



An operator inequality for weighted Bergman shift operators

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Abstract. We prove an operator inequality for the Bergman shift operator on weighted Bergman spaces of analytic functions in the unit disc with weight function controlled by a curvature parameter α assuming nonnegative integer values. This generalizes results by Shimorin, Hedenmalm and Jakobsson concerning the cases $\alpha = 0$ and $\alpha = 1$. A naturally derived scale of Hilbert space operator inequalities is studied and shown to be relaxing as the parameter $\alpha > -1$ increases. Additional examples are provided in the form of weighted shift operators.

0. Introduction

By a weight function ω in the open unit disc \mathbb{D} in the complex plane we understand a nonnegative area integrable function ω in \mathbb{D} . We shall be concerned with weight functions ω in \mathbb{D} having the additional property that the function

$$\mathbb{D} \ni z \mapsto \log(\omega(z)/(1 - |z|^2)^\alpha)$$

is subharmonic in \mathbb{D} . Here α is a real parameter and \log denotes the usual logarithm. As a matter of terminology let us call such a weight function ω log-subharmonic of order α . For $\alpha = 0$ we recover in this way the usual notion of log-subharmonicity. We mention that weight functions of this type are natural from the geometric point of view in that log-subharmonicity of order α correspond to a natural curvature condition for the metric

$$ds(z)^2 = \omega(z) |dz|^2, \quad z \in \mathbb{D},$$

as has been emphasized by Hedenmalm, Shimorin and others [7], [8], [9], and [21].

For ω as above we denote by $A_\omega^2(\mathbb{D})$ the Bergman space of functions f analytic in \mathbb{D} with finite norm

$$\|f\|_\omega^2 = \int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) < +\infty,$$

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where $dA(z) = dx dy/\pi$, $z = x + iy$, is the usual planar Lebesgue area measure normalized by a factor $1/\pi$. It is known that the Bergman space $A_\omega^2(\mathbb{D})$ is a Hilbert space of analytic functions in \mathbb{D} in the usual sense that evaluations at points in \mathbb{D} are continuous linear functionals on $A_\omega^2(\mathbb{D})$ (see Section 2 of [5] for details). It is clear that $A_\omega^2(\mathbb{D})$ contains all analytic polynomials.

We denote by $S = S_\omega$ the shift operator, which is the operator on $A_\omega^2(\mathbb{D})$ defined by

$$(Sf)(z) = zf(z), \quad z \in \mathbb{D},$$

for $f \in A_\omega^2(\mathbb{D})$. It is evident that the operator S is a contraction on $A_\omega^2(\mathbb{D})$ which is pure in the sense that the intersection of the ranges of the positive powers of S is the zero subspace $\{0\}$.

In this paper we show that the shift operator S satisfies the operator inequality

$$(0.1) \quad \|f + (\alpha + 1)Sg\|_\omega^2 \leq (\alpha + 2) (\|Sf\|_\omega^2 + \|g\|_\omega^2 + \alpha\|Sg\|_\omega^2)$$

for all $f, g \in A_\omega^2(\mathbb{D})$ whenever the weight function ω is log-subharmonic of integer order $\alpha \geq 0$ in the sense explained above (see Theorem 3.3). An interesting feature of (0.1) is that such an inequality remains stable when passing to restrictions of S to shift invariant subspaces. For $\alpha = 0$, inequality (0.1) gives the celebrated inequality

$$(0.2) \quad \|f + Sg\|_\omega^2 \leq 2 (\|Sf\|_\omega^2 + \|g\|_\omega^2), \quad f, g \in A_\omega^2(\mathbb{D}),$$

of Hedenmalm, Jakobsson and Shimorin (see Proposition 6.4 in [5]), which is valid when the weight ω is log-subharmonic in \mathbb{D} . The case $\alpha = 1$ of (0.1) is due to Shimorin (Proposition 4.7 in [19]). For integers $\alpha \geq 2$ the result appears to be new. The proof of (0.1) is accomplished by applications of Green's formula and approximation of weight functions and builds on methods and ideas from [5].

We mention that inequality (0.2) contains a significant portion of Bergman space theory for the unit disc such as contractive divisor properties for Bergman inner functions, approximation theorems of wandering subspace type, and structure formulas for normalized reproducing kernel functions. For information on such applications we refer to the papers [2], [5], [12], [13] or [19]. We mention also that related operator identities have found application in the study of Toeplitz operators [11], [16] and the calculation of operator-valued Bergman inner functions [14], [15]. A principal motivation for these developments has been the ground breaking paper [3] of Aleman, Richter and Sundberg.

Let $\alpha > -1$ be a real parameter. Motivated by (0.1) we consider bounded Hilbert space operators $T \in \mathcal{L}(\mathcal{H})$ satisfying the inequality

$$(0.3) \quad \|x + (\alpha + 1)Ty\|^2 \leq (\alpha + 2) (\|Tx\|^2 + \|y\|^2 + \alpha\|Ty\|^2)$$

for all $x, y \in \mathcal{H}$. Building on work of Shimorin [19] we show that inequality (0.3) has a certain dual formulation (see Theorem 4.1) which leads to an operator inequality for the so-called Cauchy dual $T' = T(T^*T)^{-1}$ of an operator T satisfying (0.3) (see Theorem 4.3). Furthermore, we show that inequality (0.3) relaxes as the parameter α increases (see Theorem 4.2). In the parameter range $-1 < \alpha < 0$ this analysis

gives that the Cauchy dual T' of an operator T satisfying (0.3) satisfies an inequality slightly stronger than the concavity inequality of Richter [18]. In particular this yields that such an operator T is a contraction (see Proposition 4.5).

In order to increase our supply of examples of operators satisfying (0.3) we consider the shift operator $S = S_w$ acting on a weighted ℓ^2 -space of sequences square summable with respect to a positive weight sequence $w = \{w_k\}_{k \geq 0}$. We characterize in terms of the weight sequence $w = \{w_k\}_{k \geq 0}$ when the operator $T = S_w$ satisfies (0.3) (see Theorem 5.1). Furthermore, using this characterization of weight sequences, we show that, when the operator $T = S_w$ satisfies (0.3), the weighted shift operator S_w is a hyponormal contraction with spectrum equal to the closed unit disc \mathbb{D} (see Propositions 5.3 and 5.4).

To conclude the paper we consider weight functions of the form

$$\omega_{\theta,\beta}(z) = \frac{\Gamma(\theta + \beta + 2)}{\Gamma(\theta + 1)\Gamma(\beta + 1)} |z|^{2\theta} (1 - |z|^2)^\beta, \quad z \in \mathbb{D},$$

where $\theta, \beta > -1$ are real parameters and Γ is the gamma function. The normalization factor is inserted to ensure that $\int_{\mathbb{D}} \omega_{\theta,\beta} dA = 1$. As an application of our analysis of weight sequences we show that the shift operator $S_{\theta,\beta} = S_{\omega_{\theta,\beta}}$ on the space $A^2_{\theta,\beta}(\mathbb{D}) = A^2_{\omega_{\theta,\beta}}(\mathbb{D})$ satisfies the inequality (0.1) if and only if $\beta \leq \alpha$ and $\theta + 1 \geq (\beta + 1)/(\alpha + 1)$ (see Proposition 5.5). This extends the supply of weights for which inequality (0.1) can be efficiently checked and exemplifies the fact that inequality (0.3) singles out distinct classes of operators for distinct values of the parameter $\alpha > -1$.

1. Properties of weight functions

We shall consider in this section weight functions ω in \mathbb{D} such that the function $\omega(z)/(1 - |z|^2)^\alpha$ is log-subharmonic in \mathbb{D} . We call such weight functions log-subharmonic of order α . We shall first establish pullback invariance of such weights. We denote by $\Delta = \partial\bar{\partial}$ the normalized Laplacian, where

$$\partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \quad \text{and} \quad \bar{\partial}_z = \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right)$$

are the usual complex derivatives.

