



**Partial Differential Equations** — *Birkhoff normal form for the nonlinear Schrödinger equation*, by WALTER CRAIG, ALESSANDRO SELVITELLA and YUN WANG, communicated on 11 January 2013.

ABSTRACT. — This paper is intended to highlight the differences between the nonlinear Schrödinger equation (NLS) posed on a compact manifold (such as a torus  $\mathbb{T}^d$ ) in contrast to being posed on noncompact regions such as on all of  $\mathbb{R}^d$ . The point is to indicate a number of specific facts about the behavior of solutions in the former situation, in which they have the possibility for recurrence, and the latter, in which solutions have the tendency to disperse. This is the topic of the short article by McKean [8], in which the issue of resonance for partial differential evolution equations is discussed. The aspect of this question that we describe in the present paper is that there are different normal forms for these two cases, which rephrases the question as to which of the nonlinear terms are the resonant terms, and what is the appropriate Birkhoff normal form for the NLS. We show that, at least in a neighborhood of zero of an appropriate Hilbert space, the fourth order Birkhoff normal form transformation for the NLS equation is able to eliminate all of the nonresonant terms of the Hamiltonian, and as well, all of the resonant terms. The result is a prognosis, to the negative, for the formal theory of wave turbulence for Hamiltonian partial differential equations posed in Sobolev spaces over  $\mathbb{R}^d$ .\*

KEY WORDS: Hamiltonian PDEs, nonlinear Schroedinger equations, nonlinear scattering theory.

MATHEMATICS SUBJECT CLASSIFICATION: 35Q55.

## 1. INTRODUCTION

We would like to frame a discussion of transformation theory of Hamiltonian partial differential equations (PDEs) in terms of a particular equation, which for convenience will be the nonlinear Schrödinger equation (NLS),

$$(1) \quad i\partial_t u = \frac{1}{2} \Delta_x u - \sigma |u|^2 u.$$

Denote the solution map, or *flow* of this equation by  $\varphi_t(u_0) = u(t)$ , where  $u(0) = u_0$  is the specified initial data. The defocusing case is given by the choice

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\*The results of this paper are related to the lecture that one of the Authors (W. C.) gave at the Conference “Dynamics of PDEs”, which took place at the Accademia dei Lincei on October 14, 2011.

$\sigma = +1$  and the focusing case by  $\sigma = -1$ . The energy for this PDE is given by the Hamiltonian functional

$$(2) \quad H(u) := \int \frac{1}{2} |\nabla u|^2 + \sigma \frac{1}{2} |u|^4 dx = H^{(2)} + H^{(4)},$$

with which the equation (1) is written in Hamilton's canonical form, in complex symplectic coordinates on  $L^2$ ,

$$(3) \quad \partial_t u = i \nabla_{\bar{u}} H.$$

The frequencies of the normal modes are given by the dispersion relation

$$(4) \quad \omega(k) = \frac{1}{2} |k|^2,$$

with which, using the notation  $D = -i\partial/\partial_x$  one can express solutions of the linearized equations for initial data  $u(x, 0) = u_0(x)$  in the form

$$(5) \quad \partial_t u = i \nabla_{\bar{u}} H^{(2)} = -i \frac{1}{2} \Delta u, \quad u(x, t) = \exp(i\omega(D)t) u_0(x) := \Phi_t(u_0)(x).$$

Many other physically relevant Hamiltonian PDEs can be expressed in this way, whose linearizations about the equilibrium solution  $u = 0$  take a similar form. These include the Korteweg-deVries equation, with dispersion relation  $\omega_{KdV}(k) = k^3$ , nonlinear wave equations and the Klein–Gordon equations, with dispersion relation  $\omega_{KG}(k) = \sqrt{|k|^2 + m^2}$ , Euler's equations of fluid dynamics, and the equations of free surface water waves, with dispersion relation  $\omega_w(k) = \sqrt{g|k| \tanh(h|k|)}$ .

In general, and on a formal level, a Hamiltonian PDE takes the form

$$(6) \quad \partial_t z = J \operatorname{grad}_z H := X^H(z), \quad z(x, 0) = z^0(x),$$

considered as an initial value problem. This system is posed on an appropriate *phase space*  $z \in M_0$ , which for convenience we take to be a Hilbert space with inner product  $\langle X, Y \rangle_{M_0}$ . The symplectic form is represented in terms of this inner product

$$\omega(X, Y) = \langle X, J^{-1} Y \rangle_{M_0}, \quad J^T = -J$$

for vector fields  $X, Y$ , which in turn gives the classical relationship defining the Hamiltonian vector field  $X^H$ , namely for all  $Y$

$$dH(Y) = -\omega(X^H, Y).$$

For technical reasons, because these equations are partial differential equations and we must be able to solve the initial value problem, the flow, or solution map  $z(x, t) = \varphi_t(z^0(x))$  is defined on an appropriate dense subspace  $z^0 \in M \subseteq M_0$ . The typical case for  $M_0 = L^2$  and  $M = H^s$  a Sobolev space of sufficient smoothness to solve, at least locally in time, the initial value problem.

