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L_φ–Spaces and some Related Sequence Spaces

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Abstract. In view of closed graph theorems in case of maps defined by operator–valued matrices L_φ-spaces were recently introduced by two of the present authors as a generalization of separable $FK(X)$-spaces. In this paper we study the class of L_φ-spaces and a few closely related classes of sequence spaces. It is shown that an analogue of Kalton's closed graph theorem holds for matrix mappings if we consider L_φ-spaces as range spaces, and paralleling a result of Qiu we prove that the class of L_φ-spaces is the best-possible choice here. As a consequence we show that for any L_φ-space E every matrix domain E_A is again an L_φ-space.

Keywords: *Matrix mappings, closed graph theorems, L_φ-spaces, L_r-spaces*

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1. Introduction

Let E and F be locally convex spaces and suppose that E is a Mackey space, the space $(E', \sigma(E', E))$ is sequentially complete and F is separable and B_r-complete. Then Kalton's closed graph theorem [10] states that every closed linear map $T : E \longrightarrow F$ is continuous. Subsequently, Qiu [11] has identified the maximal class of range spaces F in this result, calling its elements L_r-spaces.

Kalton's theorem was successfully applied in classical summability theory to obtain inclusion theorems for K-spaces that are important in connection with Mazur–Orlicz-type theorems (cf. [2 - 4]). In these applications F is a convergence domain c_A of some matrix A, which is always a separable Fréchet space. However, if one tries to extend these results to operator–valued matrices one encounters the problem that convergence domains are no longer separable in general. In fact, they need not even be L_r-spaces [8: Example 3.13/(b)].

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Thus a new idea was needed. Now, in summability theory one usually deals with matrix mappings between sequence spaces, which ordinarily are particular closed mappings. In a recent paper two of the present authors were able to show that if we only consider matrix mappings, then a Kalton-type result obtains for all spaces F from a new class of spaces, which they call L_φ-spaces (see [8: Theorem 4.2]). As desired, this class is large enough to contain all convergence domains of operator-valued matrices, so that one can now deduce inclusion theorems for such matrices [8: Theorem 4.4].

In this paper we study the class of L_φ-spaces and a few closely related classes of sequence spaces. We show that, indeed, Kalton's theorem and Qiu's characterization hold for L_φ-spaces if closed mappings are replaced by matrix mappings. It is also shown that for every L_φ-space E any matrix domain E_A is again an L_φ-space, answering a question in [8]. Similar results are proved for the other classes of sequence spaces considered here. For further investigations into L_φ-spaces see [6].

2. Notations and preliminaries

Throughout this paper we assume that (X,τ_X) and (Y,τ_Y) are (locally convex) Fréchet spaces. A *sequence space (over X)* is a subspace of the space $\omega(X)$ of all sequences $x=(x_k)$ in X. In particular, $c(X)$ and $\varphi(X)$ denote the spaces of convergent and finite sequences in X, respectively. The *β-dual* E^β of a sequence space E over X is defined as

$$E^\beta = \left\{ (A_k)\in\omega(X') \,\middle|\, \forall (x_k)\in E : \sum_k A_k(x_k) \text{ converges} \right\}.$$

Now suppose that the sequence space E over X is endowed with a locally convex topology τ. Then E is called a *$K(X)$-space* if the inclusion map $i: E \longrightarrow \omega(X)$ is continuous, where $\omega(X)$ carries the product topology. If, in addition, (E,τ) is a Fréchet (Banach) space, then E is called an *$FK(X)$-space* (*$BK(X)$-space*). A $K(X)$-space E is called an *AK-space* (*SAK-space*) if $(x_1,\dots,x_n,0,\dots)\longrightarrow x$ (weakly) in E as $n\longrightarrow\infty$ for all $x=(x_k)\in E$. If E is a $K(X)$-space, then every element $(A_k)\in E^\beta$ defines a linear functional on E via $(x_k)\longrightarrow \sum_k A_k(x_k)$. Hence, as usual, we can consider E^β as a subspace of E^*, the algebraic dual of E. In particular we have $\varphi(X')\subset E^*$.