Lemma 1.1. *Let φ be an analytic function mapping the unit disc \mathbb{D} into itself. If the weight ω is log-subharmonic of real order $\alpha \geq 0$, then so is $\omega \circ \varphi$.*

Proof. By straightforward calculation we find that

$$(1.1) \quad \Delta_z \log \left(\frac{\omega(\varphi(z))}{(1 - |z|^2)^\alpha} \right) = \Delta_z \log \left(\frac{\omega(\varphi(z))}{(1 - |\varphi(z)|^2)^\alpha} \right) + \alpha \left(\frac{1}{(1 - |z|^2)^2} - \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} \right)$$

for $z \in \mathbb{D}$. The first term on the right-hand side in (1.1) is nonnegative by pullback invariance of subharmonic functions (see Theorem 2.7.4 in [17]). Since $\alpha \geq 0$, the second term on the right-hand side in (1.1) is nonnegative by the invariant form of the Schwarz lemma which says that $|\varphi'(z)|/(1 - |\varphi(z)|^2) \leq 1/(1 - |z|^2)$ for $z \in \mathbb{D}$ (see Section IX.2 of [4]). □

For a weight ω in \mathbb{D} and $0 < r < 1$ the dilated weight ω_r is the weight in \mathbb{D} defined by

$$(1.2) \quad \omega_r(z) = \omega(rz), \quad z \in \mathbb{D}.$$

In particular, it follows from Lemma 1.1 that if the weight ω is log-subharmonic of real order $\alpha \geq 0$, then so is ω_r .

Lemma 1.2. *Let ω be a weight function in \mathbb{D} of the form*

$$\omega(z) = (1 - |z|^2)^\alpha u(z), \quad z \in \mathbb{D},$$

where u is subharmonic in \mathbb{D} and $\alpha > -1$. Then

$$\lim_{r \rightarrow 1} \int_{\mathbb{D}} |(1 - |z|^2)^\alpha u(rz) - \omega(z)| dA(z) = 0.$$

Proof. It is well known that the integral means of a subharmonic function increase with the radius (see Theorem 2.6.8 in [17]). By this we have that

$$\int_{\mathbb{D}} (1 - |z|^2)^\alpha u(rz) dA(z) \leq \int_{\mathbb{D}} \omega(z) dA(z)$$

for $0 < r < 1$. The conclusion of the lemma now follows by a well-known approximation result from integration theory (see Lemma 3.17 in [6]). □

We now devise a method for the approximation of log-subharmonic real non-negative order weight functions.

Theorem 1.3. *Let ω be a weight function in \mathbb{D} which is log-subharmonic of real order $\alpha \geq 0$. Then for every $\varepsilon > 0$ there exists a weight function $\tilde{\omega}$ in \mathbb{D} of the form*

$$(1.3) \quad \tilde{\omega}(z) = (1 - |z|^2)^\alpha \omega_\alpha(z), \quad z \in \mathbb{D},$$

with ω_α log-subharmonic in \mathbb{D} and C^∞ -smooth in the closed disc $\bar{\mathbb{D}}$ such that $\int_{\mathbb{D}} |\omega - \tilde{\omega}| dA < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. Following Section 4 of [5], we can approximate ω in the $L^1(\mathbb{D}, dA)$ -norm with a weight function of the form

$$\omega(z) = \int_{\text{aut}(\mathbb{D})} \Phi(\varphi) \omega \circ \varphi(z) |\varphi'(z)|^2 d\varphi,$$

where $\text{aut}(\mathbb{D})$ is the automorphism group of the unit disc, $d\varphi$ is the Haar measure on $\text{aut}(\mathbb{D})$, and Φ is an appropriate nonnegative smooth function on $\text{aut}(\mathbb{D})$. The smoothness of w in \mathbb{D} is inherited from the smoothness of Φ . In particular, we can arrange that $\int_{\mathbb{D}} |\omega - w| dA < \varepsilon/2$ with such a weight w . By Lemma 1.1 and the well-known fact that log-subharmonic functions constitute a cone (see Corollary 1.6.8 in [10]), the function w constructed in this way is log-subharmonic of order $\alpha \geq 0$. Applying the regularization procedure from Lemma 1.2 we obtain a weight $\tilde{\omega}$ of the desired form (1.3) such that $\int_{\mathbb{D}} |\tilde{\omega} - w| dA < \varepsilon/2$. Now $\int_{\mathbb{D}} |\omega - \tilde{\omega}| dA < \varepsilon$. \square

The approximation result Theorem 1.3 is inspired by the developments in Hedenmalm, Jakobsson and Shimorin (see Section 4 of [5]) and is a convenient device to have available for the proof of inequality (0.1).

2. Calculation of Laplacians

In this section we have collected some calculations needed for the proof of inequality (0.1). In order to explain matters more carefully we proceed in more generality and discuss the case of real $\alpha > -1$, when this is possible.

Recall that the Laplacian of a radial function

$$u(z) = h(|z|^2), \quad z \in \mathbb{D} \setminus \{0\},$$

takes the form

$$\Delta u(z) = |z|^2 h''(|z|^2) + h'(|z|^2), \quad z \in \mathbb{D} \setminus \{0\},$$

where $\Delta = \partial\bar{\partial}$.

Lemma 2.1. *Let $\alpha \in \mathbb{R}$ be real and consider the function*

$$u(z) = (1 - |z|^2)^{\alpha+2}, \quad z \in \mathbb{D}.$$

Then

$$\Delta u(z) = (\alpha + 2)(1 - |z|^2)^\alpha ((\alpha + 2)|z|^2 - 1), \quad z \in \mathbb{D}.$$

Proof. Straightforward calculation. \square

We next introduce a family of functions $f_{\alpha,n}$ that will be useful in our investigations.

Lemma 2.2. *Let $\alpha > -1$ be real and $n \geq 1$ a positive integer. Consider the function*

$$f_{\alpha,n}(x) = x^n \int_0^1 \frac{t^{\alpha+1}}{(1 - t(1-x))^{n+1}} dt, \quad 0 < x < 1,$$

and set

$$u(z) = (1 - |z|^2)^{\alpha+2} \frac{f_{\alpha,n}(|z|^2)}{z^n}, \quad z \in \mathbb{D} \setminus \{0\}.$$

Then

$$\Delta u(z) = (\alpha + 1) \frac{1}{z^n} (1 - |z|^2)^\alpha, \quad z \in \mathbb{D} \setminus \{0\}.$$

Proof. We shall solve the differential equation

$$\Delta_z(v(|z|^2)/z^n) = (1 - |z|^2)^\alpha/z^n, \quad z \in \mathbb{D} \setminus \{0\},$$

with boundary conditions $v(1) = 0$ and $v'(1) = 0$. Differentiating we have

$$\bar{\partial}_z(v(|z|^2)/z^n) = v'(|z|^2)/z^{n-1}, \quad z \in \mathbb{D} \setminus \{0\},$$

and

$$\Delta_z(v(|z|^2)/z^n) = (|z|^2 v''(|z|^2) - (n - 1) v'(|z|^2))/z^n, \quad z \in \mathbb{D} \setminus \{0\}.$$

We are now led to consider the ODE

$$xv''(x) - (n - 1)v'(x) = (1 - x)^\alpha, \quad 0 < x < 1.$$

Solving for v' using $v'(1) = 0$ we have

$$v'(x) = -x^{n-1} \int_x^1 \frac{(1 - t)^\alpha}{t^n} dt, \quad 0 < x < 1.$$

Solving for v using $v(1) = 0$ we have

$$v(x) = \int_x^1 \left(s^{n-1} \int_s^1 \frac{(1 - t)^\alpha}{t^n} dt \right) ds, \quad 0 < x < 1.$$

Changing the order of integration gives

$$(2.1) \quad v(x) = \frac{1}{n} \int_x^1 \left(1 - \frac{x^n}{t^n} \right) (1 - t)^\alpha dt = \frac{x^n}{\alpha + 1} \int_x^1 \frac{(1 - t)^{\alpha+1}}{t^{n+1}} dt,$$

where the last equality follows by integration by parts. The change of variables $t = 1 - s(1 - x)$ gives the representation in the lemma with $u(z) = (\alpha + 1)v(|z|^2)/z^n$ for $z \in \mathbb{D} \setminus \{0\}$. □

We mention that the function $f_{\alpha,n}$ is naturally expressed using the hypergeometric function

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} dt, \quad z \in \mathbb{D},$$

where $\Re c > \Re b > 0$ (see Subsection 15.3.1 in [1]). In fact,

$$f_{\alpha,n}(x) = x^n F(n + 1, \alpha + 2; \alpha + 3; 1 - x)/(\alpha + 2)$$

for $0 < x < 1$.