Given the flow of  $X^H$  and reasonable functionals  $F$ , the Poisson brackets are also defined in the usual way, namely

$$\{F, H\} := \langle \text{grad}_z F, J \text{grad}_z H \rangle_{M_0} = X^H(F) = \left. \frac{d}{dt} \right|_{t=0} F(\varphi_t(z)),$$

where  $\varphi_t(z)$  is of course the flow of the Hamiltonian vector field  $X^H$ .

The field of Hamiltonian PDEs focuses on the analysis of the phase space of partial differential equations, which are necessarily infinite dimensional. The aim is to go beyond the basic question of existence and uniqueness of solutions, to describe some of the important orbit structures and other dynamically relevant features, their dependence on parameters, and their stability. One of the questions of interest is to what extent these features are, or are not, similar to phenomena that occur for finite dimensional Hamiltonian systems.

Simple conservation laws for the NLS (1) are

$$M(u) := \int |u|^2 dx, \quad \text{mass}$$

$$P(u) := \text{im} \int \bar{u} \nabla u dx, \quad \text{momentum}$$

$$H(u) := \text{energy} = (2)$$

These facts can be checked by using the definition of the Poisson bracket and integrating by parts,  $\{M, H\} = 0$ ,  $\{P, H\} = 0$ . It is clear that, while we are using  $M_0 = L^2$  for the Schrödinger equation, the energy functional is only defined for  $u \in \dot{H}^1 \cap L^4 := M$  the energy space.

The intent of his article is to highlight some of the differences in behavior between the NLS (and Hamiltonian PDEs in general) posed on compact spatial domains, such as  $x \in \mathbb{T}^d$  for example, and the noncompact case of  $x \in \mathbb{R}^d$ . In particular it is to make a concrete statement in the discussion in the folklore on the topic on the difference between cases in which solutions tend to disperse, and cases in which they recur. A reference relevant to this discussion is the paper by H. P. McKean on *How real is resonance?* [8]. The idea is that there is a different normal form for the two cases  $\mathbb{T}^d$  and  $\mathbb{R}^d$ . The question we want to address is as to the nature of nonlinear resonance for a Hamiltonian partial differential equation. Namely, which of the terms of the Hamiltonian are to be considered to be the resonant terms, playing a rôle in nonlinear coherent structures in its solutions such as solitons, and in recurrence phenomena of orbits. That is, we discuss which terms are necessarily present in the appropriate normal form for the equation, and are obstructions to its complete linearization.

## 2. BIRKHOFF NORMAL FORM

A normal form is the result of a transformation that is intended to simplify the Hamiltonian, retaining only essential nonlinearities. Birkhoff normal form refers to the specific case of a normal form for a Hamiltonian systems in a neighborhood of an elliptic stationary point, which is attainable through near—identity canonical transformations. Without loss of generality assume that the stationary point in question is  $u = 0$ , and express the Hamiltonian in its Taylor series (with remainder);

$$(7) \quad H(u) := H^{(2)}(u) + H^{(3)}(u) + H^{(4)}(u) + \cdots + H^{(N)}(u) + R^{(N+1)}(u).$$

Under a canonical transformation  $v = \tau(u)$  to Birkhoff normal form, we achieve  $\tilde{H}(v) := H(u)$  in the form

$$(8) \quad \tilde{H}(v) = H^{(2)}(v) + Z^{(3)}(v) + Z^{(4)}(v) + \cdots + Z^{(N)}(v) + \tilde{R}^{(N+1)}(v).$$

The homogeneous Taylor polynomials  $Z^{(j)}(v)$  are the remaining *resonant terms*, and all nonresonant terms are eliminated by the transformation  $\tau$ . In finite dimensions the system

$$\dot{u} = J \operatorname{grad}_u H^{(2)}$$

possesses finitely many frequencies  $\{\omega(k)\}_{k=1}^n$ , the resonance conditions are finite at each order  $N$ , and Birkhoff normal form can always be achieved up to the order that is allowed given the smoothness of the Hamiltonian  $H$ . In the case of a PDE whose linearization about zero has only discrete point spectrum, such as the NLS on the torus  $\mathbb{T}^d$ , there are a countably infinite number of modes, discrete frequencies  $\{\omega(k)\}_{k \in \mathbb{Z}^d}$ , and the resonant terms in  $Z^{(N)}$  are associated with resonance conditions

$$\sum_{k \in \mathbb{Z}^d} (\omega(k)p_k - \omega(k)q_k) = \langle \omega | P - Q \rangle = 0.$$

