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RANK OF MAPPING TORI AND COMPANION MATRICES

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ABSTRACT. Given an element $\varphi \in \mathrm{GL}(d, \mathbf{Z})$, consider the mapping torus defined as the semidirect product $G = \mathbf{Z}^d \rtimes_\varphi \mathbf{Z}$. We show that one can decide whether G has rank 2 or not (i.e. whether G may be generated by two elements). When G is 2-generated, one may classify generating pairs up to Nielsen equivalence. If φ has infinite order, we show that the rank of $\mathbf{Z}^d \rtimes_{\varphi^n} \mathbf{Z}$ is at least 3 for all n large enough; equivalently, φ^n is not conjugate to a companion matrix in $\mathrm{GL}(d, \mathbf{Z})$ if n is large.

For Fritz Grunewald

1. INTRODUCTION

The *rank* of a finitely generated group is the minimum cardinality of a generating set. There are very few families of groups for which one knows how to compute the rank (see [8] and references therein), and there exists no algorithm computing the rank of a word-hyperbolic group [2].

By Grushko's theorem, rank is additive under free product. It does not behave as nicely under direct product, even when one of the factors is $\mathbf{Z}$: it can be checked that the solvable Baumslag-Solitar group $BS(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$ and the product $BS(1,2) \times \mathbf{Z}$ both have rank 2 since the latter is generated by $\{b, xa\}$ where x is the generator of $\mathbf{Z}$.

In this paper we consider semi-direct products $G = A \rtimes_\varphi \mathbf{Z}$ (also known as *mapping tori*), with the generator t of the cyclic group $\mathbf{Z}$ acting on A by some automorphism $\varphi \in \mathrm{Aut}(A)$. This was motivated by the remark that, when A is a non-abelian free group F_d of rank d and φ has finite order in $\mathrm{Out}(F_d)$, then G is a generalized Baumslag-Solitar group and its rank is computed in a forthcoming work by the first author. But we do not know how to compute the rank when φ has infinite order in $\mathrm{Out}(F_d)$. Abelianizing does not help much, so we ask:

QUESTION. *Is there an algorithm that, given $\varphi \in \mathrm{GL}(d, \mathbf{Z})$, computes the rank of $G = \mathbf{Z}^d \rtimes_\varphi \mathbf{Z}$?*

We can prove:

THEOREM 1.1. *There is an algorithm that, given $d \in \mathbf{N}$ and $\varphi \in \mathrm{GL}(d, \mathbf{Z})$, decides whether $G = \mathbf{Z}^d \rtimes_\varphi \mathbf{Z}$ has rank 2 or not.*

Here is a sketch of the proof. We show that the rank of G is 1 plus the minimum number k such that $\mathbf{Z}^d$ may be generated by k orbits of φ (i.e. there exist $g_1, \dots, g_k \in \mathbf{Z}^d$ such that the elements $\varphi^n(g_i)$, for $n \in \mathbf{Z}$ and $i = 1, \dots, k$, generate $\mathbf{Z}^d$). In particular, G has rank 2 if and only if $\mathbf{Z}^d$ may be generated by a single φ-orbit. We then show that this happens precisely when φ is conjugate in $\mathrm{GL}(d, \mathbf{Z})$ to the companion matrix M_φ having the same characteristic polynomial. This may be decided since the conjugacy problem is solvable in $\mathrm{GL}(d, \mathbf{Z})$ by Grunewald [6].

Theorem 1.1 extends to the case when φ is an automorphism of an arbitrary finitely generated nilpotent group A, by reduction to the abelian case.

When G has rank 2, one can classify generating pairs up to Nielsen equivalence. In particular:

THEOREM 1.2. *Suppose that $G = \mathbf{Z}^d \rtimes_\varphi \mathbf{Z}$ has rank 2. There are finitely many Nielsen classes of generating pairs if and only if the cyclic subgroup of $\mathrm{GL}(d, \mathbf{Z})$ generated by φ has finite index in its centralizer.*

Our next result is motivated by the following theorem due to J. Souto:

THEOREM 1.3 ([12]). *Let A be the fundamental group of a closed orientable surface of genus $g \geq 2$. Let φ be an automorphism of A representing a pseudo-Anosov mapping class. Then there exists n_0 such that the rank of $G_n = A \rtimes_{\varphi^n} \mathbf{Z}$ is $2g+1$ for all $n \geq n_0$.*

We prove:

THEOREM 1.4. *Given φ of infinite order in $\mathrm{GL}(d, \mathbf{Z})$, there exists n_0 such that the rank of $G_n = \mathbf{Z}^d \rtimes_{\varphi^n} \mathbf{Z}$ is ≥ 3 for all $n \geq n_0$.*

The theorem becomes false if the hypothesis that φ has infinite order is dropped, or if 3 is replaced by 4. We do not know hypotheses that would guarantee that the rank is $d+1$ for n large.

