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On the Nakano Individual Convergence

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Abstract. We have recently defined the notion of individual convergence for a sequence of positive elements of an Archimedean Riesz space E. In the note we complete the definition (i.e., we define the individual convergence for sequences of not necessarily positive elements of E), and we prove that our notion of individual convergence is a natural extension of the individual convergence as defined by Nakano: we will prove that if a sequence of elements of E has an individual limit in the Nakano sense, then it converges individually with respect to our definition.

Keywords: *Archimedean Riesz spaces, Dedekind completion, projection bands, Nakano individual convergence*

AMS subject classification: 46A40, 47B65, 47A35

1. Introduction

In [8] we defined a notion of individual convergence of a sequence of positive elements of an Archimedean Riesz space in order to extend the ergodic theorem of Hopf [3] (see also Krengel's book [4]):

Let E be an Archimedean Riesz space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of (not necessarily positive) elements of E. We say that $(u_n)_{n\in\mathbb{N}}$ *converges individually* if both sequences $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge individually.

In 1948 in his pioneering work [6], Nakano also defined a notion of individual convergence for sequences of elements of a countably order complete Riesz space in order to extend an ergodic theorem of Birkhoff [2]. Our goal in this paper is to show that our definition of individual convergence of a sequence of elements of an Archimedean Riesz space is a natural extension of Nakano's notion of convergence of [6]. More precisely, we will extend Nakano's individual convergence to sequences of elements of an Archimedean Riesz space, and we will show that given an Archimedean Riesz space E and a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of E, then $(u_n)_{n\in\mathbb{N}}$ converges individually in our sense whenever it converges in Nakano's sense.

Unless explicitly stated otherwise, the terminology used in this paper can be found in the books of Aliprantis and Burkinshaw [1], Luxemburg and Zaanen [5], Schaefer [7], and in our papers [8, 9].

In the next section we will recall several notions and results of [6] and [9] which will be needed throughout the paper; thus, we will review the definitions of the Nakano

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individual convergence and the Nakano individual limit in the more general setting of Archimedean Riesz spaces (rather than countably order complete Riesz spaces in which Nakano originally stated his definitions), and we will describe briefly the notion of individual convergence of [9], as well as the results which make our definition possible. In Section 3 we will define the individual limit of a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of an Archimedean Riesz space, and we will show that if there exists an individual limit of $(u_n)_{n\in\mathbb{N}}$, then $(u_n)_{n\in\mathbb{N}}$ converges individually. Finally, in the last section (Section 4) we will show that the individual limit as defined in Section 3 is nothing but a reformulation of the definition of the Nakano individual limit, as stated in Section 2.

2. Preliminaries

Let E be an Archimedean Riesz space, and let $\tilde{E}$ be the Dedekind completion of E. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E, and let $u \in E$. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ *N-converges individually to* u (converges individually to u in the sense of Nakano) if

$$\limsup_n ((u_n \wedge c_1) \vee c_2) = \liminf_n ((u_n \wedge c_1) \vee c_2) = (u \wedge c_1) \vee c_2$$

for every $c_1, c_2 \in E$, $c_2 \leq c_1$ (naturally, the limsup and the liminf are taken in $\tilde{E}$). If a sequence $(u_n)_{n\in\mathbb{N}}$ N-converges individually to u, we call u the *individual N-limit* of $(u_n)_{n\in\mathbb{N}}$. By Lemma 1.3 of Nakano [6] (note that although the lemma was stated for countably order complete Riesz spaces, it is clearly true for elements of any Riesz space), it follows that the individual N-limit of a sequence is unique whenever it exists. Using Theorem 32.2 of [5], we obtain that if E is a countably order complete Riesz space, then the individual N-convergence is exactly the individual convergence defined by Nakano in [6].

