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Extremal metrics and lower bound of the modified K-energy

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Abstract. We provide a new proof of a result of X. X. Chen and G. Tian [5]: for a polarized extremal Kähler manifold, the minimum of the modified K-energy is attained at an extremal metric. The proof uses an idea of C. Li [16] adapted to the extremal metrics using some weighted balanced metrics.

Keywords. Kähler manifolds, extremal metrics, K-energy

1. Introduction

Extremal metrics were introduced by Calabi [1]. Let (X, ω) be a Kähler manifold of complex dimension n. An *extremal metric* is a critical point of the functional

$$g \mapsto \int_X (S(g))^2 \frac{\omega_g^n}{n!}$$

defined on Kähler metrics g representing the Kähler class $[\omega]$, where $S(g)$ is the scalar curvature of the metric g. Constant scalar curvature Kähler metrics (CSCK for short), and in particular Kähler–Einstein metrics, are extremal metrics. In this work we will focus on the polarized case, assuming that there is an ample holomorphic line bundle $L \to X$ with $c_1(L) = [\omega]$. In this special case, it has been conjectured by Yau in the Kähler–Einstein case [29], and then in the CSCK case by the work of Tian [27] and Donaldson [9], that the existence of a CSCK metric representing $c_1(L)$ should be equivalent to a GIT stability of the pair (X, L). This conjecture has been extended to extremal metrics by Székelyhidi [25] and Mabuchi [20].

Let (X, L) be a polarized Kähler manifold. Donaldson [8] has shown that if X admits a CSCK metric in $c_1(L)$, and if $\mathrm{Aut}(X, L)$ is discrete, then the CSCK metric can be approximated by a sequence of balanced metrics. This approximation result implies in particular the unicity of a CSCK metric in its Kähler class. This method has been adapted

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by Mabuchi [19] to the extremal metric setting to prove unicity of an extremal metric up to automorphisms in a polarized Kähler class. Then, Chen and Tian [5] proved unicity of an extremal metric in its Kähler class up to automorphisms with no polarization assumption.

In a sequel to his work on balanced metrics [10], Donaldson shows that if $\mathrm{Aut}(X, L)$ is discrete, a CSCK metric is an absolute minimum of the Mabuchi energy E, or K-energy, introduced by Mabuchi [18]. The approximation result of Donaldson does not hold true for CSCK metrics if the automorphism group is not discrete. There are counter-examples of Ono, Yotsutani and the first author [21], or Della Vedova and Zuddas [6]. However, Li [16] managed to show that even if $\mathrm{Aut}(X, L)$ is not discrete, a CSCK metric would provide an absolute minimum of E.

By a theorem of Calabi [2], extremal metrics are invariant under a maximal connected compact subgroup G of the reduced automorphism group $\mathrm{Aut}_0(X)$ [11]. Any two such compact groups are conjugate in $\mathrm{Aut}_0(X)$ and the study of extremal metrics is done modulo one such group. In the extremal setting, the modified K-energy E^G (see Definition 2.5) plays the role of the K-energy for CSCK metrics. This functional has been introduced independently by Guan [14], Simanca [24] and Chen and Tian [5] and is defined on the space of G-invariant Kähler potentials with respect to a G-invariant metric. In [5], Chen and Tian prove that extremal metrics minimize the modified K-energy up to automorphisms of the manifold, with no polarization assumption. In this paper, we give a different proof of this result in the polarized case. We generalize Li's work to extremal metrics, using some weighted balanced metrics, which are called σ-balanced metrics (see Definition 2.6):

Theorem 1.1. *Let (X, L) be a polarized Kähler manifold and G a maximal connected compact subgroup of the reduced automorphism group* $\mathrm{Aut}_0(X)$. *Then the minimum of the modified K-energy E^G is attained at each G-invariant extremal metric representing $c_1(L)$.*

The proof relies on two observations. We will consider a sequence of Fubini–Study metrics ω_k associated to Kodaira embeddings of X into higher and higher dimension projective spaces. The first observation is that if we define ω_k to be the metric associated to an extremal metric in $c_1(L)$ by the map $Hilb_k$ (see (2.3)), then ω_k will be close to a σ-balanced metric. The second point is that σ-balanced metrics, if they exist, are minimum points of the functionals Z_k^σ (see (2.8)) that converge to the modified Mabuchi functional. Then a careful analysis of the convergence properties of the ω_k and Z_k^σ yields the proof of our main result.