Since $\mathbf{Z}^d \rtimes_\varphi \mathbf{Z}$ has rank 2 if and only if φ is conjugate to a companion matrix, an equivalent formulation of Theorem 1.4 is:

THEOREM 1.5. *Given a matrix φ of infinite order in $\mathrm{GL}(d,\mathbf{Z})$, with $d \geq 2$, there exists n_0 such that φ^n is not conjugate to a companion matrix if $n \geq n_0$.*

EXAMPLE. Let φ be the unipotent matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is obvious that φ has infinite order. Notice that $\mathbf{Z}^2 \rtimes_\varphi \mathbf{Z}$ has rank 2 since it is generated by a generator of $\mathbf{Z}$ and the element $(0,1)$ of $\mathbf{Z}^2$. The companion matrix with the same characteristic polynomial as φ is $M_\varphi = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ and one can easily confirm that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}.$$

On the other hand, $\varphi^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ has the same companion matrix as φ, but it is easy to check (by reducing modulo a prime dividing n) that φ and φ^n are not conjugate in $\mathrm{GL}(2,\mathbf{Z})$ if $n \geq 2$.

Our proof of Theorem 1.5, given in Section 5, is based on the Skolem-Mahler-Lech theorem on linear recurrent sequences [3]. There are alternative approaches based on equations in S-units and Baker's theory on linear forms in logarithms. They are due to Amoroso-Zannier [1] and yield uniformity: *one may take $n_0 = [Cd^6(\log d)^6]$ where C is a universal constant (independent of φ)*. We refer to [1] for related number-theoretic questions, for instance a discussion of a "Hasse principle".

We conclude with a few open questions.

What about ascending HNN extensions? For instance, let φ be an injective endomorphism of $\mathbf{Z}^d$ (a matrix with integral entries and non-zero determinant). Let $G = \mathbf{Z}^d *_\varphi = \langle \mathbf{Z}^d, t \mid tgt^{-1} = \varphi(g) \rangle$. Is there an algorithm that can decide whether G has rank 2?

Our analysis on $\mathbf{Z}^d$ uses the Cayley-Hamilton theorem. This is not available in a non-abelian free group F_d. Given $\varphi \in \mathrm{Aut}(F_d)$, is there an algorithm that can decide whether F_d may be generated (or normally generated) by a single φ-orbit? More basically: given $\varphi \in \mathrm{Aut}(F_d)$ and $g \in F_d$, can one decide whether the φ-orbit of g generates F_d?

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2. GENERALITIES

Let A be a finitely generated group. The letters a, b, v will always denote elements of A. We denote by i_a the inner automorphism $v \mapsto ava^{-1}$.

Given $\varphi \in \mathrm{Aut}(A)$, we let G be the *mapping torus*

$$G = A \rtimes_\varphi \mathbf{Z} = \langle A, t \mid tat^{-1} = \varphi(a) \rangle .$$

There is an exact sequence $1 \to A \to G \to \mathbf{Z} \to 1$. Up to isomorphism, G only depends on the image of φ in $\mathrm{Out}(A)$. Any $g \in G$ has unique forms at^n, $t^n a'$ with $n \in \mathbf{Z}$ and $a, a' \in A$.

If N is a characteristic subgroup of A, we denote by $\bar\varphi$ the automorphism induced on A/N. There is an exact sequence

$$1 \to N \to A \rtimes_\varphi \mathbf{Z} \to A/N \rtimes_{\bar\varphi} \mathbf{Z} \to 1 .$$

The *rank* $\mathrm{rk}(G)$ is the minimum cardinality of a generating set. We let $\mathrm{vrk}(G)$ be the minimum number of elements needed to generate a finite index subgroup: $\mathrm{vrk}(G) = \inf_H \mathrm{rk}(H)$ with the infimum taken over all subgroups of finite index. Note that one may have $\mathrm{vrk}(H) > \mathrm{vrk}(G)$ if H has finite index in G, for instance when G is free.

We say that two generating sets with the same cardinality are *Nielsen equivalent* if one can pass from one to the other by Nielsen operations: permuting the generators, replacing g_i by g_i^{-1} or $g_i g_j$. For instance, any generating set of $\mathbf{Z}$ is Nielsen equivalent to $\{0, \dots, 0, 1\}$ by the Euclidean algorithm.

The φ-orbit of $a \in A$ is $\{\varphi^n(a) \mid n \in \mathbf{Z}\}$. We denote by $\mathrm{or}(\varphi)$ the minimum number of φ-orbits needed to generate A. Clearly $\mathrm{or}(\varphi) \leq \mathrm{rk}(A)$. We also denote by $\mathrm{vor}(\varphi)$ the minimum number of φ-orbits needed to generate a finite index subgroup of A, so $\mathrm{vor}(\varphi) \leq \mathrm{vrk}(A)$.

