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# On Fundamental Solutions of the Heat Conduction Difference Operator

K. Gürlebeck and A. Hommel

**Abstract.** It is the aim of the paper to investigate fundamental solutions for the difference operator of heat conduction in the explicit and also in the implicit case. The existence of fundamental solutions will be shown in a constructive way. In both cases the convergence of the discrete fundamental solutions to the continuous fundamental solution will be investigated in the discrete $l_1$-space.

**Keywords:** *Difference operators, heat conduction, fundamental solutions*

**AMS subject classification:** Primary 31C20, secondary 39A12, 65M06

## 1. Introduction

The method to solve problems of mathematical physics using potential theory is well known. For constructive analytical considerations and also in the case of numerical applications it is necessary to have an explicit expression of the fundamental solution or of the Green function. In the most cases there will be a projection of the obtained operator equation into a finite-dimensional space to get an equation which can be solved explicitly. After the discretization one has to accept a loss of information concerning for instance algebraic properties of the operators which are very useful in potential theory.

Another method consists in the direct discretization of the partial differential equation for instance by a finite difference approximation. There are many connections between the differential equation and the finite difference equation. For a long time there was also the question if it is possible to develop a potential theory for difference operators. Of course even in the discrete case a potential theoretic approach makes sense only if it is possible to obtain explicit expressions for fundamental solutions of the partial difference operators. First answers were given already in [2], some special elliptic operators were discussed in [3] and [4]. In the fundamental paper [9] the case of more general elliptic difference operators is treated and a lot of tools for further investigations and for the representation of fundamental solutions in standard situations were given. A detailed discussion of a discrete potential theory basing on the concept of a discrete Green function is given in [7]. Further results in this direction are contained in the papers [1] and [10].

K. Gürlebeck: Techn. Univ. Chemnitz – Zwickau, Dep. Math., PSF 964, D – 09009 Chemnitz
A. Hommel: Techn. Univ. Chemnitz – Zwickau, Dep. Math., PSF 964, D – 09009 Chemnitz

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Parabolic operators has not yet received so much attention. In the paper [6] we can find a discussion of fundamental solutions of difference operators in the distribution space $\mathcal{D}'$. This space is not adapted to the discussion of a finite difference scheme where the operators and functions are connected with a fixed lattice. Other approaches to the topic use algebraic tools [12] or the authors work on networks and graphs [5]. In these cases there are some difficulties to prove convergence results.

In our paper we will prove the existence and the uniqueness of a fundamental solution for the explicit and for the implicit difference operator of heat conduction. We show that these fundamental solutions belong to the space $l_1^{loc}(\mathbb{R}_h^2 \times \mathbb{R}_{h_t})$. At the end we shall prove the convergence of the discrete fundamental solutions to the known fundamental solution of the differential operator with respect to the $l_1$-norm if time step and mesh width tend to zero.

## 2. Explicit difference equation

**2.1 Fundamental solution.** Let $\mathbb{R}_h^2 = \{x = (x_1, x_2) : x_1 = kh \text{ and } x_2 = jh\ (k,j \in \mathbb{Z})\}$ and $\mathbb{R}_{h_t} = \{t = lh_t : l \in \mathbb{N})\}$. In the following we consider functions defined on the lattices $\mathbb{R}_h^2$, $\mathbb{R}_{h_t}$, and $\mathbb{R}_h^2 \times \mathbb{R}_{h_t}$, respectively. We define discrete $l_1$-spaces in the usual way:

$$f \in l_1(\mathbb{R}_h^2) \quad \Longleftrightarrow \quad \|f\|_{l_1(\mathbb{R}_h^2)} = \sum_{x \in \mathbb{R}_h^2} |f(x)|h^2 < \infty$$

$$f \in l_1(\mathbb{R}_{h_t}) \quad \Longleftrightarrow \quad \|f\|_{l_1(\mathbb{R}_{h_t})} = \sum_{t \in \mathbb{R}_{h_t}} |f(t)|h_t < \infty.$$

Further, we use the notations

$$\delta_h(x) = \begin{cases} \frac{1}{h^2} & \text{if } x = (0,0) \\ 0 & \text{if } x \in \mathbb{R}_h^2 \setminus \{(0,0)\} \end{cases} \quad \text{and} \quad \delta_{h_t}(t) = \begin{cases} \frac{1}{h_t} & \text{if } t = 0 \\ 0 & \text{if } t \in \mathbb{R}_{h_t} \setminus \{0\} \end{cases}$$

for the discrete Delta function. The Heaviside function will be denoted by $\Theta = \Theta(t)$. We study the explicit difference equation

$$\begin{aligned} \left((-a^2\Delta_h + D_{h_t}^+)E_h\right)(x,t) &= \delta_{h,h_t}(x,t) = \delta_h(x)\delta_{h_t}(t) \\ &= \begin{cases} \frac{1}{h^2 h_t} & \text{if } (x,t) = (0,0,0) \\ 0 & \text{if } (x,t) \neq (0,0,0) \end{cases} \end{aligned} \tag{1}$$

where

$$\begin{aligned} (\Delta_h E_h)(x,t) = \frac{1}{h^2}\Big[&-4E_h(x_1,x_2,t) + E_h(x_1+h,x_2,t) \\ &+ E_h(x_1-h,x_2,t) + E_h(x_1,x_2+h,t) + E_h(x_1,x_2-h,t)\Big] \end{aligned}$$

and

$$(D_{h_t}^+ E_h)(x,t) = \frac{1}{h_t}\Big[E_h(x,t+h_t) - E_h(x,t)\Big].$$

If equation (1) has a solution $E_h$ it is called *fundamental solution* of the explicit heat conduction difference operator. We will omit here a general discussion in discrete spaces of distributions (see, e.g., [12]) because on the one hand we want to underline the analogy to the continuous case where the fundamental solution is a $L_1^{loc}$-function. On the other hand we are interested in convergence results in norms as strong as possible. Therefore we investigate if $E_h$ belongs to the space $l_1^{loc}(\mathbb{R}_h^2 \times \mathbb{R}_{h_t})$. If we assume for the moment that $E_h(\cdot,t) \in l_1(\mathbb{R}_h^2)$ for all $t \in \mathbb{R}_{h_t}$ we can apply the discrete Fourier transform (see, e.g., [9] for $l_2(\mathbb{R}_h^n)$) with respect to $x$:

$$(F_h E_h)(\xi,t) = \begin{cases} \frac{h^2}{2\pi} \sum\limits_{x\in \mathbb{R}_h^2} E_h(x,t)e^{ix\xi} & \text{in } Q_h \\ 0 & \text{in } \mathbb{R}^2 \setminus Q_h \end{cases}$$

where

$$Q_h = \left\{ \xi = (\xi_1,\xi_2) \in \mathbb{R}^2 : -\frac{\pi}{h} < \xi_1,\xi_2 < +\frac{\pi}{h} \right\}.$$

