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On Stationary Incompressible Norton Fluids and some Extensions of Korn's Inequality

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Abstract. A simple mathematical model for a so-called Norton fluid is given. We study a variational problem and make use of appropriate versions of Korn's inequality.

Keywords: *Non-Newtonian fluids, Korn's inequality*

AMS subject classification: 49S05, 76A05

1. Introduction and statement of the results

We study a variational problem modelling incompressible Norton fluids (see [4, 6]). Let Ω denote a bounded region in $\mathbb{R}^3$ and suppose that on some part Γ of $\partial\Omega$ a function $\phi : \Gamma \to \mathbb{R}^3$ is given. Then we look for a minimizer $u : \Omega \to \mathbb{R}^3$ of $F(u) = \int_\Omega |\mathcal{E}(u)|^p \, dx$ subject to the side conditions $u|_\Gamma = \phi, \operatorname{div} u = 0$ on Ω. Here $\mathcal{E}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) = \left(\frac{1}{2}(\partial_i u^j + \partial_j u^i)\right)_{1 \le i,j \le 3}$ is the symmetric part of the velocity gradient ∇u and p is a fixed real number in $(1, \infty)$. For $p = 2$ we have a Newtonian fluid and it is well known (compare [2]) how to handle the above problem with the help of Korn's inequality. On the other hand the limit case $p = 1$ corresponds to functionals with linear growth in $\mathcal{E}(u)$ (or more precisely in $\mathcal{E}^D(u) = \mathcal{E}(u) - \frac{1}{3} \operatorname{trace} \mathcal{E}(u) \cdot \mathbf{1}$) arising in plasticity theory and leads to the study of variational problems in subclasses of the space $BD(\Omega)$ (we refer the reader to the papers [1] and [7] where one also finds further references). The objective of our paper now is to give the appropriate setting for growth rates $p \in (1, \infty), p \neq 2$, which means to prove versions of Korn's inequality in $H^{1,p}(\Omega, \mathbb{R}^3)$.

From now on we assume that Ω is a domain of class $C^{3,\alpha}$ for some $0 < \alpha < 1$ and that Γ is an open portion of $\partial\Omega$ with $\mathcal{H}^2(\Gamma) > 0$. Suppose that $\phi \in H^{1,p}(\Omega, \mathbb{R}^3)$ is given such that the class

$$\mathcal{C} = \left\{ u \in H^{1,p}(\Omega, \mathbb{R}^3) : u|_\Gamma = \phi|_\Gamma, \operatorname{div} u = 0 \text{ a.e.} \right\}$$

is non-empty (here we consider $1 < p < \infty$).

Theorem 1: *The variational problem $F(u) = \int_\Omega |\mathcal{E}(u)|^p \, dx \to \min$ in $\mathcal{C}$ admits a (unique) solution $u \in \mathcal{C}$.*

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Remark: Of course the existence result of Theorem 1 can be extended to variational problems of the form

$$\int_\Omega W(x, \mathcal{E}(u)(x))\, dx \;\rightarrow\; \min$$

in the class C provided the dissipation function $W : \Omega \times \{S \in \mathbb{R}^{3\times 3} : S \text{ symmetric}\} \rightarrow \mathbb{R}$ is convex with respect to the second argument and of p growth for some power $p \in (1, \infty)$. For example a condition of the form $\lambda|S|^p \leq W(x, S) \leq \Lambda|S|^p$ (with positive numbers λ and Λ) would be sufficient. If W is in addition differentiable with respect to S, then also an analogy of the following Theorem 2 is true.

Theorem 2: *There is a function $f \in L^{p/(p-1)}_{loc}(\Omega)$ such that, for the minimizer $u \in C$ from Theorem 1, we have*

$$p \int_\Omega |\mathcal{E}(u)|^{p-2}\mathcal{E}(u)\mathcal{E}(\psi)\, dx = \int_\Omega div\,\psi \cdot f\, dx$$

for all $\psi \in C_0^\infty(\Omega, \mathbb{R}^3)$.

