions, see Propositions 1.1–1.2. On the other hand, for y^3 the Birkhoff normal form is (partially) integrable by [21] (for Dirichlet boundary conditions).
5. **Boundary conditions.** The solutions of Theorem 1.1 satisfy the Neumann boundary conditions $y_x(t, 0) = y_x(t, \pi) = 0$. For proving the existence of solutions under Dirichlet boundary conditions it would seem natural to substitute (1.6) with the oddness assumption

$$(1.13) \quad g(-x, -y, y_x, v) = -g(x, y, y_x, v),$$

so that the subspace of functions $(y, v)(x)$ odd in x is invariant under the flow evolution of (1.4). However, in order to find quasi-periodic solutions of (1.2), we need the *real-coefficients property* (2.28) which follows from (1.3) and (1.6), but not from (1.3) and (1.13). It is easy to check that (1.3), (1.13) and (2.28) imply the parity assumption (1.6). We decided to state the existence theorem in a form which requires the minimal assumptions. Of course, if a nonlinearity satisfies (1.3), (1.6) and also (1.13) we could look for quasi-periodic solutions satisfying Dirichlet boundary conditions.

6. **Derivative vs quasi-linear NLW.** It has been proved by Klainermann-Majda [16] that all classical solutions of Hamiltonian quasi-linear wave equations like

$$(1.14) \quad y_{tt} = (1 + \sigma(y_x))y_{xx}$$

with $\sigma^{(j)}(0) = 0, j = 1, \dots, p - 1, \sigma^{(p)}(0) \neq 0$, do not admit smooth, small amplitude, periodic (a fortiori quasi-periodic) solutions except the constants. Actually, any non constant solution of (1.14), with sufficiently small initial data, develops a singularity in finite time in the second derivative y_{xx} . In this respect [16] may suggest that Theorem 1.1 is optimal regarding the order of (integer) derivatives in the nonlinearity. Interestingly, the solutions of the derivative wave equation (which is a semilinear PDE) found in Theorem 1.1 are analytic in both time t and space x . Clearly the KAM approach developed in [3] fails for quasi-linear equations like (1.14) because the auxiliary vector field (whose flow defines the KAM transformations) is unbounded (of order 1). One could still ask for a KAM result for quasi-linear Klein Gordon equations (for which Delort [12] proved some steps of Birkhoff normal form). Note that adding a mass term my in the left hand side of (1.14), non constant periodic solutions of the form $y(t, x) = c(t)$ or $y(t, x) = c(x)$ may occur.

We finally complement the previous existence results with some negative results.

PROPOSITION 1.1 [3]. *Let $p \in \mathbb{N}$ be odd. The DNLW equations*

$$(1.15) \quad y_{tt} - y_{xx} = y_x^p + f(y), \quad y_{tt} - y_{xx} = \partial_x(y^p) + f(y), \quad x \in \mathbb{T},$$

have no smooth quasi-periodic solutions except for trivial periodic solutions of the form $y(t, x) = c(t)$. In particular $f \equiv 0$ implies $c(t) \equiv \text{const}$.

This result is proved in [3] showing that

$$M(y, v) := \int_{\mathbb{T}} y_x v \, dx$$

is a Lyapunov function for (1.15). For wave equations, the role of the space variable x and time variable t is symmetric. A term like y_x^p for an odd p is a friction term which destroys the existence of quasi-periodic solutions. Using

$$H(y, v) := \int_{\mathbb{T}} \frac{v^2}{2} + \frac{y_x^2}{2} - F(y) \, dx$$

as a Lyapunov function we prove that:

PROPOSITION 1.2 [3]. *Let $p \in \mathbb{N}$ be odd. The DNLW equation*

$$y_{tt} - y_{xx} = y_x^p + f(y), \quad x \in \mathbb{T},$$

has no smooth quasi-periodic solutions except for trivial periodic solutions of the form $y(t, x) = c(x)$. In particular $f \equiv 0$ implies $c(x) \equiv \text{const}$.

The mass term my is, in general, necessary to have existence of quasi-periodic solutions. The following non-existence results hold:

PROPOSITION 1.3 [3]. *The derivative NLW equation*

$$(1.16) \quad y_{tt} - y_{xx} = y_t^2, \quad x \in \mathbb{T},$$

has no smooth solutions defined for all times except the constants. Moreover, for $p, q \in \mathbb{N}$ even,

$$(1.17) \quad y_{tt} - y_{xx} = y_x^p, \quad y_{tt} - y_{xx} = y_t^p, \quad y_{tt} - y_{xx} = y_x^p + y_t^q, \quad x \in \mathbb{T},$$

have no smooth periodic/quasi-periodic solutions except the constants.