Introducing the abbreviation

$$d^2 = \frac{4}{h^2}\left(\sin^2 \frac{h\xi_1}{2} + \sin^2 \frac{h\xi_2}{2}\right)$$

we get the equation

$$\left(a^2 d^2 F_h E_h + D_{h_t}^+ F_h E_h\right)(\xi,t) = \frac{1}{2\pi}\delta_{h_t}(t)\chi_{h}(\xi)$$

which has the solution

$$(F_h E_h)(\xi,t) = \frac{1}{2\pi}\Theta(t)\left(1 - a^2 d^2 h_t\right)^{t/h_t - 1} \chi_{h}(\xi). \tag{2}$$

The above used notation $\chi_h$ stands for the characteristic function of $Q_h$. In all what follows we denote by $R_h f$ the restriction of a function $f$ defined on $\mathbb{R}^2$ to the lattice $\mathbb{R}_h^2$. We write $S_{h_t} g$ to designate the restriction of $g$ defined on $\mathbb{R}^1$ to $\mathbb{R}_{h_t}$. Then, using the inverse Fourier transform (see [9])

$$F_h^{-1} = R_h F \qquad \text{where} \quad (Fu)(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} u(\xi)e^{-ix\xi}d\xi$$

we obtain

$$E_h(x,t) = \Theta(t)\left((1 + a^2 h_t \Delta_h)^{t/h_t - 1}\delta_h\right)(x). \tag{3}$$

Obviously the support of $E_h(x,t)$ is contained only in a cone and therefore the obtained solution belongs to the above mentioned space.

**2.2 Convergence.** We shall study now the behaviour of the fundamental solution (3) if $h$ and $h_t$ tend to zero. For this purpose we rewrite equation (1) for $t > 0$ in the form

$$\begin{aligned} E_h(x_1,x_2,t+h_t) = &\left(1 - \frac{4a^2 h_t}{h^2}\right) E_h(x_1,x_2,t) \\ &+ \frac{a^2 h_t}{h^2}\Big[E_h(x_1+h,x_2,t) + E_h(x_1-h,x_2,t) \\ &+ E_h(x_1,x_2+h,t) + E_h(x_1,x_2-h,t)\Big] \end{aligned} \tag{4}$$

and we assume that $h_t/h^2 < 1/4a^2$. Because $E_h(x,t)$ is supported in a cone we can use the maximum principle (see [8]) to show that $E_h(x,t) \geq 0$ for each $x$ and $t$. Then from (3) and (4) it follows

$$\|E_h(\cdot, t+h_t)\|_{l_1(\mathbb{R}_h^2)} = \|E_h(\cdot,t)\|_{l_1(\mathbb{R}_h^2)} = \|E_h(\cdot,h_t)\|_{l_1(\mathbb{R}_h^2)} = 1.$$

Let $G \subset \mathbb{R}^2$ be a bounded domain. Then $G_h = (G \cap \mathbb{R}_h^2) \subset \mathbb{R}_h^2$ will be called *bounded discrete domain*. Further, let $T_0 = l_0 h_t$, with $l_0 \in \mathbb{N}$. By addition with respect to $t$ we get the estimation

$$\|E_h\|_{l_1(G_h \times [0,T_0])} \leq \|E_h\|_{l_1(\mathbb{R}_h^2 \times [0,T_0])} = \sum_{l=1}^{l_0} h_t = T_0. \tag{5}$$

Now we consider the continuous fundamental solution

$$E(x,t) = \frac{\Theta(t)}{4a^2\pi t} e^{-|x|^2/4a^2t}.$$

We get

$$\begin{aligned}
\|R_h E(\cdot,t)\|_{l_1(G_h)} &\leq \Theta(t) \sum_{k,j\in Z} \frac{1}{4a^2\pi t} e^{-(k^2+j^2)h^2/4a^2t} h^2 \\
&= \Theta(t) \left( \sum_{k\in Z} \frac{1}{2a\sqrt{\pi t}} e^{-k^2h^2/4a^2t} h \right)^2 \\
&= \Theta(t) \left( \frac{h}{2a\sqrt{\pi t}} + 2\sum_{k=1}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-k^2h^2/4a^2t} h \right)^2 \\
&\leq \Theta(t) \left( \frac{h}{2a\sqrt{\pi t}} + 2\int_0^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-y^2/4a^2t} dy \right)^2 \\
&\leq \Theta(t) \left( \frac{h}{2a\sqrt{\pi t}} + 1 \right)^2
\end{aligned}$$

and furthermore, under the assumption $h^2/h_t < C_1$,

$$\begin{aligned}
\|S_{h_t} R_h E\|_{l_1(G_h \times [0,T_0])} &\leq T_0 + \frac{h\sqrt{h_t}}{a\sqrt{\pi}} + \frac{h}{a\sqrt{\pi}} \int_{h_t}^{T_0} \frac{1}{\sqrt{t}}\, dt + \frac{h^2}{4a^2\pi} + \frac{h^2}{4a^2\pi} \int_{h_t}^{T_0} \frac{1}{t}\, dt \\
&\leq T_0 + \frac{2h}{a\sqrt{\pi}} \sqrt{T_0} + \frac{h^2}{4a^2\pi} + \frac{h^2}{4a^2\pi} \ln \frac{T_0}{h_t} \\
&\leq T_0 + \frac{2h}{a\sqrt{\pi}} \sqrt{T_0} + \frac{C_1}{4a^2\pi} T_0.
\end{aligned} \tag{6}$$

From (5) and (6) we conclude

$$\left\| E_h - S_{h_t} R_h E \right\|_{l_1(G_h \times [0,T_0])} \leq 2T_0 + \frac{2h}{a\sqrt{\pi}} \sqrt{T_0} + \frac{C_1}{4a^2\pi} T_0. \tag{7}$$

The inequality (7) describes the approximation error of the fundamental solution (3) for small values of the time variable.