The existence result Theorem 1 follows from the following

Theorem 3 *(Korn's inequality) : There is a constant $c = c(p, \Omega, \Gamma)$ with the property that*

$$||v||_{H^{1,p}(\Omega)} \leq c\,||\mathcal{E}(v)||_{L^p(\Omega)}$$

holds for all $v \in H^{1,p}(\Omega, \mathbb{R}^3)$ such that $v|_\Gamma = 0$.

In Section 2 we prove Korn's inequality with the help of several lemmas, the Euler equation will be discussed in Section 3.

Accepting Theorem 3 we select a minimizing sequence $\{u_n\} \subset C$ and deduce

$$||u_n - \phi||_{H^{1,p}(\Omega)} \leq c\,[F(u_n) + F(\phi)]^{1/p}$$

so that $\sup_n ||u_n||_{H^{1,p}(\Omega)} < \infty$ and, for some element u, $u_n \rightharpoonup u$ in $H^{1,p}(\Omega, \mathbb{R}^3)$ at least for a subsequence. Since F is weakly lower semicontinuous $F(u) \leq \inf_C F$. Clearly $u|_\Gamma = \phi$ and $\operatorname{div} u = 0$ follows from

$$\int_\Omega \operatorname{div} u \cdot \eta\, dx = \lim_{n\to\infty} \int_\Omega \operatorname{div} u_n \cdot \eta\, dx = 0$$

for all $\eta \in C_0^\infty(\Omega)$, hence u is minimizing. If $\tilde{u} \in C$ is also minimizing, then we must have $\mathcal{E}(u) = \mathcal{E}(\tilde{u})$ and from Theorem 3 we infer $||u - \tilde{u}||_{H^{1,p}(\Omega)} = 0$. This proves Theorem 1.

2. Korn's inequality in the space $H^{1,p}(\Omega, \mathbb{R}^3), 1 < p < \infty$

We here assume that $\Omega \subset \mathbb{R}^3$ is a domain satisfying the assumptions of [8: Satz I.5.2].

Lemma 1: *There exists a positive constant $c = c(p, \Omega)$ with the following property: If $f \in L^p(\Omega)$ satisfies $\int_\Omega f\,dx = 0$, then we find a vector field $U \in \mathring{H}^{1,p}(\Omega, \mathbb{R}^3)$ such that $\operatorname{div} U = f$ on Ω and $\|U\|_{H^{1,p}(\Omega)} \le c\|f\|_{L^p(\Omega)}$.*

Remark: 1. The strong smoothness properties imposed on $\partial\Omega$ enter our arguments only in the proof of Lemma 1. So Theorems 1 - 3 can be extended to precisely those class of bounded domains $\Omega \subset \mathbb{R}^3$ for which Lemma 1 is true. **2.** For balls Ω we have $c = c(p)$.

Proof of Lemma 1: We select a sequence of functions $f_n \in C_0^\infty(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$. According to [8: Satz I.5.2] there exists $U_n \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \cap C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ such that

$$\operatorname{div} U_n = f_n - (f_n)_\Omega \ \text{ on } \Omega, \quad U_n|_{\partial\Omega} = 0, \quad \|\nabla U_n\|_{L^p(\Omega)} \le c\|f_n - (f_n)_\Omega\|_{L^p(\Omega)}$$

for a constant c as in Lemma 1. Hence $U_n \in H^{1,p}(\Omega, \mathbb{R}^3)$ and [5: Theorems 3.6.2 and 3.6.3] imply $U_n \in \mathring{H}^{1,p}(\Omega, \mathbb{R}^3)$. From the above estimate we deduce that $\{U_n\}$ is a Cauchy sequence in $\mathring{H}^{1,p}(\Omega, \mathbb{R}^3)$ and for the limit U we obtain the equation $\operatorname{div} U = f - (f)_\Omega = f$. Here and in the following $(f)_\Omega = ⨍_\Omega f\,dx$ denotes the mean value ∎

Lemma 2: *For Ω as above and $1 < q < \infty$ consider a distribution $T : C_0^\infty(\Omega) \to \mathbb{R}$ with the property $T, \partial_i T \in (\mathring{H}^{1,q}(\Omega))^*$ for $i = 1, 2, 3$ where * means the dual space, i.e.*

$$|T(\varphi)| + \sum_{i=1}^{3} |T(\partial_i\varphi)| \le C\,\|\varphi\|_{H^{1,q}(\Omega)} \qquad \textit{for all } \varphi \in C_0^\infty(\Omega).$$

Then $T(\varphi) = \int_\Omega u \cdot \varphi\,dx$ for a function $u \in L^{q'}(\Omega), q' = q/(q-1)$.