LEMMA 2.1. *Given* $a, a_1, \dots, a_k \in A$, *the intersection*

$$A' = \langle a_1, \dots, a_k, at \rangle \cap A$$

is generated by the $(i_a \circ \varphi)$*-orbits of* $a_1, \dots, a_k$.

The $(i_a \circ \varphi)$*-orbits of* $a_1, \dots, a_k$ *generate* A *if and only if* $a_1, \dots, a_k, at$ *generate* G.

Proof. One has $(i_a \circ \varphi)^n(v) = (at)^n v (at)^{-n}$ for $v \in A$ and $n \in \mathbf{Z}$. This shows that the $(i_a \circ \varphi)$-orbit of a_i is contained in A'. Conversely, if $v \in A'$, write it in terms of $a_1, \dots, a_k, at$. The exponent sum of t is 0, so v is a product of elements of the form $(at)^n a_i^{\pm 1} (at)^{-n}$.

If $A' = A$, then $\langle a_1, \dots, a_k, at \rangle$ contains A and at, so equals G. □

COROLLARY 2.2. $\mathrm{rk}(G) = 1 + \min_{a \in A} \mathrm{or}(i_a \circ \varphi)$.

Proof. $\leq$ is clear. For the converse, apply Euclid's algorithm modulo A to see that any finite generating set of G is Nielsen equivalent to a set $\{a_1, \dots, a_k, at\}$. □

COROLLARY 2.3. $\mathrm{vrk}(G) = 1 + \min_{a \in A, n \neq 0} \mathrm{vor}(i_a \circ \varphi^n)$.

Proof. If $n \neq 0$ and the $(i_a \circ \varphi^n)$-orbits of $a_1, \dots, a_k$ generate a finite index subgroup of A, the subgroup of G generated by $a_1, \dots, a_k, at^n$ has finite index because it maps onto $n\mathbf{Z}$ and it meets A in a subgroup of finite index. This shows that $\mathrm{vrk}(G) \leq 1 + \min_{a \in A, n \neq 0} \mathrm{vor}(i_a \circ \varphi^n)$.

For the opposite inequality, note that any finite subset of G generating a finite index subgroup is Nielsen equivalent to $\{a_1, \dots, a_k, at^n\}$ with $n \neq 0$, and the $(i_a \circ \varphi^n)$-orbits of $a_1, \dots, a_k$ generate a finite index subgroup of A. □

COROLLARY 2.4. *Suppose that* A *is abelian.*

(1) $\mathrm{rk}(G) = 1 + \mathrm{or}(\varphi)$ *and* $\mathrm{vrk}(G) = 1 + \mathrm{vor}(\varphi)$.

(2) G *has rank* ≤ 2 *if and only if* A *is generated by a single* φ*-orbit. A pair* (a_1, at) *generates* G *if and only if the* φ*-orbit of* a_1 *generates* A.

(3) $\mathrm{vrk}(G)$ *is computable.*

Proof. i_a is the identity and $\mathrm{vor}(\varphi) \leq \mathrm{vor}(\varphi^n)$, so (1) follows from previous results. (2) is clear.

For (3), first suppose $A = \mathbf{Z}^d$. View φ as an automorphism of the vector space $\mathbf{Q}^d$. Then $\mathrm{vor}(\varphi)$ is the minimum number of φ-orbits needed to generate $\mathbf{Q}^d$. This is computable (it is the number of blocks in the rational canonical form of φ). In general, if T is the torsion subgroup of A, then $A/T \simeq \mathbf{Z}^d$ for some d. Let $\bar{\varphi}$ be the automorphism induced on $\mathbf{Z}^d$. Then $\operatorname{vor}(\varphi) = \operatorname{vor}(\bar{\varphi})$ is computable. □

3. Computability

Suppose $A = \mathbf{Z}^d$ with $d \geq 1$. We view $\varphi \in \operatorname{Aut}(A)$ as an automorphism of $\mathbf{Z}^d$ or as a matrix in $\mathrm{GL}(d, \mathbf{Z})$. Its *companion matrix* M_φ is the unique matrix of the form

$$\begin{pmatrix} 0 & & & & * \\ 1 & 0 & & & * \\ & \ddots & \ddots & & * \\ & & 1 & 0 & * \\ & & & 1 & * \end{pmatrix}$$

having the same characteristic polynomial as φ (the empty triangles are filled with 0's, and $*$ denotes an arbitrary integer).