In the following we study

$$\left\| E_h(\cdot,t) - S_{h_t} R_h E(\cdot,t) \right\|_{l_1(G_h)}$$

for $t > T_0$. Let

$$A(G_h) = \sum_{x \in G_h} h^2.$$

Then we have

$$\begin{aligned}
&\left\| E_h(\cdot,t) - S_{h_t} R_h E(\cdot,t) \right\|_{l_1(G_h)} \\
&\qquad \le A(G_h) \max_{x \in G_h} \left| E_h(x,t) - S_{h_t} R_h E(x,t) \right| \\
&\qquad = A(G_h) \max_{x \in G_h} \left| (R_h F F_h E_h)(x,t) - (S_{h_t} R_h F F^{-1} E)(x,t) \right| \\
&\qquad \le \frac{1}{2\pi} A(G_h) \left\| (F_h E_h)(\cdot,t) - (S_{h_t} F^{-1} E)(\cdot,t) \right\|_{L_1(\mathbb{R}^2)}.
\end{aligned} \tag{8}$$

In $\mathbb{R}^2 \setminus Q_h$ it follows from inequality (8)

$$\begin{aligned}
\left\| (F_h E_h)(\cdot,t) - (S_{h_t} F^{-1} E)(\cdot,t) \right\|_{L_1(\mathbb{R}^2 \setminus Q_h)} &= \left\| (S_{h_t} F^{-1} E)(\cdot,t) \right\|_{L_1(\mathbb{R}^2 \setminus Q_h)} \\
&= \frac{\Theta(t)}{2\pi} \left\| e^{-a^2 |\xi|^2 t} \right\|_{L_1(\mathbb{R}^2 \setminus Q_h)} \\
&\le \frac{\Theta(t)}{2a^2 t} e^{-a^2 \pi^2 t / h^2}.
\end{aligned} \tag{9}$$

In $Q_h$ we use the estimation

$$\begin{aligned}
&\left\| (F_h E_h)(\cdot,t) - (S_{h_t} F^{-1} E)(\cdot,t) \right\|_{L_1(Q_h)} \\
&\qquad = \frac{\Theta(t)}{2\pi} \left\| (1 - a^2 d^2 h_t)^{t/h_t - 1} - e^{-a^2 |\xi|^2 t} \right\|_{L_1(Q_h)} \\
&\qquad \le \frac{\Theta(t)}{2\pi} \Big( \left\| (1 - a^2 d^2 h_t)^{t/h_t - 1} - e^{-a^2 d^2 t} \right\|_{L_1(Q_h)} \\
&\qquad\qquad + \left\| e^{-a^2 d^2 t} - e^{-a^2 |\xi|^2 t} \right\|_{L_1(Q_h)} \Big).
\end{aligned} \tag{10}$$

First we consider the expression

$$\left\| e^{-a^2 d^2 t} - e^{-a^2 |\xi|^2 t} \right\|_{L_1(Q_h)}.$$

From

$$|\xi|^2 - d^2 = |\xi|^2 - \frac{4}{h^2} \left( \sin^2 \frac{h\xi_1}{2} + \sin^2 \frac{h\xi_2}{2} \right) \le \frac{h^2}{12} (\xi_1^4 + \xi_2^4) \le \frac{h^2}{12} |\xi|^4$$

and $d^2 \geq (4/\pi^2)|\xi|^2$ we obtain

$$\left|e^{-a^2d^2t} - e^{-a^2|\xi|^2t}\right| < a^2t\left(|\xi|^2 - d^2\right)e^{-a^2d^2t} < \frac{a^2th^2}{12}|\xi|^4 e^{-4a^2|\xi|^2t/\pi^2}$$

and

$$\left\|e^{-a^2d^2t} - e^{-a^2|\xi|^2t}\right\|_{L_1(Q_h)} \leq \frac{a^2th^2}{3}\int_{r=0}^{\sqrt{2}\pi/h}\int_{\varphi=0}^{\pi/2} r^5 e^{-4a^2r^2t/\pi^2}\,d\varphi dr \leq C_2\frac{h^2}{t^2}. \qquad (11)$$

Now we deal with the expression

$$\left\|(1 - a^2d^2h_t)^{t/h_t-1} - e^{-a^2d^2t}\right\|_{L_1(Q_h)}$$

in inequality (10). From the inequality $(1+\frac{1}{x})^x < e < (1+\frac{1}{x})^{x+1}$ for real numbers $x > 0$ we come to

$$\left(1 - a^2d^2h_t\right)^{t/h_t} < e^{-a^2d^2t} < \left(1 - a^2d^2h_t\right)^{t/h_t - a^2d^2t} < \left(1 - a^2d^2h_t\right)^{t/h_t - a^2|\xi|^2t} \qquad (12)$$

provided that $h_t < h^2/8a^2$. We mention that

$$a^2d^2h_t < a^2d^2\frac{h^2}{8a^2} = \frac{h^2}{8}\frac{4}{h^2}\left(\sin^2\frac{h\xi_1}{2} + \sin^2\frac{h\xi_2}{2}\right) \leq 1.$$

First we restrict our consideration to the case $|\xi|^2 \leq 1/a^2t$. Then the inequality

$$\left(1 - a^2d^2h_t\right)^{t/h_t - a^2|\xi|^2t} \leq \left(1 - a^2d^2h_t\right)^{t/h_t-1}$$

is valid. From (12) it follows

$$\begin{aligned}\left|\left(1 - a^2d^2h_t\right)^{t/h_t-1} - e^{-a^2d^2t}\right| &= \left(1 - a^2d^2h_t\right)^{t/h_t-1} - e^{-a^2d^2t} \\ &= e^{(t/h_t-1)\ln(1-a^2d^2h_t)} - e^{-a^2d^2t} \leq e^{-(t/h_t-1)a^2d^2h_t} - e^{-a^2d^2t} \\ &= e^{-a^2d^2t}\sum_{n=1}^{\infty}\frac{(a^2d^2h_t)^n}{n!} \leq \frac{a^2d^2h_t}{1 - a^2d^2h_t}e^{-a^2d^2t}.\end{aligned}$$