Proof: Let G denote a subregion of Ω with the same regularity properties. For $\varepsilon < \varepsilon_0$, $\varepsilon_0 = \operatorname{dist}(G, \partial\Omega)$, we let $\Phi_\varepsilon(z) := \varepsilon^{-3}\Phi(\frac{z}{\varepsilon}) \in C_0^\infty(B_\varepsilon(0))$ for a mollifier $\Phi \in C_0^\infty(B_1(0))$. Then, according to [3], for $\varphi \in C_0^\infty(G)$ we have

$$T_\varepsilon(\varphi) := T(\Phi_\varepsilon * \varphi) = \int_G \varphi(y) T_x(\Phi_\varepsilon(x-y))dy \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} T(\Phi_\varepsilon * \varphi) = T(\varphi)$$

(clearly $\operatorname{spt}(x \mapsto \Phi_\varepsilon(x-y)) \subset \Omega$ for arbitrary $y \in G$ so that $u_\varepsilon(y) := T_x(\Phi_\varepsilon(x-y))$ is well defined). We have

$$\begin{aligned}
\left|\int_G \partial_i\varphi \cdot u_\varepsilon\,dx\right| &= \left|T(\Phi_\varepsilon * \partial_i\varphi)\right| \\
&= \left|T(\partial_i(\Phi_\varepsilon * \varphi))\right| = \left|(\partial_i T)(\Phi_\varepsilon * \varphi)\right| \\
&\le C\,\|\Phi_\varepsilon * \varphi\|_{H^{1,q}(\Omega)} \le C\,\|\varphi\|_{H^{1,q}(G)} \qquad (\varphi \in C_0^\infty(G)).
\end{aligned}$$

For the last inequality one has to observe

$$\operatorname{spt}(\Phi_\varepsilon * \varphi) \subset G_\varepsilon = \{x : \operatorname{dist}(x, G) < \varepsilon\}$$

which implies $\|\Phi_\varepsilon * \varphi\|_{H^{1,q}(G_\varepsilon)} \le \|\varphi\|_{H^{1,q}(G_{2\varepsilon})}$ by a standard property of mollifiers. But obviously $\|\varphi\|_{H^{1,q}(G_{2\varepsilon})} = \|\varphi\|_{H^{1,q}(G)}$.

By approximation

$$\left| \int_G u_\varepsilon \cdot \partial_i \varphi \, dx \right| \le C \, \|\varphi\|_{H^{1,q}(G)}$$

extends to all $\varphi \in \mathring{H}^{1,q}(G)$. Consider $f \in L^q(G)$ such that $(f)_G = 0$. Lemma 1 yields the existence of a field $\psi \in \mathring{H}^{1,q}(G, \mathbb{R}^3)$ such that $\operatorname{div} \psi = f$ and

$$\|\psi\|_{H^{1,q}(G)} \le c \, \|f\|_{L^q(G)}, \qquad c = c(q, G).$$

This implies

$$\left| \int_G u_\varepsilon \cdot f \, dx \right| = \left| \int_G u_\varepsilon \operatorname{div} \psi \, dx \right| \le c \, \|f\|_{L^q(G)}$$

for all $f \in L^q(G)$ with vanishing mean value or

$$\left| \int_G (u_\varepsilon - (u_\varepsilon)_G) \cdot f \, dx \right| \le c \, \|f\|_{L^q(G)} \qquad (f \in L^q(G))$$

(observe that $\int_G (u_\varepsilon - (u_\varepsilon)_G) \cdot f \, dx = \int_G u_\varepsilon \cdot (f - (f)_G) \, dx$ and $\|f - (f)_G\|_{L^q(G)} \le 2 \, \|f\|_{L^q(G)}$). If we take the supremum over all $f \in L^q(G)$ with $\|f\|_{L^q(G)} \le 1$, we have shown

$$\|u_\varepsilon - (u_\varepsilon)_G\|_{L^{q'}(G)} \le c\,(q, G) < \infty$$

for all small $\varepsilon > 0$, and there exists $v^G \in L^{q'}(G)$ such that

$$u_\varepsilon - (u_\varepsilon)_G \rightharpoonup v^G \qquad \text{in } L^{q'}(G) \text{ as } \varepsilon \downarrow 0.$$