Lemma 3.1. *Let $\varphi \in \mathrm{GL}(d, \mathbf{Z})$, with $d \geq 1$.*

(1) *The following are equivalent:*

  (a) *$G = \mathbf{Z}^d \rtimes_\varphi \mathbf{Z}$ has rank 2;*

  (b) *$\mathbf{Z}^d$ may be generated by a single φ-orbit;*

  (c) *there exists $a \in \mathbf{Z}^d$ such that $\{a, \varphi(a), \dots, \varphi^{p-1}(a)\}$ is a basis of $\mathbf{Z}^d$;*

  (d) *φ is conjugate to its companion matrix M_φ in $\mathrm{GL}(d, \mathbf{Z})$.*

(2) *Suppose that the φ-orbit of a generates $\mathbf{Z}^d$. Then the φ-orbit of b generates $\mathbf{Z}^d$ if and only if $b = h(a)$ where $h \in \mathrm{GL}(d, \mathbf{Z})$ commutes with φ.*

Proof. We already know that (a) is equivalent to (b). If a is the first element of a basis of $\mathbf{Z}^d$ in which φ is represented by the matrix M_φ, then the basis is $\{a, \varphi(a), \dots, \varphi^{d-1}(a)\}$ and the φ-orbit of a generates $\mathbf{Z}^d$, so (d)⇒(c)⇒(b).

Conversely, note that the φ-orbit of any element a is generated by $\{a, \varphi(a), \dots, \varphi^{d-1}(a)\}$ as a consequence of the Cayley-Hamilton theorem. So if (b) holds for the orbit of a, we obtain (c). Finally (c) clearly implies (d).

To prove (2), suppose that h commutes with φ, and define $b = h(a)$. The image of the basis $(a, \varphi(a), \dots, \varphi^{d-1}(a))$ by h is $(b, \varphi(b), \dots, \varphi^{d-1}(b))$, so the orbit of b generates. Conversely, if the orbit of b generates, define h as the automorphism of $\mathbf{Z}^d$ taking $(a, \varphi(a), \dots, \varphi^{d-1}(a))$ to $(b, \varphi(b), \dots, \varphi^{d-1}(b))$. It commutes with φ because M_φ represents φ in both bases. $\square$

PROPOSITION 3.2. *Let A be a finitely generated nilpotent group. There is an algorithm which, given $\varphi \in \mathrm{Aut}(A)$, decides whether $G = A \rtimes_\varphi \mathbf{Z}$ has rank 2 or not.*

Proof. If $A = \mathbf{Z}^d$, one has to decide whether φ is conjugate to its companion matrix M_φ in $\mathrm{GL}(d, \mathbf{Z})$. This is possible because the conjugacy problem is algorithmically solvable in $\mathrm{GL}(d, \mathbf{Z})$ by [6] (see Remark 3.4).

We now assume that A is abelian. It fits in an exact sequence

$$0 \to T \to A \to \mathbf{Z}^d \to 0$$

with T finite. We denote by $a \mapsto \bar{a}$ the map $A \to \mathbf{Z}^d$, and by $h \mapsto \bar{h}$ the natural epimorphism $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathbf{Z}^d)$. They each have finite kernel.

We have to decide whether A may be generated by a single φ-orbit. We first check whether the matrix of $\bar{\varphi}$ is conjugate to its companion matrix. If not, the answer to our question is no. If yes, [6] yields a conjugator and therefore an explicit $u \in \mathbf{Z}^d$ whose $\bar{\varphi}$-orbit generates $\mathbf{Z}^d$.

We claim that A may be generated by a single φ-orbit if and only if there exist $a \in A$ mapping onto u, and $\psi \in \mathrm{Aut}(A)$ of the form $h\varphi h^{-1}$ with $h \in \mathrm{Aut}(A)$ and $[\bar{h}, \bar{\varphi}] = 1$, such that the ψ-orbit of a generates A.

The "if" direction is clear. Conversely, suppose that the φ-orbit of b generates A. Then the $\bar{\varphi}$-orbit of $\bar{b}$ generates $\mathbf{Z}^d$, so by Lemma 3.1 there exists $\theta \in \mathrm{Aut}(\mathbf{Z}^d)$ commuting with $\bar{\varphi}$ and mapping $\bar{b}$ to u. Let h be any lift of θ to $\mathrm{Aut}(A)$. Defining $a = h(b)$ and $\psi = h\varphi h^{-1}$, it is easy to check that the ψ-orbit of a generates A. This proves the claim.

We now explain how to decide whether a and ψ as above exist. Note that a and ψ must belong to explicit finite sets: a belongs to the preimage A_u of u, and ψ belongs to the preimage X_φ of $\bar{\varphi}$ in $\mathrm{Aut}(A)$.

By Theorem C of [6], the centralizer of $\bar{\varphi}$ in $\mathrm{Aut}(\mathbf{Z}^d)$ is a finitely generated subgroup and one can compute a finite generating set. The same is true of $D = \{h \in \mathrm{Aut}(A) \mid [\bar{h}, \bar{\varphi}] = 1\}$, so we can list the elements ψ in the orbit $D\varphi$ of φ for the action of D on X_φ by conjugation.

By the claim proved above, A may be generated by a single φ-orbit if and only if there exist $a \in A_u$ and $\psi \in D\varphi$ such that the ψ-orbit of a

generates A. To decide this, we enumerate the pairs (a,ψ) with $a \in A_u$ and $\psi \in D\varphi$. For each pair, we consider the increasing sequence of subgroups $A_N = \langle \psi^{-N}(a), \dots, \psi^{-1}(a), a, \psi(a), \dots \psi^N(a) \rangle$. It stabilizes and we check whether $A_N = A$ for N large.

