Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 13 (1994), No. 3, 463–476

The Smoothness of Solutions to Nonlinear Weakly Singular Integral Equations

A. Pedas and G. Vainikko

Abstract. The differential properties of a solution of a nonlinear multidimensional weakly singular integral equation of the Uryson type on an open bounded set $G \subset \mathbb{R}^n$ are examined. Showing that the solution belongs to special weighted space of smooth functions, the growth of the derivatives near the boundary is described.

Keywords: *Smoothness of solutions, weakly singular integral equations*

AMS subject classification: 45M05, 45G10

1. Introduction

The construction of effective numerical methods for solving weakly singular integral equations in a region $G \subset \mathbb{R}^n$ is impossible without taking into account the singularities of the derivatives of the solution near the boundary ∂G. The presence of singularities is an elementary fact, but significant difficulties are encountered in describing them precisely and proving the corresponding assertions. The case of one-dimensional integral equations was analyzed by Richter [9] , Pedas [6], Schneider [10], Vainikko and Pedas [14], Graham [1], Vainikko, Pedas and Uba [15], Kaneko, Noren and Xu [2], and Kangro [3]. The case of multidimensional integral equation was analyzed by Pitkäranta [7, 8], Vainikko [11 - 13], and Kangro [4, 5].

In [11 - 13] estimates for derivatives of a solution to the linear multidimensional weakly singular integral equation are derived. In many cases these estimates are sharp. In [12, 13] the main results were extended to nonlinear equations, too, but the proofs were outlined only on the idea level. In this paper we present a full proof (Sections 4 - 5); the formulation of the main result is given in Section 3. Note that we treat the Uryson equation which is more general than the Hammerstein equation considered in [2]. Compared to [2], our result is more complete.

For a linear equation $u = Tu + f$, there are at least three different ideas how to show that the solution belongs to special weighted spaces of smooth functions. Pitkäranta [8] examined step by step the improving properties of the weakly singular operator T and obtained that a power of T maps $L^\infty(G)$ (or even $L^1(G)$) into a special weighted space; this idea may be implemented in the case of nonlinear equations, too. The authors of this paper (see [6, 11, 14, 15]) used another idea proving that T is compact in appropriate

A. Pedas: Univ. Tartu, Dep. Math., Liivi 2, 202400 Tartu, Estonia
G. Vainikko: Univ. Tartu, Dep. Math., Liivi 2, 202400 Tartu, Estonia

ISSN 0232-2064 / $ 2.50

weighted spaces; this idea does not work in the case of nonlinear equations. The third idea elaborated in [13] is based on the "smallness" of $(T_\Omega u)(x) = \int_\Omega \mathcal{K}(x,y)u(y)dy$, $x \in \Omega$, where $\Omega \subset G$ is a small subregion; the integral over $G \setminus \Omega$ is treated as a part of the inhomogeneity. This idea can be extended to the case of nonlinear equation, and we pursue it in the present paper.

2. Integral equation

Consider the nonlinear integral equation

$$u(x) = \int_G \mathcal{K}(x,y,u(y))dy + f(x) \qquad (x \in G) \tag{1}$$

where $G \subset \mathbb{R}^n$ is an open bounded set. The kernel $\mathcal{K} = \mathcal{K}(x,y,u)$ is assumed to be m times ($m \geq 1$) continuously differentiable with respect to x, y and u for $x \in G$, $y \in G$ ($x \neq y$) and $u \in \mathbb{R}$ whereby there exists a real number $\nu \in (-\infty, n)$ such that, for any $k \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ with $k + |\alpha| + |\beta| \leq m$, the inequalities

$$\left| D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} \mathcal{K}(x,y,u) \right| \leq b_1(|u|) \begin{cases} 1 & \text{if } \nu + |\alpha| < 0 \\ 1 + |\log|x-y|| & \text{if } \nu + |\alpha| = 0 \\ |x-y|^{-\nu-|\alpha|} & \text{if } \nu + |\alpha| > 0 \end{cases} \tag{2}$$

and

$$\left| D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} \mathcal{K}(x,y,u_1) - D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} \mathcal{K}(x,y,u_2) \right|$$
$$\leq b_2(\max\{|u_1|,|u_2|\})|u_1 - u_2| \begin{cases} 1 & \text{if } \nu + |\alpha| < 0 \\ 1 + |\log|x-y|| & \text{if } \nu + |\alpha| = 0 \\ |x-y|^{-\nu-|\alpha|} & \text{if } \nu + |\alpha| > 0 \end{cases} \tag{3}$$

hold. The functions $b_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $b_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are assumed to be monotonically increasing. Here the following standard conventions are adopted:

$$\mathbb{R}_+ = [0,\infty), \qquad \mathbb{Z}_+ = \{0,1,2,\ldots\}$$
$$|\alpha| = \alpha_1 + \ldots + \alpha_n \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$$
$$|x| = \sqrt{x_1^2 + \ldots + x_n^2} \quad \text{for} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$
$$D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$
$$D_{x+y}^\beta = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)^{\beta_n}.$$

Note that asymmetry of (2) and (3) with respect to x and y is only seeming: using the equality $\partial/\partial y_i = (\partial/\partial x_i + \partial/\partial y_i) - \partial/\partial x_i$ we can deduce from (2) and (3) similar estimates for $D_y^\alpha D_{x+y}^\beta \partial^k \mathcal{K}(x,y,u)/\partial u^k$.

Putting $k = |\alpha| = |\beta| = 0$, inequality (2) yields

$$|\mathcal{K}(x,y,u)| \le b_1(|u|) \begin{cases} 1 & \text{if } \nu < 0 \\ 1 + |\log|x-y|| & \text{if } \nu = 0 \\ |x-y|^{-\nu} & \text{if } \nu > 0 \end{cases}.$$

Thus the kernel $\mathcal{K}$ may have a weak singularity ($\nu < n$). In the case $\nu < 0$, the kernel $\mathcal{K}$ is bounded but its derivatives may be singular. In the case of a linear integral equation $\mathcal{K}(x,y,u) = \mathcal{K}_1(x,y)u$, and conditions (2) and (3) reduce to a condition for $\mathcal{K}_1 = \mathcal{K}_1(x,y)$ from [11, 13].