Lemma 2.3. *Let $f_{\alpha,n}$ be as in Lemma 2.2. Then $\lim_{x \rightarrow 0} f_{\alpha,n}(x) = 1/n$ and $\lim_{x \rightarrow 1} f_{\alpha,n}(x) = 1/(\alpha + 2)$.*

Proof. By dominated convergence we have that

$$\lim_{x \rightarrow 1} f_{\alpha,n}(x) = \int_0^1 t^{\alpha+1} dt = \frac{1}{\alpha + 2}.$$

Recall next the formula (2.1) from the proof of Lemma 2.2, which gives the representation

$$(1 - x)^{\alpha+2} f_{\alpha,n}(x) = \frac{\alpha + 1}{n} \int_x^1 \left(1 - \frac{x^n}{t^n}\right) (1 - t)^\alpha dt, \quad 0 < x < 1.$$

By dominated convergence we have that

$$\lim_{x \rightarrow 0} f_{\alpha,n}(x) = \frac{\alpha + 1}{n} \int_0^1 (1 - t)^\alpha dt = \frac{1}{n}.$$

This completes the proof of the lemma. □

We next derive a first order differential equation for the function $f_{\alpha,n}$.

Lemma 2.4. *Let $f_{\alpha,n}$ be as in Lemma 2.2. Then*

$$f'_{\alpha,n}(x) = \left(\frac{n}{x} + \frac{\alpha + 2}{1 - x}\right) f_{\alpha,n}(x) - \frac{1}{x} - \frac{1}{1 - x}, \quad 0 < x < 1.$$

Proof. Recall Lemma 2.2. Differentiating under the integral we have that

$$(2.2) \quad f'_{\alpha,n}(x) = \frac{n}{x} f_{\alpha,n}(x) - (n + 1)x^n \int_0^1 \frac{t^{\alpha+2}}{(1 - t(1 - x))^{n+2}} dt.$$

An integration by parts now gives

$$f'_{\alpha,n}(x) = \frac{n}{x} f_{\alpha,n}(x) - \frac{1}{x(1 - x)} + (\alpha + 2) \frac{x^n}{1 - x} \int_0^1 \frac{t^{\alpha+1}}{(1 - t(1 - x))^{n+1}} dt,$$

which yields the conclusion of the lemma. □

We next estimate the function $f_{\alpha,n}$.

Lemma 2.5. *Let $f_{\alpha,n}$ be as in Lemma 2.2. Then*

$$\min(1/n, 1/(2 + \alpha)) \leq f_{\alpha,n}(x) \leq \max(1/n, 1/(2 + \alpha))$$

for $0 < x < 1$. In particular, if $n = \alpha + 2$ is an integer, then $f_{\alpha,n}(x) = 1/(\alpha + 2)$ for all $0 < x < 1$.

Proof. Recall Lemma 2.3. If $x \in (0, 1)$ is a critical point of $f_{\alpha,n}$, then by Lemma 2.4 we have that

$$f_{\alpha,n}(x) = \left(\frac{1}{x} + \frac{1}{1 - x}\right) / \left(\frac{n}{x} + \frac{\alpha + 2}{1 - x}\right) = \frac{1}{n(1 - x) + (\alpha + 2)x}.$$

Observe that

$$\min(1/n, 1/(2 + \alpha)) \leq \frac{1}{n(1 - x) + (\alpha + 2)x} \leq \max(1/n, 1/(2 + \alpha))$$

for $0 < x < 1$. This gives the bounds for $f_{\alpha,n}(x)$. The last assertion is an immediate consequence of these bounds. \square

For the sake of completeness we include also the following lemma, which we have used for the purpose of numerical computation.

Lemma 2.6. *Let $f_{\alpha,n}$ be as in Lemma 2.2. Then*

$$f'_{\alpha,n}(1) = \lim_{x \rightarrow 1} \frac{f_{\alpha,n}(x) - 1/(\alpha + 2)}{x - 1} = \frac{1}{\alpha + 3} \left(\frac{n}{\alpha + 2} - 1 \right).$$

Proof. Recall formula (2.2) from the proof of Lemma 2.4. Passing to the limit using monotone convergence and Lemma 2.3, we have that

$$\lim_{x \rightarrow 1} f'_{\alpha,n}(x) = \frac{n}{\alpha + 2} - \frac{n + 1}{\alpha + 3}.$$

This yields the conclusion of the lemma. \square

We need to calculate one more Laplacian.

Lemma 2.7. *Let $f_{\alpha,n}$ be as in Lemma 2.2 and consider the function*

$$u(z) = (1 - |z|^2)^{\alpha+2} \frac{f_{\alpha,n}(|z|^2)^2}{|z|^{2n}}, \quad z \in \mathbb{D} \setminus \{0\}.$$

Then

$$\Delta u(z) = \frac{(1 - |z|^2)^\alpha}{|z|^{2(n+1)}} g_{\alpha,n}(|z|^2), \quad z \in \mathbb{D} \setminus \{0\},$$

where

$$(2.3) \quad g_{\alpha,n}(x) = (l_{\alpha,n}(x)f_{\alpha,n}(x) - 1)^2 + (\alpha + 2)x(f_{\alpha,n}(x) - 1/(\alpha + 2))^2 + 1 - x/(\alpha + 2), \quad 0 < x < 1,$$

and $l_{\alpha,n}$ is the linear function

$$l_{\alpha,n}(x) = n(1 - x) + (\alpha + 2)x, \quad 0 < x < 1.$$

Proof. Let

$$h(x) = \frac{(1 - x)^{\alpha+2}}{x^n} f_{\alpha,n}(x)^2, \quad 0 < x < 1.$$

We need to show that

$$xh''(x) + h'(x) = \frac{(1 - x)^\alpha}{x^{n+1}} g_{\alpha,n}(x), \quad 0 < x < 1.$$

Observe first that

$$\frac{d}{dx} \left(\frac{(1-x)^{\alpha+2}}{x^n} \right) = -\frac{(1-x)^{\alpha+1}}{x^{n+1}} l_{\alpha,n}(x)$$

as follows by straightforward differentiation. Using this observation we now compute using Lemma 2.4 that

$$\begin{aligned} h'(x) &= -\frac{(1-x)^{\alpha+1}}{x^{n+1}} l_{\alpha,n}(x) f_{\alpha,n}(x)^2 \\ &\quad + \frac{(1-x)^{\alpha+2}}{x^n} 2f_{\alpha,n}(x) \left(\frac{l_{\alpha,n}(x)}{x(1-x)} f_{\alpha,n}(x) - \frac{1}{x(1-x)} \right) \\ &= \frac{(1-x)^{\alpha+1}}{x^{n+1}} (l_{\alpha,n}(x) f_{\alpha,n}(x)^2 - 2f_{\alpha,n}(x)). \end{aligned}$$

In particular,

$$(2.4) \quad xh'(x) = \frac{(1-x)^{\alpha+1}}{x^n} (l_{\alpha,n}(x) f_{\alpha,n}(x)^2 - 2f_{\alpha,n}(x))$$

for $0 < x < 1$.