Now we fix some small ball $B_\varrho(x_0) \subset G$ and pick

$$\varphi_1 \in C_0^\infty(B_\varrho(x_0), [0,1]) \qquad \text{with} \qquad \int_{B_\varrho(x_0)} \varphi_1 \, dx = 1.$$

From

$$T(\varphi_1) = \lim_{\varepsilon \downarrow 0} \int_G u_\varepsilon \cdot \varphi_1 \, dx = \lim_{\varepsilon \downarrow 0} \left\{ \int_G \varphi_1 \cdot (u_\varepsilon - (u_\varepsilon)_G) \, dx + (u_\varepsilon)_G \right\}$$

we deduce the existence of $\xi^G := \lim_{\varepsilon \downarrow 0} (u_\varepsilon)_G$ with value $T(\varphi_1) - \int_G v^G \cdot \varphi_1 \, dx$ and since

$$T(\varphi) = \lim_{\varepsilon \downarrow 0} \left\{ \int_G \varphi \cdot (u_\varepsilon - (u_\varepsilon)_G) \, dx + (u_\varepsilon)_G \int_G \varphi \, dx \right\}$$

holds for arbitrary $\varphi \in C_0^\infty(G)$ we end up with the representation

$$T(\varphi) = \int_G u^G \cdot \varphi \, dx \qquad \text{where } u^G = v^G + \xi^G \in L^{q'}(G)$$

being valid on the space $C_0^\infty(G)$. The inclusions $G \subset G' \subset \Omega$ clearly imply the equality $u^G = u^{G'}$ almost everywhere on G.

Observe next that

$$\begin{aligned}||u^G||_{L^{q'}(G)} &\le ||v^G||_{L^{q'}(G)} + |\xi^G|\,\mathcal{L}^3(G)^{1/q'} \\ &\le c(q,G) + \mathcal{L}^3(G)^{1/q'}\left\{|T(\varphi_1)| + \int_G |v^G|\cdot\varphi_1\,dx\right\} \\ &\le c(q,G) + \mathcal{L}^3(G)^{1/q'}\left\{C\,||\varphi_1||_{H^{1,q}(B_\varrho(x_0))} + ||v^G||_{L^{q'}(G)}||\varphi_1||_{L^q(B_\varrho(x_0))}\right\}.\end{aligned}$$

Here the constant $c(q,G)$ has the form cC with c from Lemma 1 and C denotes the bound for $T, \partial_i T$ in $(\mathring{H}^{1,q}(\Omega))^*$.

In a final step we replace G by an increasing sequence $\{G_n\}$ of regular domains exhausting Ω and define $u \in L^{q'}_{loc}(\Omega)$ through $u(x) = u^{G_n}(x)$ if $x \in G_n$. By construction u represents the distribution T, moreover $c(q,G_n)$ is bounded independent of n so that $u \in L^{q'}(\Omega)$ on account of the above estimates ∎

Now we are in the position to prove versions of Korn's inequality.

Lemma 3: *For $1 < p < \infty$ and $\Omega \subset \mathbb{R}^3$ as in Lemma 1 there exists a constant $c(p,\Omega)$ such that*

$$||u||_{H^{1,p}(\Omega)} \le c(p,\Omega)\left[||u||_{L^p(\Omega)} + ||\mathcal{E}(u)||_{L^p(\Omega)}\right]$$

for all $u \in H^{1,p}(\Omega, \mathbb{R}^3)$.