Remark 1.2. We mention that Guan [14] shows that extremal metrics are local minimum points, assuming the existence of C^2-geodesics in the space of Kähler potentials.

1.1. Plan of the paper

In Section 2, we review the definition of extremal metrics and recall quantization of CSCK metrics. We then introduce σ-balanced metrics and the relevant functionals. Then in Section 3, we prove the main theorem. In the Appendix, we collect some facts and proofs of properties of σ-balanced metrics.

2. Extremal metrics and quantization

2.1. Quantization

Let (X, L) be a polarized Kähler manifold of complex dimension n. Let $\mathcal{H}$ be the space of smooth Kähler potentials with respect to a fixed Kähler form $\omega \in c_1(L)$:

$$\mathcal{H} = \{\phi \in C^\infty(X) \mid \omega_\phi := \omega + \sqrt{-1}\,\partial\bar{\partial}\phi > 0\}.$$

For each k, we can consider the space $\mathcal{H}_k$ of Hermitian metrics on $L^{\otimes k}$. To each $h \in \mathcal{H}_k$ one associates a metric $-\sqrt{-1}\,\partial\bar{\partial}\log(h)$ on X, identifying the spaces $\mathcal{H}_k$ to $\mathcal{H}$. Write ω_h for the curvature of the Hermitian metric h on L. If we fix a base metric h_0 in $\mathcal{H}_1$ such that $\omega = \omega_{h_0}$, the correspondence reads

$$\omega_\phi = \omega_{e^{-\phi}h_0} = \omega + \sqrt{-1}\,\partial\bar{\partial}\phi.$$

We denote by $\mathcal{B}_k$ the space of positive definite Hermitian forms on $H^0(X, L^{\otimes k})$. Let $N_k = \dim(H^0(X, L^k))$. The space $\mathcal{B}_k$ is identified with $\mathrm{GL}_{N_k}(\mathbb{C})/U(N_k)$ using the base metric h_0^k. This symmetric space comes with a metric d_k defined by the Riemannian metric:

$$(H_1, H_2)_h = \mathrm{Tr}(H_1 H^{-1} \cdot H_2 H^{-1}).$$

There are maps

$$Hilb_k : \mathcal{H} \to \mathcal{B}_k, \qquad FS_k : \mathcal{B}_k \to \mathcal{H}$$

defined by

$$\forall h \in \mathcal{H},\ s \in H^0(X, L^{\otimes k}), \qquad \|s\|^2_{Hilb_k(h)} = \int_X |s|^2_{h^k}\, d\mu_h$$

and

$$\forall H \in \mathcal{B}_k, \qquad FS_k(H) = \frac{1}{k}\log\left(\frac{1}{N_k}\sum_\alpha |s_\alpha|^2_{h_0^k}\right),$$

where $\{s_\alpha\}$ is an orthonormal basis of $H^0(X, L^{\otimes k})$ with respect to H and $d\mu_h = \omega_h^n/n!$ is the volume form. Note that $\omega_{FS_k(H)}$ is the pull-back of the Fubini–Study metric on $\mathbb{CP}^{N_k-1}$ under the projective embedding induced by $\{s_\alpha\}$. A result of Tian [26] states that any Kähler metric ω_ϕ in $c_1(L)$ can be approximated by projective metrics, namely

$$\lim_{k\to\infty} FS_k \circ Hilb_k(\phi) = \phi$$

where the convergence is uniform on $C^2(X, \mathbb{R})$ bounded subsets of $\mathcal{H}$.

Metrics satisfying

$$FS_k \circ Hilb_k(\phi) = \phi$$

are called *balanced* metrics, and the existence of such metrics is equivalent to the Chow stability of (X, L^k) by the results of Zhang [31] and Wang [28]. Let $\mathrm{Aut}(X, L)$ be the group of automorphisms of the pair (X, L). From the work of Donaldson [8], if X admits a CSCK metric in the Kähler class $c_1(L)$, and if $\mathrm{Aut}(X, L)$ is discrete, then there are balanced metrics ω_{ϕ_k} for k sufficiently large with

$$FS_k \circ Hilb_k(\phi_k) = \phi_k,$$

and these metrics converge to the CSCK metric on $C^\infty(X, \mathbb{R})$ bounded subsets of $\mathcal{H}$.