We shall next differentiate (2.4). Observe first that

$$(2.5) \quad \frac{d}{dx} \left(\frac{(1-x)^{\alpha+1}}{x^n} \right) = -\frac{(1-x)^\alpha}{x^{n+1}} l_{\alpha,n}(x) + \frac{(1-x)^\alpha}{x^n},$$

which follows by straightforward differentiation. By Lemma 2.4 we have that

$$\begin{aligned} (2.6) \quad (l_{\alpha,n} f_{\alpha,n}^2 - 2f_{\alpha,n})'(x) &= (\alpha + 2 - n) f_{\alpha,n}(x)^2 + 2(l_{\alpha,n}(x) f_{\alpha,n}(x) - 1) f'_{\alpha,n}(x) \\ &= (\alpha + 2 - n) f_{\alpha,n}(x)^2 + \frac{2}{x(1-x)} (l_{\alpha,n}(x) f_{\alpha,n}(x) - 1)^2 \end{aligned}$$

for $0 < x < 1$. Now differentiating (2.4) using (2.5) and (2.6) we have that

$$\begin{aligned} &xh''(x) + h'(x) \\ &= \left(-\frac{(1-x)^\alpha}{x^{n+1}} l_{\alpha,n}(x) + \frac{(1-x)^\alpha}{x^n} \right) (l_{\alpha,n}(x) f_{\alpha,n}(x)^2 - 2f_{\alpha,n}(x)) \\ &\quad + \frac{(1-x)^{\alpha+1}}{x^n} \left((\alpha + 2 - n) f_{\alpha,n}(x)^2 + \frac{2}{x(1-x)} (l_{\alpha,n}(x) f_{\alpha,n}(x) - 1)^2 \right) \\ &= \frac{(1-x)^\alpha}{x^{n+1}} g_{\alpha,n}(x) \end{aligned}$$

for $0 < x < 1$, where

$$\begin{aligned} g_{\alpha,n}(x) &= (-l_{\alpha,n}(x) + x) (l_{\alpha,n}(x) f_{\alpha,n}(x)^2 - 2f_{\alpha,n}(x)) \\ &\quad + (\alpha + 2 - n) x(1-x) f_{\alpha,n}(x)^2 + 2(l_{\alpha,n}(x) f_{\alpha,n}(x) - 1)^2 \end{aligned}$$

for $0 < x < 1$.

Straightforward calculations now give that

$$\begin{aligned}
 g_{\alpha,n}(x) &= -(l_{\alpha,n}(x)^2 f_{\alpha,n}(x)^2 - 2l_{\alpha,n}(x)f_{\alpha,n}(x)) + xl_{\alpha,n}(x)f_{\alpha,n}(x)^2 - 2xf_{\alpha,n}(x) \\
 &\quad + (\alpha + 2 - n)x(1 - x)f_{\alpha,n}(x)^2 + 2(l_{\alpha,n}(x)f_{\alpha,n}(x) - 1)^2 \\
 &= (l_{\alpha,n}(x)f_{\alpha,n}(x) - 1)^2 + 1 \\
 &\quad + (xl_{\alpha,n}(x) + (\alpha + 2 - n)x(1 - x))f_{\alpha,n}(x)^2 - 2xf_{\alpha,n}(x) \\
 &= (l_{\alpha,n}(x)f_{\alpha,n}(x) - 1)^2 + (\alpha + 2)xf_{\alpha,n}(x)^2 - 2xf_{\alpha,n}(x) + 1 \\
 &= (l_{\alpha,n}(x)f_{\alpha,n}(x) - 1)^2 + (\alpha + 2)x(f_{\alpha,n}(x) - 1/(\alpha + 2))^2 + 1 - x/(\alpha + 2)
 \end{aligned}$$

for $0 < x < 1$. This completes the proof of the lemma. □

We next specialize to the case when $n = \alpha + 2$ is an integer.

Corollary 2.8. *Let $g_{\alpha,n}$ be as in Lemma 2.7 and assume that $n = \alpha + 2$ is an integer. Then*

$$g_{\alpha,n}(x) = 1 - x/(\alpha + 2), \quad 0 < x < 1.$$

Proof. By Lemma 2.5 the function $f_{\alpha,n}$ is identically $1/(\alpha + 2)$. The result now follows by Lemma 2.7. □

Numerical experiments suggest that the quadratic terms in (2.3) are negligible compared to the linear term $1 - x/(\alpha + 2)$ when n is close to $\alpha + 2$. For the purpose of numerical experiments we have used the computer software package Octave, which is freely available under the GNU license agreement.

3. The operator inequality for non-radial weights

In this section we shall prove the operator inequality (0.1) for a general weight function ω which is log-subharmonic of integer order $\alpha \geq 0$. For such a weight function ω we denote by $P_\omega^2(\mathbb{D})$ the closure in $A_\omega^2(\mathbb{D})$ of the space of analytic polynomials. It is apparent that $P_\omega^2(\mathbb{D})$ is invariant under the shift operator.

Theorem 3.1. *Let α be a nonnegative integer and let ω be a weight function in \mathbb{D} which is log-subharmonic of order α . Then the shift operator $S = S_\omega$ on the weighted Bergman space $P_\omega^2(\mathbb{D})$ satisfies the operator inequality*

$$(3.1) \quad \|f + Sf_1\|_\omega^2 \leq (\alpha + 2)\|Sf\|_\omega^2 + \frac{\alpha + 2}{(\alpha + 1)^2} \|f_1\|_\omega^2 + \frac{\alpha(\alpha + 2)}{(\alpha + 1)^2} \|Sf_1\|_\omega^2$$

for all $f, f_1 \in P_\omega^2(\mathbb{D})$.

Proof. We consider first a weight function ω of the form

$$\omega(z) = (1 - |z|^2)^\alpha \omega_\alpha(z), \quad z \in \mathbb{D},$$

where $\omega_\alpha \in C^\infty(\bar{\mathbb{D}})$ is log-subharmonic in \mathbb{D} and smooth in the closed disc $\bar{\mathbb{D}}$.

Let f and f_1 be polynomials and let $n \geq 1$ be a positive integer. By log-subharmonicity we have that

$$\Delta_z(|f(z) - \lambda z^{n+1} f_1(z)|^2 \omega_\alpha(z)) \geq 0, \quad z \in \mathbb{D},$$

for all $\lambda \in \mathbb{C}$. Expanding this inequality we get

$$\Delta_z(|f(z)|^2 \omega_\alpha(z)) - 2\Re(\lambda \Delta_z(z^{n+1} f_1(z) \overline{f(z)}) \omega_\alpha(z)) + |\lambda|^2 \Delta_z(|z^{n+1} f_1(z)|^2 \omega_\alpha(z)) \geq 0$$

for all $\lambda \in \mathbb{C}$, where the symbol \Re denotes the real part. Now set $\lambda = c f_{\alpha,n}(|z|^2)/z^n$, where $f_{\alpha,n}$ is as in Lemma 2.2 and $c \in \mathbb{R}$ is a real constant to be specified below. Integration gives

$$\begin{aligned} (3.2) \quad & \int_{\mathbb{D}} (1 - |z|^2)^{\alpha+2} \Delta_z(|f(z)|^2 \omega_\alpha(z)) \, dA(z) \\ & - 2c \Re \left(\int_{\mathbb{D}} (1 - |z|^2)^{\alpha+2} f_{\alpha,n}(|z|^2)/z^n \Delta_z(z^{n+1} f_1(z) \overline{f(z)}) \omega_\alpha(z) \, dA(z) \right) \\ & + c^2 \int_{\mathbb{D}} (1 - |z|^2)^{\alpha+2} f_{\alpha,n}(|z|^2)^2 / |z|^{2n} \Delta_z(|z^{n+1} f_1(z)|^2 \omega_\alpha(z)) \, dA(z) \geq 0. \end{aligned}$$

Notice that the integrands of the rightmost two integrals are both continuous at the origin as follows by cancellation and Lemma 2.3. We proceed to analyze the individual terms in (3.2).