Here  $|P| + |Q| = N$ , where we are introducing multi-index notation for  $P = (p_k)_{k \in \mathbb{Z}^d}$ ,  $Q = (q_k)_{k \in \mathbb{Z}^d}$ . In the case in which the sequence  $\{\omega(k)\}_{k \in \mathbb{Z}^d}$  is as non-resonant as is possible for a Hamiltonian system, meaning that its only resonances stem from the case of multi-indices  $P = Q$ , then the normal form can be expressed, at least formally, in terms of action variables alone,  $Z_j = Z_j(I)$ ,  $j = 3 \dots N$ , and the truncated Hamiltonian  $H^{(2)}(I) + Z^{(3)}(I) + \cdots + Z^{(N)}(I)$  is integrable. It of course remains an analytic issue as to the function space mapping properties of the associated normal forms transformation, whether this normal form has any rigorous implications for the dynamics of orbits of the equations or not. In the case of the NLS in one dimension, with Dirichlet boundary conditions, S. Kuksin and J. Pöschel [7] showed that the Birkhoff normal form up to fourth order is achieved in a neighborhood of the origin by a biholomorphic transformation, and that it is nonresonant in the sense given above. For the

NLS posed on  $\mathbb{T}^d$  the analog Birkhoff normal forms is resonant, and the Birkhoff normal forms transformation is given by C. Procesi and M. Procesi [9].

We contrast the discrete case with the example of the NLS posed on all of  $\mathbb{R}^d$ , a situation that can occur for which the linearized operator  $J \operatorname{grad}_u H^{(2)}$  has purely continuous spectrum,

$$(9) \quad i\partial_t u = \frac{1}{2} \Delta_x u - \sigma |u|^2 u, \quad u \in M \subseteq L^2(\mathbb{R}^d).$$

Its Hamiltonian (2) can be expressed in terms of the Fourier transform

$$\begin{aligned} H &= \int_{\mathbb{R}_x^d} \frac{1}{2} |\nabla u|^2 + \sigma \frac{1}{2} |u|^4 dx \\ &= \int_{\mathbb{R}_k^d} \omega(k) |\hat{u}(k)|^2 dk + \frac{\sigma}{2(2\pi)^d} \iiint_{k_1+k_2=k_3+k_4} \hat{u}(k_1) \hat{u}(k_2) \bar{\hat{u}}(k_3) \bar{\hat{u}}(k_4) dk_1 dk_2 dk_3. \end{aligned}$$

We note that the Fourier transform  $u(x) \mapsto \hat{u}(k)$  is a canonical transformation, and we recall that the frequencies are given by  $\omega(k) = \frac{1}{2} |k|^2$ .

Following the outline of Birkhoff’s recipe, a fourth order normal form for  $H$  is given through a solution of the cohomological equation

$$(10) \quad \{H^{(2)}, G^{(4)}\} = H^{(4)},$$

which has a solution in this case

$$G^{(4)} = -i \frac{\sigma}{(2\pi)^d} \iiint_{k_1+k_2=k_3+k_4} \frac{\hat{u}(k_1) \hat{u}(k_2) \bar{\hat{u}}(k_3) \bar{\hat{u}}(k_4)}{\frac{1}{2} (|k_1|^2 + |k_2|^2 - |k_3|^3 - |k_4|^2)} dk_1 dk_2 dk_3.$$

Rewrite the denominator to exhibit the convolution nature of the integral kernel,

$$\frac{1}{2} (|k_1|^2 + |k_2|^2 - |k_3|^3 - |k_4|^2) = -(k_1 - k_3) \cdot (k_2 - k_3).$$

The actual transformation is given by the time one flow of the Hamiltonian vector field, which we express in spatial variables:

$$\begin{aligned} X^{G^{(4)}}(u) &:= J \operatorname{grad}_{\bar{u}} G^{(4)}(u) \\ &= -\sigma \iint_{\mathbb{R}^{2d}} u(x_1) u(x_2) \bar{u}(x - x_1 - x_2) \operatorname{sgn}((x - x_1) \cdot (x - x_2)) dx_1 dx_2. \end{aligned}$$

Subsequent to this somewhat formal discussion, the main outstanding question is whether the flow  $\psi_s(u)$  of the Hamiltonian vector field  $X^{G^{(4)}}$  exists, and on which Banach spaces does it give a well defined transformation of a neighborhood of  $u = 0$ . For  $r \geq 0$  define the Sobolev spaces  $H^{r,r}(\mathbb{R}^d) := \{u \in L^2 : \partial_x^\alpha u, x^\beta u \in L^2 \forall |\alpha| = r, |\beta| = r\}$ .