In the proof of these results, the density of state function plays a central role. For any $\phi \in \mathcal{H}$ and $k > 0$, let $\{s_\alpha\}$ be an orthonormal basis of $H^0(X, L^k)$ with respect to $Hilb_k(\phi)$. The k^{th} *Bergman function* of ϕ is defined to be

$$\rho_k(\phi) = \sum_\alpha |s_\alpha|^2_{h^k}.$$

It is well known that a metric $\phi \in Hilb_k(\mathcal{H})$ is balanced if and only if $\rho_k(\phi)$ is constant. A key result in the study of balanced metrics is the following expansion:

Theorem 2.1 ([3], [23], [26], [30]). *The following uniform expansion holds:*

$$\rho_k(\phi) = k^n + A_1(\phi)k^{n-1} + A_2(\phi)k^{n-2} + \cdots$$

where $A_1(\phi) = \frac{1}{2}S(\phi)$ is half of the scalar curvature of the Kähler metric ω_ϕ and for any l and $R \in \mathbb{N}$, there is a constant $C_{l,R}$ such that

$$\left\| \rho_k(\phi) - \sum_{j \le R} A_j k^{n-j} \right\|_{C^l} \le C_{l,R} k^{n-R}.$$

As a corollary, if $\phi_k = FS_k \circ Hilb_k(\phi)$, then $\phi_k - \phi = \frac{1}{k}\log(\rho_k(\phi)) \to 0$ as $k \to \infty$. In particular we have the convergence of metrics,

$$\omega_{\phi_k} = \omega_\phi + \mathcal{O}(k^{-2}). \tag{2.1}$$

By integration over X we also deduce

$$\int_X \rho_k(\phi)\, d\mu_\phi = k^n \int_X d\mu_\phi + k^{n-1}\frac{1}{2}\int_X S(\phi)\, d\mu_\phi + \mathcal{O}(k^{n-2})$$

where $S(\phi)$ is the scalar curvature of the metric g_ϕ associated to the Kähler form ω_ϕ. Thus

$$N_k = k^n \operatorname{Vol}(X) + \tfrac{1}{2}\operatorname{Vol}(X)\underline{S}k^{n-1} + \mathcal{O}(k^{n-2}), \tag{2.2}$$

where

$$\underline{S} = 2n\pi \frac{c_1(L) \cup [\omega]^{n-1}}{[\omega]^n}$$

is the average of the scalar curvature and $\operatorname{Vol}(X)$ is the volume of $(X, c_1(L))$.

2.2. *The relative setup*

In order to find a canonical representative of a Kähler class, Calabi suggested [1] to look for minima of the functional

$$Ca : \mathcal{H} \to \mathbb{R}, \quad \phi \mapsto \int_X (S(\phi) - \underline{S})^2 \, d\mu_\phi.$$

In fact, critical points for this functional are local minimum points, called extremal metrics. The associated Euler–Lagrange equation is equivalent to the fact that $\mathrm{grad}_{\omega_\phi}(S(\phi))$ is a holomorphic vector field, and constant scalar curvature metrics, CSCK for short, are extremal metrics.

By a theorem of Calabi [2], the connected component of the identity of the isometry group of an extremal metric is a maximal compact connected subgroup of $\mathrm{Aut}_0(X)$. As all these maximal subgroups are conjugate, the quest for extremal metrics can be done modulo a fixed group action. Note that $\mathrm{Aut}_0(X)$ is isomorphic to $\mathrm{Aut}_0(X, L)$, the connected component of the identity of $\mathrm{Aut}(X, L)$. As we will see later, it will be useful to consider a less restrictive setup, working modulo a circle action. We then define the relevant functionals in a general situation, and we fix a compact subgroup G of $\mathrm{Aut}_0(X, L)$ and denote by $\mathfrak{g}$ its Lie algebra.

2.2.1. Space of potentials. We extend the quantization tools to the extremal metrics setup.

Replacing L by a sufficiently large tensor power if necessary, we can assume that $\mathrm{Aut}_0(X, L)$ acts on L (see e.g. [15]). Then the G-action on X induces a G-action on the space $H^0(X, L^k)$ of sections. This action in turn provides a G-action on the space $\mathcal{B}_k$ of positive definite Hermitian forms on $H^0(X, L^k)$, and we define $\mathcal{B}_k^G$ to be the subspace of G-invariant elements. The space $\mathcal{B}_k^G$ is totally geodesic in $\mathcal{B}_k$ for the distance d_k. Define $\mathcal{H}^G$ to be the space of G-invariant potentials with respect to a G-invariant base point ω. We see from their definitions that we have the induced maps

$$Hilb_k : \mathcal{H}^G \to \mathcal{B}_k^G, \quad FS_k : \mathcal{B}_k^G \to \mathcal{H}^G. \tag{2.3}$$