We calculate the first integral in (3.2) using Green’s formula and Lemma 2.1 as

$$\begin{aligned} (3.3) \quad I_1 &= \int_{\mathbb{D}} (1 - |z|^2)^{\alpha+2} \Delta_z(|f(z)|^2 \omega_\alpha(z)) \, dA(z) \\ &= \int_{\mathbb{D}} \Delta_z((1 - |z|^2)^{\alpha+2}) |f(z)|^2 \omega_\alpha(z) \, dA(z) \\ &= (\alpha + 2) \int_{\mathbb{D}} (1 - |z|^2)^\alpha ((\alpha + 2)|z|^2 - 1) |f(z)|^2 \omega_\alpha(z) \, dA(z) \\ &= (\alpha + 2)((\alpha + 2) \|Sf\|_\omega^2 - \|f\|_\omega^2). \end{aligned}$$

Observe that the boundary terms vanish since the function $(1 - |z|^2)^{\alpha+2}$ vanishes on $\mathbb{T} = \partial\mathbb{D}$ to order $\alpha + 2 > 1$.

We next calculate the middle integral in (3.2) using Lemma 2.2 and Green’s formula. Removing a small disc around the origin we have that

$$\begin{aligned} I_2 &= \int_{\mathbb{D} \setminus \varepsilon\mathbb{D}} (1 - |z|^2)^{\alpha+2} f_{\alpha,n}(|z|^2)/z^n \Delta_z(z^{n+1} f_1(z) \overline{f(z)}) \omega_\alpha(z) \, dA(z) + o(1) \\ &= \int_{\mathbb{D} \setminus \varepsilon\mathbb{D}} \Delta_z((1 - |z|^2)^{\alpha+2} f_{\alpha,n}(|z|^2)/z^n) z^{n+1} f_1(z) \overline{f(z)} \omega_\alpha(z) \, dA(z) + o(1) \\ &= (\alpha + 1) \int_{\mathbb{D} \setminus \varepsilon\mathbb{D}} (1 - |z|^2)^\alpha / z^n \cdot z^{n+1} f_1(z) \overline{f(z)} \omega_\alpha(z) \, dA(z) + o(1) \\ &= (\alpha + 1) \int_{\mathbb{D}} z f_1(z) \overline{f(z)} \omega_\alpha(z) \, dA(z) = (\alpha + 1) \langle Sf_1, f \rangle_\omega \end{aligned}$$

as $\varepsilon \rightarrow 0$.

By polarization we have that

$$(3.4) \quad 2\Re(I_2) = (\alpha + 1)(\|f + Sf_1\|_\omega^2 - \|f\|_\omega^2 - \|Sf_1\|_\omega^2).$$

To analyze the third integral in (3.2) we again apply Green’s formula and calculate using Lemma 2.7 that

$$\begin{aligned} I_3 &= \int_{\mathbb{D} \setminus \varepsilon\mathbb{D}} (1 - |z|^2)^{\alpha+2} f_{\alpha,n}(|z|^2)^2 / |z|^{2n} \Delta_z(|z^{n+1} f_1(z)|^2 \omega_\alpha(z)) \, dA(z) + o(1) \\ &= \int_{\mathbb{D} \setminus \varepsilon\mathbb{D}} \Delta_z((1 - |z|^2)^{\alpha+2} f_{\alpha,n}(|z|^2)^2 / |z|^{2n}) |z^{n+1} f_1(z)|^2 \omega_\alpha(z) \, dA(z) + o(1) \\ &= \int_{\mathbb{D}} (1 - |z|)^\alpha / |z|^{2n+2} \cdot g_{\alpha,n}(|z|^2) |z^{n+1} f_1(z)|^2 \omega_\alpha(z) \, dA(z) \\ &= \int_{\mathbb{D}} g_{\alpha,n}(|z|^2) |f_1(z)|^2 \omega(z) \, dA(z) \end{aligned}$$

as $\varepsilon \rightarrow 0$, where $g_{\alpha,n}$ is as in Lemma 2.7. For $n = \alpha + 2$ we have that

$$(3.5) \quad I_3 = \int_{\mathbb{D}} (1 - |z|^2 / (\alpha + 2)) |f_1(z)|^2 \omega(z) \, dA(z) = \|f_1\|_\omega^2 - \|Sf_1\|_\omega^2 / (\alpha + 2)$$

by Corollary 2.8.

By (3.2) and (3.3)–(3.5) we have

$$\begin{aligned} (\alpha + 2)((\alpha + 2)\|Sf\|_\omega^2 - \|f\|_\omega^2) - c(\alpha + 1)(\|f + Sf_1\|_\omega^2 - \|f\|_\omega^2 - \|Sf_1\|_\omega^2) \\ + c^2(\|f_1\|_\omega^2 - \|Sf_1\|_\omega^2 / (\alpha + 2)) \geq 0, \end{aligned}$$

which by a rearrangement of terms leads to the inequality

$$(3.6) \quad c(\alpha + 1)\|f + Sf_1\|_\omega^2 \leq (\alpha + 2)^2\|Sf\|_\omega^2 + (c(\alpha + 1) - (\alpha + 2))\|f\|_\omega^2 \\ + c^2\|f_1\|_\omega^2 + c((\alpha + 1) - c/(\alpha + 2))\|Sf_1\|_\omega^2$$

We now set $c = (\alpha + 2)/(\alpha + 1)$ to arrive at (3.1).

The passage to a general weight function ω in inequality (3.1) follows by approximation using Theorem 1.3. By another approximation the inequality (3.1) remains true for $f, f_1 \in P_\omega^2(\mathbb{D})$. \square

Remark 3.2. The exact value of the constant $c \neq 0$ in (3.6) is irrelevant with respect to the strength of the resulting inequality of the form (3.1). Indeed, in expanded form (3.6) reads as

$$\begin{aligned} 2c(\alpha + 1)\Re(f, Sf_1)_\omega \\ \leq (\alpha + 2)^2\|Sf\|_\omega^2 - (\alpha + 2)\|f\|_\omega^2 + c^2(\|f_1\|_\omega^2 - \|Sf_1\|_\omega^2 / (\alpha + 2)), \end{aligned}$$

which makes evident that the constant c can be incorporated in the function f_1 . Another natural choice is $c = (\alpha + 2)(\alpha + 1)$ in (3.6) which leads to the equivalent form

$$\|f + Sf_1\|^2 \leq \frac{\alpha + 2}{(\alpha + 1)^2}\|Sf\|^2 + \frac{\alpha(\alpha + 2)}{(\alpha + 1)^2}\|f\|^2 + (\alpha + 2)\|f_1\|^2$$

of (3.1).

We next extend the validity of inequality (3.1) to the full Bergman space $A^2_\omega(\mathbb{D})$.