This completes the proof for A abelian. If A is nilpotent, let B be its abelianization and let $\rho\colon B \to B$ be the automorphism induced by φ. If $G_\varphi = A \rtimes_\varphi \mathbf{Z}$ has rank 2, so does its quotient $G_\rho = B \rtimes_\rho \mathbf{Z}$. Conversely, if G_ρ has rank 2, it is generated by t and some $b \in B$ whose ρ-orbit generates B. Let a be any lift of b to A. The subgroup of A generated by the φ-orbit of a maps surjectively to B, so equals A by a classical fact about nilpotent groups (see e.g. Theorem 2.2.3(d) of [9]). Thus G_φ has rank 2. □

COROLLARY 3.3. *If $A = \mathbf{Z}^2$ or $A = F_2$, one can compute the rank of G.*

Proof. The rank is 2 or 3, so this is clear from the proposition if $A = \mathbf{Z}^2$. Recall that the natural map $\mathrm{Out}(F_2) \to \mathrm{Out}(\mathbf{Z}^2) = \mathrm{Aut}(\mathbf{Z}^2)$ is an isomorphism (both groups are isomorphic to $\mathrm{GL}(2,\mathbf{Z})$). Given $G = F_2 \rtimes_\varphi \mathbf{Z}$, let ρ be the image of φ in $\mathrm{Aut}(\mathbf{Z}^2)$. Consider $G_\rho = \mathbf{Z}^2 \rtimes_\rho \mathbf{Z}$. We prove that G and G_ρ have the same rank.

Clearly $2 \le \mathrm{rk}(G_\rho) \le \mathrm{rk}(G) \le 3$. If G_ρ has rank 2, Lemma 3.1 lets us assume that ρ is of the form $\begin{pmatrix} 0 & \pm 1 \\ 1 & n \end{pmatrix}$. Since G only depends on the class of φ in $\mathrm{Out}(F_2)$, it is isomorphic to

$$\langle a, b, t \mid tat^{-1} = b, tbt^{-1} = a^{\pm 1} b^n \rangle,$$

so has rank 2. □

REMARK 3.4. Grunewald's solution to the conjugacy problem is entirely algorithmic. Given two matrices $T_1, T_2 \in \mathrm{GL}(d,\mathbf{Z})$, there is an algorithm which decides whether there exists a matrix $X \in \mathrm{GL}(d,\mathbf{Z})$ such that $XT_1X^{-1} = T_2$. If the answer is yes, the algorithm constructs such an X. In fact, Grunewald's algorithm decomposes each T_i into the sum of two matrices $T_i = S_i + U_i$, where S_i is a rational semisimple matrix and U_i is a rational nilpotent matrix. Then the conjugation question between the T_i's reduces to conjugation questions between the S_i's and U_i's. In turn these questions are transformed into problems about isomorphisms of modules over quotient rings of a subring of finite index in a ring of integers of an algebraic number field. Arguments are rather involved.

4. Nielsen equivalence

Proposition 4.1. *Suppose that A is abelian and $G = A \rtimes_\varphi \mathbf{Z}$ has rank 2.*

(1) *Any generating pair of G is Nielsen equivalent to a pair (a,t) with $a \in A$.*

(2) *Two generating pairs (a,t) and (b,t), with $a,b \in A$, are Nielsen equivalent if and only if b belongs to the φ-orbit of a or a^{-1}.*

Proof. Given $x, y \in A$, and n, write

$$(x, ty) \sim ((ty)^n x (ty)^{-n}, ty) = (\varphi^n(x), ty)$$

and

$$(x, ty) \sim (\varphi^n(x), ty) \sim (\varphi^n(x), ty\varphi^n(x)) \sim (x, ty\varphi^n(x)),$$

where $\sim$ denotes Nielsen equivalence.

Every generating pair is equivalent to some (a, ty), with the φ-orbit of a generating A. But $(a,ty) \sim (a, ty\varphi^n(a))$ so by an easy induction $(a,ty) \sim (a,t)$. This proves (1).

If $b = \varphi^n(a^\varepsilon)$ with $\varepsilon = \pm 1$, then

$$(b,t) = (\varphi^n(a^\varepsilon), t) = (t^n a^\varepsilon t^{-n}, t) \sim (a,t).$$

The converse follows from Theorem 2.1 of [7]. We give a proof for completeness. If $(b,t) \sim (a,t)$, we can write $b = w(a,t)$ with w a primitive word with exponent sum 0 in t. Such a word is conjugate to $a^{\pm 1}$ in the free group $F(a,t)$, so b is conjugate to $a^{\pm 1}$ in G. Since A is abelian, b belongs to the φ-orbit of $a^{\pm 1}$. □

Remark 4.2. More generally, if A is abelian, any generating set of G is Nielsen equivalent to a set of the form $\{a_1, \dots, a_k, t\}$.