2.2.2. Modified K-energy. For a fixed metric g, we say that a vector field V is a *Hamiltonian vector field* if there is a real valued function f such that

$$V = J\nabla_g f \quad \text{or equivalently} \quad \omega(V, \cdot) = -df.$$

For any $\phi \in \mathcal{H}^G$, let P_ϕ^G be the space of normalized (i.e. mean value zero) Killing potentials with respect to g_ϕ whose corresponding Hamiltonian vector fields lie in $\mathfrak{g}$, and let Π_ϕ^G be the orthogonal projection from $L^2(X, \mathbb{R})$ to P_ϕ^G given by the inner product

$$(f, g) \mapsto \int fg \, d\mu_\phi.$$

Note that G-invariant metrics satisfying $S(\phi) - \underline{S} - \Pi_\phi^G S(\phi) = 0$ are extremal.

Definition 2.2 ([13, Section 4.13]). The *reduced scalar curvature S^G with respect to G* is defined by

$$S^G(\phi) = S(\phi) - \underline{S} - \Pi^G_\phi S(\phi).$$

The *extremal vector field V^G with respect to G* is defined by the equation

$$V^G = \nabla_g(\Pi^G_\phi S(\phi))$$

for any ϕ in $\mathcal{H}^G$ and does not depend on ϕ (see e.g. [13, Proposition 4.13.1]).

Remark 2.3. Note that by definition the extremal vector field is real-holomorphic and lies in $J\mathfrak{g}$ where J is the almost complex structure of X, while JV^G lies in $\mathfrak{g}$.

Remark 2.4. When $G = \{1\}$ we recover the normalized scalar curvature. When G is a maximal compact connected subgroup, or a maximal torus of $\mathrm{Aut}_0(X)$, we find the reduced scalar curvature and the usual extremal vector field initially defined by Futaki and Mabuchi [12].

We are now able to define the relative Mabuchi K-energy, introduced by Guan [14], Chen and Tian [5], and Simanca [24]:

Definition 2.5 ([13, Section 4.13]). The *modified Mabuchi K-energy E^G* (relative to G) is defined, up to a constant, as the primitive of the following one-form on $\mathcal{H}^G$:

$$\phi \mapsto -S^G(\phi)\, d\mu_\phi.$$

If $\phi \in \mathcal{H}^G$, then the modified K-energy admits the expression

$$E^G(\phi) = -\int_X \phi\left(\int_0^1 S^G(t\phi)\, d\mu_{t\phi}\, dt\right).$$

As for CSCK metrics, G-invariant extremal metrics whose extremal vector field lie in $J\mathfrak{g}$ are critical points of the relative Mabuchi energy.

2.2.3. The σ-balanced metrics. We present a generalization of balanced metrics adapted to the relative setting of extremal metrics.

Definition 2.6. Let $\sigma_k(t)$ be a one-parameter subgroup of $\mathrm{Aut}_0(X, L^k)$. Let $\phi \in \mathcal{H}$. Then ϕ is a k^{th} *σ_k-balanced metric* if

$$\omega_{kFS_k \circ Hilb_k(\phi)} = \sigma_k(1)^* \omega_{k\phi}. \tag{2.4}$$

Conjecturally, the σ-balanced metrics would provide a generalization of the notion of balanced metric and would approximate an extremal Kähler metric. Indeed, in one direction, assume that we are given σ_k-balanced metrics ω_{ϕ_k} with $\sigma_k \in \mathrm{Aut}_0(X, L^k)$ such that the ω_k converge to ω_∞. Suppose that the vector fields $k\frac{d}{dt}\big|_{t=0}\sigma_k(t)$ converge to a vector field $V_\infty \in \mathfrak{h}_0$. A simple calculation implies that ω_∞ must be extremal.