Corollary: *Consider the Banach space*

$$V = \left\{u \in L^p(\Omega, \mathbb{R}^3) : \mathcal{E}_{ij}(u) = \frac{1}{2}(\partial_i u^j + \partial_j u^i) \in L^p(\Omega) \ \ (i,j,=1,2,3)\right\}$$

equipped with the norm

$$||u||_V = ||u||_{L^p(\Omega)} + ||\mathcal{E}(u)||_{L^p(\Omega)}$$

where $\mathcal{E}_{ij}(u)$ is defined in the sense of distributions. Then $V = H^{1,p}(\Omega, \mathbb{R}^3)$ and the norms $||\cdot||_V$ and $||\cdot||_{H^{1,p}(\Omega)}$ are equivalent.

Proof: Consider the continuous embedding $I : H^{1,p}(\Omega, \mathbb{R}^3) \ni u \mapsto u \in V$ and take $v \in V$. Then in the weak sense

$$\partial_j\partial_k v^i = \partial_j\mathcal{E}_{ik}(v) + \partial_k\mathcal{E}_{ij}(v) - \partial_i\mathcal{E}_{jk}(v) \qquad (i,j,k = 1,2,3).$$

Since we assume $\mathcal{E}(v) \in L^p(\Omega, \mathbb{R}^{3\times 3})$ the above relation yields $\partial_j\partial_k v^i \in (\mathring{H}^{1,p'}(\Omega))^*$ (where * indicates the dual space) and $v \in L^p(\Omega, \mathbb{R}^3)$ implies $\partial_k v^i \in (\mathring{H}^{1,p'}(\Omega))^*$ so that $\partial_k v^i \in L^p(\Omega)$ by Lemma 2, that is $v \in H^{1,p}(\Omega, \mathbb{R}^3)$ which shows surjectivity of the embedding I. Hence $V = H^{1,p}(\Omega, \mathbb{R}^3)$ and the desired estimate follows from the closed graph theorem, i.e. the continiuty of I^{-1} ∎

We now come to the

Proof of Theorem 3: According to Lemma 3 it remains to show

$$||v||_{L^p(\Omega)} \leq c\,||\mathcal{E}(v)||_{L^p(\Omega)}$$

for a suitable constant $c = c(p,\Omega,\Gamma)$ and all $v \in H^{1,p}(\Omega, \mathbb{R}^3)$ with $v|_\Gamma = 0$. We assume that the statement is wrong, hence there is a sequence $\{v_n\}$ in $H^{1,p}(\Omega,\mathbb{R}^3)$ with $v_n|_\Gamma = 0$ such that, without loss of generality, $||v_n||_{L^p(\Omega)} = 1$ and $1 > n\,||\mathcal{E}(v_n)||_{L^p(\Omega)}$, i.e. $\mathcal{E}(v_n) \to 0$ in $L^p(\Omega, \mathbb{R}^{3\times 3})$ as $n \to \infty$. Quoting Lemma 3 we have $v_n \rightharpoondown v$ in $H^{1,p}(\Omega,\mathbb{R}^3)$ (at least for a subsequence) with v satisfying $||v||_{L^p(\Omega)} = 1$, $v|_\Gamma = 0$ and $\mathcal{E}(v) = 0$ (by the weak lower semicontinuity of $||\mathcal{E}(\cdot)||_{L^p(\Omega)}$). On the other hand we know $H^{1,p}(\Omega,\mathbb{R}^3) \subset BD(\Omega)$ so that [1: Corollary 1.11] implies

$$||w||_{L^{3/2}(\Omega)} \leq c\int_\Omega |\mathcal{E}(w)|\,dx = 0,$$

hence $w = 0$ contradicting $||w||_{L^p(\Omega)} = 1$ ∎

3. The existence of a pressure function

Suppose that $u \in \mathcal{C}$ is the minimizer obtained in Theorem 1. For a suitable field $U \in L^{p'}(\Omega, \mathbb{R}^{3\times 3})$ we have

$$p\int_\Omega |\mathcal{E}(u)|^{p-2}\mathcal{E}(u)\mathcal{E}(\psi)\,dx = \int_\Omega U\nabla\psi\,dx$$

on the space $\mathring{H}^{1,p}(\Omega,\mathbb{R}^3)$, especially $\int_\Omega U\nabla\psi\,dx = 0$ if $\operatorname{div}\psi = 0$. Consider a region G as in the proof of Lemma 2; for $\varepsilon < \operatorname{dist}(G,\partial\Omega)$ we define $U_\varepsilon = \Phi_\varepsilon * U$. Then