Remark 4.3. The proposition does not extend to nilpotent groups. Let A be the Heisenberg group $\langle a,b,c \mid [a,b] = c, [a,c] = [b,c] = 1 \rangle$. Let φ map a to ab and b to b. The generating pairs (a,t) and (ac^{-1}, t) are Nielsen equivalent (even conjugate) but ac^{-1} does not belong to the φ-orbit of $a^{\pm 1}$. Moreover, (a, tc) is a generating pair which is not Nielsen equivalent to a pair (x,t) with $x \in A$. Indeed, if it were, then t would be conjugate to tca^k for some $k \in \mathbf{Z}$ by [7]. Counting exponent sum in a yields $k = 0$. But t and tc are not conjugate.

Corollary 4.4. *Let $A = \mathbf{Z}^d$. If G has rank 2, the number of Nielsen classes of generating pairs is equal to the (possibly infinite) index of the group generated by φ and $-Id$ in the centralizer of φ in* $\mathrm{GL}(d, \mathbf{Z})$.

Proof. By Proposition 4.1 we need only consider generating pairs of the form (a,t). Fix one. To any $b \in \mathbf{Z}^d$ such that (b,t) generates G we associate the automorphism ψ_b of $\mathbf{Z}^d$ taking the basis $\{a, \varphi(a), \dots, \varphi^{d-1}(a)\}$ to the basis $\{b, \varphi(b), \dots, \varphi^{d-1}(b)\}$. By Lemma 3.1, the image of this map $b \mapsto \psi_b$ is the centralizer of φ in $\mathrm{GL}(d, \mathbf{Z})$. By Proposition 4.1, $(b,t) \sim (a,t)$ if and only if ψ_b is $\pm\varphi^n$ for some $n \in \mathbf{Z}$. □

EXAMPLE. If $A = \mathbf{Z}^2$ and G has rank 2, the number of Nielsen classes of generating pairs is always finite. If

$$\varphi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

this number is infinite.

5. POWERS

Fix $\varphi \in \mathrm{GL}(d, \mathbf{Z})$. Say that $v \in \mathbf{Z}^d$ is *φ-cyclic* if its φ-orbit generates $\mathbf{Z}^d$, or equivalently if $\{v, \varphi(v), \dots, \varphi^{d-1}(v)\}$ is a basis of $\mathbf{Z}^d$. The existence of such a v is equivalent to φ being conjugate to its companion matrix, and also to G having rank 2. If v is φ^n-cyclic for some $n \geq 2$, it is φ-cyclic since its φ^n-orbit is contained in its φ-orbit.

If v is φ-cyclic, we denote by δ_n the index of the subgroup of $\mathbf{Z}^d$ generated by the φ^n-orbit of v. It does not depend on the choice of v since φ always has matrix M_φ in the basis $\{v, \varphi(v), \dots, \varphi^{d-1}(v)\}$. Also note that $\delta_1 = 1$. The group $G_n = \mathbf{Z}^d \rtimes_{\varphi^n} \mathbf{Z}$ has rank 2 (equivalently, φ^n is conjugate to its companion matrix) if and only if $\delta_n = 1$.

THEOREM 5.1. *If $\varphi \in \mathrm{GL}(2, \mathbf{Z})$ has infinite order, the rank of $G_n = \mathbf{Z}^2 \rtimes_{\varphi^n} \mathbf{Z}$ is 3 for all $n \geq 3$ (and also for $n = 2$ unless $\det(\varphi) = -1$ and $\mathrm{trace}(\varphi) = \pm 1$).*

Proof. If G_n has rank 2 for some n, there exists a φ^n-cyclic element v. Such a v is also φ-cyclic. In the basis $\{v, \varphi(v)\}$, the matrix of φ has the form $M = \begin{pmatrix} 0 & \varepsilon \\ 1 & \tau \end{pmatrix}$ with $\varepsilon = \pm 1$. If finite, the index δ_n is the absolute value of the determinant c_n of the matrix expressing the family $\{v, \varphi^n(v)\}$ in the basis $\{v, \varphi(v)\}$. We prove the theorem by showing that $|c_n| > 1$ for $n \geq 3$.

The number c_n is determined by the equation $M^n = c_n M + d_n I$. It follows from the Cayley-Hamilton theorem that the sequence c_n satisfies the recurrence relation $c_{n+2} - \tau c_{n+1} - \varepsilon c_n = 0$.

If $\varepsilon = -1$ one has

$$c_n = \prod_{k=1}^{n-1} (\tau - 2\cos\frac{k\pi}{n}),$$

because c_n is a monic polynomial of degree $n-1$ in τ which vanishes for $\tau = 2\cos\frac{k\pi}{n}$ (one also has $c_n = U_{n-1}(\tau/2)$, with U_{n-1} a Chebyshev polynomial of the second kind).