Now let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E such that $u_n \geq 0$ for every $n \in \mathbb{N}$. As in [9], let $B_\infty((u_n)_{n\in\mathbb{N}})$ be the largest band in $\tilde{E}$ on which $(u_n)_{n\in\mathbb{N}}$ is unbounded. Set (see [9])

$$B_{OS}((u_n)_{n\in\mathbb{N}}) = \left\{ u \in \tilde{E} \;\middle|\; \begin{array}{l} u = 0, \text{ or } u \neq 0 \text{ and, for every } v \in \tilde{E} \text{ with } 0 \leq v \leq |u|,\ v \neq 0 \text{ there exist} \\ w \in \tilde{E} \text{ with } 0 \leq w \leq v, w \neq 0 \text{ and } \alpha, \beta \in \mathbb{R} \text{ with } 0 < \beta < \alpha \text{ such that} \\ \left(\limsup_n (((u_n - \beta w)^-) \wedge w)\right) \wedge \left(\limsup_n (((u_n - \alpha w)^+) \wedge w)\right) \neq 0 \end{array} \right\}$$

and

$$B_{NOS}((u_n)_{n\in\mathbb{N}}) = \left\{ u \in \tilde{E} \;\middle|\; \begin{array}{l} u = 0, \text{ or } u \neq 0 \text{ and, for every } v \in \tilde{E} \text{ with } 0 \leq v \leq |u|,\ v \neq 0 \\ \text{there exists } w \in \tilde{E} \text{ with } 0 \leq w \leq v, w \neq 0 \text{ such that} \\ \left(\limsup_n (((u_n - \beta s)^-) \wedge s)\right) \wedge \left(\limsup_n (((u_n - \alpha s)^+) \wedge s)\right) = 0 \\ \text{for every } s \in \tilde{E} \text{ with } 0 \leq s \leq w \text{ and } \alpha, \beta \in \mathbb{R} \text{ with } 0 < \beta < \alpha \end{array} \right\}.$$

By Proposition 2 of [9], $B_{OS}((u_n)_{n\in\mathbb{N}})$ and $B_{NOS}((u_n)_{n\in\mathbb{N}})$ are projection bands in $\tilde{E}$, and $\tilde{E}$ is the order direct sum of $B_{OS}((u_n)_{n\in\mathbb{N}})$ and $B_{NOS}((u_n)_{n\in\mathbb{N}})$. As in [9], let $B_d((u_n)_{n\in\mathbb{N}})$ be the projection band in $\tilde{E}$ generated by $B_\infty((u_n)_{n\in\mathbb{N}})\cup B_{OS}((u_n)_{n\in\mathbb{N}})$. We say (see [9]) that $(u_n)_{n\in\mathbb{N}}$ *converges individually* on $\tilde{E}$ if $B_d((u_n)_{n\in\mathbb{N}}) = 0$.

As we mentioned in Introduction, if $(u_n)_{n\in\mathbb{N}}$ is a sequence of (not necessarily positive) elements of E, then we say that the sequence $(u_n)_{n\in\mathbb{N}}$ converges individually if both sequences $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge individually.

3. The Individual Limit of a Sequence

As in the previous section, we assume to be given an Archimedean Riesz space E and its Dedekind completion $\tilde{E}$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of E. We say that $(u_n)_{n\in\mathbb{N}}$ *converges individually* to zero if $\limsup_n(u_n\wedge v) = 0$ for every $v\in\tilde{E}$, $v\geq 0$ ($\limsup_n(u_n\wedge v)$ is evaluated in $\tilde{E}$, and it exists always since $\tilde{E}$ is order complete). Now let $(u_n)_{n\in\mathbb{N}}$ be a sequence of (not necessarily positive) elements of E. We say that $(u_n)_{n\in\mathbb{N}}$ *converges individually to zero* if the sequences $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge individually to zero. Given a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of E and $u\in E$, we say that $(u_n)_{n\in\mathbb{N}}$ *converges individually to* u if the sequences $(u_n^+ - u^+)_{n\in\mathbb{N}}$ and $(u_n^- - u^-)_{n\in\mathbb{N}}$ converge individually to zero.

Proposition 1. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E, and let $u\in E$. If the sequence $(u_n)_{n\in\mathbb{N}}$ converges individually to u, then $(u_n)_{n\in\mathbb{N}}$ converges individually.*

Proof. We have to prove that $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge individually; that is, we have to prove that $B_d((u_n^+)_{n\in\mathbb{N}}) = 0$ and $B_d((u_n^-)_{n\in\mathbb{N}}) = 0$.