$$\int_\Omega U_\varepsilon\nabla\psi\,dx = \int_\Omega U\nabla(\Phi_\varepsilon * \psi)\,dx = 0$$

for all $\psi \in C_0^\infty(G,\mathbb{R}^3)$ with $\operatorname{div}\psi = 0$, since $\operatorname{div}(\Phi_\varepsilon * \psi) = \Phi_\varepsilon * \operatorname{div}\psi$ and $\operatorname{spt}(\Phi_\varepsilon * \psi) \subset \Omega$. Let h_ε denote the unique element in $\mathring{H}^{1,2}(G,\mathbb{R}^3)$ representing U_ε with respect to the Dirichlet scalar product, i.e.

$$\langle h_\varepsilon, \psi\rangle := \int_G \nabla h_\varepsilon\nabla\psi\,dx = \int_G U_\varepsilon\nabla\psi\,dx \qquad (\psi \in \mathring{H}^{1,2}(\Omega,\mathbb{R}^3)).$$

Then the above calculations show that this element h_ε is orthogonal to the kernel of the operator $\operatorname{div} : \mathring{H}^{1,2}(G,\mathbb{R}^3) \to L^2(G)$, hence there exists $f_\varepsilon \in L^2(G)$ such that $-\Delta h_\varepsilon = \nabla f_\varepsilon$ which means

$$\int_G U_\varepsilon\nabla\psi\,dx = \int_G f_\varepsilon \cdot \operatorname{div}\psi\,dx \qquad (\psi \in C_0^\infty(G,\mathbb{R}^3)).$$

Without changing the above identity we may suppose $(f_\varepsilon)_G = 0$. Next we select $g \in C_0^\infty(G)$ and choose $\psi \in C^{1,\alpha}(G,\mathbb{R}^3) \cap C^{0,\alpha}(\overline{\Omega},\mathbb{R}^3)$ such that

$$\psi|_{\partial G} = 0, \quad \operatorname{div}\psi = g - (g)_G, \quad ||\nabla\psi||_{L^p(G)} \leq c\,||g-(g)_G||_{L^p(G)} \leq c\,||g||_{L^p(G)}.$$

Then $(f_\varepsilon)_G = 0$ yields

$$\int_G f_\varepsilon \cdot g\,dx = \int_G f_\varepsilon \cdot (g - (g)_G)\,dx = \int_G f_\varepsilon \cdot \operatorname{div}\psi\,dx$$
$$= \int_G U_\varepsilon \nabla\psi\,dx \leq c\,||U||_{L^{p'}(\Omega)}\,||g||_{L^p(G)}$$

which implies $||f_\varepsilon||_{L^{p'}(G)} \leq c(p,G) < \infty$ independent of ε. After passing to the limit we find $f_G \in L^{p'}(G)$ such that $f_\varepsilon \rightharpoonup f_G$ weakly in $L^{p'}(G)$ and

$$\int_\Omega U\nabla\psi\,dx = \int_G f_G \cdot \operatorname{div}\psi\,dx \qquad \text{for all } \psi \in C_0^\infty(G, \mathbb{R}^3). \tag{1}$$

As before let $\{G_n\}$ denote an increasing sequence of domains such that $\bigcup_{n=1}^\infty G_n = \Omega$. For each $n \in \mathbb{N}$ we take a function f_n satisfying (1) on $G = G_n$ (note that (1) fixes f_G only up to an additive constant). Then $f_{n+1} - f_n \equiv a_n$ on G_n, hence the definition

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in G_1 \\ f_2(x) - a_1 & \text{for } x \in G_2 \\ \vdots & \\ f_n(x) - \sum_{k=1}^{n-1} a_k & \text{for } x \in G_n \end{cases}$$

leads to a well defined function $f \in L^{p'}_{loc}(\Omega)$ satisfying (1) on Ω, i.e. for all $\psi \in C_0^\infty(\Omega, \mathbb{R}^3)$. From our construction we deduce

$$||f_n||_{L^{p'}(G_n)} \leq c(p, G_n),$$

$c(p, G_n)$ defined in Lemma 1 and bounded independent of n.

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