Theorem 3.3. *Let ω be a weight function in \mathbb{D} which is log-subharmonic of integer order $\alpha \geq 0$, and denote by $S = S_\omega$ the shift operator on $A^2_\omega(\mathbb{D})$. Then*

$$\|f + (\alpha + 1)Sg\|_\omega^2 \leq (\alpha + 2)(\|Sf\|_\omega^2 + \|g\|_\omega^2 + \alpha\|Sg\|_\omega^2) \quad \text{for all } f, g \in A^2_\omega(\mathbb{D}).$$

Proof. Let $0 < r < 1$ and fix $f, g \in A^2_\omega(\mathbb{D})$. By Lemma 1.1 the dilated weight ω_r defined by (1.2) is log-subharmonic of integer order $\alpha \geq 0$. Applying Theorem 3.1 with the weight ω_r in place of ω , the dilated function f_r in place of f , and $f_1 = (\alpha + 1)g_r$ we have that

$$\begin{aligned} & \int_{\mathbb{D}} |f(rz) + (\alpha + 1)zg(rz)|^2 \omega(rz) dA(z) \\ & \leq (\alpha + 2) \int_{\mathbb{D}} |zf(rz)|^2 \omega(rz) dA(z) + (\alpha + 2) \int_{\mathbb{D}} |g(rz)|^2 \omega(rz) dA(z) \\ & \quad + \alpha(\alpha + 2) \int_{\mathbb{D}} |zg(rz)|^2 \omega(rz) dA(z). \end{aligned}$$

By a change of variables we obtain

$$\begin{aligned} & \int_{r\mathbb{D}} |f(z) + (\alpha + 1)(z/r)g(z)|^2 \omega(z) dA(z) \\ & \leq (\alpha + 2) \int_{r\mathbb{D}} |(z/r)f(z)|^2 \omega(z) dA(z) + (\alpha + 2) \int_{r\mathbb{D}} |g(z)|^2 \omega(z) dA(z) \\ & \quad + \alpha(\alpha + 2) \int_{r\mathbb{D}} |(z/r)g(z)|^2 \omega(z) dA(z) \end{aligned}$$

after cancellation of a factor $1/r^2$. A passage to the limit as $r \rightarrow 1$ now yields the result. □

4. Dual considerations

Let $\alpha > -1$ be a real parameter. We shall discuss in this section some general observations concerning a bounded Hilbert space operator $T \in \mathcal{L}(\mathcal{H})$ such that

$$(4.1) \quad \|x + (\alpha + 1)Ty\|^2 \leq (\alpha + 2)(\|Tx\|^2 + \|y\|^2 + \alpha\|Ty\|^2)$$

for all $x, y \in \mathcal{H}$.

We first provide a dual reformulation of (4.1).

Theorem 4.1. *Let $\alpha > -1$, and let $T \in \mathcal{L}(\mathcal{H})$ be a bounded Hilbert space operator. Then (4.1) holds if and only if the operator T is left-invertible, the operator $I + \alpha T^*T$ is positive and invertible and*

$$(4.2) \quad (T^*T)^{-1} + (\alpha + 1)^2 T(I + \alpha T^*T)^{-1} T^* \leq (\alpha + 2)I$$

in $\mathcal{L}(\mathcal{H})$, where I is the identity operator.

Proof. Assume first that (4.1) holds. Setting $y = 0$ in (4.1) we have $\|x\|^2 \leq (\alpha + 2)\|Tx\|^2$ for $x \in \mathcal{H}$, which shows that T is left-invertible. Setting $x = 0$ in (4.1) we see that

$$(\alpha + 1)^2 \|Ty\|^2 \leq (\alpha + 2) \langle (I + \alpha T^*T)y, y \rangle$$

for $y \in \mathcal{H}$. Since T is left-invertible this gives that $I + \alpha T^*T$ is positive and invertible by standard spectral theory.

We now proceed to prove (4.2). Substituting $x = (T^*T)^{-1/2}x_1$ and $y = (I + \alpha T^*T)^{-1/2}y_1$ in (4.1) we obtain

$$\| (T^*T)^{-1/2}x_1 + (\alpha + 1)T(I + \alpha T^*T)^{-1/2}y_1 \|^2 \leq (\alpha + 2) (\|x_1\|^2 + \|y_1\|^2)$$

for all $x_1, y_1 \in \mathcal{H}$, where the superscript $1/2$ indicates a positive square root. This last inequality equivalently means that the operator

$$X(x, y) = (T^*T)^{-1/2}x + (\alpha + 1)T(I + \alpha T^*T)^{-1/2}y, \quad x, y \in \mathcal{H},$$

is bounded from $\mathcal{H} \oplus \mathcal{H}$ into \mathcal{H} with norm bound $\|X\|^2 \leq \alpha + 2$. A straightforward calculation shows that the adjoint operator X^* from \mathcal{H} into $\mathcal{H} \oplus \mathcal{H}$ acts as $X^*x = (y_1, y_2)$, where $y_1 = (T^*T)^{-1/2}x$ and $y_2 = (\alpha + 1)(I + \alpha T^*T)^{-1/2}T^*x$. It is now clear that

$$XX^* = (T^*T)^{-1} + (\alpha + 1)^2 T(I + \alpha T^*T)^{-1}T^* \leq (\alpha + 2)I$$

in $\mathcal{L}(\mathcal{H})$, which gives (4.2).

It is straightforward to check that the above argument is reversible to the extent that (4.2) implies (4.1). □

We next show that the operator inequality (4.1) relaxes as α increases.

Theorem 4.2. *Assume that $T \in \mathcal{L}(\mathcal{H})$ satisfies (4.1) for some $\alpha = \alpha_0 > -1$. Then T satisfies (4.1) for all $\alpha \geq \alpha_0$.*

Proof. Recall Theorem 4.1. Consider the operator-valued function

$$f(\alpha) = \alpha I - (\alpha + 1)^2 T(I + \alpha T^*T)^{-1}T^*, \quad \alpha \geq \alpha_0.$$

Observe that f is well-defined since the operator $I + \alpha T^*T$ is positive and invertible for $\alpha \geq \alpha_0$. To prove the theorem it suffices to show that f is increasing on the half-axis $[\alpha_0, \infty)$. Differentiating we have that

$$f'(\alpha) = I - 2(\alpha + 1)T(I + \alpha T^*T)^{-1}T^* + (\alpha + 1)^2 T(I + \alpha T^*T)^{-2}T^*TT^*$$

for $\alpha \geq \alpha_0$. Another differentiation gives that

$$\begin{aligned} f''(\alpha) &= -2T(I + \alpha T^*T)^{-1}T^* + 2(\alpha + 1)T(I + \alpha T^*T)^{-2}T^*TT^* \\ &\quad + 2(\alpha + 1)T(I + \alpha T^*T)^{-2}T^*TT^* - 2(\alpha + 1)^2 T(I + \alpha T^*T)^{-3}(T^*T)^2T^* \\ &= -2T(I + \alpha T^*T)^{-1}T^* + 4(\alpha + 1)T(I + \alpha T^*T)^{-2}T^*TT^* \\ &\quad - 2(\alpha + 1)^2 T(I + \alpha T^*T)^{-3}(T^*T)^2T^* \end{aligned}$$

for $\alpha \geq \alpha_0$. A further calculation gives that

$$\begin{aligned} f''(\alpha) &= -2T(I + \alpha T^*T)^{-1}(I - 2(\alpha + 1)(I + \alpha T^*T)^{-1}T^*T \\ &\quad + (\alpha + 1)^2(I + \alpha T^*T)^{-2}(T^*T)^2)T^* \\ &= -2T(I + \alpha T^*T)^{-1}(I - (\alpha + 1)(I + \alpha T^*T)^{-1}T^*T)^2T^* \end{aligned}$$

for $\alpha \geq \alpha_0$. By this last formula it is evident that $f''(\alpha) \leq 0$ for $\alpha \geq \alpha_0$. In particular, the function f' is decreasing on (α_0, ∞) . A limit calculation shows that

$$\lim_{\alpha \rightarrow \infty} f'(\alpha) = I - 2T(T^*T)^{-1}T^* + T(T^*T)^{-2}T^*T^*T^* = I - T(T^*T)^{-1}T^*.$$

It is straightforward to check that the operator $P = I - T(T^*T)^{-1}T^*$ is the orthogonal projection onto the wandering subspace $\mathcal{H} \ominus T(\mathcal{H})$ for T (see Section 2 of [19]). Indeed, the operator $T(T^*T)^{-1}T^*$ is self-adjoint, idempotent and has range equal to $T(\mathcal{H})$, which proves the claim. As a consequence we have that $f'(\alpha) \geq P \geq 0$ for $\alpha \geq \alpha_0$. This shows that the function f is increasing. \square

Let $T \in \mathcal{L}(\mathcal{H})$ be a left-invertible operator. The so-called Cauchy dual T' of T is the operator defined by $T' = T(T^*T)^{-1}$.