Clearly, in order to show $B_d((u_n^+)_{n\in\mathbb{N}}) = 0$, we have to prove that $B_\infty((u_n^+)_{n\in\mathbb{N}}) = 0$, and that $B_{OS}((u_n^+)_{n\in\mathbb{N}}) = 0$. Assume that $B_\infty((u_n^+)_{n\in\mathbb{N}}) \neq 0$. Taking into consideration that $B_\infty((u_n)_{n\in\mathbb{N}})$ is a band in $\tilde{E}$, we obtain that there exists $v\in B_\infty((u_n^+)_{n\in\mathbb{N}})$, $v\geq 0, v\neq 0$. Let $B(v)$ be the principal (projection) band in $\tilde{E}$ generated by the singleton $\{v\}$. Then, the sequence $(u_n^+)_{n\in\mathbb{N}}$ is unbounded on $B(v)$. By Lemma 3 of [9], the sequence $((u_n^+ - u^+)^+)_{n\in\mathbb{N}}$ is also unbounded on $B(v)$. By Lemma 4 of [8], it follows that the sequence $((u_n^+ - u^+)^+)_{n\geq k}$ is unbounded on $B(v)$ for every $k\in\mathbb{N}$. We obtain that $\sup_{n\geq k}(((u_n^+ - u^+)^+)\wedge v) = v$ for every $k\in\mathbb{N}$; hence, $\limsup_n(((u_n^+ - u^+)^+)\wedge v) = v\neq 0$. We have obtained a contradiction since $((u_n^+ - u^+)^+)_{n\in\mathbb{N}}$ converges individually to zero. It follows that $B_\infty((u_n^+)_{n\in\mathbb{N}}) = 0$.

Assume now that $B_{OS}((u_n^+)_{n\in\mathbb{N}}) \neq 0$, and let $0\neq s\in B_{OS}((u_n^+)_{n\in\mathbb{N}})$. Then there exist $w\in\tilde{E}$, $0\leq w\leq |s|, w\neq 0$ and $\alpha,\beta\in\mathbb{R}$, $0<\beta<\alpha$ such that

$$\left(\limsup_n(((u_n^+ - \beta w)^-)\wedge w)\right)\wedge\left(\limsup_n(((u_n^+ - \alpha w)^+)\wedge w)\right)\neq 0.$$

Using a consequence of the decomposition property in Riesz spaces (see [7: Corollary on p. 53]), and taking into consideration that $(u_n)_{n\in\mathbb{N}}$ converges individually to u, we obtain

$$
\begin{aligned}
0 &\le \left(\limsup_n(((u_n^+ - \beta w)^-) \wedge w)\right) \wedge \left(\limsup_n(((u_n^+ - \alpha w)^+) \wedge w)\right) \\
&\le \left(\limsup_n(((\beta w - u^+)^+ + (u^+ - u_n^+)^+) \wedge w)\right) \\
&\quad \wedge \left(\limsup_n(((u_n^+ - u^+)^+ + (u^+ - \alpha w)^+) \wedge w)\right) \\
&\le \left(\limsup_n(((\beta w - u^+)^+ \wedge w) + ((u^+ - u_n^+)^+ \wedge w))\right) \\
&\quad \wedge \left(\limsup_n(((u_n^+ - u^+)^+ \wedge w) + ((u^+ - \alpha w)^+ \wedge w))\right) \\
&= \left(((u^+ - \beta w)^- \wedge w) + \left(\limsup_n((u_n^+ - u^+)^- \wedge w)\right)\right) \\
&\quad \wedge \left(((u^+ - \alpha w)^+ \wedge w) + \left(\limsup_n((u_n^+ - u^+)^+ \wedge w)\right)\right) \\
&= (u^+ - \beta w)^- \wedge w \wedge (u^+ - \alpha w)^+ \\
&\le (u^+ - \alpha w)^- \wedge w \wedge (u^+ - \alpha w)^+ \\
&= 0.
\end{aligned}
$$

It follows that

$$
\left(\limsup_n(((u_n^+ - \beta w)^-) \wedge w)\right) \wedge \left(\limsup_n(((u_n^+ - \alpha w)^+) \wedge w)\right) = 0;
$$

that is, we have obtained a contradiction. Accordingly, $B_{OS}((u_n^+)_{n\in\mathbb{N}}) = 0$. We have therefore proved that $B_d((u_n^+)_{n\in\mathbb{N}}) = 0$.