**THEOREM 2.1.** *For any  $r > n/2$  the vector field  $X^{G^{(4)}}$  is holomorphic on  $H^{r,r}$  in the variables  $(u, \bar{u})$ . Therefore the Hamiltonian flow  $\psi_s(u)$  exists, locally in  $s$ , and for sufficiently small  $R$  gives rise to a biholomorphic canonical transformation  $\tau^{(4)} := \psi_s|_{s=1}$  on a neighborhood  $B_R(0) \subseteq H^{r,r}$ .*

This result will follow by standard arguments, once we have established a Lipschitz estimate for the vector field  $X^{G^{(4)}}$ .

**LEMMA 2.2.** *Consider  $u_1, u_2 \in B_R(0) \subseteq H^{r,r}$ , then the Hamiltonian vector field of  $G^{(4)}$  satisfies the estimate*

$$\|X^{G^{(4)}}(u_1) - X^{G^{(4)}}(u_2)\|_{r,r} \leq CR^2 \|u_1 - u_2\|_{r,r}.$$

The proof of this lemma is given in [4].

Several remarks are in order at this point of the discussion. The first is the fact that the canonical transformation we have constructed, namely  $\tau^{(4)}(u) := \psi_s(u)|_{s=1}$  removes all of the nonresonant terms of the original fourth order Hamiltonian. But it also succeeds in removing all of the resonant terms of the Hamiltonian. This is closely related to the fact that a singular integral kernel can give rise to a bounded integral operator, when acting in the setting of continuous spectrum. When performing the same formal exercise for the periodic case  $x \in \mathbb{T}^d$ , Fourier transform variables are sequences, and an operator on sequences is definitely not bounded if one of its matrix elements is infinite.

There are straightforward sufficient conditions for regular behavior of a singular integral kernel which is designed to remove the term  $\hat{u}^P \hat{u}^Q$  in the Hamiltonian. For the  $N^{th}$  order term of the Hamiltonian  $H^{(N)}$ , these conditions are that on the resonant set

$$(11) \quad \mathcal{R}_N(P, Q) := \{(k_1, \dots, k_N) \in \mathbb{R}^{Nd} \mid \langle P - Q | k \rangle = 0 \text{ and } \langle P - Q | \omega(k) \rangle = 0\},$$

the resonance relation vanishes only to first order, namely for all  $j = 1 \dots N$ ,

$$(12) \quad (p_j \partial_k \omega(k_j) - q_j \omega(k_j)) \neq 0.$$

In [5] the variety  $\mathcal{R}_N := \bigcup_{P, Q} \mathcal{R}_N(P, Q)$  that consists of the union of all of the above resonant sets (11) is called the *time resonant set* of order  $N$ . Denote by  $\mathcal{P}_N(P, Q)$  the set on which (12) fails for a given  $(P, Q)$  and for at least one  $1 \leq j \leq N$ , namely

$$\mathcal{P}_N(P, Q) := \{(k_1, \dots, k_N) \in \mathbb{R}^{Nd} \mid (p_j \partial_k \omega(k_j) - q_j \omega(k_j)) = 0, \text{ for some } 1 \leq j \leq N\}.$$

Setting  $\mathcal{P}_N := \bigcup_{P, Q} \mathcal{P}_N(P, Q)$ , which is called the *space resonant set* in [5], the difficult case lies in the intersection  $\mathcal{P}_N \cap \mathcal{R}_N$ . In the situation of the cubic nonlinear Schrödinger equation, and considering the normal forms transformation for  $H^{(4)}$ , this intersection only occurs on the coordinate axes  $\{(k_1 - k_3) = 0\} \cap \{(k_2 - k_3) = 0\}$ , where the special product structure of the kernel ensures that the singular integral operator remains bounded.

There remain a number of questions at this point, on the domain and the behavior of the normal forms transformation  $\tau^{(4)}$ , including the following. Question

(1) Is the normal form transformation defined globally on the space  $H^{r,r}$ , or just locally in a neighborhood which includes the ball  $B_R(0)$ ? This is relevant because Schrödinger flow, both for the linear Schrödinger equation and for the NLS, preserve the function space  $H^{r,r}$ , but not the ball  $B_R(0) \subseteq H^{r,r}$ . Indeed typically the norm  $\|\varphi_t(u)\|_{r,r}$  is growing for large time. Question (2) Is the Birkhoff normal forms transformation  $\tau^{(4)}$  defined on a larger space, such as the more natural  $\dot{H}^1 \cap L^4$ , which is the energy space for the NLS equation? And Question (3) we have only touched upon the boundedness of the fourth order normal form; although we are formally in the position to treat the higher transformations, we have not done this so far.