Theorem 4.3. *Let $\alpha > -1$ and let $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying (4.1). Then the Cauchy dual T' of T satisfies the operator inequality*

$$(4.3) \quad (T'^*)^2 T'^2 - 2T'^* T' + I \leq \alpha(T'^* T' + \alpha I)^{-1}(T'^* T' - I)^2 \quad \text{in } \mathcal{L}(\mathcal{H}).$$

Proof. Recall Theorem 4.1. Observe that $T'^*T' = (T^*T)^{-1}$ and also that

$$(I + \alpha T^* T)^{-1} = T'^* T' (T'^* T' + \alpha I)^{-1}.$$

By (4.2) we have that

$$T'^* T' + (\alpha + 1)^2 T T'^* T' (T'^* T' + \alpha I)^{-1} T^* \leq (\alpha + 2)I$$

in $\mathcal{L}(\mathcal{H})$. We now multiply this last inequality from the left by T'^* and from the right by T' to obtain

$$(4.4) \quad (T'^*)^2 T'^2 + (\alpha + 1)^2 (T'^* T' + \alpha I)^{-1} T'^* T' \leq (\alpha + 2) T'^* T'$$

since $T^*T' = I$. The calculation

$$I + \alpha T'^* T' - (\alpha + 1)^2 (T'^* T' + \alpha I)^{-1} T'^* T' = \alpha (T'^* T' + \alpha I)^{-1} (T'^* T' - I)^2$$

shows that (4.4) is equivalent to (4.3). This completes the proof of the theorem. \square

Remark 4.4. It is easy to see that condition (4.3) relaxes as α increases. Letting $\alpha \rightarrow \infty$, we obtain in the limit the operator inequality $T'^{*2} T'^2 \leq (T'^* T')^2$ for T' .

Observe that for $-1 < \alpha < 0$, the inequality (4.3) is a stronger form of the concavity inequality

$$T'^{*2} T'^2 + I \leq 2 T'^* T'^*$$

for T' .

The interest in concave operators originates from an approximation result by Richter [18], later elaborated on in the papers [13] and [19].

Proposition 4.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying (4.1) for some α in the range $-1 < \alpha \leq 0$. Then T is a contraction.*

Proof. By Theorem 4.3 we have that T' is concave. It is well known that every concave operator is expansive (see for instance Proposition 2.1 in [13]). Expansiveness of T' is equivalent to T being a contraction (see Proposition 1.1 in [13]). \square

It should be mentioned here that for the parameter values $\alpha = 0, 1$ the results of Theorems 4.1 and 4.3 above are from Shimorin (see the proof of Theorem 3.6 in [19] and Lemma 4.8 in [19]).

5. Weighted shifts

In this section we shall study inequality (0.3) in the context of weighted shift operators. Let $w = \{w_k\}_{k \geq 0}$ be a positive weight sequence and denote by $\ell^2(w)$ the standard Hilbert space of complex sequences $a = \{a_k\}_{k \geq 0}$ with finite norm

$$(5.1) \quad \|a\|_w^2 = \sum_{k \geq 0} |a_k|^2 w_k.$$

The shift operator $S = S_w$ on the space $\ell^2(w)$ is defined by $Sa = \{b_k\}_{k \geq 0}$, where $b_0 = 0$, $b_k = a_{k-1}$ for $k \geq 1$ and $a = \{a_k\}_{k \geq 0} \in \ell^2(w)$. Notice that the operator S is bounded on $\ell^2(w)$ if and only if the quotients $q_k = w_k/w_{k-1}$ are bounded and that in this case the operator norm of S_w is given by $\|S_w\|^2 = \sup_{k \geq 1} w_k/w_{k-1}$. We proceed to investigate when the operator $S = S_w$ satisfies the inequality

$$(5.2) \quad \|a + (\alpha + 1)Sb\|_w^2 \leq (\alpha + 2)(\|Sa\|_w^2 + \|b\|_w^2 + \alpha\|Sb\|_w^2) \quad \text{for } a, b \in \ell^2(w).$$

Theorem 5.1. *Let $\alpha > -1$ and let $w = \{w_k\}_{k \geq 0}$ be a positive weight sequence. Then the shift operator $S = S_w$ satisfies (5.2) if and only if the inequalities*

$$(5.3) \quad w_{k-1}/(\alpha + 2) \leq w_k \leq (\alpha + 2)w_{k-1}$$

$$(5.4) \quad \text{and} \quad \frac{1}{w_{k-1}} + \frac{1}{w_{k+1}} - \frac{2}{w_k} \leq \alpha \left(\frac{1}{w_k} - \frac{w_k}{w_{k-1}w_{k+1}} \right)$$

hold for all $k \geq 1$.

Proof. We shall use the well-known fact that a quadratic form

$$Q(x, y) = Ax^2 - 2Bxy + Cy^2, \quad x, y \in \mathbb{R},$$

is positive semidefinite if and only if $A \geq 0$, $C \geq 0$ and $B^2 \leq AC$, which follows by completion of squares. Evaluating norms we see that (5.2) is equivalent to

$$(5.5) \quad |a_0|^2 w_0 + \sum_{k \geq 1} \{ |a_k|^2 w_k + (\alpha + 1)^2 |b_{k-1}|^2 w_k + 2(\alpha + 1) \Re(a_k \bar{b}_{k-1}) w_k \} \\ \leq (\alpha + 2) |a_0|^2 w_1 + (\alpha + 2) \sum_{k \geq 1} \{ |a_k|^2 w_{k+1} + |b_{k-1}|^2 (w_{k-1} + \alpha w_k) \}$$

whenever $\{a_k\}_{k \geq 0}, \{b_k\}_{k \geq 0} \in \ell^2(w)$. Varying the a_k 's and b_k 's we see that (5.5) holds if and only if $w_0 \leq (\alpha + 2)w_1$ and the quadratic form

$$Q_k(x, y) = ((\alpha + 2)w_{k+1} - w_k)x^2 - 2(\alpha + 1)w_kxy + ((\alpha + 2)w_{k-1} - w_k)y^2,$$

$x, y \in \mathbb{R}$, is positive semidefinite for every $k \geq 1$. By the above criterion for positive semidefiniteness we have that Q_k is positive semidefinite if and only if

$$(5.6) \quad (\alpha + 2)w_{k+1} - w_k \geq 0, \quad (\alpha + 2)w_{k-1} - w_k \geq 0$$

and

$$(\alpha + 1)^2 w_k^2 \leq ((\alpha + 2)w_{k+1} - w_k)((\alpha + 2)w_{k-1} - w_k).$$

A straightforward calculation shows that this last inequality is equivalent to (5.4). The remaining conditions (5.6) for $k \geq 1$ and $w_0 \leq (\alpha + 2)w_1$ are equivalent to (5.3) for $k \geq 1$. This completes the proof of the theorem. \square

Remark 5.2. We observe that for $\alpha \geq 0$ inequality (5.4) implies that $w_{k-1}/(\alpha + 2) \leq w_k$ for $k \geq 2$ and $w_k \leq (\alpha + 2)w_{k-1}$ for $k \geq 1$. Indeed, writing (5.4) in the form

$$\frac{1}{w_{k-1}} + \frac{1}{w_{k+1}} \left(1 + \alpha \frac{w_k}{w_{k-1}}\right) \leq (\alpha + 2) \frac{1}{w_k}$$

an estimation gives $1/w_{k+1} + 1/w_{k-1} \leq (\alpha + 2)/w_k$, which yields the claim.