3. Main result

Introduce the weight functions

$$w_\lambda(x) = \begin{cases} 1 & \text{if } \lambda < 0 \\ (1 + |\log \rho(x)|)^{-1} & \text{if } \lambda = 0 \\ \rho(x)^\lambda & \text{if } \lambda > 0 \end{cases} \qquad (x \in G, \lambda \in \mathbb{R}) \tag{4}$$

where $G \subset \mathbb{R}^n$ is an open bounded set with the boundary ∂G and

$$\rho(x) = \rho^G(x) = \inf_{y \in \partial G} |x-y| \qquad (x \in G) \tag{5}$$

is the distance from x to ∂G. Define the space $C^{m,\nu}(G)$ as the collection of all m times continuously differentiable functions $u : G \to \mathbb{R}$ (or $u : G \to \mathbb{C}$) such that

$$\|u\|_{m,\nu} := \sum_{|\alpha| \le m} \sup_{x \in G} \left(w_{|\alpha|-(n-\nu)}(x) |D^\alpha u(x)| \right) < \infty. \tag{6}$$

In other words, an m times continuously differentiable function u on G belongs tothe space $C^{m,\nu}(G)$ if the growth of its derivatives near the boundary can be estimated as

$$|D^\alpha u(x)| \le \text{const} \begin{cases} 1 & \text{if } |\alpha| < n - \nu \\ 1 + |\log \rho(x)| & \text{if } |\alpha| = n - \nu \\ \rho(x)^{n-\nu-|\alpha|} & \text{if } |\alpha| > n - \nu \end{cases} \qquad (x \in G, |\alpha| \le m).$$

The space $C^{m,\nu}(G)$, equipped with norm (6), is complete (is a Banach space).

Our main result is contained in the following theorem.

Theorem 1. *Let $G \subset \mathbb{R}^n$ be an open bounded set, $f \in C^{m,\nu}(G)$, and let the kernel $\mathcal{K} = \mathcal{K}(x,y,u)$ satisfy conditions (2) and (3). If the integral equation (1) has a solution $u \in L^\infty(G)$, then $u \in C^{m,\nu}(G)$.*

This theorem was formulated and partly (for $m = 2$) proved in [13]. A full proof of Theorem 1 is given in Section 5. Section 4 contains necessary preliminaries for the proof.

Remark. In Theorem 1 we have not assumed a global or local uniqueness of the solution to equation (1).

4. Differentiation of the weakly singular integral

We use the notations

$$B(x,r)=\left\{y\in I\!R^n : |x-y|<r\right\} \qquad \text{and} \qquad S(x,r)=\left\{y\in I\!R^n : |x-y|=r\right\}$$

for an open ball and a sphere, respectively, in $I\!R^n$. First we present some inequalities that follow from (2). For $k+|\alpha|+|\beta|\le m$, $x,\overline{x}\in G$, $|\overline{x}-x|\le r$, $|u|\le d$, $r>0$ we have the inequalities

$$\int\limits_{G\cap B(\overline{x},r)} \left|D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k}\mathcal{K}(x,y,u)\right| dy \le c_1 b_1(d)\begin{cases} r^n & \text{if } \nu+|\alpha|<0 \\ r^n(1+|\log r|) & \text{if } \nu+|\alpha|=0 \\ r^{n-\nu-|\alpha|} & \text{if } 0<\nu+|\alpha|<n \end{cases} \tag{7}$$

and

$$\int\limits_{G\setminus B(x,r)} \left|D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k}\mathcal{K}(x,y,u)\right| dy \le c_2 b_1(d)\begin{cases} 1 & \text{if } \nu+|\alpha|<n \\ 1+|\log r| & \text{if } \nu+|\alpha|=n \\ r^{n-\nu-|\alpha|} & \text{if } \nu+|\alpha|>n \end{cases} \tag{8}$$

where the constants c_1 and c_2 depend only on n, ν and on n, ν, diam$G$, respectively (by diam G we denote the diameter of G; we assume that $r\le \operatorname{diam} G$).

Let $\Omega\subseteq G$ be a domain with a piecewise smooth boundary $\partial\Omega$, $u\in C(\overline{\Omega})\cap C^1(\Omega)$, $\partial u/\partial x_i\in L^1(\Omega)$, and let the kernel $\mathcal{K}=\mathcal{K}(x,y,u)$ satisfy conditions (2) and (3) with $m=1$. Then (see [11, 13]), for $x\in\Omega$,

$$\frac{\partial}{\partial x_i}\int\limits_\Omega \mathcal{K}(x,y,u(y))\,dy = \int\limits_\Omega\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}\right)\mathcal{K}(x,y,u(y))\,dy + \int\limits_{\partial\Omega}\mathcal{K}(x,y,u(y)\omega_i(y))\,dS_y \tag{9}$$

where $\omega(y)=(\omega_1(y),\dots,\omega_n(y))$ is the unit inner normal to $\partial\Omega$ at $y\in\partial\Omega$.

Now we fix an arbitrary point $\overline{x}\in G$ and take a sufficiently small $\delta>0$ such that $B(\overline{x},\delta)\subset G$. Let the kernel $\mathcal{K}=\mathcal{K}(x,y,u)$ satisfy conditions (2) and (3). Using (9) with $\Omega=B(\overline{x},\delta)$ we have for any $u\in C^{m,\nu}(G)$

$$\begin{aligned}\frac{\partial}{\partial x_i}\int\limits_{B(\overline{x},\delta)}\mathcal{K}(x,y,u(y))\,dy &= \int\limits_{B(\overline{x},\delta)}\left(\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}\right)\mathcal{K}(x,y,u)\right)\Bigg|_{u=u(y)} dy \\ &+ \int\limits_{B(\overline{x},\delta)}\frac{\partial}{\partial u}\mathcal{K}(x,y,u(y))\frac{\partial u(y)}{\partial y_i}\,dy \\ &+ \int\limits_{S(\overline{x},\delta)}\mathcal{K}(x,y,u(y))\,\omega_i(y)\,dS_y\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\int\limits_{B(\overline{x},\delta)} \mathcal{K}(x,y,u(y))\,dy \\
&\qquad = \int\limits_{B(\overline{x},\delta)} \mathcal{K}^2_{i,j}(x,y,u(y))\,dy \\
&\qquad\quad + \int\limits_{B(\overline{x},\delta)} \frac{\partial}{\partial u}\mathcal{K}^1_i(x,y,u(y))\frac{\partial u(y)}{\partial y_j}\,dy \\
&\qquad\quad + \int\limits_{B(\overline{x},\delta)} \frac{\partial}{\partial u}\mathcal{K}^1_j(x,y,u(y))\frac{\partial u(y)}{\partial y_i}\,dy \\
&\qquad\quad + \int\limits_{B(\overline{x},\delta)} \frac{\partial^2}{\partial u^2}\mathcal{K}(x,y,u(y))\frac{\partial u(y)}{\partial y_i}\frac{\partial u(y)}{\partial y_j}\,dy \\
&\qquad\quad + \int\limits_{B(\overline{x},\delta)} \frac{\partial}{\partial u}\mathcal{K}(x,y,u(y))\frac{\partial^2 u(y)}{\partial y_j \partial y_i}\,dy \\
&\qquad\quad + \int\limits_{S(\overline{x},\delta)} \mathcal{K}^1_i(x,y,u(y))\,\omega_j(y)\,dS_y \\
&\qquad\quad + \int\limits_{S(\overline{x},\delta)} \frac{\partial}{\partial u}\mathcal{K}(x,y,u(y))\frac{\partial u(y)}{\partial y_i}\,\omega_j(y)\,dS_y \\
&\qquad\quad + \int\limits_{S(\overline{x},\delta)} \frac{\partial}{\partial x_j}\mathcal{K}(x,y,u(y))\,\omega_i(y)\,dS_y
\end{aligned}$$

where

$$\mathcal{K}^p_{i_1,\cdots,i_p}(x,y,u) = \left(\frac{\partial}{\partial x_{i_1}} + \frac{\partial}{\partial y_{i_1}}\right)\cdots\left(\frac{\partial}{\partial x_{i_p}} + \frac{\partial}{\partial y_{i_p}}\right)\mathcal{K}(x,y,u). \tag{10}$$