The result of Theorem 2.1 reflects negatively on the efforts for a rigorous theory of wave turbulence, making it a more difficult mathematical objective. The reason is that many physicists view the theory of wave turbulence as an averaging theory, in which details of individual orbits and the phases of each active mode are not as important as more slowly varying quantities such as the action variables and other averaged quantities, for which the essential nonlinear interactions in the Hamiltonian are retained. Normal forms transformations play an important rôle in these ideas, seeking to eliminate as much as possible a rapid phase evolution resulting from the dependence of the Hamiltonian on angle variables. To make a rigorous analytic formulation for this transformation theory, is necessary to ascribe boundary conditions for the ensemble of wave fields that one wishes to address as the realizations in a theory of turbulence. This can be phrased in terms of the choice of phase space for the problem. Candidates include  $u(\cdot) \in L^2(\mathbb{T}_{2\pi}^d)$ , where  $\mathbb{T}_{2\pi}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$ . However the class of periodic functions is considered to be too limited to appropriately represent a very complicated wave field, and as well the periodic structure doesn't admit long wave spatial scalings, so it is ruled out by many wave turbulence theorists. A second choice might be  $u(\cdot) \in L^2(\mathbb{R}^d)$ , however the above Theorem 2.1 shows that there is not a sufficient phenomenon of nonlinear resonance and recurrence for there to be sustained turbulence. Indeed the dispersion is strong in the problem, all sufficiently small solutions linearly scatter and decay to zero, and the only invariant measure is a delta function measure supported on  $u = 0$ . This leads to the third suggestion, which is that one takes as phase space  $u \in \lim_{L \rightarrow +\infty} L^2(\mathbb{T}_L^d)$ , a thermodynamic limit of tori of increasingly large size. This is a subset of  $L_{loc}^2$ , functions with infinite energy but finite energy density. It is however a Fréchet space, and therefore it represents a more difficult setting in which to analyse solutions of these equations and their flows.

### 3. ENVELOPE FORMULATION

The concept of an envelope formulation is naturally tied to a variation of the classical Duhamel's principle, which we describe here. Consider a Hamiltonian PDE for a quantity  $u(x, t)$ , with Hamiltonian given as

$$H := H^{(2)} + R,$$

where  $H^{(2)}$  is quadratic in  $u$  and describes the linear part of the equation,  $R$  is considered a higher order perturbation, and  $u = 0$  is an elliptic point of equilibrium. The linearized solution describes the tangent space approximation

$$\dot{u} = X^{H^{(2)}}(u).$$

Denote the linear flow of this equation by  $\Phi_t(u)$ . The classical *envelope function* is defined by

$$(13) \quad M(t) = \Phi_{-t}(u(t)),$$

which satisfies Duhamel's principle

$$(14) \quad \dot{M} = \Phi_{-t}X^R(\Phi_t(M)).$$

This is usually described in terms of the integral equation

$$M(t) = M(0) + \int_0^t \Phi_{-s}X^R(\Phi_s(M)) ds.$$

Consider instead a more general system, with Hamiltonian  $H + R$ , where the flow  $\varphi_t$  for the Hamiltonian vector field  $X^H$  is known. To study

$$(15) \quad \dot{u} = X^{H+R}(u),$$

define the *generalized envelope*  $m(t)$  using this flow,

$$(16) \quad m(t) = \varphi_{-t}(m(t)).$$

Then

$$(17) \quad \begin{aligned} \dot{u} &= \dot{\varphi}_t(m) + \partial_u \varphi_t(m) \dot{m} \\ &= J \operatorname{grad}_u H(u) + J \operatorname{grad}_u R(u), \end{aligned}$$

and solving for  $\dot{m}$ ,

$$(18) \quad \begin{aligned} \dot{m} &= (\partial_u \varphi_t(m))^{-1} (J \operatorname{grad}_u H(u) + J \operatorname{grad}_u R(u) - \dot{\varphi}_t(m)) \\ &= (\partial_u \varphi_t(m))^{-1} (J \operatorname{grad}_u R(u)), \end{aligned}$$

since  $u = \varphi_t(m)$ . Because the flow  $\varphi_t(m)$  is a canonical map,  $\partial_u \varphi_t(m) J \partial_u \varphi_t(m)^T = J$ , therefore

$$(19) \quad \begin{aligned} \dot{m} &= (\partial_u \varphi_t(m))^{-1} (J \operatorname{grad}_u R(u)) \\ &= J (\partial_u \varphi_t(m))^T (\operatorname{grad}_u R(u)) \\ &= J \operatorname{grad}_m R(\varphi_t(m)). \end{aligned}$$

The functional  $R(\varphi_t(m))$  is a new non-autonomous Hamiltonian describing the perturbed system for  $m$ , which is in many interesting cases a higher order non-linearity than the original system (15).