When the lower limit $R^2 = \liminf_{k \rightarrow \infty} w_k^{1/k}$ is positive the space $\ell^2(w)$ is naturally identified by means of Fourier transformation with the Hilbert space $A_w^2 = \mathcal{F}\ell^2(w)$ of functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D}_R,$$

analytic in the disc $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ with finite norm $\|f\|_w^2 = \sum_{k \geq 0} |a_k|^2 w_k$ given by (5.1). It is straightforward to check that the kernel function K_w for A_w^2 has the form $K_w(z, \zeta) = k_w(\bar{\zeta}z)$, where $k_w(z) = \sum_{k \geq 0} z^k/w_k$ for $z \in \mathbb{D}_R$.

We proceed to investigate weighted shifts satisfying (5.2).

Proposition 5.3. *Let $S = S_w$ be a weighted shift operator satisfying (5.2) for some $\alpha > -1$. Then the weight sequence $w = \{w_k\}_{k \geq 0}$ is log-convex: $w_k^2 \leq w_{k-1}w_{k+1}$ for $k \geq 1$. As a consequence, the operator S_w is hyponormal.*

Proof. By Theorem 4.2 the operator inequality (5.2) relaxes as α increases. Letting $\alpha \rightarrow \infty$ in (5.4) we see that

$$\frac{1}{w_k} - \frac{w_k}{w_{k-1}w_{k+1}} \geq 0$$

for $k \geq 1$, which yields the log-convexity of the weight sequence. The condition $SS^* \leq S^*S$ of hyponormality means precisely that for a weighted shift $S = S_w$ the weight sequence $\{w_k\}_{k \geq 0}$ is log-convex. \square

We next calculate the limit $\lim_{k \rightarrow \infty} w_k/w_{k-1}$.

Proposition 5.4. *Let $S = S_w$ be a weighted shift operator satisfying (5.2) for some $\alpha > -1$. Then $\lim_{k \rightarrow \infty} w_k/w_{k-1} = 1$. As a consequence, the operator S_w is a contraction with spectrum equal to the closed unit disc \mathbb{D} .*

Proof. Consider the quotients $q_k = w_k/w_{k-1}$ for $k \geq 1$. By Proposition 5.3 the sequence $\{q_k\}$ is increasing: $q_k \leq q_{k+1}$ for $k \geq 1$. By (5.3) we have the upper bound $q_k \leq \alpha + 2$ for $k \geq 1$. It is now clear that the limit $q = \lim_{k \rightarrow \infty} q_k$ exists as a positive real number. By (5.4) we have that

$$q_k + 1/q_{k+1} - 2 \leq \alpha(1 - q_k/q_{k+1})$$

for $k \geq 1$. Letting $k \rightarrow \infty$ in this last inequality we obtain $q + 1/q - 2 \leq 0$, which yields that $q = 1$. Thus the quotients $q_k = w_k/w_{k-1}$ increase to 1 as $k \rightarrow \infty$.

By the result of the previous paragraph we have that the weight sequence decreases: $w_k \geq w_{k+1}$ for $k \geq 0$. This shows that S_w is a contraction.

We consider next the spectrum $\sigma(S_w)$ of S_w . Using that $\lim_{k \rightarrow \infty} w_k/w_{k-1} = 1$ it is straightforward to check that A_w^2 is a Hilbert space of analytic functions in the unit disc \mathbb{D} . The standard property

$$S_w^* K_w(\cdot, \zeta) = \bar{\zeta} K_w(\cdot, \zeta), \quad \zeta \in \mathbb{D},$$

of the kernel function makes evident that every point in \mathbb{D} is an eigenvalue for S_w^* showing that $\mathbb{D} \subset \sigma(S_w^*)$. Since S_w^* is a contraction we conclude that $\sigma(S_w^*) = \bar{\mathbb{D}}$, which by passage to adjoints gives that $\sigma(S_w) = \mathbb{D}$. □

We shall next evaluate Theorem 5.1 on a weight sequence of the form

$$w_{\theta, \beta; k} = \frac{\Gamma(\theta + \beta + 2)}{\Gamma(\theta + 1)} \frac{\Gamma(k + \theta + 1)}{\Gamma(k + \theta + \beta + 2)}, \quad k \geq 0,$$

where $\theta, \beta > -1$ are real parameters and Γ is the gamma function. The weight sequence $\{w_{\theta, \beta; k}\}_{k \geq 0}$ is the sequence of moments

$$w_{\theta, \beta; k} = \int_{\mathbb{D}} |z|^{2k} \omega_{\theta, \beta}(z) dA(z), \quad k \geq 0,$$

for the measure $\omega_{\theta, \beta} dA$, where

$$\omega_{\theta, \beta}(z) = \frac{\Gamma(\theta + \beta + 2)}{\Gamma(\theta + 1)\Gamma(\beta + 1)} |z|^{2\theta} (1 - |z|^2)^\beta, \quad z \in \mathbb{D}.$$

Indeed, this latter fact is straightforward to check using the beta integral

$$\int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y > 0$ (see Chapter 6 of [1]). For notation simplicity we write $A_{\theta, \beta}^2(\mathbb{D}) = A_{\omega_{\theta, \beta}}^2(\mathbb{D})$ and $S_{\theta, \beta} = S_{\omega_{\theta, \beta}}$. Observe that the space $A_{\theta, \beta}^2(\mathbb{D})$ is the standard weighted Bergman space with weight parameter $\beta > -1$.

Proposition 5.5. *Let $\theta, \alpha, \beta > -1$. Then the shift operator $S = S_{\theta, \beta}$ on $A_{\theta, \beta}^2(\mathbb{D})$ satisfies inequality (5.2) if and only if $\beta \leq \alpha$ and $\theta + 1 \geq (\beta + 1)/(\alpha + 1)$.*

Proof. Recall Theorem 5.1. We first investigate condition (5.3). A straightforward calculation gives that

$$q_k = \frac{w_{\theta,\beta;k}}{w_{\theta,\beta;k-1}} = \frac{k + \theta}{k + \theta + \beta + 1} = 1 - \frac{\beta + 1}{k + \theta + \beta + 1}$$

for $k \geq 1$. In particular, the sequence $\{q_k\}$ is increasing and $q_k \leq 1$ for $k \geq 1$. Also, $q_k \geq q_1 = (\theta + 1)/(\theta + \beta + 2)$ for $k \geq 1$. It is straightforward to check that the inequality $(\theta + 1)/(\theta + \beta + 2) \geq 1/(\alpha + 2)$ is equivalent to $\theta + 1 \geq (\beta + 1)/(\alpha + 1)$.

We next analyze condition (5.4). Observe that (5.4) can be equivalently formulated as saying that

$$q_k + 1/q_{k+1} - 2 \leq \alpha(1 - q_k/q_{k+1})$$

for $k \geq 1$ using the quotients $\{q_k\}$. By straightforward calculation we have

$$q_k + 1/q_{k+1} - 2 = \frac{\beta(\beta + 1)}{(k + \theta + 1)(k + \theta + \beta + 1)}$$

and similarly that

$$1 - q_k/q_{k+1} = \frac{\beta + 1}{(k + \theta + 1)(k + \theta + \beta + 1)}$$

for $k \geq 1$. By these calculations we see that (5.4) holds if and only if $\beta \leq \alpha$. \square

We mention that the kernel function $K_{\theta,\beta}$ for $A_{\theta,\beta}^2(\mathbb{D})$ is naturally expressed in terms of the hypergeometric function. In fact,

$$K_{\theta,\beta}(z, \zeta) = F(\theta + \beta + 2, 1; \theta + 1; \bar{\zeta}z), \quad (z, \zeta) \in \mathbb{D}^2,$$

where

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k \geq 0} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}, \quad z \in \mathbb{D},$$

is the hypergeometric function (see Chapter 15 of [1]). In this context we wish to mention also the interesting paper of Shimorin [20] providing an integral representation for kernel functions in the radially weighted log-subharmonic case.

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