Clearly, in order to prove $B_d((u_n^-)_{n\in\mathbb{N}}) = 0$, we have to prove that $B_\infty((u_n^-)_{n\in\mathbb{N}}) = 0$ and $B_{OS}((u_n^-)_{n\in\mathbb{N}}) = 0$.

Assume that $B_\infty((u_n^-)_{n\in\mathbb{N}}) \neq 0$. Then there exists $v \in B_\infty((u_n^-)_{n\in\mathbb{N}})$, $v \ge 0, v \neq 0$. Let $B(v)$ be the principal (projection) band in $\tilde{E}$ generated by the singleton $\{v\}$. Taking into consideration that $(u_n^-)_{n\in\mathbb{N}}$ is unbounded on $B(v)$, and using Lemma 3 of [9] and Lemma 4 of [8] (like in the proof of the fact that $B_\infty((u_n^+)_{n\in\mathbb{N}}) = 0$), we obtain that $\sup_{n\ge k}(((u_n^- - u^-)^+) \wedge v) = v$ for every $k \in \mathbb{N}$. Thus, it follows that $\limsup_n(((u_n^- - u^-)^+) \wedge v) = v \neq 0$; hence, we have obtained a contradiction.

Assume that $B_{OS}((u_n^-)_{n\in\mathbb{N}}) \neq 0$. Then there exists $s \in B_{OS}((u_n^-)_{n\in\mathbb{N}})$, $s \neq 0$. We obtain that there exist $w \in \tilde{E}$, $0 \le w \le |s|, w \neq 0$ and $\alpha, \beta \in \mathbb{R}$, $0 < \beta < \alpha$ such that

$$
\left(\limsup_n(((u_n^- - \beta w)^-) \wedge w)\right) \wedge \left(\limsup_n(((u_n^- - \alpha w)^+) \wedge w)\right) \neq 0.
$$

It follows (by arguments similar to the ones used in order to prove that $B_{OS}((u_n^+)_{n\in\mathbb{N}}) =$

0) that

$$\begin{aligned}
0 &\le \left(\limsup_n (((u_n^- - \beta w)^-) \wedge w)\right) \wedge \left(\limsup_n (((u_n^- - \alpha w)^+) \wedge w)\right) \\
&\le \left(\limsup_n (((\beta w - u^-)^+ \wedge w) + ((u^- - u_n^-)^+ \wedge w))\right) \\
&\quad \wedge \left(\limsup_n (((u_n^- - u^-)^+ \wedge w) + ((u^- - \alpha w)^+ \wedge w))\right) \\
&= \left(((\beta w - u^-)^+ \wedge w) + \left(\limsup_n ((u_n^- - u^-)^- \wedge w)\right)\right) \\
&\quad \wedge \left(((u^- - \alpha w)^+ \wedge w) + \left(\limsup_n ((u_n^- - u^-)^+ \wedge w)\right)\right) \\
&= ((\beta w - u^-)^+) \wedge w \wedge ((u^- - \alpha w)^+) \\
&\le ((u^- - \alpha w)^-) \wedge w \wedge ((u^- - \alpha w)^+) \\
&= 0.
\end{aligned}$$

We have obtained a contradiction; accordingly, $B_{OS}((u_n^-)_{n\in I\!N}) = 0$ ∎

4. Individual convergence and N-convergence

As mentioned in Introduction, our goal in this section is to discuss the relationships among the various types of individual convergence described earlier. As always in this note, E is a given Archimedean Riesz space, and $\tilde{E}$ is the Dedekind completion of E.

Proposition 2. *Let $(u_n)_{n\in I\!N}$ be a sequence of elements of E such that $u_n \ge 0$ for every $n \in I\!N$. Then, $(u_n)_{n\in I\!N}$ converges individually to zero if and only if $(u_n)_{n\in I\!N}$ N-converges individually to zero.*