Let $A=(A_{nk})$ be a matrix with entries $A_{nk}\in B(X,Y)$, i.e., continuous linear operators $A_{nk}: X\longrightarrow Y$. A is called *row-finite* if each sequence $(A_{nk})_k$ ($n\in\mathbb{N}$) is finite. For a sequence space E over Y the *matrix domain* E_A is defined as

$$E_A = \left\{ x\in\omega(X) \,\middle|\, \forall n\in\mathbb{N} : \sum_k A_{nk}(x_k) \text{ converges and } \left(\sum_k A_{nk}(x_k)\right)_n \in E \right\}.$$

Here, the convergence of $\sum_k A_{nk}(x_k)$ is taken in the topology τ_Y. If, instead, we only require convergence with respect to $\sigma(Y,Y')$, then the corresponding sequence space is called a *weak matrix domain*, denoted by E_{A_w}. For any $x\in E_{A_w}$ we put $Ax := (\sum_k A_{nk}(x_k))_n$. If F is a sequence space over X with $F\subset E_A$ ($F\subset E_{A_w}$), then the mapping $A : F \longrightarrow E, x \longrightarrow Ax$, is called a *(weak) matrix mapping.* The space $\omega(Y)_A$ is an $FK(X)$-space by [5: Theorem 2.14], and the matrix domain E_A becomes an $FK(X)$-space when it is endowed with the strongest topology that makes the matrix mappings $A : E_A \longrightarrow E$, $x \longrightarrow Ax$ and $i : E_A \longrightarrow \omega(Y)_A$, $x \longrightarrow x$ continuous [1: Proposition 2.4].

The terminology from the theory of locally convex spaces is standard. We follow Wilansky [12]. For the theory of $FK(X)$-spaces and operator–valued matrix domains we refer to [1] and [5].

3. L_φ–K–spaces and some related K–spaces

Let (E,τ) be a locally convex space with topological dual E' and algebraic dual E^*. For any subspace S of E^*, $S < E^*$, we use the notations

$$\overset{\sqcap}{S} := \left\{ g \in E^* \mid \exists\, (g_n) \text{ in } S : g_n \longrightarrow g\ (\sigma(E^*,E)) \right\}$$
$$\overset{\sqcup}{S} := \bigcap \left\{ V < E^* \mid S \subset V = \overset{\sqcap}{V} \right\}$$
$$\overset{\sqcap}{S}{}^{1} := \overbracket{S \cap E'} \quad \text{and} \quad \overset{\sqcap}{S}{}^{j+1} := \overbracket{\overset{\sqcap}{S}{}^{j}}{}^{1} = \overbracket{\overset{\sqcap}{S}{}^{j}} \cap E' \quad (j \in \mathbb{N}).$$

Following J. Qiu [11] we define E to be an L_r*-space* if $E' \subset \overset{\sqcup}{S}$ for any $\sigma(E',E)$-dense subspace S of E'.

In case of $K(X)$-spaces E we note that $\varphi(X')$ is $\sigma(E',E)$-dense in E' [8: Theorem 3.4] and introduce the following notations (see also [8]).

Definition and Remarks 3.1 . Let E be a $K(X)$-space and $j \in \mathbb{N}$. E is called an

L_φ*-space* if $E' \subset \overset{\sqcup}{\varphi(X')}$

$L_\varphi(j)$*-space* if $E' \subset \overbracket{\varphi(X')}^{j}$

$L_\beta(j)$*-space* if $E' \subset \overbracket{E^\beta}^{j}$.

In [8] $L_\varphi(1)$–spaces and $L_\beta(1)$–spaces are called *spaces having φ–sequentially dense dual* and *β–sequentially dense dual*, respectively.

E is an L_φ–space if and only if $E' \subset \overset{\sqcup}{E^\beta}$ since $E^\beta \subset \overset{\sqcup}{\varphi(X')}$. In fact, we even have $E^\beta \subset \overbracket{\varphi(X')}$.

The above definitions depend only on the dual pair (E,E') and not on the particular topology compatible with this dual pair. Obviously, for each $j \in \mathbb{N}$ we have

$$L_\varphi(j)\text{–space} \Rightarrow L_\beta(j)\text{–space} \Rightarrow L_\varphi(j+1)\text{–space} \Rightarrow L_\varphi\text{–space} \Leftarrow L_r\text{–space}.$$

Remarks 3.2 . Let E be any sequence space over X and let H with $\varphi(X') < H < \overset{\sqcup}{E^{\beta}}$ be given.