We now define the functionals that play the role of finite-dimensional versions of the modified Mabuchi K-energy on $\mathcal{B}_k^G$ and $FS_k(\mathcal{B}_k^G)$ respectively. First define $I_k = \log \circ \det$ on $\mathcal{B}_k^G$. This functional is defined up to an additive constant when we see $\mathcal{B}_k^G$ as a space of positive Hermitian matrix once a suitable basis of $H^0(X, L^k)$ is fixed. It is shown in [4] that I_k gives a quantization of the Aubin functional I. However, in the extremal case, we need a modified version of the Aubin functional defined by the first author in order to fit with the balanced metrics. Let $V \in Lie(\mathrm{Aut}_0(X, L))$ and denote by $\sigma(t)$ the associated one-parameter subgroup of $\mathrm{Aut}_0(X, L)$. Define up to a constant for each $\phi \in \mathcal{H}$ the function $\psi_{\sigma,\phi}$ by

$$\sigma(1)^*\omega_\phi = \omega_\phi + \sqrt{-1}\,\partial\bar{\partial}\psi_{\sigma,\phi}. \tag{2.5}$$

We will see how to suitably choose a normalization constant for these potentials. We then consider a modified I functional defined up to a constant by its differential:

$$\delta I^\sigma(\phi)(\delta\phi) = \int_X \delta\phi(1 + \Delta_\phi)e^{\psi_{\sigma,\phi}}\, d\mu_\phi$$

where $\Delta_\phi = -g_\phi^{i\bar{j}} \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}$ is the complex Laplacian of g_ϕ. We will also need to consider the potentials ϕ as metrics on the tensor powers $L^{\otimes k}$, we thus consider the normalized vector fields $V_k = -V/(4k)$ and the associated one-parameter groups $\sigma_k(t)$. We choose the normalization

$$\int_X \exp(\psi_{\sigma_k,\phi})\, d\mu_\phi = \frac{N_k}{k^n}. \tag{2.6}$$

Then we define, for each k,

$$\delta I_k^\sigma(\phi)(\delta\phi) = \int_X k\delta\phi(1 + \Delta_\phi/k)e^{\psi_{\sigma_k,\phi}} k^n\, d\mu_\phi.$$

Remark 2.7. If σ is the identity, we recover the usual Aubin functional.

Remark 2.8. This one-form integrates along paths in $\mathcal{H}^G$ to a functional $I_k^\sigma(\phi)$ on $\mathcal{H}^G$, which is independent of the path used from 0 to ϕ. The proof of this fact is given in the Appendix (Proposition 4.1).

We define $\mathcal{L}_k^\sigma$ on $\mathcal{H}^G$ and Z_k^σ on $\mathcal{B}_k^G$ by

$$\mathcal{L}_k^\sigma = I_k \circ Hilb_k + I_k^\sigma, \tag{2.7}$$

$$Z_k^\sigma = I_k^\sigma \circ FS_k + I_k - k^n \log(k^n)\,\mathrm{Vol}(X). \tag{2.8}$$

We will show in the following that these functionals converge to the modified K-energy in some sense. Note also that σ_k-balanced metrics are critical points for $\mathcal{L}_k^\sigma$ (Proposition 3.4), and if $FS_k(H_k)$ is a σ_k-balanced metric for some $H_k \in \mathcal{B}_k^G$, then H_k is a minimum point for Z_k^σ (Proposition 3.9).

3. Minima of the modified K-energy

The aim of this section is to prove Theorem 1.1. For the convenience of the reader we first give a sketch of the proof.

We will choose a special group G corresponding to the Killing field JV^* associated to the extremal vector field V^* of the extremal Kähler metric $\omega^* = \omega_{\phi^*}$. We know that the metrics $\omega_k^* = \omega + \sqrt{-1}\,\partial\bar{\partial}\phi_k^*$ with Kähler potentials $\phi_k^* = FS_k \circ Hilb_k(\phi^*)$ converge to ω^* ([26], [3] and [30]). We begin our proof by showing that the functionals $\mathcal{L}_k^\sigma$ converge to the modified Mabuchi functional on the space $\mathcal{H}^G$. Then we show that $Z_k^\sigma \circ Hilb_k$ and $\mathcal{L}_k^\sigma$ converge to the same functional, thus Z_k^σ gives a quantization of the modified Mabuchi functional and we reduce our problem to studying the minimum points of Z_k^σ. However, the metrics ω_k^* constructed above are not in general critical points of Z_k^σ, as there is no reason for these metrics to be σ_k-balanced. We use instead an idea of Li [16] relying on the Bergman kernel expansion to show that these metrics ω_k^* are almost σ_k-balanced in the sense that $Hilb_k(\omega_k^*)$ is a minimum point of the functional Z_k^σ up to an error which goes to zero when k tends to infinity.