For

$$\xi \in Q_{1,h} = \{(\xi_1,\xi_2) \in Q_h : |\xi|^2 \leq 1/a^2t\}$$

we use the estimations

$$d^2 \leq |\xi|^2 \qquad \text{and} \qquad d^2 \geq (4/\pi^2)|\xi|^2.$$

If we suppose $h_t < h^2/3a^2\pi^2 < h^2/8a^2$, then we can show

$$a^2d^2h_t < a^2d^2\frac{h^2}{3a^2\pi^2} = \frac{h^2}{3\pi^2}\frac{4}{h^2}\left(\sin^2\frac{h\xi_1}{2} + \sin^2\frac{h\xi_2}{2}\right) < \frac{8}{3\pi^2} < \frac{3}{\pi^2}$$

and

$$\frac{1}{1-a^2d^2h_t} < \frac{\pi^2}{\pi^2-3} < 2 \tag{13}$$

such that

$$\begin{aligned}\int_{Q_{1,h}} \left|\left(1-a^2d^2h_t\right)^{t/h_t-1} - e^{-a^2d^2t}\right| d\xi &< 2a^2h_t \int_{Q_{1,h}} e^{-4a^2|\xi|^2t/\pi^2}|\xi|^2 d\xi \\ &< 8a^2h_t \int_{r=0}^{1/a\sqrt{t}} \int_{\varphi=0}^{\pi/2} e^{-4a^2r^2t/\pi^2} r^3 d\varphi dr \\ &< C_3\frac{h_t}{t^2}.\end{aligned} \tag{14}$$

Now we pass on to the case $|\xi|^2 > 1/a^2t$. Then the inequality

$$\left(1-a^2d^2h_t\right)^{t/h_t} < \left(1-a^2d^2h_t\right)^{t/h_t-1} < \left(1-a^2d^2h_t\right)^{t/h_t-a^2|\xi|^2t}$$

is valid. From (12) we obtain

$$\begin{aligned}&\left|\left(1-a^2d^2h_t\right)^{t/h_t-1} - e^{-a^2d^2t}\right| \\ &\quad\le \left(1-a^2d^2h_t\right)^{t/h_t-a^2|\xi|^2t} - \left(1-a^2d^2h_t\right)^{t/h_t} \\ &\quad= \left(1-a^2d^2h_t\right)^{t/h_t-a^2|\xi|^2t}\left(1-\left(1-a^2d^2h_t\right)^{a^2|\xi|^2t}\right) \\ &\quad< -\left(1-a^2d^2h_t\right)^{t/h_t-a^2|\xi|^2t}\ln\left(\left(1-a^2d^2h_t\right)^{a^2|\xi|^2t}\right) \\ &\quad= a^2|\xi|^2t\left(1-a^2d^2h_t\right)^{t/h_t-a^2|\xi|^2t}\sum_{n=1}^{\infty}\frac{(a^2d^2h_t)^n}{n} \\ &\quad< a^2|\xi|^2t\frac{a^2d^2h_t}{1-a^2d^2h_t}\left(1-a^2d^2h_t\right)^{t/h_t-a^2|\xi|^2t}.\end{aligned}$$

Using the inequalities $d^2 \le |\xi|^2$ and $d^2 \ge (4/\pi^2)|\xi|^2$ in $Q_h$ and (13) for $h_t < h^2/3\pi^2a^2$

we get

$$
\begin{aligned}
&\int\limits_{Q_h \backslash Q_{1,h}} \left|\left(1-a^2d^2h_t\right)^{t/h_t-1} - e^{-a^2d^2t}\right| d\xi \\
&\qquad < 2a^4th_t \int\limits_{Q_h \backslash Q_{1,h}} e^{(t/h_t - a^2|\xi|^2 t)\ln(1-a^2d^2h_t)} |\xi|^4\, d\xi \\
&\qquad < 2a^4th_t \int\limits_{Q_h \backslash Q_{1,h}} e^{-(t/h_t - a^2|\xi|^2 t)a^2d^2h_t} |\xi|^4\, d\xi \\
&\qquad < 2a^4th_t \int\limits_{Q_h \backslash Q_{1,h}} e^{(-(4/\pi^2)a^2|\xi|^2 t + (3/\pi^2)a^2|\xi|^2 t)} |\xi|^4\, d\xi \\
&\qquad < 8a^4th_t \int\limits_{r=1/a\sqrt{t}}^{\sqrt{2}\pi/h} \int\limits_{\varphi=0}^{\pi/2} e^{-a^2r^2t/\pi^2} r^5\, d\varphi dr \\
&\qquad < C_4 \frac{h_t}{t^2}.
\end{aligned}
\tag{15}
$$

The inequalities (14) and (15) prove

$$
\left\|\left(1-a^2d^2h_t\right)^{t/h_t-1} - e^{-a^2d^2t}\right\|_{L_1(Q_h)} \leq C_5 \frac{h_t}{t^2} \tag{16}
$$

and from (10), (11) and (16) we get the result

$$
\left\|(F_hE_h)(\cdot,t) - (F^{-1}E)(\cdot,t)\right\|_{L_1(Q_h)} \leq \frac{\Theta(t)}{2\pi}\left(C_2\frac{h^2}{t^2} + C_5\frac{h_t}{t^2}\right).
$$

Finally, by the help of (8) and (9) we conclude

$$
\left\|E_h(\cdot,t) - (R_hE)(\cdot,t)\right\|_{l_1(G_h)} \leq \frac{\Theta(t)}{4\pi^2} A(G_h)\left(C_2\frac{h^2}{t^2} + C_5\frac{h_t}{t^2} + \frac{\pi}{a^2t}e^{-a^2\pi^2t/h^2}\right).
$$

A summing-up with respect to the time variable leads to the estimation

$$
\begin{aligned}
&\left\|E_h - S_{h_t}R_hE\right\|_{l_1(G_h\times(T_0,\infty))} \\
&\quad < \frac{A(G_h)}{4\pi^2}\left(\left(C_5 + C_2\frac{h^2}{h_t}\right)\sum_{l=l_0+1}^{\infty}\left(\frac{1}{l-1}\frac{1}{l}\right) + \frac{\pi}{a^2l_0}\sum_{l=l_0+1}^{\infty} e^{-a^2\pi^2lh_t/h^2}\right) \\
&\quad \leq \frac{A(G_h)}{4\pi^2}\left(\left(C_5 + C_2\frac{h^2}{h_t}\right)\frac{1}{l_0} + \frac{\pi}{a^2l_0}e^{-a^2\pi^2T_0/h^2}\sum_{n=1}^{\infty}\left(e^{-a^2\pi^2h_t/h^2}\right)^n\right) \\
&\quad < \frac{A(G_h)}{4\pi^2}\left(C_5\frac{h_t}{T_0} + C_2\frac{h^2}{T_0} + \frac{\pi}{a^2l_0}\frac{h^2}{a^2\pi^2h_t}e^{-a^2\pi^2T_0/h^2}\right) \\
&\quad < \frac{A(G_h)}{4\pi^2T_0}(C_5h_t + C_6h^2) \\
&\quad = C_7\frac{h_t}{T_0} + C_8\frac{h^2}{T_0}.
\end{aligned}
\tag{17}
$$