Proof. Assume first that $(u_n)_{n\in I\!N}$ converges individually to zero. Let $c_1, c_2 \in E$ be such that $c_1 \le c_2$. The sequence $(u_n \wedge c_1)_{n\in I\!N}$ is order bounded since $-c_1^- \le u_n \wedge c_1 \le c_1$ for every $n \in I\!N$; hence, $\limsup_n (u_n \wedge c_1)$ exists in $\tilde{E}$. Let $B(|c_1|)$, $B(c_1^+)$, $B(c_1^-)$ be the principal projection bands in $\tilde{E}$ generated by the singletons $\{|c_1|\}$, $\{c_1^+\}$, $\{c_1^-\}$, and let $P_{|c_1|}$, $P_{c_1^+}$, $P_{c_1^-}$ be the band projections associated with $B(|c_1|)$, $B(c_1^+)$, $B(c_1^-)$, respectively. Since $0 \le P_{c_1^+} u_n \le u_n$ for every $n \in I\!N$, and since $(u_n)_{n\in I\!N}$ converges individually to zero, it follows that

$$0 \le \limsup_n ((P_{c_1^+} u_n) \wedge c_1^+) \le \limsup_n (u_n \wedge c_1^+) = 0.$$

Therefore,

$$\limsup_n ((P_{c_1^+} u_n) \wedge c_1^+) = 0. \tag{2.1}$$

Taking into consideration $P_{|c_1|} = P_{c_1^+} + P_{c_1^-}$ and using (2.1), we obtain

$$\begin{aligned}\limsup_n (u_n \wedge c_1) &= \limsup_n (P_{|c_1|}(u_n \wedge c_1)) \\ &= \limsup_n \Big((P_{c_1^+} + P_{c_1^-})(u_n \wedge (c_1^+ - c_1^-))\Big) \\ &= \limsup_n \Big(P_{c_1^+}(u_n \wedge (c_1^+ - c_1^-)) + P_{c_1^-}(u_n \wedge (c_1^+ - c_1^-))\Big) \\ &= \limsup_n \Big(((P_{c_1^+}u_n) \wedge c_1^+) + (P_{c_1^-}(u_n \wedge (-c_1^-)))\Big) \\ &= \Big(\limsup_n ((P_{c_1^+}u_n) \wedge c_1^+)\Big) - c_1^- \\ &= -c_1^-.\end{aligned}$$

It follows that

$$\limsup_n ((u_n \wedge c_1) \vee c_2) = (\limsup_n (u_n \wedge c_1)) \vee c_2 = (-c_1^-) \vee c_2 = (0 \wedge c_1) \vee c_2.$$

Note that $u_n \wedge c_1 \geq -c_1^-$ for every $n \in \mathbb{N}$; hence, $\liminf_n (u_n \wedge c_1) \geq -c_1^-$. Taking into consideration $\limsup_n (u_n \wedge c_1) = -c_1^-$, we obtain $\liminf_n (u_n \wedge c_1) = -c_1^-$.

Accordingly,

$$\liminf_n ((u_n \wedge c_1) \vee c_2) = (\liminf_n (u_n \wedge c_1)) \vee c_2 = (-c_1^-) \vee c_2 = (0 \wedge c_1) \vee c_2.$$

It follows that $(u_n)_{n\in\mathbb{N}}$ N-converges individually to 0. Conversely, assume that $(u_n)_{n\in\mathbb{N}}$ N-converges individually to zero. Then

$$\limsup_n (u_n \wedge v) = \limsup_n ((u_n \wedge v) \vee 0) = (0 \wedge v) \vee 0 = 0$$

for every $v \in E$, $v \geq 0$. Let $v \in \tilde{E}$, $v \geq 0$. Clearly, $\limsup_n (u_n \wedge v) = 0$ whenever $v = 0$. If $v \neq 0$, then by [5: Theorem 32.6/p. 195] there exists $w \in E$, $v \leq w$. Thus, $0 \leq \limsup_n (u_n \wedge v) \leq \limsup_n (u_n \wedge w) = 0$. We conclude that $\limsup_n (u_n \wedge v) = 0$ for every $v \in \tilde{E}$, $v \geq 0$. Accordingly, the sequence $(u_n)_{n\in\mathbb{N}}$ converges individually to zero ∎

Lemma 3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E, and let $u \in E$. Then $(u_n)_{n\in\mathbb{N}}$ as a sequence of elements of E converges individually to u if and only if $(u_n)_{n\in\mathbb{N}}$, thought of as a sequence of elements of $\tilde{E}$, converges individually to u.*

Proof. The proof is obvious in view of the definition of individual convergence to a given element, and in view of the fact that the Dedekind completion of $\tilde{E}$ coincides with $\tilde{E}$.