(a) Then $(E, \tau(E,H))$ is an L_φ-space. (The proof of the Inclusion Theorem in [7] shows us that we may be interested in L_φ-spaces $(E, \tau(E,H))$ where H is a very small subspace of $\overset{\sqcup}{E^{\beta}}$ containing $\varphi(X')$.)

(b) The statement in (a) remains true for any topology τ (instead of $\tau(E,H)$) that is compatible with the dual pair (E,H).

(c) Obviously, $\tau(E, \overset{\sqcup}{E^{\beta}})$ is the strongest locally convex topology τ such that (E,τ) is an L_φ-space.

(d) If $j \in \mathbb{N}$ and τ is any topology that is compatible with the dual pair (E,H) such that (E,τ) is an $L_\varphi(j)$-space ($L_\beta(j)$-space), then $(E, \tau(E, \overset{\sqcap}{H}))$ is an $L_\varphi(j+1)$-space ($L_\beta(j+1)$-space).

Examples 3.3 . (a) Each separable $FK(X)$-space, more generally each *subWCG*–$FK(X)$-space, is an L_φ-space (see [8: Theorem 3.3]). Here, a *subWCG*-space is a (topological) subspace of a weakly compactly generated locally convex space.

(b) Every SAK-$K(X)$-space, in particular every AK-$K(X)$-space, is an $L_\varphi(1)$-space.

(c) The $BK(m)$-space $c(m)$ is an $L_\varphi(1)$-space, however, in general it is not separable and no L_r-space. (See [8: Example 3.13/(b)].)

(d) Based on an example of P. Erdös and G. Piranian [9] in [8: Example 3.12] a regular (real-valued) matrix A is given such that the domain c_A is an $L_\beta(1)$-space but no $L_\varphi(1)$-space. In Remark 4.2 below we will give an example of an $L_\varphi(2)$-space that is no $L_\beta(1)$-space. We do not know if the $L_\beta(2)$-spaces and the $L_\varphi(2)$-spaces coincide.

The following result will be needed in the next section. For sake of brevity we put $\overset{\sqcap}{S}^{0} = S$ (not to be confused with the polar of $\overset{\sqcap}{S}$).

Proposition 3.4 . *Let E and F be locally convex spaces, $U < E^*$, $S < F^*$ and $i, j \in \mathbb{N}_0$. Let $T : E \longrightarrow F$ be a continuous linear mapping such that*

$$f \circ T \in \overset{\sqcap}{U}^{i} \qquad \textit{whenever} \qquad f \in \overset{\sqcap}{S}^{0}.$$

Then

$$g \circ T \in \overset{\sqcap}{U}^{i+j} \qquad \textit{whenever} \qquad g \in \overset{\sqcap}{S}^{j}.$$

Proof. We can assume $j > 0$. Let $g \in \overset{\sqcap}{S}^{j}$. Then there are elements $f_{\nu_{i+1}\ldots\nu_{i+j}} \in S \cap F'$ for $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ such that:

(a) For $i+1 \le \rho < i+j$ and all $\nu_{\rho+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ the mappings

$$y \longrightarrow \lim_{\nu_\rho} \ldots \lim_{\nu_{i+1}} f_{\nu_{i+1}\ldots\nu_{i+j}}(y) \qquad (y \in F)$$

exist and belong to F'.

(b) For all $y \in F$ we have

$$g(y) = \lim_{\nu_{i+j}} \dots \lim_{\nu_{i+1}} f_{\nu_{i+1}\dots\nu_{i+j}}(y).$$

From our assumption we know that $f_{\nu_{i+1}\dots\nu_{i+j}} \circ T \in \overline{U}^{i}$ for each $\nu_{i+1}, \dots, \nu_{i+j} \in \mathbb{N}$. This implies that there are elements $g_{\nu_1\dots\nu_i\nu_{i+1}\dots\nu_{i+j}} \in U \cap E'$ for $\nu_1, \dots, \nu_{i+j} \in \mathbb{N}$ such that:

(c) For $1 \leq \sigma < i$ and all $\nu_{\sigma+1}, \dots, \nu_i, \nu_{i+1}, \dots, \nu_{i+j} \in \mathbb{N}$ the mappings

$$x \longrightarrow \lim_{\nu_\sigma} \dots \lim_{\nu_1} g_{\nu_1\dots\nu_i\nu_{i+1}\dots\nu_{i+j}}(x) \qquad (x \in E)$$

exist and belong to E'.