Using the inequalities (7) and (17) we obtain the general estimation

$$\left\|E_h - S_{h_t} R_h E\right\|_{l_1(G_h\times \mathbb{R}_{h_t})} \le 2T_0 + \frac{2h}{a\sqrt{\pi}}\sqrt{T_0} + C_1\frac{1}{4a^2\pi}T_0 + C_7\frac{h_t}{T_0} + C_8\frac{h^2}{T_0}. \quad (18)$$

For the purpose of our convergence theorem we require $h \le h_0$, where $h_0$ is an arbitrary constant. Now we can formulate the following convergence theorem.

**Theorem 1:** *Let $1/C_1 < h_t/h^2 < 1/3\pi^2 a^2$. Then there is valid the convergence*

$$\left\|E_h - S_{h_t} R_h E\right\|_{l_1(G_h\times[0,\infty))} \longrightarrow 0 \qquad \text{for } h\to 0,\ h_t \to 0.$$

**Proof:** We prove that for arbitrary $\varepsilon > 0$ there exists a constant $h^* > 0$ such that for all $h < \min\{h^*, h_0\}$ and for all $h_t < h^2/3\pi^2a^2$ it follows

$$\left\|E_h - S_{h_t} R_h E\right\|_{l_1(G_h\times[0,\infty))} < \varepsilon.$$

We choose

$$T_0 = \frac{\left(\sqrt{\frac{h_0^2}{a^2\pi} + \frac{\varepsilon}{4}\left(2+\frac{C_1}{4a^2\pi}\right)} - \frac{h_0}{a\sqrt{\pi}}\right)^2}{\left(2+\frac{C_1}{4a^2\pi}\right)^2}$$

and

$$h^* = \min\left\{\sqrt{\frac{\varepsilon}{4}\,\frac{T_0}{\frac{C_7}{3\pi^2a^2}+C_8}},\ \sqrt{3\pi^2a^2\frac{T_0}{2}}\right\}.$$

$T_0$ is not necessary a point of the lattice $\mathbb{R}_{h_t}$. We define

$$T_0^+ = T_0 + \alpha h_t \qquad \text{and} \qquad T_0^- = T_0 - (1-\alpha)h_t \qquad \text{with } \alpha \in [0,1)$$

such that $T_0^+ \in \mathbb{R}_{h_t}$ and $T_0^- \in \mathbb{R}_{h_t}$. Obviously, we have

$$\left\|E_h - S_{h_t} R_h E\right\|_{l_1(G_h\times[0,\infty))} \le \left\|E_h - S_{h_t} R_h E\right\|_{l_1(G_h\times[0,T_0^+])} + \left\|E_h - S_{h_t} R_h E\right\|_{l_1(G_h\times(T_0^-,\infty))}.$$

Now a simple estimation using (7) and (17) shows that the right-hand side of the last inequality is bounded by $\varepsilon$ ∎

**Remark 1:** Let $h_t < h^2/3\pi^2a^2$. In addition to the above considerations we get

$$\left\|E_h - S_{h_t} R_h E\right\|_{l_1(\mathbb{R}_h^2\times[0,T_1])} < \infty$$

for each fixed $T_1 \in \mathbb{R}_{h_t}$.

**Proof:** Using (5) we have $\|E_h\|_{l_1(\mathbb{R}_h^2\times[0,T_1])} = T_1$. Repeating the considerations which lead to (6) we receive

$$\|R_hE(\cdot,t)\|_{l_1(\mathbb{R}_h^2)} = \Theta(t)\sum_{k,j\in Z}\frac{1}{4a^2\pi t}e^{-(k^2+j^2)h^2/4a^2t}h^2 \le \Theta(t)\left(\frac{h}{2a\sqrt{\pi t}}+1\right)^2$$

hence

$$\|S_{h_t}R_hE\|_{l_1(\mathbb{R}_h^2\times[0,T_1])} \le T_1 + \frac{2h}{a\sqrt{\pi}}\sqrt{T_1} + C_1\frac{1}{4a^2\pi}T_1.$$

Combining these results we can finish the proof ∎

## 3. Implicit difference equation

In this section we consider the implicit difference equation

$$(-a^2\Delta_h E_h)(x,t+h_t) + (D^+_{h_t} E_h)(x,t) = \delta_h(x)\delta_{h_t}(t).$$

Using the discrete Fourier transform again we find the solution

$$(F_h E_h)(\xi,t) = \frac{1}{2\pi}\Theta(t)\left(1+a^2d^2h_t\right)^{-t/h_t}\chi_h(\xi)$$

in analogy to (2). Finally, we get the following system of equations to calculate $E_h(x,t)$:

$$\begin{aligned} E_h(x,0) &= 0 && \text{for } x \in \mathbb{R}^2_h \\ \left(\left(1-a^2h_t\Delta_h\right)E_h\right)(x,h_t) &= \delta_h(x) && \text{for } x \in \mathbb{R}^2_h \\ \left(\left(1-a^2h_t\Delta_h\right)E_h\right)(x,t+h_t) &= E_h(x,t) && \text{for } x \in \mathbb{R}^2_h, t \in \mathbb{R}_{h_t}, t \geq h_t. \end{aligned} \tag{19}$$

We note that it is also possible to describe the fundamental solution by application of $R_hF$:

$$E_h(x,t) = R_hF\left(\frac{1}{2\pi}\Theta(t)\left(1+a^2d^2h_t\right)^{-t/h_t}\chi_h(\xi)\right)(x,t).$$

Of course, this is only a formal description of $E_h$ and we shall now investigate existence and regularity more precisely.