We continue the differentation using (9). By induction we obtain the formula for higher order derivatives:

$$\begin{aligned}
&\frac{\partial^p}{\partial x_{i_p}\cdots\partial x_{i_1}}\int\limits_{B(\overline{x},\delta)} \mathcal{K}(x,y,u(y))\,dy \\
&\qquad = \int\limits_{B(\overline{x},\delta)} \mathcal{D}^p_{i_1,\ldots,i_p}\mathcal{K}(x,y,u(y))\,dy \\
&\qquad\quad + \sum_{k=1}^{p}\int\limits_{S(\overline{x},\delta)} \frac{\partial^{p-k}}{\partial x_{i_p}\cdots\partial x_{i_{k+1}}}\mathcal{D}^{k-1}_{i_1,\ldots,i_{k-1}}\mathcal{K}(x,y,u(y))\,\omega_{i_k}(y)\,dS_y
\end{aligned} \tag{11}$$

$(1 \le p \le m)$. Here $\omega(y) = (\omega_1(y),\ldots,\omega_n(y)) = (\overline{x} - y)/\delta$ is the unit inner normal to

$\partial B(\overline{x},\delta)=S(\overline{x},\delta)$ at $y\in S(\overline{x},\delta)$, $u\in C^{m,\nu}(G)$, and we have used the notation

$$\begin{aligned}
&\mathcal{D}^p_{i_1,\dots,i_p}\mathcal{K}(x,y,u(y))\\
&\quad=\mathcal{K}^p_{i_1,\dots,i_p}(x,y,u(y))\\
&\qquad+\sum_{j=1}^{p}\frac{\partial u(y)}{\partial y_{i_j}}\frac{\partial}{\partial u}\mathcal{K}^{p-1}_{\{i_1,\dots,i_p\}\setminus\{i_j\}}(x,y,u(y))\\
&\qquad+\sum_{\substack{j,k\\1\le j<k\le p}}\left(\frac{\partial^2 u(y)}{\partial y_{i_k}\partial y_{i_j}}\frac{\partial}{\partial u}\mathcal{K}^{p-2}_{\{i_1,\dots,i_p\}\setminus\{i_j,i_k\}}(x,y,u(y))\right.\\
&\qquad\left.+\frac{\partial u(y)}{\partial y_{i_j}}\frac{\partial u(y)}{\partial y_{i_k}}\frac{\partial^2}{\partial u^2}\mathcal{K}^{p-2}_{\{i_1,\dots,i_p\}\setminus\{i_j,i_k\}}(x,y,u(y))\right)\\
&\qquad+\sum_{\substack{j,k,l\\1\le j<k<l\le p}}\left(\frac{\partial^3 u(y)}{\partial y_{i_l}\partial y_{i_k}\partial y_{i_j}}\frac{\partial}{\partial u}\mathcal{K}^{p-3}_{\{i_1,\dots,i_p\}\setminus\{i_j,i_k,i_l\}}(x,y,u(y))\right.\\
&\qquad+\left(\frac{\partial^2 u(y)}{\partial y_{i_l}\partial y_{i_k}}\frac{\partial u(y)}{\partial y_{i_j}}+\frac{\partial^2 u(y)}{\partial y_{i_l}\partial y_{i_j}}\frac{\partial u(y)}{\partial y_{i_k}}\right.\\
&\qquad\left.+\frac{\partial^2 u(y)}{\partial y_{i_k}\partial y_{i_j}}\frac{\partial u(y)}{\partial y_{i_l}}\right)\frac{\partial^2}{\partial u^2}\mathcal{K}^{p-3}_{\{i_1,\dots,i_p\}\setminus\{i_j,i_k,i_l\}}(x,y,u(y))\\
&\qquad\left.+\frac{\partial u(y)}{\partial y_{i_j}}\frac{\partial u(y)}{\partial y_{i_k}}\frac{\partial u(y)}{\partial y_{i_l}}\frac{\partial^3}{\partial u^3}\mathcal{K}^{p-3}_{\{i_1,\dots,i_p\}\setminus\{i_j,i_k,i_l\}}(x,y,u(y))\right)+\cdots\\
&\qquad+\left(\frac{\partial^p u(y)}{\partial y_{i_p}\cdots\partial y_{i_1}}\frac{\partial}{\partial u}\mathcal{K}(x,y,u(y))\right.\\
&\qquad+\left(\frac{\partial^{p-1}u(y)}{\partial y_{i_p}\cdots\partial y_{i_2}}\frac{\partial u(y)}{\partial y_{i_1}}+\frac{\partial^{p-1}u(y)}{\partial y_{i_p}\cdots\partial y_{i_3}\partial y_{i_1}}\frac{\partial u(y)}{\partial y_{i_2}}+\cdots\right.\\
&\qquad+\frac{\partial^{p-1}u(y)}{\partial y_{i_{p-1}}\cdots\partial y_{i_1}}\frac{\partial u(y)}{\partial y_{i_p}}+\frac{\partial^{p-2}u(y)}{\partial y_{i_p}\cdots\partial y_{i_3}}\frac{\partial^2 u(y)}{\partial y_{i_2}\partial y_{i_1}}\\
&\qquad+\frac{\partial^{p-2}u(y)}{\partial y_{i_p}\cdots\partial y_{i_4}\partial y_{i_1}}\frac{\partial^2 u(y)}{\partial y_{i_3}\partial y_{i_2}}+\cdots\\
&\qquad\left.+\frac{\partial^{p-q}u(y)}{\partial y_{i_{p-q}}\cdots\partial y_{i_1}}\frac{\partial^q u(y)}{\partial y_{i_p}\cdots\partial y_{i_{p-q+1}}}\right)\frac{\partial^2}{\partial u^2}\mathcal{K}(x,y,u(y))+\cdots\\
&\qquad+\left(\frac{\partial u(y)}{\partial y_{i_1}}\cdots\frac{\partial u(y)}{\partial y_{i_{p-2}}}\frac{\partial^2 u(y)}{\partial y_{i_p}\partial y_{i_{p-1}}}\right.\\
&\qquad+\frac{\partial u(y)}{\partial y_{i_1}}\cdots\frac{\partial u(y)}{\partial y_{i_{p-3}}}\frac{\partial u(y)}{\partial y_{i_{p-1}}}\frac{\partial^2 u(y)}{\partial y_{i_p}}\partial y_{i_{p-2}}+\cdots\\
&\qquad\left.+\frac{\partial u(y)}{\partial y_{i_3}}\cdots\frac{\partial u(y)}{\partial y_{i_p}}\frac{\partial^2 u(y)}{\partial y_{i_2}\partial y_{i_1}}\right)\frac{\partial^{p-1}}{\partial u^{p-1}}\mathcal{K}(x,y,u(y))\\
&\qquad\left.+\frac{\partial u(y)}{\partial y_{i_1}}\cdots\frac{\partial u(y)}{\partial y_{i_p}}\frac{\partial^p}{\partial u^p}\mathcal{K}(x,y,u(y))\right)
\end{aligned}\tag{12}$$