If $\varepsilon = 1$ one has

$$c_n = \prod_{k=1}^{n-1} (\tau - 2i\cos\frac{k\pi}{n}).$$

Since φ is assumed to have infinite order, one has $\tau \neq 0$ if $\varepsilon = 1$, and $|\tau| \geq 2$ if $\varepsilon = -1$. One checks that $|c_n| > 1$ for $n \geq 3$ (for $n \geq 2$ if $\varepsilon = -1$ or $|\tau| \geq 2$). □

THEOREM 5.2. *Suppose that $\varphi \in \mathrm{GL}(d, \mathbf{Z})$ has infinite order.*

(1) *There exists n_0 such that $G_n = \mathbf{Z}^d \rtimes_{\varphi^n} \mathbf{Z}$ has rank ≥ 3 for every $n \geq n_0$. Equivalently: φ^n is not conjugate to its companion matrix for $n \geq n_0$.*

(2) *More precisely, the minimum index of 2-generated subgroups of G_n goes to infinity with n.*

Note that there are arbitrarily large values of n for which the rank of G_n is $d+1$ (whenever φ^n is the identity modulo some prime number). As already mentioned, it is proved in [1] that n_0 may be chosen to depend only on d.

The key step in the proof of Theorem 5.2 is the following result.

PROPOSITION 5.3. *If φ has infinite order and v is φ-cyclic, then the index δ_n of the subgroup of $\mathbf{Z}^d$ generated by the φ^n-orbit of v goes to infinity with n.*

REMARK. This proposition remains true if v is not assumed to be φ-cyclic, provided δ_n is defined as the index of the subgroup generated by the φ^n-orbit of v in the subgroup generated by the φ-orbit of v.

Proof of the theorem from the proposition. As above, if G_n has rank 2 for some n, there exists a φ-cyclic element v. For n large one has $\delta_n > 1$, so G_n has rank > 2. Assertion 1 is proved.

For Assertion 2, suppose that there are arbitrarily large values of n such that G_n contains a 2-generated subgroup H_n of index $\leq C$, for some fixed C. This subgroup has a generating pair of the form (a_n, t_n) with $a_n \in \mathbf{Z}^d$, and the intersection of H_n with $\mathbf{Z}^d$ is generated by the φ^{nm_n}-orbit of a_n for some $m_n \geq 1$. It has index $\leq C$ in $\mathbf{Z}^d$.

The subgroup of $\mathbf{Z}^d$ generated by the φ-orbit of a_n has index $\leq C$, so we can assume that it does not depend on n. Call it J. It is φ-invariant so we can apply the proposition to the action of φ on J, with $v = a_n$. This gives the required contradiction. □

Proof of Proposition 5.3. When $d = 2$, one easily checks that c_n, as computed above, goes to infinity with n. The proof in the general case is more involved.

Define numbers $u_k(i)$, for $k = 0, \ldots, d-1$ and $i \geq 0$, by

$$\varphi^i(v) = \sum_{k=0}^{d-1} u_k(i) \varphi^k(v) \ .$$

The sequences $u_0, \ldots, u_{d-1}$ form a basis for the space $\mathcal{S}$ of sequences satisfying the linear recurrence associated to the characteristic polynomial of φ (the recurrence is $\sum_{j=0}^d a_j u_k(i+j) = 0$ if the characteristic polynomial is $\sum_{j=0}^d a_j X^j$).

The index δ_n is the absolute value of the determinant c_n of the matrix $(u_k(ni))_{0 \leq i,k \leq d-1}$ (unless the determinant is 0, in which case δ_n is infinite). We have to prove that, given $c \neq 0$, the set of n's such that $c_n = c$ is finite. We assume it is not and we work towards a contradiction.

A sequence satisfies a linear recurrence if and only if it is a finite sum of polynomials times exponentials, so c_n also is a recurrent sequence. The Skolem-Mahler-Lech theorem [3] then implies that $c_n = c$ for all n in an arithmetic progression $\mathbf{N}_0 \subset \mathbf{N}$.

We shall now replace the basis u_k of $\mathcal{S}$ by another basis w_k depending on the eigenvalues of φ. We then assume that $D_n := \det(w_k(ni))_{0 \leq i,k \leq d-1} = c' \neq 0$ for $n \in \mathbf{N}_0$.

We sort the eigenvalues λ_k of φ so that $0 < |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_d|$. First suppose that the eigenvalues are all distinct. We then choose $w_k(i) = (\lambda_{k+1})^i$. In this case D_n is a Vandermonde determinant, for instance

$$D_n = \begin{vmatrix} 1 & 1 & 1 \\ (\lambda_1)^n & (\lambda_2)^n & (\lambda_3)^n \\ (\lambda_1)^{2n} & (\lambda_2)^{2n} & (\lambda_3)^{2n} \end{vmatrix}$$

for $d = 3$, so $D_n = \prod_{1 \le k < m \le d} \big((\lambda_m)^n - (\lambda_k)^n\big)$.