**3.1 Existence of the fundamental solution.** First we have the following three lemmas.

**Lemma 1:** *Let $f_h$ be an arbitrary bounded function. Then the equation*

$$\left(1-a^2h_t\Delta_h\right)v_h(x) = f_h(x) \qquad \textit{for all } x \in \mathbb{R}^2_h \tag{20}$$

*has a unique solution $v_h$.*

**Proof:** By the help of discrete Fourier transform we get the fundamental solution $e_h$ of the operator $(1-a^2h_t\Delta_h)$ in the form

$$e_h(x) = \frac{1}{(2\pi)^2}\int_{[-\pi/h,+\pi/h]^2}\frac{1}{1+a^2d^2h_t}e^{-ix\xi}d\xi.$$

Because $1+a^2d^2h_t \neq 0$ for all $\xi \in \mathbb{R}^2$ we can deduce from the corresponding result in [1] that

$$|e_h(x)| \leq Ke^{-c|x|} \tag{21}$$

where the constants $K>0$ and $c>0$ may depend on $h_t$. Furthermore, for bounded $f_h$ we get the unique bounded solution of equation (20) in the form $v_h(x) = e_h(x) * f_h(x) = \sum_{y\in\mathbb{R}^2_h} e_h(y)f_h(x-y)h^2$ ∎

**Lemma 2:** *If $f_h \in l_1(\mathbb{R}_h^2)$, then $v_h \in l_1(\mathbb{R}_h^2)$.*

**Proof:** Let $\|f_h\|_{l_1(\mathbb{R}_h^2)} \le K_1$. Then we can estimate

$$\begin{aligned}\sum_{x\in\mathbb{R}_h^2} |v_h(x)|h^2 &= \sum_{x\in\mathbb{R}_h^2} \left| \sum_{y\in\mathbb{R}_h^2} e_h(y) f_h(x-y)h^2 \right| h^2 \\ &\le \sum_{y\in\mathbb{R}_h^2} \left( |e_h(y)|h^2 \sum_{x\in\mathbb{R}_h^2} |f_h(x-y)|h^2 \right) \\ &= \sum_{y\in\mathbb{R}_h^2} \left( |e_h(y)|h^2 \sum_{z\in\mathbb{R}_h^2} |f_h(z)|h^2 \right) \\ &\le K_1 \sum_{y\in\mathbb{R}_h^2} |e_h(y)|h^2.\end{aligned}$$

From (21) it follows

$$\begin{aligned}K_1 \sum_{y\in\mathbb{R}_h^2} |e_h(y)|h^2 &\le K K_1 \sum_{y\in\mathbb{R}_h^2} e^{-c|y|}h^2 \\ &= K_2 \left( 4 \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} e^{-c|y|}h^2 + 4 \sum_{y_1=1}^{\infty} e^{-cy_1}h^2 + h^2 \right) \\ &\le K_2 \left( 4 \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} e^{-c|y|} dy_2 dy_1 + 4h \int_{y_1=0}^{\infty} e^{-cy_1} dy_1 + h^2 \right) \\ &< K_3\end{aligned}$$

and the statement is proved ∎

We remark that $(1-a^2h_t\Delta_h)$ is a Toeplitz operator and that we proved the inclusion $(1-a^2h_t\Delta_h)^{-1} \in \mathcal{L}(l_1(\mathbb{R}_h^2), l_1(\mathbb{R}_h^2))$.

**Lemma 3:** *If $|f_h(x)| \le K_1 e^{-c_1|x|}$ and $|e_h(x)| \le K_2 e^{-c_2|x|}$ with $0 < c_1 < c_2$, then $|v_h(x)| \le K_5 e^{-(c_1-\varepsilon)|x|}$ for all $\varepsilon > 0$.*

**Proof:** We start with

$$\begin{aligned}e^{c_1|x|}|v_h(x)| &= e^{c_1|x|} \left| \sum_{y\in\mathbb{R}_h^2} e_h(y) f_h(x-y)h^2 \right| \\ &\le e^{c_1|x|} \left( \sum_{y:|y|\le|x|} |e_h(y)|\,|f_h(x-y)|h^2 + \sum_{y:|y|>|x|} |e_h(y)|\,|f_h(x-y)|h^2 \right).\end{aligned}$$

In case $|y| \le |x|$ it follows from the assumption that

$$
\begin{aligned}
e^{c_1|x|} \sum_{y:|y|\le|x|} |e_h(y)|\,|f_h(x-y)|h^2
&\le K_1 \sum_{y:|y|\le|x|} |e_h(y)|e^{c_1(|x|-|x-y|)}h^2 \le K_1 \sum_{y:|y|\le|x|} |e_h(y)|e^{c_1|y|}h^2 \\
&\le K_1K_2 \sum_{y:|y|\le|x|} e^{|y|(-c_2+c_1)}h^2 \le K_1K_2 \sum_{y:|y|\le|x|} h^2 \\
&\le K_1K_2 \int_{\varphi=0}^{2\pi} \int_{r=0}^{|x|} r\,dr d\varphi = K_3|x|^2.
\end{aligned}
$$

In case $|y| > |x|$ we have

$$
\begin{aligned}
e^{c_1|x|} \sum_{y:|y|>|x|} |e_h(y)|\,|f_h(x-y)|h^2
&\le K_2 \sum_{y:|y|>|x|} e^{c_1|x|}e^{-c_2|y|}|f_h(x-y)|h^2 \\
&\le K_2 \sum_{y:|y|>|x|} e^{c_2(|x|-|y|)}|f_h(x-y)|h^2 \\
&\le K_2 \sum_{y:|y|>|x|} |f_h(x-y)|h^2 \le K_2\|f_h\|_{l_1(\mathbb{R}_h^2)} = K_4
\end{aligned}
$$

such that

$$|v_h(x)| \le K_3|x|^2e^{-c_1|x|} + K_4e^{-c_1|x|}.$$

Finally, we are looking for constants $\alpha_1$ and $\alpha_2$ such that $|x|^2e^{-c_1|x|} < \alpha_1 e^{-\alpha_2|x|}$. Let us take $\alpha_2 = c_1 - \varepsilon$ with $\varepsilon > 0$. Then we have only to fulfil the inequality $|x|^2e^{-\varepsilon|x|} < \alpha_1$ with a suitably chosen $\alpha_1$. This is easy to prove and we get the desired estimation $|v_h(x)| \le K_5e^{-(c_1-\varepsilon)|x|}$ ∎

**Theorem 2:** *The system* (19) *has a unique solution $E_h$ and for arbitrary $T < \infty$ it holds $E_h \in l_1(\mathbb{R}_h^2 \times [0,T])$.*