where $q = \begin{cases} p/2 & \text{if } p \text{ is even} \\ (p-1)/2 & \text{if } p \text{ is odd} \end{cases}$ and $\mathcal{D}^0_\emptyset \mathcal{K}(x,y,u) = \mathcal{K}^0_\emptyset(x,y,u) = \mathcal{K}(x,y,u)$.

5. Proof of Theorem 1

Let conditions (2) and (3) be fulfilled, $f \in C^{m,\nu}(G)$, and let $u_0 \in L^\infty(G)$ be a solution to equation (1). We have to prove that $u_0 \in C^{m,\nu}(G)$.

Fix an arbitrary point $x^0 \in G \subset \mathbb{R}^n$ and introduce the set $\Omega = B(x^0,r) \cap G$, where the ball $B(x^0,r) \subset \mathbb{R}^n$ has a small radius r, $0 < r \le \rho(x_0)/2$. Thus $\Omega = B(x^0,r) \subset G$. We consider an integral equation on Ω:

$$u(x) = \int_\Omega \mathcal{K}(x,y,u(y))\,dy + f_\Omega(x) \qquad (x \in \Omega) \tag{13}$$

where

$$f_\Omega(x) = f(x) + \int_{G\setminus\Omega} \mathcal{K}(x,y,u_0(y))\,dy \qquad (x \in \Omega). \tag{14}$$

Clearly, u_0 solves this equation. But we will show also that equation (13) is uniquely solvable and the solution is as smooth in Ω as asserted in Theorem 1. Since $x^0 \in G$ is arbitrary, it finally proves that $u_0 \in C^{m,\nu}(G)$.

Let us define (cf. (4))

$$w^\Omega_\lambda(x) = \begin{cases} 1 & \text{for } \lambda < 0 \\ (1 + |\log \rho^\Omega(x)|)^{-1} & \text{for } \lambda = 0 \\ (\rho^\Omega(x))^\lambda & \text{for } \lambda > 0 \end{cases} \qquad (x \in \Omega)$$

where $\rho^\Omega(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$ (cf. (5)). Introduce the space $C^{m,\nu}(\Omega)$ with the norm $\|\cdot\|_{m,\nu,\Omega}$ in a similar way as in Section 3: $u \in C^{m,\nu}(\Omega)$ if

$$\|u\|_{m,\nu,\Omega} := \sum_{|\alpha|\le m} \sup_{x\in\Omega} \Big(w^\Omega_{|\alpha|-(n-\nu)}(x) |D^\alpha u(x)| \Big) < \infty.$$

An important observation is that $f_\Omega \in C^{m,\nu}(\Omega)$. Indeed, for $x \in \Omega$ and $y \in G \setminus \Omega$ we have $|x-y| \ge \rho^\Omega(x) > 0$, and we may differentiate the function $\int_{G\setminus\Omega} \mathcal{K}(x,y,u_0(y))\,dy$ under the integral sign. The result of differentation

$$D^\alpha_x \int_{G\setminus\Omega} \mathcal{K}(x,y,u_0(y))\,dy = \int_{G\setminus\Omega} D^\alpha_x \mathcal{K}(x,y,u_0(y))\,dy \qquad (|\alpha| \le m)$$

is a continuous function on Ω. Further, using (8) we estimate

$$\begin{aligned} &\left| D^\alpha_x \int_{G\setminus\Omega} \mathcal{K}(x,y,u_0(y))\,dy \right| \\ &\qquad \le \int_{G\setminus B(x,\rho^\Omega(x))} |D^\alpha_x \mathcal{K}(x,y,u_0(y))|\,dy \\ &\qquad \le \text{const}_{u_0} \begin{cases} 1 & \text{if } |\alpha| < n-\nu \\ 1 + |\log \rho^\Omega(x)| & \text{if } |\alpha| = n-\nu \\ \rho^\Omega(x)^{n-\nu-|\alpha|} & \text{if } |\alpha| > n-\nu \end{cases} \qquad (x \in \Omega, |\alpha| \le m). \end{aligned}$$

We see that the second term on the right-hand side of (14) belongs to $C^{m,\nu}(\Omega)$, and thus $f_\Omega \in C^{m,\nu}(\Omega)$.

Further, define the operators T_Ω and S_Ω, for $x \in \Omega$, by

$$(T_\Omega u)(x) = \int_\Omega \mathcal{K}(x, y, u(y))\, dy \qquad \text{and} \qquad (S_\Omega u)(x) = (T_\Omega u)(x) + f_\Omega(x).$$

Denote $\|u\|_{0,\Omega} = \sup_{x\in\Omega} |u(x)|$. Using (2) and (3) with $k = |\alpha| = |\beta| = 0$ and (7) it is easy to check that, for any $u, v \in L^\infty(\Omega)$ with $\|u\|_{0,\Omega} \le d$ and $\|v\|_{0,\Omega} \le d$, we have

$$\|T_\Omega u\|_{0,\Omega} \le b_1(d)\varepsilon_r \qquad \text{and} \qquad \|T_\Omega u - T_\Omega v\|_{0,\Omega} \le b_2(d)\varepsilon_r \|u - v\|_{0,\Omega} \tag{15}$$

where $\varepsilon_r \to 0$ as $r \to 0$. A consequence is that, for sufficiently small $r > 0$, the operator S_Ω maps the ball

$$\mathcal{B}_{0,\Omega,d} = \left\{ u \in L^\infty(\Omega) : \|u\|_{0,\Omega} \le d \right\} \qquad (d > \|f_\Omega\|_{0,\Omega})$$

into itself and is contractive on it:

$$\|S_\Omega u - S_\Omega v\|_{0,\Omega} \le q\|u - v\|_{0,\Omega} \qquad \text{for all } u, v \in \mathcal{B}_{0,\Omega,d} \qquad (q < 1).$$