The outstanding question is for which cases do we know the flow  $\varphi_t$  of  $X^H$ . The classical situation of an envelope equation is to take  $H = H^{(2)}$  which gives the linear flow, where one is reduced to the study of (14). Another possibility is when to use  $H = H^{(2)}(I) + Z(I)$  a nonlinear but integrable flow, which is available for example from a nonresonant Birkhoff normal forms transformation. A third possibility is the situation in which  $Z \equiv 0$ , which will occur in the case of the NLS posed on  $\mathbb{R}^d$  with small initial data, as we are describing above.

#### 4. SCATTERING THEORY

There is another setting for transformations that simplify the flow  $\varphi_t$  of  $X^H$ , namely that of nonlinear scattering theory. In this section let us work with a critical or a subcritical nonlinearity, which restricts us in the case of the cubic Schrödinger equations to dimensions  $2 \leq d \leq 4$ . Solutions of (1) with  $\sigma = -1$ , with initial data which is small in energy norm  $\|u\|_{H^1}$ , disperse to zero as  $t \mapsto \pm\infty$  and behave asymptotically as solutions of the linear Schrödinger equation (5). The same is true for the defocusing case  $\sigma = +1$  without restriction on the size of the data in energy norm; this is the content of the following result of Ginibre & Velo [6].

**THEOREM 4.1** [Ginibre & Velo (1985)]. *Let  $\sigma = +1$ , the defocusing case. For all  $u \in H^1(\mathbb{R}^d)$  the limits*

$$\lim_{t \rightarrow \pm\infty} \Phi_{-t}(\varphi_t(u)) = u_{\pm} := \Omega_{\pm}(u)$$

*exist, and are continuous in  $u$ . In the case of a focusing nonlinearity  $\sigma = -1$  then the limits exist for all  $u \in B_R(0) \subseteq H^1(\mathbb{R}^d)$  and are continuous in  $u$ , for a constant  $R$  given by the  $L^2$  norm of the nonlinear ground state.*

The mappings  $u \mapsto \Omega_{\pm}(u)$  are called the forward and backwards *nonlinear scattering maps*. The existence of these limits implies that as  $t \mapsto +\infty$  the nonlinear solutions  $\varphi_t(u)$  tend to the evolution of a linear solution  $\Phi_t(u_+)$  with initial data  $u_+$ . There is a similar description of the behavior of solutions as  $t \mapsto -\infty$ . The theory of nonlinear scattering has been a very active area of research in nonlinear PDEs for at least two decades, with improvements on the nonlinearities for which scattering behavior is known, and with contributions to the regularity of the scattering maps  $\Omega_{\pm}(u)$ . Of particular interest for this paper is the fact that the scattering maps are transformations of the nonlinear flow to a linear flow. Indeed, we have the relationship of conjugacy

$$(21) \quad \Omega_{\pm}(\varphi_t(u)) = \Phi_t(\Omega_{\pm}(u)),$$

as one can check from the uniqueness of the solution map. Namely, for  $\Omega_+$  we have

$$\Phi_{-s}(\varphi_s(\varphi_t(u))) = \Phi_t(\Phi_{-(s+t)}(\varphi_{s+t}(u))),$$

and taking  $s \mapsto +\infty$  (respectively  $s + t \mapsto +\infty$ ) the LHS has limit  $\Omega_+(\varphi_t(u))$  while the RHS has limit  $\Phi_t(\Omega_+(u))$ . Incidentally, under the involution of time reversal and complex conjugation,

$$t \mapsto -t, \quad u \mapsto \bar{u}$$

the scattering maps are exchanged;

$$\Omega_-(\bar{u}) = \overline{\Omega_+(u)}.$$

A short proof of the above Theorem 4.1 on scattering is available, this is adapted from [2] and [3], and we restrict ourselves to the case  $d = 3$  (and a cubic nonlinearity). The solution map  $\Phi_t(u)$  for the linear Schrödinger equation (5) preserves all Sobolev norms, and in particular the norms in  $L^2$  and  $H^1$ . The solution map of the nonlinear Schrödinger equation (1) satisfies the integral equation

$$u(t) = \varphi_t(u) = \Phi_t(u) + i\sigma \int_0^t \Phi_{t-s}(|\varphi_s(u)|^2 \varphi_s(u)) ds,$$

which is to say that

$$\Phi_{-t}(\varphi_t(u)) = u + i\sigma \int_0^t \Phi_{-s}(|\varphi_s(u)|^2 \varphi_s(u)) ds.$$

In order for the LHS to have a limit in  $H^1$ , the improper integral in the above expression should have a limit in  $H^1$ . This is implied by the following lemma.