The blow-up result for (1.16) is proved by projecting the equation on the constants. The non-existence results for (1.17) may be obtained simply by integrating the equations in (t, x) .

2. IDEAS OF PROOF: THE ABSTRACT KAM THEOREM

The proof of Theorem 1.1 is based on an abstract KAM Theorem for reversible infinite dimensional systems (Theorem 4.1 in [3]) which proves the existence of elliptic invariant tori and provides a reducible normal form around them. We now explain the main ideas and techniques of proof.

Complex formulation. We extend (1.4) as a first order system with complex valued variables $(y, v) \in \mathbb{C}^n \times \mathbb{C}^n$. In the unknowns

$$u^+ := \frac{1}{\sqrt{2}}(Dy + iv), \quad u^- := \frac{1}{\sqrt{2}}(Dy - iv), \quad D := \sqrt{-\partial_{xx} + m}, \quad i := \sqrt{-1},$$

system (1.4) becomes the first order system

$$(2.18) \quad \begin{cases} u_t^+ = -iDu^+ + ig(u^+, u^-) \\ u_t^- = iDu^- - ig(u^+, u^-) \end{cases}$$

where

$$g(u^+, u^-) = \frac{1}{\sqrt{2}}g\left(x, D^{-1}\left(\frac{u^+ + u^-}{\sqrt{2}}\right), D^{-1}\left(\frac{u_x^+ + u_x^-}{\sqrt{2}}\right), \frac{u^+ - u^-}{i\sqrt{2}}\right).$$

In (2.18), the dynamical variables (u^+, u^-) are independent. However, since g is real analytic (real on real), the real subspace

$$(2.19) \quad \mathbf{R} := \{\overline{u^+} = u^-\}$$

is invariant under the flow evolution of (2.18), since

$$(2.20) \quad \overline{\mathbf{g}(u^+, u^-)} = \mathbf{g}(u^+, u^-), \quad \forall (u^+, u^-) \in \mathbf{R},$$

and the second equation in (2.18) reduces to the complex conjugated of the first one. Clearly, this corresponds to real valued solutions (y, v) of the real system (1.4). We say that system (2.18) is “real-on-real”. For systems satisfying this property it is customary to use also the shorter notation

$$(u^+, u^-) = (u, \bar{u}).$$

Moreover the subspace of even functions

$$(2.21) \quad \mathbf{E} := \{u^+(x) = u^+(-x), u^-(x) = u^-(-x)\}$$

(see (1.9)) is invariant under the flow evolution of (2.18), by (1.6). System (2.18) is reversible with respect to the involution

$$(2.22) \quad S(u^+, u^-) := (u^-, u^+),$$

(which is nothing but (1.5) in the variables (u^+, u^-)).

Dynamical systems formulation. We introduce infinitely many coordinates by Fourier transform

$$(2.23) \quad u^+ = \sum_{j \in \mathbb{Z}} u_j^+ e^{ijx}, \quad u^- = \sum_{j \in \mathbb{Z}} u_j^- e^{-ijx}.$$

Then (2.18) becomes the infinite dimensional dynamical system

$$(2.24) \quad \begin{cases} \dot{u}_j^+ = -i\lambda_j u_j^+ + i\mathbf{g}_j^+(\dots, u_h^+, u_h^-, \dots) \\ \dot{u}_j^- = i\lambda_j u_j^- - i\mathbf{g}_j^-(\dots, u_h^+, u_h^-, \dots) \end{cases} \quad \forall j \in \mathbb{Z},$$

where

$$(2.25) \quad \lambda_j := \sqrt{j^2 + m}$$

are the eigenvalues of D and

$$\mathbf{g}_j^+ = \frac{1}{2\pi} \int_{\mathbb{T}} \mathbf{g} \left(\sum_{h \in \mathbb{Z}} u_h^+ e^{ihx}, \sum_{h \in \mathbb{Z}} u_h^- e^{-ihx} \right) e^{-ijx} dx, \quad \mathbf{g}_j^- := \mathbf{g}_{-j}^+.$$

By (2.23), the “real” subset \mathbf{R} in (2.19) reads $\overline{u_j^+} = u_j^-$ (this is the motivation for the choice of the signs in (2.23)) and, by (2.20), the second equation in (2.24) is the complex conjugated of the first one, namely

$$\overline{\mathbf{g}_j^+} = \mathbf{g}_j^- \quad \text{when} \quad \overline{u_j^+} = u_j^-, \quad \forall j.$$