Due to the Banach fixed point theorem, S_Ω has a unique fixed point in $\mathcal{B}_{0,\Omega,d}$; we know this fixed point – it is u_0, the solution of equation (1) under consideration. A more serious consequence of (2) and (3) is that, for sufficiently small $r > 0$, the operator S_Ω maps a closed subset $\mathcal{B}_{m,\nu,\Omega,d,d'}$ of the space $C^{m,\nu}(\Omega)$ into itself and is contractive on it with respect to a norm $\|u\|'_{m,\nu,\Omega}$ which is equivalent to the usual norm $\|u\|_{m,\nu,\Omega}$ of $C^{m,\nu}(\Omega)$:

$$\|S_\Omega u - S_\Omega v\|'_{m,\nu,\Omega} \le q\|u - v\|'_{m,\nu,\Omega} \qquad \text{for all } u, v \in \mathcal{B}_{m,\nu,\Omega,d,d'} \quad (q < 1). \tag{16}$$

We soon present the definitions of the subset $\mathcal{B}_{m,\nu,\Omega,d,d'}$ as well of the norm $\|u\|'_{m,\nu,\Omega}$. In order to prove these properties of S_Ω we have to study various derivatives of the weakly singular integral. As a first step we note that for any $u \in C^{m,\nu}(\Omega)$ the singularities of the terms

$$\frac{\partial u}{\partial y_{i_1}} \cdots \frac{\partial u}{\partial y_{i_p}}$$
$$\frac{\partial u}{\partial y_{i_1}} \cdots \frac{\partial u}{\partial y_{i_{p-2}}} \frac{\partial^2 u}{\partial y_{i_p} \partial y_{i_{p-1}}} \cdots$$
$$\frac{\partial u}{\partial y_{i_1}} \frac{\partial^{p-1} u}{\partial y_{i_p} \cdots \partial y_{i_2}}$$

in (12) are weaker than the singularity allowed for $\partial^p u / \partial y_{i_p} \cdots \partial y_{i_1}$ by the definition of the space $C^{m,\nu}(\Omega)$:

$$\left| \frac{\partial u(y)}{\partial y_{i_1}} \cdots \frac{\partial u(y)}{\partial y_{i_p}} \right| \le \text{const} \begin{cases} 1 & \text{if } \nu < n-1 \\ (1 + |\log \rho^\Omega(y)|)^p & \text{if } \nu = n-1 \\ (\rho^\Omega(y))^{p(n-\nu)-p} & \text{if } \nu > n-1 \end{cases}$$

and, for $p \geq 2$,

$$\left|\frac{\partial u(y)}{\partial y_{i_1}}\cdots\frac{\partial u(y)}{\partial y_{i_{p-2}}}\frac{\partial^2 u(y)}{\partial y_{i_p}\partial y_{i_{p-1}}}\right| \leq \text{const}\begin{cases}1 & \text{if } \nu < n-2\\ 1+|\log\rho^\Omega(y)| & \text{if } \nu = n-2\\ (\rho^\Omega(y))^{n-\nu-2} & \text{if } n-2<\nu<n-1\\ (\rho^\Omega(y))^{n-\nu-2}(1+|\log\rho^\Omega(y)|)^{p-2} & \text{if } \nu = n-1\\ (\rho^\Omega(y))^{(p-1)(n-\nu)-p} & \text{if } \nu > n-1\end{cases}$$

and, for $p \geq 3$,

$$\left|\frac{\partial u(y)}{\partial y_{i_1}}\frac{\partial^{p-1} u(y)}{\partial y_{i_p}\cdots\partial y_{i_2}}\right| \leq \text{const}\begin{cases}1 & \text{if } \nu < n-p+1\\ 1+|\log\rho^\Omega(y)| & \text{if } \nu = n-p+1\\ (\rho^\Omega(y))^{n-\nu-p+1} & \text{if } n-p+1<\nu<n-1\\ (\rho^\Omega(y))^{n-\nu-p+1}(1+|\log\rho^\Omega(y)|) & \text{if } \nu = n-1\\ (\rho^\Omega(y))^{2(n-\nu)-p} & \text{if } \nu > n-1\end{cases}$$

and

$$\left|\frac{\partial^p u(y)}{\partial y_{i_p}\cdots\partial y_{i_1}}\right| \leq \text{const}\begin{cases}1 & \text{if } \nu < n-p\\ 1+|\log\rho^\Omega(y)| & \text{if } \nu = n-p\\ (\rho^\Omega(y))^{n-\nu-p} & \text{if } \nu > n-p\end{cases}.$$

Therefore it is sufficient to analyze only the terms with $\partial^p u/\partial y_{i_p}\cdots\partial y_{i_1}$ in (12) and (13). Consider any point $\overline{x} \in \Omega = B(x^0, r)$ and take the ball $B(\overline{x}, \delta) \subset \Omega$ with

$$\delta = \frac{\rho^\Omega(\overline{x})}{2}. \tag{17}$$

Then, for $y \in B(\overline{x},\delta) \cup S(\overline{x},\delta)$, $\rho^\Omega(\overline{x}) \leq 2\rho^\Omega(y) \leq 3\rho^\Omega(\overline{x})$, and $w_\lambda^\Omega(\overline{x})$ and $w_\lambda^\Omega(y)$ are of the same order, i.e.

$$\left(\frac{1}{2}\right)^\lambda w_\lambda^\Omega(\overline{x}) \leq w_\lambda^\Omega(y) \leq \left(\frac{3}{2}\right)^\lambda w_\lambda^\Omega(\overline{x}) \qquad (\lambda > 0). \tag{18}$$

Let us estimate the terms on the right-hand side of (11) for $u \in C^{m,\nu}(G) \subset C^{m,\nu}(\Omega)$, $\|u\|_{0,\Omega} \leq d$, $x \in B(\overline{x},\delta) \subset \Omega$. First, it follows from (7) and (10) that

$$w^\Omega_{p-(n-\nu)}(\overline{x})\left|\int_{B(\overline{x},\delta)} \mathcal{K}^p_{i_1,\ldots,i_p}(x,y,u(y))\,dy\right| \leq \text{const}\, b_1(d)\begin{cases}\delta^n & \text{if } \nu < 0\\ \delta^n(1+|\log\delta|) & \text{if } \nu = 0\\ \delta^{n-\nu} & \text{if } \nu > 0\end{cases}. \tag{19}$$