LEMMA 4.2. *Suppose that  $\sigma = +1$ , or that  $\sigma = -1$  and  $u \in B_R(0) \subseteq H^1(\mathbb{R}^d)$ . For all  $\varepsilon > 0$  there is a time  $T = T_\varepsilon$  such that for  $t_1, t_2 > T$  then*

$$\left\| \int_{t_1}^{t_2} \Phi_{-s}(|\varphi_s(u)|^2 \varphi_s(u)) ds \right\|_{H^1} < \varepsilon.$$

PROOF. The proof relies on two inequalities, the ‘interaction’ Morawetz estimate for (1) and certain Strichartz estimates. The first inequality is that over any time interval  $I \subseteq \mathbb{R}$ , solutions obey

$$\int_I \int_{\mathbb{R}^3} |\varphi_t(u)(x)|^4 dx dt \leq C_1 \|u\|_{L^2}^2 \left( \sup_{t \in I} \|\varphi_t(u)\|_{\dot{H}^{1/2}} \right)^2.$$

The leading factor on the RHS is a conservation law, and depends only upon the initial data. The second factor is bounded in the defocusing case, and also for small  $H^1$  data in the focusing case.

The second inequality is the Strichartz estimates for the linear Schrödinger flow. Following [2],

$$\begin{aligned} \left\| \int_I \Phi_{-s}(|\varphi_s(u)|^2 \varphi_s(u)) ds \right\|_{H^1} &\leq C_2 \|\varphi_t(u)\|_{L_{tx}^5(I \times \mathbb{R}^3)}^2 \|\varphi_t(u)\|_{L_t^{10/3}(I; W_x^{1,10/3}(\mathbb{R}^3))} \\ &\leq C_3 (\|\varphi_t(u)\|_{L_{tx}^4(I \times \mathbb{R}^3)}^\beta \|\varphi_t(u)\|_{L_t^{10}(I; W_x^{1,30/13}(\mathbb{R}^3))}^{1-\beta})^2 \\ &\quad \times \|\varphi_t(u)\|_{L_t^{10}(I; W_x^{1,10/3}(\mathbb{R}^3))}. \end{aligned}$$

The indices  $(10/3, 10/3)$  and  $(10, 30/13)$  are admissible Strichartz pairs, so the second two norms are bounded, while by the Morawetz estimate the  $L_{xt}^4$  norm is arbitrarily small if  $I = [t_1, t_2]$  is chosen so that  $t_1, t_2 > T$  sufficiently large.  $\square$

With regards to the regularity of the scattering map, there is the following recent theorem

**THEOREM 4.3** (Carles & Gallagher (2009) [1]). *The scattering maps  $\Omega_{\pm}(u)$  are holomorphic in  $(u, \bar{u})$  in the space  $H^{1,1}$ .*

We therefore have three normal forms for the NLS, namely

$$u_+ = \Omega_+(u), \quad u_- = \Omega_-(u) = \overline{\Omega_+(\bar{u})}, \quad v = \tau(u),$$

where  $\tau = \tau^{(4)}$ , the Birkhoff normal forms transformation of Section 2, or indeed a higher order normal forms transformation if it is available. Each of these transformations is holomorphic on the appropriate phase space. It naturally brings up the question of the relationship between these transformations, and the more pointed question of why one should study the Birkhoff normal forms transformation in the first place. One answer is that the Birkhoff normal form gives insight into the rate of scattering of solutions.

### 5. RATES OF SCATTERING

The Birkhoff normal forms transformation is at least a concrete transformation, well approximated in a neighborhood of the origin by Poisson brackets with a specific Hamiltonian, namely  $G^{(4)}$ . It furthermore can be used in the study of rates of scattering, which we now explain. The scattering map satisfies Duhamel’s principle, which we state in terms of the quadratic Hamiltonian  $H^{(2)}$  and the perturbation  $R = H^{(4)}$  given in (2);

$$\Omega_+(u) - \Phi_{-t}(\varphi_t(u)) = \int_t^{+\infty} \Phi_{-s}(X^{H^{(4)}}(\varphi_s(u)) ds.$$