Moreover, the invariant subspace E of even functions in (2.21) reads, under Fourier transform,

$$(2.26) \quad E := \{u_j^+ = u_{-j}^+, u_j^- = u_{-j}^-, \forall j \in \mathbb{Z}\}.$$

By (2.23) the involution (2.22) reads

$$(2.27) \quad S : (u_j^+, u_j^-) \rightarrow (u_{-j}^-, u_{-j}^+), \quad \forall j \in \mathbb{Z}.$$

Finally, since g is real analytic, the assumptions (1.3) and (1.6) imply the key property

$$(2.28) \quad g_j^\pm(\dots, u_j^+, u_j^-, \dots) \quad \text{has real Taylor coefficients}$$

in the variables (u_j^+, u_j^-) .

REMARK 2.1. *The previous property is compatible with oscillatory phenomena for (2.24), excluding friction phenomena. This is another strong motivation for assuming (1.3) and (1.6).*

Abstract KAM theorem. For every choice of symmetric tangential sites

$$\mathcal{J} = \mathcal{J}^+ \cup (-\mathcal{J}^+) \quad \text{with } \mathcal{J}^+ \subset \mathbb{N} \setminus \{0\}, \quad \#\mathcal{J} = n,$$

the linearized system (2.24) where $g_j^\pm = 0$ has the invariant tori

$$\{u_j \bar{u}_j = \xi_{|j|} > 0, \text{ for } j \in \mathcal{J}, u_j = \bar{u}_j = 0 \text{ for } j \notin \mathcal{J}\}$$

parametrized by the actions $\xi = (\xi_j)_{j \in \mathcal{J}^+}$. They correspond to the quasi-periodic solutions in (1.11).

We first analyze the nonlinear dynamics of (2.24) close to the origin, via a Birkhoff normal form reduction (see section 7 of [3]). This step depends on the nonlinearity g and on the fact that the mass $m > 0$. Here we use (1.8) to ensure that the third order Birkhoff normalized system is (partially) integrable and that the ‘‘twist condition’’ holds (the frequency-to-action map is a diffeomorphism).

Then we introduce action-angle coordinates on the tangential variables:

$$(2.29) \quad \begin{aligned} u_j^+ &= \sqrt{\xi_{|j|} + y_j} e^{ix_j}, & u_j^- &= \sqrt{\xi_{|j|} + y_j} e^{-ix_j}, & j \in \mathcal{J}, \\ (u_j^+, u_j^-) &= (z_j^+, z_j^-) \equiv (z_j, \bar{z}_j), & j \notin \mathcal{J}, \end{aligned}$$

where $|y_j| < \xi_{|j|}$. Then, system (2.24) transforms into a parameter dependent family of analytic systems of the form

$$(2.30) \quad \begin{cases} \dot{x} = \omega(\xi) + P^{(x)}(x, y, z, \bar{z}; \xi) \\ \dot{y} = P^{(y)}(x, y, z, \bar{z}; \xi) \\ \dot{z}_j = -i\Omega_j(\xi)z_j + P^{(z_j)}(x, y, z, \bar{z}; \xi) \\ \dot{\bar{z}}_j = i\Omega_j(\xi)\bar{z}_j + P^{(\bar{z}_j)}(x, y, z, \bar{z}; \xi), & j \in \mathbb{Z} \setminus \mathcal{J}, \end{cases}$$

where $(x, y) \in \mathbb{T}_s^n \times \mathbb{C}^n$, z, \bar{z} are infinitely many variables, $\omega(\xi) \in \mathbb{R}^n$, $\Omega(\xi) \in \mathbb{R}^\infty$. The frequencies $\omega_j(\xi)$, $\Omega_j(\xi)$ are close to the unperturbed frequencies λ_j in (2.25) and satisfy $\omega_{-j} = \omega_j$, $\Omega_{-j} = \Omega_j$. The vector field X in system (2.30) has the form

$$X = \mathcal{N} + P,$$

where the normal form is

$$(2.31) \quad \mathcal{N} := \omega(\xi)\partial_x - i\Omega_j(\xi)z_j\partial_{z_j} + i\Omega_j(\xi)\bar{z}_j\partial_{\bar{z}_j}$$

(we use the differential geometry notation for vector fields). The following properties hold:

1. REVERSIBLE. The vector field $X = (X^{(x)}, X^{(y)}, X^{(z)}, X^{(\bar{z})})$ is REVERSIBLE, namely

$$X \circ S = -S \circ X,$$

with respect to the involution

$$S : (x_j, y_j, z_j, \bar{z}_j) \mapsto (-x_{-j}, y_{-j}, \bar{z}_{-j}, z_{-j}), \quad \forall j \in \mathbb{Z}, \quad S^2 = I,$$

which is nothing but (2.27) in the variables (2.29).