Using (7) and (18), for $1 \le j \le p \le m$ and $\{i_{l_1}, \dots, i_{l_j}\} \subset \{i_1, \dots, i_p\}$ we have

$$w^{\Omega}_{p-(n-\nu)}(\overline{x}) \left| \int\limits_{B(\overline{x},\delta)} \frac{\partial}{\partial u} \mathcal{K}^{p-j}_{\{i_1,\dots,i_p\}\setminus\{i_{l_1},\dots,i_{l_j}\}}(x, y, u(y)) \frac{\partial^j u(y)}{\partial y_{i_{l_j}} \cdots \partial y_{i_{l_1}}}\, dy \right| \le \operatorname{const} b_1(d) \begin{cases} \delta^n & \text{if } \nu < 0 \\ \delta^n(1 + |\log\delta|) & \text{if } \nu = 0 \\ \delta^{n-\nu} & \text{if } \nu > 0 \end{cases} \times \sup_{y \in B(\overline{x},\delta)} w^{\Omega}_{j-(n-\nu)}(y) \left| \frac{\partial^j u(y)}{\partial y_{i_{l_j}} \cdots \partial y_{i_{l_1}}} \right|. \tag{20}$$

Here we roughly estimated $w^{\Omega}_{p-(n-\nu)}(\overline{x})$ and $w^{\Omega}_{p-(n-\nu)}(\overline{x})/w^{\Omega}_{j-(n-\nu)}(\overline{x})$, respectively, by a constant. The area of $S(\overline{x}, \delta)$ is equal to $\sigma_n \delta^{n-1}$, where σ_n is the area of the unit sphere in $\mathbb{R}^n$. Using (2) for $x = \overline{x} \in B(\overline{x}, \delta)$ we find

$$\left| \int\limits_{S(\overline{x},\delta)} \frac{\partial^{p-1}}{\partial x_{i_p} \cdots \partial x_{i_2}} \mathcal{K}(x, y, u(y))\, \omega_{i_1}(y)\, dS_y \right| \le c\, b_1(d)\, \delta^{n-1} \begin{cases} 1 & \text{if } \nu + p - 1 < 0 \\ 1 + |\log\delta| & \text{if } \nu + p - 1 = 0 \\ \delta^{-\nu-(p-1)} & \text{if } \nu + p - 1 > 0 \end{cases} \le c' b_1(d) (w^{\Omega}_{p-(n-\nu)}(\overline{x}))^{-1}$$

and, after the multiplication by $w^{\Omega}_{p-(n-\nu)}(\overline{x})$, we obtain

$$w^{\Omega}_{p-(n-\nu)}(\overline{x}) \left| \int\limits_{S(\overline{x},\delta)} \frac{\partial^{p-1}}{\partial x_{i_p} \cdots \partial x_{i_2}} \mathcal{K}(x, y, u(y))\, \omega_{i_1}(y)\, dS_y \right| \le \operatorname{const} b_1(d). \tag{21}$$

Further, using (2) and (18), we estimate, for $x = \overline{x} \in B(\overline{x}, \delta)$, $2 \le k \le p \le m$, $1 \le j \le k-1$ and $\{i_{l_1}, \dots, i_{l_j}\} \subset \{i_1, \dots, i_{k-1}\}$,

$$\left| \int\limits_{S(\overline{x},\delta)} \frac{\partial^{p-k}}{\partial x_{i_p} \cdots \partial x_{i_{k+1}}} \frac{\partial}{\partial u} \mathcal{K}^{k-1-j}_{\{i_1,\dots,i_{k-1}\}\setminus\{i_{l_1},\dots,i_{l_j}\}}(x, y, u(y))\, \omega_{i_k}(y) \frac{\partial^j u(y)}{\partial y_{i_{l_j}} \cdots \partial y_{i_{l_1}}}\, dS_y \right| \le c\, b_1(d)\, \delta^{n-1} \begin{cases} 1 & \text{if } \nu + p - k < 0 \\ 1 + |\log\delta| & \text{if } \nu + p - k = 0 \\ \delta^{-\nu-(p-k)} & \text{if } \nu + p - k > 0 \end{cases} \times \sup_{y \in S(\overline{x},\delta)} \left| \frac{\partial^j u(y)}{\partial y_{i_{l_j}} \cdots \partial y_{i_{l_1}}} \right|$$

and

$$
\begin{aligned}
& w^{\Omega}_{p-(n-\nu)}(\overline{x}) \Bigg| \int\limits_{S(\overline{x},\delta)} \frac{\partial^{p-k}}{\partial x_{i_p} \cdots \partial x_{i_{k+1}}} \frac{\partial}{\partial u} \\
& \qquad \mathcal{K}^{k-1-j}_{\{i_1,\dots,i_{k-1}\}\setminus\{i_{l_1},\dots,i_{l_j}\}}(x,y,u(y))\omega_{i_k}(y) \frac{\partial^j u(y)}{\partial y_{i_{l_j}} \cdots \partial y_{i_{l_1}}}\, dS_y \Bigg| \\
& \qquad \le c' b_1(d) \begin{cases} \delta^{n-1} & \text{if } \nu+p-k<0 \\ \delta^{n-1}(1+|\log\delta|) & \text{if } \nu+p-k=0 \\ \delta^{n-1-\nu-p+k} & \text{if } \nu+p-k>0 \end{cases} \\
& \qquad \times \frac{w^{\Omega}_{p-(n-\nu)}(\overline{x})}{w^{\Omega}_{j-(n-\nu)}(\overline{x})} \sup_{y\in S(\overline{x},\delta)} w^{\Omega}_{j-(n-\nu)}(y) \left| \frac{\partial^j u(y)}{\partial y_{i_{l_j}} \cdots \partial y_{i_{l_1}}} \right| :
\end{aligned} \tag{21}
$$