(d) For all $x \in E$ and $\nu_{i+1}, \dots, \nu_{i+j} \in \mathbb{N}$ we have

$$\left(f_{\nu_{i+1}\dots\nu_{i+j}} \circ T\right)(x) = \lim_{\nu_i} \dots \lim_{\nu_1} g_{\nu_1\dots\nu_i\nu_{i+1}\dots\nu_{i+j}}(x).$$

We thus have found elements $g_{\nu_1\dots\nu_{i+j}} \in U \cap E'$ for $\nu_1, \dots, \nu_{i+j} \in \mathbb{N}$ with the following properties:

(a') For $1 \leq \rho < i + j$ and all $\nu_{\rho+1}, \dots, \nu_{i+j} \in \mathbb{N}$ the mappings

$$x \longrightarrow \lim_{\nu_\rho} \dots \lim_{\nu_1} g_{\nu_1\dots\nu_{i+j}}(x) \qquad (x \in E)$$

exist and belong to E' (this is just (c) in case $\rho < i$; for $\rho = i$ it follows from (d) and for $\rho > i$ from (a) if we note that T is continuous).

(b') For all $x \in E$ we have

$$(g \circ T)(x) = \lim_{\nu_{i+j}} \dots \lim_{\nu_1} g_{\nu_1\dots\nu_{i+j}}(x)$$

(this follows from (b) and (d)).

But (a') and (b') together imply that $g \circ T \in \overline{U}^{i+j}$ ∎

Remark 3.5 . Using the adjoint $T' : F' \longrightarrow E'$ of the mapping T, the assertion of the proposition can be put more concisely as

$$T'\left(\overline{S}^{0}\right) \subset \overline{U}^{i} \qquad \textit{implies} \qquad T'\left(\overline{S}^{j}\right) \subset \overline{U}^{i+j}.$$

4. Domains of operator–valued matrices

From [8: Theorems 3.9 and 3.10] it is known that the domain $c(Y)_A$ of an operator–valued matrix A is an $L_\beta(1)$–space, and that E_A is an L_φ–space whenever E is an $L_\beta(1)$–space. Here we are going to improve these results.

Theorem 4.1 . *Let E be a $K(Y)$-space, $A = (A_{nk})$ a matrix with $A_{nk} \in B(X,Y)$ and let $j \in \mathbb{N}$.*

(a) *If E is an $L_\varphi(j)$-space, then E_A is an $L_\beta(j)$-space.*

(b) *If E is an $L_\beta(j)$-space, then E_A is an $L_\beta(j+1)$-space.*

Suppose that in addition A is row-finite. Then:

(a') *If E is an $L_\varphi(j)$-space, then E_A is an $L_\varphi(j)$-space.*

(b') *If E is an $L_\beta(j)$-space, then E_A is an $L_\varphi(j+1)$-space.*

Special case (see [8: Theorem 3.9]): *$c(Y)_A$ is an $L_\beta(1)$-space, and even an $L_\varphi(1)$-space if A is row-finite.*

Remark 4.2 . Example 3.3/(d) tells us that, in general, we cannot replace '$L_\beta(j)$-space' by '$L_\varphi(j)$-space' in statement (a). Assertion (a') is obviously best-possible, while in statement (b') we cannot replace '$L_\varphi(j+1)$-space' by '$L_\beta(j)$-space' in general: In [8: Example 3.14] there is an example of a (real-valued) row-finite matrix A and an $L_\beta(1)$-space E such that the domain E_A is no $L_\beta(1)$-space. (From statement (b') above we see that it is an $L_\varphi(2)$-space.) We do not know if one can replace '$L_\beta(j+1)$-space' in statement (b) by '$L_\varphi(j+1)$-space'.