2. REAL-COEFFICIENTS. The components of the vector field

$$X^{(x)}, iX^{(y)}, iX^{(z_j)}, iX^{(\bar{z}_j)}$$

have *real* Taylor-Fourier coefficients in the variables (x, y, z, \bar{z}) , see (2.28).

3. REAL-ON-REAL.

$$X^{(x)}(v) = \overline{X^{(x)}(v)}, \quad X^{(y)}(v) = \overline{X^{(y)}(v)}, \quad X^{(z^-)}(v) = \overline{X^{(z^+)}(v)},$$

for all $v = (x, y, z^+, z^-)$ such that $x \in \mathbb{T}^n$, $y \in \mathbb{R}^n$, $\overline{z^+} = z^-$.

4. EVEN. The vector field $X : E \rightarrow E$ where

$$E := \{x_j = x_{-j}, y_j = y_{-j}, j \in \mathcal{J}, z_j = z_{-j}, \bar{z}_j = \bar{z}_{-j}, j \in \mathbb{Z} \setminus \mathcal{J}\}$$

is nothing but (2.26) in the variables (2.29). Hence the subspace E is invariant under the flow evolution of (2.30).

In system (2.30) we think $x_j, y_j, j \in \mathcal{J}, z_j^\pm, j \in \mathbb{Z} \setminus \mathcal{J}$, as *independent* variables and then we look for solutions in the invariant subspace E , which means solutions of (2.18) even in x . We proceed in this way because, in a phase space of functions even in x , the notion of *momentum* (see (2.36)) is *not* well defined, as the Neumann boundary conditions break the translation invariance*. On the

* In a more technical language we may see the above difficulty by noting that, if $z_{-j}^\pm \equiv z_j^\pm$, the vector fields $z_{-j}\partial_{z_j}$ and $z_j\partial_{z_j}$, that have DIFFERENT momentum, would be identified.

other hand, the concept of momentum is essential in order to verify the properties of quasi-Töplitz vector fields, as explained after (2.37)–(2.38). That is why in [3] we actually work in a phase space of 2π -periodic functions, where the notion (2.36) of momentum is well defined, and then we look for solutions in the invariant subspace (2.21) of even functions.

A difficulty that arises working in the whole phase space of periodic functions is that the linear frequencies $\omega_{-j} = \omega_j$, $j \in \mathcal{J}$, $\Omega_{-j} = \Omega_j$, $j \in \mathbb{Z} \setminus \mathcal{J}$, are *resonant*. Therefore, along the KAM iteration, the monomial vector fields of the perturbation

$$e^{ik \cdot x} \partial_{x_j}, \quad e^{ik \cdot x} y^i \partial_{y_j}, \quad k \in \mathbb{Z}_{\text{odd}}^n, \quad |i| = 0, 1, \quad j \in \mathcal{J},$$

$$e^{ik \cdot x} z_{\pm j} \partial_{z_j}, \quad e^{ik \cdot x} \bar{z}_{\pm j} \partial_{\bar{z}_j}, \quad \forall k \in \mathbb{Z}_{\text{odd}}^n, \quad j \in \mathbb{Z} \setminus \mathcal{J},$$

where $\mathbb{Z}_{\text{odd}}^n := \{k \in \mathbb{Z}^n : k_{-j} = -k_j, \forall j \in \mathcal{J}\}$, can *not* be averaged out. On the other hand, on the invariant subspace E , where we look for the quasi-periodic solutions, the above terms can be replaced by the constant coefficients monomial vector fields, obtained setting $x_{-j} = x_j$, $z_{-j}^{\pm} = z_j^{\pm}$. More precisely, we proceed as follows (section 5 of [3]): we replace the nonlinear vector field perturbation P with its symmetrized $\mathcal{S}P$ defined, by linearity, as

$$(2.32) \quad \mathcal{S}(e^{ik \cdot x} \partial_{x_j}) := \partial_{x_j}, \quad \mathcal{S}(e^{ik \cdot x} y^i \partial_{y_j}) := y^i \partial_{y_j}, \quad \forall k \in \mathbb{Z}_{\text{odd}}^n, \quad |i| = 0, 1, \quad j \in \mathcal{J},$$