Lemma 4. *Lemma 3 remains true if we replace the individual convergence by the individual N-convergence.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E, and let $u \in E$. Clearly, if $(u_n)_{n\in\mathbb{N}}$, thought of as a sequence of elements of $\tilde{E}$, N-converges individually to u, then the same is true of $(u_n)_{n\in\mathbb{N}}$ as a sequence of elements of E.

Conversely, assume that $(u_n)_{n\in\mathbb{N}}$ N-converges individually to u as a sequence of elements of E. Thus,

$$\limsup_n ((u_n \wedge c_1) \vee c_2) = \liminf_n ((u_n \wedge c_1) \vee c_2) = (u \wedge c_1) \vee c_2$$

for every $c_1, c_2 \in E$, $c_2 \le c_1$. We have to prove that

$$\limsup_n ((u_n \wedge x) \vee y) = \liminf_n ((u_n \wedge x) \vee y) = (u \wedge x) \vee y$$

for every $x, y \in \tilde{E}$, $y \le x$. To this end, let $x, y \in \tilde{E}$, $y \le x$. Clearly,

$$\begin{aligned} &\sup\left\{(u \wedge c_1) \vee c_2 \,\middle|\, c_1, c_2 \in E,\ c_1 \le x,\ c_2 \le y,\ c_2 \le c_1\right\} \\ &\qquad \le \sup\left\{(u \wedge c_1) \vee c_2 \,\middle|\, c_1, c_2 \in E,\ c_1 \le x,\ c_2 \le y\right\} \end{aligned}$$

(naturally, the suprema are taken in $\tilde{E}$). Now, let $r_1, r_2 \in E$, $r_1 \le x$ and $r_2 \le y$. Set $r_1' = r_1 \vee r_2$. Then $r_1' \le x$, so it follows

$$\begin{aligned} (u \wedge r_1) \vee r_2 &\le (u \wedge r_1') \vee r_2 \\ &\le \sup\left\{(u \wedge c_1) \vee c_2 \,\middle|\, c_1, c_2 \in E,\ c_1 \le x,\ c_2 \le y,\ c_2 \le c_1\right\}. \end{aligned}$$

We conclude that

$$\begin{aligned} &\sup\left\{(u \wedge c_1) \vee c_2 \,\middle|\, c_1, c_2 \in E,\ c_1 \le x,\ c_2 \le y\right\} \\ &\qquad \le \sup\left\{(u \wedge c_1) \vee c_2 \,\middle|\, c_1, c_2 \in E,\ c_1 \le x,\ c_2 \le y,\ c_2 \le c_1\right\}. \end{aligned}$$

Accordingly,

$$\begin{aligned} &\sup\left\{(u \wedge c_1) \vee c_2 \,\middle|\, c_1, c_2 \in E,\ c_1 \le x,\ c_2 \le y\right\} \\ &\qquad = \sup\left\{(u \wedge c_1) \vee c_2 \,\middle|\, c_1, c_2 \in E,\ c_1 \le x,\ c_2 \le y,\ c_2 \le c_1\right\}. \end{aligned} \tag{4.1}$$

For every $r_1, r_2 \in E$, $r_1 \ge x$ and $r_2 \ge y$, it follows that

$$\begin{aligned} &\inf\left\{(u \wedge d_1) \vee d_2 \,\middle|\, d_1, d_2 \in E,\ d_1 \ge x,\ d_2 \ge y,\ d_2 \le d_1\right\} \\ &\qquad \le (u \wedge r_1) \vee (r_1 \wedge r_2) \\ &\qquad \le (u \wedge r_1) \vee r_2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\inf\left\{(u \wedge d_1) \vee d_2 \,\middle|\, d_1, d_2 \in E,\ d_1 \ge x,\ d_2 \ge y,\ d_2 \le d_1\right\} \\ &\qquad \le \inf\left\{(u \wedge d_1) \vee d_2 \,\middle|\, d_1, d_2 \in E,\ d_1 \ge x,\ d_2 \ge y\right\}. \end{aligned}$$

Since the converse inequality is obviously true, we obtain

$$\begin{aligned}\inf\Big\{(u\wedge d_1)\vee d_2\Big|\ d_1,d_2\in E,\ d_1\geq x,\ d_2\geq y,\ d_2\leq d_1\Big\}\\ =\inf\Big\{(u\wedge d_1)\vee d_2\Big|\ d_1,d_2\in E,\ d_1\geq x,\ d_2\geq y\Big\}.\end{aligned}\tag{4.2}$$