Proof of Theorem 4.1. Let E be a $K(Y)$-space, and let $f \in E_A'$ be given. Then we may choose elements $g \in E'$ and $h \in \omega(Y)_A^\beta = \omega(Y)_A'$ with $f = g \circ A + h$ (see [1: Proposition 2.10] and [5: Theorem 2.14/(b)]). Since $E_A \subset \omega(Y)_A$, we have $h \in E_A^\beta \subset \overline{\varphi(X')}^{\,j} \subset \overline{E_A^\beta}^{\,j}$ for all $j \in \mathbb{N}$. Hence in order to prove the various statements of the theorem we need only show that $g \circ A$ belongs to $\overline{E_A^\beta}^{\,j}$, $\overline{E_A^\beta}^{\,j+1}$, $\overline{\varphi(X')}^{\,j}$ and $\overline{\varphi(X')}^{\,j+1}$, respectively. To this end we apply Proposition 3.4 to the mapping $A : E_A \longrightarrow E$.

(a) Let E be an $L_\varphi(j)$-space. Then $g \in E' \subset \overline{\varphi(Y')}^{\,j}$. Here we choose $U = E_A^\beta$, $S = \varphi(Y')$ and $i = 0$. If $\Phi = (\Phi_n)_{n=1}^N \in \varphi(Y')$, then we have for $x \in E_A$

$$(\Phi \circ A)(x) = \sum_{n=1}^{N} \Phi_n \left(\sum_{k=1}^{\infty} A_{nk}(x_k) \right) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{N} \Phi_n \circ A_{nk} \right)(x_k)$$

so that $\Phi \circ A \in E_A^\beta$. Hence the hypothesis of Proposition 3.4 holds, so that $g \circ A \in \overline{E_A^\beta}^{\,j}$, as desired.

(b) Let E be an $L_\beta(j)$-space. Then $g \in E' \subset \overline{E^\beta}^{\,j}$. Here we choose $U = E_A^\beta$, $S = E^\beta$ and $i = 1$. If $\Phi = (\Phi_n) \in E^\beta$, then we have for $x \in E_A$

$$(\Phi \circ A)(x) = \lim_{m\to\infty} \sum_{n=1}^{m} \Phi_n \left(\sum_{k=1}^{\infty} A_{nk}(x_k) \right) = \lim_{m\to\infty} \sum_{k=1}^{\infty} \left(\sum_{n=1}^{m} \Phi_n \circ A_{nk} \right)(x_k)$$

so that $\Phi \circ A \in \overline{E_A^\beta}^{\,1}$. Proposition 3.4 implies that $g \circ A \in \overline{E_A^\beta}^{\,j+1}$.

Now suppose that A is row-finite.

(a') Let E be an $L_\varphi(j)$-space. Then $g \in E' \subset \overline{\varphi(Y')}^{\nu}$. Here we choose $U = \varphi(X')$, $S = \varphi(Y')$ and $i = 0$. If $\Phi = (\Phi_n)_{n=1}^N \in \varphi(Y')$, then we have for $x \in E_A$

$$(\Phi \circ A)(x) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{N} \Phi_n \circ A_{nk} \right) (x_k)$$

and hence $\Phi \circ A \in \varphi(X')$. Now Proposition 3.4 implies that $g \circ A \in \overline{\varphi(X')}^{\nu}$.

(b') This follows from statement (a') since every $L_\beta(j)$-space is also an $L_\varphi(j+1)$-space ∎

5. Matrix maps into L_φ–K–spaces

The aim of this section is to show that the class of L_φ-spaces is the complete analogue of Qiu's L_r-spaces if closed linear mappings are replaced by matrix mappings. We also prove that the matrix domain E_A of an operator–valued matrix is an L_φ-space whenever E is an L_φ-space. This result may be considered as a generalization of the classical fact that the matrix domain E_A of a scalar-valued matrix is separable if E is a separable FK-space.

Our first result is the analogue for matrix mappings of Qiu's extension of Kalton's closed graph theorem. It generalizes the results in Theorem 4.2 and Theorem 4.4./(a) $\Rightarrow$ (b) of [8].

Theorem 5.1 . *Let E be a $K(X)$-space and F a $K(Y)$-space. If E is a Mackey space, $(E', \sigma(E', E))$ is sequentially complete and F is an L_φ-space, then every (weak) matrix mapping $A : E \longrightarrow F$ is continuous.*

Proof. We put

$$D_A^\bullet := (A')^{-1}(E') = \left\{ f \in F^\bullet \mid f \circ A \in E' \right\}$$

and $D_A := D_A^\bullet \cap F'$. If we can show that $D_A = F'$, then A is weakly continuous hence continuous as E is a Mackey space.