$$\mathcal{S}(e^{ik \cdot x} z_{\pm j} \partial_{z_j}) := z_j \partial_{z_j}, \quad \mathcal{S}(e^{ik \cdot x} \bar{z}_{\pm j} \partial_{\bar{z}_j}) := \bar{z}_j \partial_{\bar{z}_j}, \quad \forall k \in \mathbb{Z}_{\text{odd}}^n, \quad j \in \mathbb{Z} \setminus \mathcal{J},$$

and \mathcal{S} is the identity on the other monomial vector fields. Since

$$P|_E = (\mathcal{S}P)|_E$$

the two vector fields determine the same dynamics on the invariant subspace E (Corollary 5.1 of [3]). Moreover $\mathcal{S}P$ is reversible as well, and its weighted (see (2.37)) and quasi-Töplitz norms are (essentially) the same[†] as those of P (Proposition 5.2 of [3]).

The homological equations for a *symmetric* and *reversible* vector field perturbation can now be solved (Lemma 5.1 of [3]) and the remaining resonant term is a diagonal, constant coefficients correction of the normal form (2.31) (also using the *real coefficients* property). This procedure allows the KAM iteration to be carried out. Note that, after this composite KAM step, the correction to the normal frequencies described in (2.35) comes out from the symmetrized vector field $\mathcal{S}P$ and not P itself.

As in the Hamiltonian case [2], a major difficulty of the KAM iteration is to fulfill, at each iterative step, the second order Melnikov non-resonance condi-

[†]This is due to the fact that the symmetrization procedure in (2.32) does not increase the momentum, see (2.36).

tions. Actually, following the formulation of the KAM theorem given in [2] it is sufficient to verify

$$(2.33) \quad |\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| \geq \frac{\gamma}{1 + |k|^\tau}, \quad \gamma > 0,$$

only for the “final” frequencies $\omega^\infty(\xi)$ and $\Omega^\infty(\xi)$ and not along the inductive iteration.

As in [2] the key idea for verifying the second order Melnikov non-resonance conditions (2.33) for DNLW is to prove the higher order asymptotic decay estimate

$$(2.34) \quad \Omega_j^\infty(\xi) = j + a(\xi) + \frac{m}{2j} + O\left(\frac{\gamma^{2/3}}{j}\right) \quad \text{for } j \geq O(\gamma^{-1/3})$$

where $a(\xi)$ is a constant independent of j .

This property follows by introducing the notion of *quasi-Töplitz vector field*, see Definition 3.4 in [3]. The new normal frequencies for a symmetric perturbation $P = \mathcal{L}P$ are $\Omega_j^+ = \Omega_j + iP^{z_j, z_j}$ where the corrections P^{z_j, z_j} are the diagonal entries of the matrix defined by

$$(2.35) \quad P^{z, z} z \partial_z := \sum_{i, j} P^{z_i, z_j} z_j \partial_{z_i}, \quad P^{z_i, z_j} := \int_{\mathbb{T}^n} (\partial_{z_j} P^{(z_i)})(x, 0, 0, 0; \xi) dx.$$

Note that thanks to the *real-coefficients* property the corrections iP^{z_j, z_j} are real. We say that a matrix $P = P^{z, z}$ is quasi-Töplitz if it has the form

$$P = T + R$$

where T is a Töplitz matrix (i.e. constant on the diagonals) and R is a “small” remainder, satisfying in particular $R_{jj} = O(1/j)$. Then (2.34) follows with the constant $a := T_{jj}$ which is independent of j .

The definition of quasi-Töplitz vector field is actually simpler than that of quasi-Töplitz function, used in the Hamiltonian context [2], [22]. In turn, the notion of quasi-Töplitz function is weaker than the Töplitz-Lipschitz property, introduced by Eliasson-Kuksin in [13].

The quasi-Töplitz nature of the perturbation is preserved along the KAM iteration (with slightly modified parameters) because the class of quasi-Töplitz vector fields is closed with respect to

1. Lie bracket $[\cdot, \cdot]$ (Proposition 3.1 of [3]),
2. Lie series (Proposition 3.2 of [3]),
3. Solution of the homological equation (Proposition 5.3 of [3]),

namely the operations which are used along the KAM iterative scheme.