**Proof:** The assertion follows from Lemmas 1 - 3. We remark that the considerations in Lemma 3 can be repeated as long as necessary. An estimation of the $l_1$-norm with respect to $t$ is possible because the number of time steps is bounded ∎

**3.2 Convergence.** For $t = 0$ we can write the difference equation in the form

$$
\left(1 + \frac{4a^2h_t}{h^2}\right) E_h(x_1,x_2,h_t) = \delta_h(x) + \frac{a^2h_t}{h^2}\Big(E_h(x_1+h,x_2,h_t) + E_h(x_1-h,x_2,h_t) + E_h(x_1,x_2+h,h_t) + E_h(x_1,x_2-h,h_t)\Big).
$$

Using Lemma 2 we get

$$\left(1+\frac{4a^2h_t}{h^2}\right)\sum_{x\in\mathbb{R}_h^2}|E_h(x,h_t)|h^2 \le 1+\frac{4a^2h_t}{h^2}\sum_{x\in\mathbb{R}_h^2}|E_h(x,h_t)|h^2$$

which implies

$$\|E_h(\cdot,h_t)\|_{l_1(\mathbb{R}_h^2)} \le 1.$$

In the same way we prove the inequality

$$\|E_h(x,t+h_t)\|_{l_1(\mathbb{R}_h^2)} \le \|E_h(x,t)\|_{l_1(\mathbb{R}_h^2)}$$

for each $t \ge h_t$ starting with the corresponding equations in (19). We obtain

$$\|E_h\|_{l_1(G_h\times[0,T_0])} \le \|E_h\|_{l_1(\mathbb{R}_h^2\times[0,T_0])} \le \sum_{l=1}^{l_0} h_t = T_0.$$

From (6), under the assumption $h^2/h_t < C_1$, there follows

$$\left\|E_h - S_{h_t}R_hE\right\|_{l_1(G_h\times[0,T_0])} \le 2T_0 + \frac{2h}{a\sqrt{\pi}}\sqrt{T_0} + C_1\frac{1}{4\pi a^2}T_0. \tag{22}$$

Now we study

$$\left\|E_h(\cdot,t) - R_hE(\cdot,t)\right\|_{l_1(G_h)}$$

for $t > T_0$. In order to estimate the right-hand side of (8) we consider

$$\left\|F_hE_h(\cdot,t) - F^{-1}E(\cdot,t)\right\|_{L_1(Q_h)}.$$

We get

$$\begin{aligned}&\left\|(F_hE_h)(\cdot,t) - (F^{-1}E)(\cdot,t)\right\|_{L_1(Q_h)}\\ &\le \frac{\Theta(t)}{2\pi}\left(\left\|\left(1+a^2d^2h_t\right)^{-t/h_t} - e^{-a^2d^2t}\right\|_{L_1(Q_h)}\right.\\ &\quad\left.+\left\|e^{-a^2d^2t} - e^{-a^2|\xi|^2t}\right\|_{L_1(Q_h)}\right)\end{aligned} \tag{23}$$

in analogy to (10). We now estimate

$$\left\|(1+a^2d^2h_t)^{-t/h_t} - e^{-a^2d^2t}\right\|_{L_1(Q_h)}.$$

From

$$\left(1+a^2d^2h_t\right)^{-t/h_t} = e^{-(t/h_t)\ln(1+a^2d^2h_t)} \ge e^{-a^2d^2t}$$

it follows

$$\begin{aligned}
&\left|\left(1+a^2d^2h_t\right)^{-t/h_t} - e^{-a^2d^2t}\right| \\
&= \left(1+a^2d^2h_t\right)^{-t/h_t} - e^{-a^2d^2t} \\
&= e^{-a^2d^2t}\sum_{n=1}^{\infty}\frac{(-\frac{t}{h_t}\ln(1+a^2d^2h_t)+a^2d^2t)^n}{n!} \\
&\le e^{-a^2d^2t}\left(-\frac{t}{h_t}\ln\left(1+a^2d^2h_t\right)+a^2d^2t\right)e^{(-(t/h_t)\ln(1+a^2d^2h_t)+a^2d^2t)} \\
&= \left(-\frac{t}{h_t}\ln\left(1+a^2d^2h_t\right)+a^2d^2t\right)\left(1+a^2d^2h_t\right)^{-t/h_t} \\
&\le \left(-\frac{t}{h_t}\frac{a^2d^2h_t}{1+a^2d^2h_t}+a^2d^2t\right)\left(1+a^2d^2h_t\right)^{-t/h_t} \\
&= a^4d^4h_tt\left(1+a^2d^2h_t\right)^{-t/h_t-1}.
\end{aligned}$$

Using the inequalities $d^2 \le |\xi|^2$ and $d^2 \ge (4/\pi^2)|\xi|^2$ in $Q_h$ we get for $t > 2h_t$

$$\begin{aligned}
&\int_{Q_h}\left|\left(1+a^2d^2h_t\right)^{-t/h_t} - e^{-a^2d^2t}\right| d\xi \\
&\qquad \le a^4h_tt\int_{Q_h}|\xi|^4\left(1+a^2\frac{4}{\pi^2}|\xi|^2h_t\right)^{-t/h_t-1} d\xi \\
&\qquad \le 4a^4h_tt\int_{r=0}^{\sqrt{2}\pi/h}\int_{\varphi=0}^{\pi/2}\left(1+a^2\frac{4}{\pi^2}r^2h_t\right)^{-t/h_t-1} r^5\,d\varphi dr \\
&\qquad < C_9\frac{h_t}{(t-h_t)(t-2h_t)}
\end{aligned}$$

and by the help of (8), (9), (11) and (23) we find

$$\begin{aligned}
&\left\|E_h(\cdot,t)-(S_{h_t}R_hE)(\cdot,t)\right\|_{l_1(G_h)} \\
&\qquad \le \frac{\Theta(t)}{4\pi^2}A(G_h)\left(C_2\frac{h^2}{t^2}+C_9\frac{h_t}{(t-h_t)(t-2h_t)}+\frac{\pi}{a^2t}e^{-a^2\pi^2t/h^2}\right).
\end{aligned}$$