To that end let $f \in F^\bullet$ and (f_n) in $F^\bullet$ with $f_n \circ A \in E'$ and $f_n \longrightarrow f$ in $\left(F^\bullet, \sigma(F^\bullet, F)\right)$ be given. Then we have $f_n \circ A \longrightarrow f \circ A$ in $\left(E^\bullet, \sigma(E^\bullet, E)\right)$. Since $\left(E', \sigma(E', E)\right)$ is sequentially complete, this shows that $f \circ A \in E'$, so that $f \in D_A^\bullet$. Thus $D_A^\bullet$ is $\sigma(F^\bullet, F)$-sequentially closed, which implies that $\underline{D_A} \subset D_A^\bullet$, hence $\underline{D_A} \cap F' = D_A$.

We next show that $\varphi(Y') \subset D_A$. For this it suffices to prove that for each $g \in Y'$ and $n \in \mathbb{N}$ the mapping $x \longrightarrow g(\sum_{k=1}^\infty A_{nk}(x_k))$ belongs to E'. But since we have

$$g\left(\sum_{k=1}^{\infty} A_{nk}(x_k) \right) = \lim_m \sum_{k=1}^{m} (g \circ A_{nk})(x_k)$$

for all $x \in E$, this follows from the weak sequential completeness of E'.

In conclusion, $\overset{\llcorner\lrcorner}{D_A} \cap F' = D_A$, $\varphi(Y') \subset D_A$ and the fact that F is an L_φ-space imply that

$$F' = \overset{\llcorner\lrcorner}{\varphi(Y')} \cap F' \subset \overset{\llcorner\lrcorner}{D_A} \cap F' = D_A,$$

which had to be shown ∎

Remark 5.2 . The proof shows that the theorem remains true for any linear mapping $A = (A_n) : E \longrightarrow F$ with the property that $\varphi(Y') \subset D_A$, which is equivalent to the continuity of each mapping $A_n : E \longrightarrow Y$ ($n \in \mathbb{N}$).

The next result is the analogue to Qiu's characterization of L_r-spaces [11]. It shows that the class of L_φ-spaces is the maximal class of range spaces in Theorem 5.1.

Theorem 5.3 . *Let F be a $K(X)$-space. Then the following statements are equivalent:*

(a) *F is an L_φ-space.*

(b) *For each $K(X)$-space E that is a Mackey space such that $(E', \sigma(E', E))$ is sequentially complete every matrix mapping $A : E \longrightarrow F$ is continuous.*

Proof. The implication (a) $\Rightarrow$ (b) is contained in Theorem 5.1. The converse implication follows immediately from the following remark ∎

Remark 5.4 . Let F be a $K(X)$-space. If the inclusion map

$$i : \left(F, \tau\left(F, \overset{\llcorner\lrcorner}{F^\beta}\right)\right) \longrightarrow F$$

is continuous, then F is an L_φ-space. (Namely, in this situation we have $F' \subset \overset{\llcorner\lrcorner}{F^\beta} = \overset{\llcorner\lrcorner}{\varphi(X')}$.)

Using the last remark we can now obtain a permanence result for L_φ-spaces under the formation of matrix domains, answering a question in [8].

Theorem 5.5 . *Let $A = (A_{nk})$ be a matrix with $A_{nk} \in B(X, Y)$. If E is an L_φ-$K(Y)$-space, then E_A is an L_φ-$K(X)$-space.*

Proof. By Remark 5.4 we have to prove the continuity of

$$i : \left(E_A, \tau\left(E_A, \overset{\llcorner\lrcorner}{E_A^\beta}\right)\right) \longrightarrow E_A,$$

which is equivalent to the continuity of the inclusion map

$$i_\omega : \left(E_A, \tau\left(E_A, \overset{\llcorner\lrcorner}{E_A^\beta}\right)\right) \longrightarrow \omega(Y)_A$$

and of the map

$$A : \left(E_A, \tau\left(E_A, \overset{\llcorner\lrcorner}{E_A^\beta}\right)\right) \longrightarrow E\,, \; x \longrightarrow Ax\,.$$

However, since in both cases the range space is an L_φ-space (note that $\omega(Y)_A$ is an AK-space by [5: Theorem 2.14]), this is an immediate corollary of Theorem 5.1 ∎

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