An important difference with respect to [2] is that we do *not* require the conservation of momentum, and so our KAM theorem applies to the DNLW

equation (1.2) where the nonlinearity g may depend on the space variable x . At a first insight this is a serious problem because the properties of quasi-Töplitz functions as introduced in [22] and [2], strongly rely on the conservation of momentum.

We remark that, anyway, the concept of *momentum* of a vector field is well defined (and useful) also if it is not conserved (prime integral), and so for PDEs which are not-invariant under x -translations (but with $x \in \mathbb{T}$). The momentum of a monomial vector field

$$m_{k,i,\alpha,\beta;\nu} := e^{ik \cdot x} y^i z^\alpha \bar{z}^\beta \partial_\nu, \quad \nu \in \{x, y, z_j, \bar{z}_j\},$$

(with multi-indices notation $z^\alpha \bar{z}^\beta := \prod_{j \in \mathbb{Z} \setminus \mathcal{J}} z_j^{\alpha_j} \bar{z}_j^{\beta_j}$, $\alpha_j, \beta_j \in \mathbb{N}$) is defined by

$$(2.36) \quad \pi(k, \alpha, \beta; \nu) := \begin{cases} \pi(k, \alpha, \beta) & \text{if } \nu \in \{x_1, \dots, x_n, y_1, \dots, y_n\} \\ \pi(k, \alpha, \beta) - \sigma j & \text{if } \nu = z_j^\sigma, \sigma = \pm, \end{cases}$$

where

$$\pi(k, \alpha, \beta) := \sum_{i=1}^n j_i k_i + \sum_{j \in \mathbb{Z} \setminus \mathcal{J}} (\alpha_j - \beta_j) j,$$

and $\mathcal{J} := \{j_1, \dots, j_n\}$ are the tangential sites. The monomial vector fields $m_{k,i,\alpha,\beta;\nu}$ are nothing but the eigenvectors, with eigenvalues $i\pi(k, \alpha, \beta; \nu)$, of the adjoint action of the momentum vector field $X_M := (j, 0, \dots, ijz_j, \dots, -ij\bar{z}_j, \dots)$, namely

$$[m_{k,i,\alpha,\beta;\nu}, X_M] = i\pi(k, \alpha, \beta; \nu) m_{k,i,\alpha,\beta;\nu},$$

see Lemma 2.1 of [3]. This is why it is convenient to use the exponential basis in the Fourier decomposition (2.23) instead of the cosine basis $\{\cos(jx)\}_{j \geq 0}$.

REMARK 2.2. *For a PDE which is translation invariant (namely the nonlinearity g is x -independent), all the monomials of the corresponding vector field X have momentum equal to zero.*

Then we overcome the impasse of the non-conservation of the momentum introducing the a -weighted majorant norm for vector fields

$$(2.37) \quad \|X\|_{s,r,a} := \sup_{|y|_1 < r^2, \|z\|_{a,p}, \|\bar{z}\|_{a,p} < r} \left\| \left(\sum_{k,i,\alpha,\beta} e^{a|\pi(k,\alpha,\beta;\nu)|} |X_{k,i,\alpha,\beta}^{(\nu)}| e^{|k|s} |y^i| |z^\alpha| |\bar{z}^\beta| \right)_{\nu \in \mathbf{V}} \right\|_{s,r}$$

where $r, s, a > 0$, $\mathbf{V} := \{x, y, z_j, \bar{z}_j\}$, $j \in \mathbb{Z} \setminus \mathcal{J}$, and

$$\|(x, y, z, \bar{z})\|_{s,r} := \frac{|x|_\infty}{s} + \frac{|y|_1}{r^2} + \frac{\|z\|_{a,p}}{r} + \frac{\|\bar{z}\|_{a,p}}{r}, \quad \|z\|_{a,p}^2 := \sum_{j \in \mathbb{Z} \setminus \mathcal{J}} |z_j|^2 e^{2a|j|} \langle j \rangle^{2p},$$

$a \geq 0$, $p > 1/2$ fixed (analytic spaces). The $\|\cdot\|_{s,r,a}$ -norm penalizes the high-momentum monomials

$$(2.38) \quad \|\Pi_{|\pi| \geq K} X\|_{s,r,a'} \leq e^{-K(a-a')} \|X\|_{s,r,a}, \quad \forall 0 \leq a' \leq a,$$

so that only the *low-momentum* monomials vector fields with $|\pi| \leq K$ are relevant (slightly decreasing $a' < a$). This fact is crucial, in particular, in order to prove that the class of quasi-Töplitz vector fields is closed with respect to Lie brackets (Proposition 3.1 of [3]).

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