Set

$$\underline{L}(a,b)=\liminf_n((u_n\wedge a)\vee b)\qquad\text{and}\qquad\overline{L}(a,b)=\limsup_n((u_n\wedge a)\vee b)$$

for every $a,b\in E$, $b\leq a$. Then, using (4.1) and (4.2), we obtain

$$\begin{aligned}(u\wedge x)\vee y&=\sup\Big\{(u\wedge c_1)\vee c_2\Big|\ c_1,c_2\in E,\ c_1\leq x,\ c_2\leq y\Big\}\\&=\sup\Big\{(u\wedge c_1)\vee c_2\Big|\ c_1,c_2\in E,\ c_1\leq x,\ c_2\leq y,\ c_2\leq c_1\Big\}\\&=\sup\Big\{\underline{L}(c_1,c_2)\Big|\ c_1,c_2\in E,\ c_1\leq x,\ c_2\leq y,\ c_2\leq c_1\Big\}\\&\leq\liminf_n((u_n\wedge x)\vee y)\\&\leq\limsup_n((u_n\wedge x)\vee y)\\&\leq\inf\Big\{\overline{L}(d_1,d_2)\Big|\ d_1,d_2\in E,\ d_1\leq x,\ d_2\leq y,\ d_2\leq d_1\Big\}\\&=\inf\Big\{(u\wedge d_1)\vee d_2\Big|\ d_1,d_2\in E,\ d_1\leq x,\ d_2\leq y,\ d_2\leq d_1\Big\}\\&=\inf\Big\{(u\wedge d_1)\vee d_2\Big|\ d_1,d_2\in E,\ d_1\leq x,\ d_2\leq y\Big\}\\&=(u\wedge x)\vee y.\end{aligned}$$

Thus the statement is proved ∎

The results discussed in this section enable us to show that the definition of the individual convergence to a given element is just a reformulation of the definition of individual N-convergence.

Theorem 5. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E, and let $u\in E$. Then $(u_n)_{n\in\mathbb{N}}$ converges individually to u if and only if $(u_n)_{n\in\mathbb{N}}$ N-converges individually to u.*

Proof. In view of Lemma 3 and Lemma 4, it is clear that we may assume E to be an order complete Riesz space (that is, we may assume E coinciding with its Dedekind completion $\hat{E}$). The sequence $(u_n)_{n\in\mathbb{N}}$ converges individually to u if and only if the sequences

$$\begin{aligned}&((u_n^+-u^+)^+)_{n\in\mathbb{N}},\qquad((u_n^+-u^+)^-)_{n\in\mathbb{N}}\\&((u_n^--u^-)^+)_{n\in\mathbb{N}},\qquad((u_n^--u^-)^-)_{n\in\mathbb{N}}\end{aligned}\tag{5.1}$$

converge individually to zero. By Proposition 2, the four sequences (5.1) converge individually to zero if and only if they N-converge individually to zero. By Theorem 1.4 of [6], the sequences (5.1) N-converge individually to zero if and only if $(u_n)_{n\in\mathbb{N}}$

N-converges individually to u. Indeed, if all the sequences (5.1) converge individually to zero, then

$$(u_n^+ - u^+)_{n\in\mathbb{N}} = ((u_n^+ - u^+)^+ - (u_n^+ - u^+)^-)_{n\in\mathbb{N}}$$
$$(u_n^- - u^-)_{n\in\mathbb{N}} = ((u_n^- - u^-)^+ - (u_n^- - u^-)^-)_{n\in\mathbb{N}}$$

N-converge individually to zero. Therefore,

$$(u_n)_{n\in\mathbb{N}} = ((u_n^+ - u^+) - (u^- - u^-) + u)_{n\in\mathbb{N}}$$

N-converges individually to u. Conversely, if $(u_n)_{n\in\mathbb{N}}$ converges individually to u, then $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ N-converge individually to u^+ and u^-, respectively. Hence, the sequences (5.1) N-converge individually to zero ∎

Theorem 5 and Proposition 1 have the following obvious consequence.

Corollary 6. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of E, and let $u \in E$. If $(u_n)_{n\in\mathbb{N}}$ N-converges individually to u, then $(u_n)_{n\in\mathbb{N}}$ converges individually.*

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