Some trouble causes only the case $\nu + p - k > 0$ (the third row in estimation (21)). If thereby $n - 1 - \nu - p + k \ge 0$, then we have again no difficulty in estimation (21) (estimating $w^{\Omega}_{p-(n-\nu)}(\overline{x})/w^{\Omega}_{j-(n-\nu)}(\overline{x})$ coarsely by a constant). Consider the case $n - 1 - \nu - p + k < 0$. Together with the inequality $k \ge 2$ we have then $n - \nu + 1 < p$. Therefore $w^{\Omega}_{p-(n-\nu)}(\overline{x}) = (\rho^{\Omega}(\overline{x}))^{p-(n-\nu)}$ and, due to (17),

$$
\begin{aligned}
& \delta^{n-1-\nu-(p-k)} \frac{w^{\Omega}_{p-(n-\nu)}(\overline{x})}{w^{\Omega}_{j-(n-\nu)}(\overline{x})} \\
& \quad \le \delta^{n-1-\nu-(p-k)} \begin{cases} (\rho^{\Omega}(\overline{x}))^{p-(n-\nu)} & \text{if } j<n-\nu \\ (\rho^{\Omega}(\overline{x}))^{p-(n-\nu)}(1+|\log\rho^{\Omega}(\overline{x})|) & \text{if } j=n-\nu \\ (\rho^{\Omega}(\overline{x}))^{p-j} & \text{if } j>n-\nu \end{cases} \\
& \quad \le c'' \begin{cases} \delta^{k-1} & \text{if } j<n-\nu \\ \delta^{k-1}(1+|\log\delta|) & \text{if } j=n-\nu \\ \delta^{n-\nu+k-1-j} & \text{if } j>n-\nu \end{cases} \\
& \quad \le c''' \begin{cases} \delta & \text{if } j<n-\nu \\ \delta(1+|\log\delta|) & \text{if } j=n-\nu \\ \delta^{n-\nu} & \text{if } j>n-\nu \end{cases} .
\end{aligned} \tag{22}
$$

It follows from (11), (12) and (19) - (22) that, for $x = \overline{x} \in B(\overline{x},\delta) \subset \Omega$,

$$
\begin{aligned}
& w^{\Omega}_{p-(n-\nu)}(\overline{x}) \left| \frac{\partial^p}{\partial x_{i_p} \cdots \partial x_{i_1}} \int\limits_{B(\overline{x},\delta)} \mathcal{K}(x,y,u(y))\, dy \right| \\
& \qquad \le \mathrm{const}\, b_1(d) \left(1 + \begin{cases} \delta^{\min\{1,n-\nu\}} \|u\|_{p,\nu,\Omega} & \text{if } \nu \text{ is a fraction} \\ \delta(1+|\log\delta|)\|u\|_{p,\nu,\Omega} & \text{if } \nu \text{ is an integer} \end{cases} \right).
\end{aligned}
$$

Since $\overline{x} \in \Omega = B(x^0,r)$ is arbitrary and $\delta = \rho^{\Omega}(\overline{x})/2$, we finally find that, for any $u \in C^{m,\nu}(G)$ with $\|u\|_{0,\Omega} \le d$,

$$
w^{\Omega}_{|\alpha|-(n-\nu)}(x)|D^{\alpha}_x(T_{\Omega}u)(x)| \le b_1(d)(c' + \varepsilon'_r \|u\|_{|\alpha|,\nu,\Omega}) \tag{23}
$$

where $x \in \Omega = B(x^0, \frac{r}{2}), 1 \le |\alpha| \le m, \varepsilon'_r \to 0$ as $r \to 0$ and the constant $c' > 0$ is independent of x, u and d.

Using (2) and (3) we find in a similar way that, for any $u, v \in C^{m,\nu}(G)$ with $\|u\|_{0,\Omega} \le d$ and $\|v\|_{0,\Omega} \le d$,

$$w^{\Omega}_{|\alpha|-(n-\nu)}(x)|D^{\alpha}_x(T_\Omega u - T_\Omega v)(x)| \\ \le c'' b_2(d)\|u - v\|_{0,\Omega} + b_1(d)\varepsilon''_r \|u - v\|_{|\alpha|,\nu,\Omega} \tag{24}$$

where $x \in \Omega = B(x^0, \frac{r}{2})$, $1 \le |\alpha| \le m$, $\varepsilon''_r \to 0$ as $r \to 0$ and the constant $c'' > 0$ is independent of x, u, v and d.

Inequalities (23) and (24) may be extended to any $u, v \in C^{m,\nu}(\Omega)$ with $\|u\|_{0,\Omega} \le d$ and $\|v\|_{0,\Omega} \le d$. Introduce the norm $\|\cdot\|'_{m,\nu,\Omega}$ in $C^{m,\nu}(\Omega)$:

$$\|u\|'_{m,\nu,\Omega} = M\|u\|_{0,\Omega} + \|u\|_{m,\nu,\Omega} \\ = (M+1)\|u\|_{0,\Omega} + \sum_{1 \le |\alpha| \le m} \sup_{x \in \Omega} \left(w^{\Omega}_{|\alpha|-(n-\nu)}(x)|D^{\alpha}_x u(x)| \right)$$

where $M \ge 1$ is a sufficiently large constant. We fix

$$M > \max\{c' b_1(d), c'' b_2(d)\} \sum_{1 \le |\alpha| \le m} 1.$$

It is clear that the norms $\|\cdot\|_{m,\nu,\Omega}$ and $\|\cdot\|'_{m,\nu,\Omega}$ are equivalent:

$$\|u\|_{m,\nu,\Omega} \le \|u\|'_{m,\nu,\Omega} \le (M+1)\|u\|_{m,\nu,\Omega}.$$

Introduce the set

$$\mathcal{B}_{m,\nu,\Omega,d,d'} = \left\{ u \in C^{m,\nu}(\Omega) : \|u\|'_{m,\nu,\Omega} \le d' \quad \text{and} \quad \|u\|_{0,\Omega} \le d \right\}$$

where $d > \|f_\Omega\|_{0,\Omega}$ and $d' > \|f_\Omega\|'_{m,\nu,\Omega} + M$. It is clear that $\mathcal{B}_{m,\nu,\Omega,d,d'} \subset \mathcal{B}_{0,\Omega,d}$ and $\mathcal{B}_{m,\nu,\Omega,d,d'}$ is closed in $C^{m,\nu}(\Omega)$. A consequence of (15), (23) and (24) is that, for sufficiently small $r > 0$, the operator S_Ω maps $\mathcal{B}_{m,\nu,\Omega,d,d'}$ into itself and satisfies (16) with $q = 1/2$. Indeed, using the first inequality in (15) and (23), for $u \in \mathcal{B}_{m,\nu,\Omega,d,d'}$ we have

$$\|S_\Omega u\|_{0,\Omega} \le b_1(d)\varepsilon_r + \|f_\Omega\|_{0,\Omega}$$

and

$$\|S_\Omega u\|'_{m,\nu,\Omega} = (M+1)\|S_\Omega u\|_{0,\Omega} + \sum_{1 \le |\alpha| \le m} \sup_{x \in \Omega} \left(w^{\Omega}_{|\alpha|-(n-\nu)}(x)|D^{\alpha}_x (S_\Omega u)(x)| \right) \\ \le (M+1) b_1(d)\varepsilon_r + \|f_\Omega\|'_{m,\nu,\Omega} + \sum_{1 \le |\alpha| \le m} b_1(d)\left(c' + \varepsilon'_r \|u\|_{|\alpha|,\nu,\Omega}\right).$$