If all moduli $|\lambda_k|$ are distinct, then $|D_n|$ goes to infinity with n because its diagonal term

$$(\lambda_2)^n(\lambda_3)^{2n} \dots (\lambda_d)^{(d-1)n} = \big(\lambda_2(\lambda_3)^2 \dots (\lambda_d)^{(d-1)}\big)^n$$

has modulus bigger than all others.

If the λ_k's are distinct but their moduli are not, we write each of the $d!$ terms in the standard expansion of D_n in the form $\varepsilon_j {\mu_j}^n$ (with $\varepsilon_j = \pm 1$). Now there may be several (possibly cancelling) terms for which $|\mu_j|$ takes its maximal value $K = |\lambda_2(\lambda_3)^2 \dots (\lambda_d)^{(d-1)}|$. Note that $K > 1$ because otherwise all λ_k's have modulus 1, hence are roots of unity by a classical result of Kronecker ([11], [5, Proposition 1.2.1]), and φ has finite order.

Since $D_n = c'$ for $n \in \mathbf{N}_0$ and $K > 1$, one has $\sum_{|\mu_j| = K} \varepsilon_j {\mu_j}^n = 0$ for $n \in \mathbf{N}_0$. Call this sum $D_{n,K}$. Recall that $D_n = \prod_{1 \le k < m \le d} \big((\lambda_m)^n - (\lambda_k)^n\big)$. To expand this product, one chooses one of $(\lambda_m)^n$ or $(\lambda_k)^n$ for each couple k, m. The corresponding term contributes to $D_{n,K}$ if and only if one always chooses a term of maximal modulus. In other words, $D_{n,K} = \prod_{1 \le k < m \le p} E_{k,m}$ with $E_{k,m} = (\lambda_m)^n - (\lambda_k)^n$ if $|\lambda_m| = |\lambda_k|$ and $E_{k,m} = (\lambda_m)^n$ if $|\lambda_m| > |\lambda_k|$. Since the λ_k's are non-zero, $D_{n,K} = 0$ implies $(\lambda_k)^n = (\lambda_m)^n$ for some k, m with $k \neq m$, so that $D_n = 0$, a contradiction.

This completes the proof when the eigenvalues of φ are distinct. In the remaining case, the basis w_k must have a different form: if λ is an eigenvalue of multiplicity r, we use the sequences $\lambda^i, i\lambda^i, \dots, i^{r-1}\lambda^i$. For instance,

$$D_n = \begin{vmatrix} 1 & 0 & 0 & 1 \\ (\lambda_1)^n & n(\lambda_1)^n & n^2(\lambda_1)^n & (\lambda_4)^n \\ (\lambda_1)^{2n} & 2n(\lambda_1)^{2n} & (2n)^2(\lambda_1)^{2n} & (\lambda_4)^{2n} \\ (\lambda_1)^{3n} & 3n(\lambda_1)^{3n} & (3n)^2(\lambda_1)^{3n} & (\lambda_4)^{3n} \end{vmatrix}$$

when $d = 4$ and $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$.

Calling $\nu_1, \dots, \nu_q$ the distinct eigenvalues of φ, there exist integers a, b, c_k, d_{mk} (depending only on the multiplicities of the eigenvalues) such that

$$D_n = an^b \prod_{k=1}^{q} (\nu_k)^{nc_k} \prod_{1 \le k < m \le q} \big((\nu_m)^n - (\nu_k)^n\big)^{d_{mk}}$$

(see [4] or Theorem 21 in [10]). For instance, D_n as displayed above equals $2n^3(\lambda_1)^{3n}((\lambda_4)^n - (\lambda_1)^n)^3$.

If $K > 1$, we conclude as in the previous case. If $K = 1$, all eigenvalues are roots of unity and $D_n = n^b E_n$ where E_n only takes finitely many values and $b > 0$ (an eigenvalue ν_j of multiplicity $r \geq 2$ contributes $1 + \cdots + (r-1)$ to b). Such a product cannot take a non-zero value infinitely often. $\square$

COROLLARY 5.4. *If A is abelian, and $\varphi \in \mathrm{Aut}(A)$ has infinite order, then $G_n = A \rtimes_{\varphi^n} \mathbf{Z}$ has rank ≥ 3 for n large. The minimum index of 2-generated subgroups of G_n goes to infinity with n.*

This follows readily from Theorem 5.2, writing $A/T \simeq \mathbf{Z}^d$ with T finite. The analogous result for nilpotent groups is false, as the following example shows. Let A be the Heisenberg group as in Remark 4.3. If φ maps a to bc, b to ac^2, and c to c^{-1}, then $\varphi^{2n+1}(a) = bc^{1-n}$, so G_{2n+1} has rank 2 since a and $\varphi^{2n+1}(a)$ generate A. The automorphism induced by φ on the abelianization of A has order 2.

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