Furthermore, we can estimate in the same way as in (17)

$$\begin{aligned}
&\left\|E_h-S_{h_t}R_hE\right\|_{l_1(G_h\times[T_0,\infty))} \\
&\qquad \le \frac{A(G_h)}{4\pi^2}\left(\frac{C_6h^2}{T_0}+C_9\sum_{l=l_0+1}^{\infty}\frac{1}{(l-1)(l-2)}\right) \\
&\qquad < \frac{A(G_h)}{4\pi^2}\left(\frac{C_6h^2}{T_0}+C_9\frac{1}{l_0-1}\right) \\
&\qquad = C_{10}\frac{h^2}{T_0}+C_{11}\frac{h_t}{T_0-h_t}.
\end{aligned} \tag{24}$$

From (22) and (24) we obtain

$$\begin{aligned}\Big\| E_h - S_{h_t} R_h E \Big\|_{l_1(G_h \times [0,\infty))} \\ \leq 2T_0 + \frac{2h}{a\sqrt{\pi}}\sqrt{T_0} + \frac{C_1}{4a^2\pi} T_0 + C_{10}\frac{h^2}{T_0} + C_{11}\frac{h_t}{T_0 - h_t}.\end{aligned} \tag{25}$$

If we require $h \leq h_0$ and $h^2/h_t < C_1$, then we can prove

**Theorem 3:** *For $h \to 0$, $h_t \to 0$ we have the convergence*

$$\Big\| E_h - S_{h_t} R_h E \Big\|_{l_1(G_h \times [0,\infty))} \longrightarrow 0.$$

**Proof:** We have to show that for arbitrary $\varepsilon > 0$ there exist $h^* > 0$ and $h_t^* > 0$ such that for all $h < \min(h^*, h_0)$ and $h_t < h_t^*$ it follows

$$\Big\| E_h - S_{h_t} R_h E \Big\|_{l_1(G_h \times [0,\infty))} < \varepsilon.$$

We take $T_0$ as in the proof of Theorem 1, and

$$h_t^* = \min\left\{ \frac{\varepsilon}{4} \frac{T_0}{C_{11} + \varepsilon/2}, \frac{T_0}{2} \right\} \qquad \text{and} \qquad h^* = \sqrt{\frac{\varepsilon}{8} \frac{T_0}{C_{10}}}.$$

Using (22) for $T_0^+ = T_0 + \alpha h_t$ and (24) for $T_0^- = T_0 - (1-\alpha)h_t$ with $\alpha \in [0,1)$ such that $T_0^+, T_0^- \in I\!R_{h_t}$ we obtain the desired result ∎

## 4. Final remarks

The difference between the convergence results in Theorem 1 and Theorem 3 is based on the condition

$$h_t < h^2/3\pi^2 a^2.$$

This condition is stronger than the known stability condition (see, e.g., [8]) for the explicit difference scheme. In our case one has to take into consideration that we solved an initial value problem in an unbounded domain. The condition $h_t < h^2/3\pi^2a^2$ can be improved. Therefore we have to use the restriction $|\xi|^2 < 1/4a^2t$ in the proof of (13). For the sake of brevity we omitted this consideration.

The technical condition $h^2/h_t < C_1$ does not restrict numerical calculations.

Using the above norms it is difficult to describe the order of convergence near the point $t = 0$. Nevertheless, the global result can be improved. We choose $T_0 = j_0\sqrt{h_t}$ with a sufficiently large $j_0 \in I\!N$ and take into consideration $h^2/h_t \leq C_1$. Then we follow the same way as in the proofs of Theorem 1 and Theorem 3 and we obtain

$$\Big\| E_h - S_{h_t} R_h E \Big\|_{l_1(G_h \times I\!R_{h_t})} = O\Big(h + \sqrt{h_t}\Big).$$

If we return to the idea of a fixed $T$, then from (17) and (24) it follows for an arbitrary $T > 0$

$$\left\| E_h - S_{h_t} R_h E \right\|_{l_1(G_h \times (T,\infty))} = O\left(h^2 + h_t\right).$$

Furthermore, using uniform norms we can prove by the help of (8), (9), (11) and (16) that

$$\max_{(x,t)\in \mathbb{R}_h^2 \times ([T,\infty)\cap \mathbb{R}_{h_t})} \left| E_h(x,t) - E(x,t) \right| \leq \frac{1}{4\pi^2 T^2} \left( C_2 h^2 + C_5 h_t + \frac{h^2}{a^4 \pi} \right)$$
$$= O\left(h^2 + h_t\right).$$

First numerical tests underline the theoretical results.

If we apply discrete potential theory to the solution of a boundary value or an initial value problem the result will be a discrete potential with known density defined on the lattice. At the end we have to compare the approximate solution with the exact solution in continuous spaces. In the language of projection methods an extension (by interpolation) of lattice functions to functions defined in $G \times [0,\infty)$ is necessary. The main problem in our case is to find an appropriate extension of the discrete fundamental solution. Therefore we use the interpolation operator $I_h = FF_h$ investigated in [9] and the above mentioned property $R_h F = (F_h)^{-1}$. We obtain that $FF_h E_h$ is an (entire analytic) extension of $E_h$. Then, using $A(G) = \int_G dx$ we get similar as in (8)

$$\left\| FF_h E_h(\cdot,t) - E(\cdot,t) \right\|_{L_1(G)} \leq A(G) \max_{x\in G} \left| FF_h E_h(x,t) - FF^{-1} E(x,t) \right|$$
$$\leq \frac{1}{2\pi} A(G) \left\| (F_h E_h)(\cdot,t) - (F^{-1}E)(\cdot,t) \right\|_{L_1(\mathbb{R}^2)}.$$

Starting from (8) we can repeat the considerations which prove (16) and we arrive at the inequality

$$\left\| FF_h E_h(\cdot,t) - E(\cdot,t) \right\|_{L_1(G)} \leq \frac{\Theta(t)}{4\pi^2} A(G) \left( C_2 \frac{h^2}{t^2} + C_5 \frac{h_t}{t^2} + \frac{\pi}{a^2 t} e^{-a^2\pi^2 t/h^2} \right).$$

A simple integration with respect to $t$ leads to the estimate

$$\left\| FF_h E_h - E \right\|_{L_1(G\times(T_0,\infty))} \leq C_7' \frac{h_t}{T_0} + C_8' \frac{h^2}{T_0}.$$

Hence, the rate of convergence in the continuous $L_1$-norm coincides with the rate of convergence in the discrete $l_1$-norm.

It is also possible to consider the convergence in $l_p$ and $L_p$, respectively. In this case we can replace the domain $G_h$ by $R_h^2$ but one has to accept a loss of convergence rate.

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