When  $d \geq 2$  and in the setting of solutions which disperse sufficiently strongly so that they scatter, a basic decay estimate for the solution,

$$\|\varphi_t(u)\|_{L^\infty} \leq \frac{C_d}{\langle t \rangle^{d/2}}$$

gives an estimate for the remainder, namely

$$\begin{aligned} \|\Omega_+(u) - \Phi_{-t}(\varphi_t(u))\|_{H^1} &\leq \int_t^{+\infty} \|\Phi_{-s}(i|\varphi_s(u)|^2\varphi_s(u))\|_{H^1} ds \\ &\leq \int_t^{+\infty} \left(\frac{C_d}{\langle s \rangle^{d/2}}\right)^2 ds \sim \frac{C_d^2}{\langle t \rangle^{d-1}}. \end{aligned}$$

Under Birkhoff normal form, the nonlinearity is of higher order; suppose that we have succeeded in transforming the NLS through a succession of normal forms  $v = \tau(u)$  so that

$$\partial_t v = J \operatorname{grad}(H^{(2)} + R), \quad R = R^{(2N+2)}(v)$$

with flow denoted by  $\tilde{\varphi}_t(v)$ . In this article we only analyse the fourth order Birkhoff normal forms transformation, but one could imagine that the higher order transformations are also bounded, in a correctly chosen neighborhood of the origin. This form of equation also exhibits scattering, with its scattering map given by

$$\tilde{\Omega}_\pm(v) = \lim_{t \rightarrow \pm\infty} \Phi_{-t}(\tilde{\varphi}_t(v)).$$

Therefore the flow  $\tilde{\varphi}_t(v)$  exhibits a faster rate of scattering, indeed

$$\begin{aligned} (22) \quad \|\tilde{\Omega}_+(v) - \Phi_{-t}(\tilde{\varphi}_t(v))\|_{H^1} &\leq \int_t^{+\infty} \|\Phi_{-s}(X^{R^{(2N+2)}}(\tilde{\varphi}_s(v)))\|_{H^1} ds \\ &\leq \int_t^{+\infty} C_N \|\tilde{\varphi}_s(v)\|_{L^\infty}^{2N} \|\tilde{\varphi}_s(v)\|_{H^1} ds \leq \frac{C_N C_d^{2N}}{\langle t \rangle^{Nd-1}}, \end{aligned}$$

a more rapid decay rate.

The flows  $\varphi_t(u)$  and  $\tilde{\varphi}_t(v)$  are conjugate to each other via the normal forms transformation  $v = \tau(u)$ ;

$$v(t) = \tilde{\varphi}_t(v) = \tau(\varphi_t(u)) = \tau(\varphi_t(\tau^{(-1)}(v))).$$

Furthermore the Birkhoff normal forms transformation is a near identity mapping, so we can write

$$\begin{aligned} v &= \tau(u) = u + \mathcal{F}(u), \quad \mathcal{F}(u) \sim \mathcal{O}(u^3) \\ u &= \tau^{(-1)}(v) = v + \tilde{\mathcal{F}}(v), \quad \tilde{\mathcal{F}}(v) \sim \mathcal{O}(v^3). \end{aligned}$$

Using this information, we write

$$\begin{aligned} u_+ - \Phi_{-t}(\varphi_t(u)) &= u_+ - \Phi_{-t}(\tau^{(-1)}(\tilde{\varphi}_t(v))) \\ &= (u_+ - v_+) + v_+ - \Phi_{-t}(\tilde{\varphi}_t(v)) - \Phi_{-t}\tilde{\mathcal{F}}(\tilde{\varphi}_t(v)) \\ &= (u_+ - v_+) - \Phi_{-t}\tilde{\mathcal{F}}(\Phi_t(v_+)) + E_+(t), \end{aligned}$$

where the error term is

$$E_+(t) := v_+ - \Phi_{-t}(\tilde{\varphi}_t(v)) - \Phi_{-t}\partial_u\tilde{\mathcal{F}}(\cdot)\Phi_t(e_t(v_+)),$$

with  $e_t(v_+) := v_+ - \Phi_{-t}(\tilde{\varphi}_t(v))$ . Therefore  $E_+(t)$ , satisfying the better decay rate (22), namely

$$\|e_t(v_+)\|_{H^1} \leq \mathcal{O}(\langle t \rangle^{-Nd+1}).$$

This is to say, the asymptotic decay rate of the flow of  $\varphi_t(u)$  to the scattering state  $\Phi_t(u_+)$  is described, up to errors of order  $\mathcal{O}(\langle t \rangle^{-Nd+1})$ , by the expression

$$(23) \quad u_+ - \Phi_{-t}(\varphi_t(u)) \simeq (u_+ - v_+) - \Phi_{-t}\tilde{\mathcal{F}}(\Phi_t(v_+)),$$

that is, in terms of a fixed transformation in terms of the Birkhoff normal forms Hamiltonian  $G^{(2N)}$  and the flow of the linear Schrödinger equation  $\Phi_t$ , which are able to be analysed by the techniques of harmonic analysis.

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