For sufficiently small $r = r_1 > 0$ it follows that $S_\Omega u \in \mathcal{B}_{m,\nu,\Omega,d,d'}$:

$$\|S_\Omega u\|_{0,\Omega} \le d \qquad \text{and} \qquad \|S_\Omega u\|'_{m,\nu,\Omega} \le d'.$$

Similarly, using the second inequality in (15) and (24) we find that, for $u, v \in \mathcal{B}_{m,\nu,\Omega,d,d'}$,

$$
\begin{aligned}
\|S_\Omega u - S_\Omega v\|'_{m,\nu,\Omega}
&= (M+1)\|S_\Omega u - S_\Omega v\|_{0,\Omega} \\
&\quad + \sum_{1\le|\alpha|\le m} \sup_{x\in\Omega}\left(w^\Omega_{|\alpha|-(n-\nu)}(x)|D^\alpha_x(S_\Omega u - S_\Omega v)(x)|\right) \\
&\le (M+1)b_2(d)\varepsilon_r\|u-v\|_{0,\Omega} \\
&\quad + \sum_{1\le|\alpha|\le m}\left(c''b_2(d)\|u-v\|_{0,\Omega} + b_1(d)\varepsilon''_r\|u-v\|_{|\alpha|,\nu,\Omega}\right) \\
&\le \left(b_2(d)\varepsilon_r + \frac{(c''b_2(d)+b_1(d)\varepsilon''_r)\sum_{1\le|\alpha|\le m}1}{M+1}\right)(M+1)\|u-v\|_{0,\Omega} \\
&\quad + b_1(d)\varepsilon''_r\left(\sum_{1\le|\alpha|\le m}1\right)\sum_{1\le|\alpha|\le m}\sup_{x\in\Omega}\left(w^\Omega_{|\alpha|-(n-\nu)}(x)|D^\alpha_x(u-v)(x)|\right).
\end{aligned}
$$

We see that for sufficently small $r = r_2 > 0$ (such that the inequalities $b_2(d)\varepsilon_{r_2} \le 1/4$ and $b_1(d)\varepsilon''_{r_2}\left(\sum_{1\le|\alpha|\le m}1\right) \le 1/2$ are fulfilled) and for sufficiently large M (such that the inequality $(c''b_2(d)+b_1(d)\varepsilon''_{r_2})\sum_{1\le|\alpha|\le m}1/(M+1) \le 1/4$ is fulfilled) the operator S_Ω satisfies (16) with $q = 1/2$.

Thus, for $\Omega = B(x^0, r_0/2)$, $r_0 = \min\{r_1, r_2, \rho(x^0)\} > 0$, (16) is valid. Using again the Banach fixed point theorem we see that equation (13) is uniquely solvable in $\mathcal{B}_{m,\nu,\Omega,d,d'}$. The solution coincides with the unique solution u_0 of equation (13) (equation (1)) in $\mathcal{B}_{0,\Omega,d}$. In other words, the restriction to Ω of u_0, the solution to equation (1) under consideration, belongs to $\mathcal{B}_{m,\nu,\Omega,d,d'} \subset C^{m,\nu}(\Omega)$. Especially, for the point $x^0 \in \Omega = G \cap B(x^0, r_0/2)$ we have

$$
|D^\alpha u_0(x^0)| \le \text{const}\begin{cases} 1 & \text{if } |\alpha| < n-\nu \\ 1 + |\log\rho^\Omega(x^0)| & \text{if } |\alpha| = n-\nu \\ (\rho^\Omega(x^0))^{n-\nu-|\alpha|} & \text{if } |\alpha| > n-\nu \end{cases}
$$

with a constant which is independent of $x^0 \in G$. Since $x^0 \in G$ is arbitrary and, for $\Omega = B(x^0, r_0/2)$, $\rho^\Omega(x^0) = \min\{\rho(x^0)/2, r_0/2\}$,

$$
\frac{r_0}{2\operatorname{diam} G}\rho(x^0) \le \rho^\Omega(x^0) \le \rho(x^0)
$$

we obtain that $u_0 \in C^{m,\nu}(G)$. Theorem 1 is proved.

References

[1] Graham, I.G.: *Singularity expansion for the solutions of second kind Fredholm integral equations with weakly singular convolution kernels.* J. Int. Equ. 4 (1982), 1 - 30.

[2] Kaneko, H., Noren, R. and Y. Xu: *Regularity of the solution of Hammerstein equations with weakly singular kernel.* Int. Equ. Oper. Theory 13 (1990), 660 - 670.

[3] Kangro, R.: *On the smoothness of solutions to an integral equation with a kernel having a singularity on a curve.* Acta Comm. Univ. Tartuensis 913 (1990), 24 - 37.

[4] Kangro, U.: *The smoothness of the solution of a two-dimensional integral equations with logarithmic kernel* (in Russian). Proc. Eston. Acad. Sci., Phys.-Math. 39 (1990), 196 - 204.

[5] Kangro, U.: *The smoothness of the solution to a two-dimensional integral equation with logarithmic kernel.* Z. Anal. Anw. 12 (1993), 305 - 318.

[6] Pedas, A.: *On the smoothness of the solution of an integral equation with weakly singular kernel* (in Russian). Acta Comm. Univ. Tartuensis 492 (1979), 56 - 68.

[7] Pitkäranta, J.: *On the differential properties of solutions to Fredholm equations with weakly singular kernels.* J. Inst. Math. Appl. 24 (1979), 109 - 119.

[8] Pitkäranta, J.: *Estimates for the derivatives of solutions to weakly singular Fredholm integral equations.* SIAM J. Math. Anal. 11(1980), 952 - 968.

[9] Richter, G. R.: *On weakly singular Fredholm integral equations with displacement kernels.* J. Math. Anal. Appl. 55 (1976), 32 - 42.

[10] Schneider, C.: *Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind.* Int. Equ. Oper. Theory 2 (1979), 62 - 68.

[11] Vainikko, G. M.: *On the smoothness of the solution of the multidimensional weakly singular integral equation* (Russian original 1989). Math. USSR Sbornik 68 (1991), 585 - 600.

[12] Vainikko, G.: *Estimations of derivatives of a solution to a nonlinear weakly singular integral equation.* In: Problems of Pure and Applied Mathematics (ed.: M. Rahula). Tartu: Publ. Univ. Tartu 1990, pp. 212 - 215.

[13] Vainikko, G.: *Multidimensional weakly singular integral equations.* Lect. Notes Math. 1549 (1993), 1 - 158.

[14] Vainikko, G. and A. Pedas: *The properties of solutions of weakly singular integral equations.* J. Austral. Math. Soc. (Ser B) 22 (1981), 419 - 430.

[15] Vainikko, G., Pedas, A. and P. Uba: *Methods for Solving Weakly Singular Integral Equations* (in Russian). Tartu: Univ. Tartu 1984.

Received 30.12.1993