Let V^* be the extremal vector field of the class $c_1(L)$. In the polarized case, the vector field JV^* generates a periodic action [12] by a one-parameter subgroup of automorphisms of (X, L). Fix G to be the one-parameter subgroup of $\mathrm{Aut}(X, L)$ associated to JV^*. This group is isomorphic to S^1 or trivial by the theorem of Futaki and Mabuchi [12]. This will be a group of isometries for each of our metrics.

Remark 3.1. The modified K-energy E^{G_m} is defined to be the modified Mabuchi functional with respect to a maximal compact connected subgroup G_m of $\mathrm{Aut}(X, L)$. Assume that G is contained in such a G_m. Then E^{G_m} is equal to E^G when restricted to the space of G_m-invariant potentials. Indeed, the projection of any G_m-invariant scalar curvature to the space of holomorphy potentials of $Lie(G_m)$ gives a potential for the extremal vector field by definition. Thus a minimum point of E^G which is invariant under the G_m-action, such as an extremal metric, will be a minimum point of the usual modified Mabuchi functional.

Let σ_k be the element of $\mathrm{Aut}(X, L)$ associated to the vector field $-V^*/(4k)$. We will also need to define for each ϕ in $\mathcal{H}^G$ the function $\theta(\phi)$ to be the normalized (i.e. mean value zero) holomorphy potential of the vector field V^* with respect to the metric ω_ϕ:

$$g_\phi(V^*, \cdot) = d\theta(\phi)$$

or

$$\theta(\phi) = \Pi_\phi^G(S(\phi)).$$

3.1. The functionals $\mathcal{L}_k^\sigma$ converge to E^G

In this section we prove the following fact:

Proposition 3.2. *There are constants c_k such that*

$$\frac{2}{k^n}\mathcal{L}_k^\sigma + c_k \to E^G$$

as $k \to \infty$, where the convergence is uniform on $C^l(X, \mathbb{R})$ bounded subsets of $\mathcal{H}^G$.

Proof. We show that

$$\frac{2}{k^n}\delta\mathcal{L}_k^\sigma \to \delta E^G$$

uniformly on $C^l(X,\mathbb{R})$ bounded subsets of $\mathcal{H}^G$. First we compute $\delta\mathcal{L}_k^\sigma$. Following [10], we have

$$\delta(I_k \circ Hilb_k)_\phi(\delta\phi) = -\int_X \delta\phi(\Delta_\phi + k)\rho_k(\phi)\, d\mu_\phi$$

and by definition

$$\delta(I_k^\sigma)_\phi(\delta\phi) = k^n \int_X \delta\phi(k + \Delta_\phi)e^{\psi_k(\phi)}\, d\mu_\phi$$

where we set $\psi_k(\cdot) = \psi_{\sigma_k,\cdot}$. Then

$$\delta(\mathcal{L}_k^\sigma)_\phi(\delta\phi) = -\int_X \delta\phi(\Delta_\phi + k)(\rho_k(\phi) - k^n e^{\psi_k(\phi)})\, d\mu_\phi. \tag{3.1}$$

We need an expansion for the potential ψ_k:

$$\psi_k(\phi) = \frac{\theta(\phi) + \underline{S}}{2k} + \mathcal{O}_0(k^{-1}),$$

whose proof is postponed to Lemma 3.3. Then by the expansions of $\psi_k(\phi)$ and $\rho_k(\phi)$,

$$\begin{aligned}(\Delta_\phi + k)(\rho_k(\phi) - k^n e^{\psi_k(\phi)}) &= k^n(\Delta_\phi + k)\\ &\quad\cdot\left(1 + \frac{S(\phi)}{2k} + \mathcal{O}(k^{-2}) - 1 - \frac{\theta(\phi) + \underline{S}}{2k} + \mathcal{O}_0(k^{-1})\right)\\ &= k^n\left(\frac{S(\phi) - \underline{S} - \theta(\phi)}{2} + \mathcal{O}(k^{-1})\right),\end{aligned}$$

and

$$\frac{\delta(\mathcal{L}_k^\sigma)_\phi}{k^n} \to \frac{1}{2}\delta E_\phi^G.$$

As the expansions of $\psi_k(\phi)$ and $\rho_k(\phi)$ are uniform on bounded subsets of $C^l(X,\mathbb{R})$, the result follows. □

The following lemma will be useful:

Lemma 3.3. *The following expansion holds uniformly in $C^l(X,\mathbb{R})$ for $l \gg 1$:*

$$\psi_k(\phi) = \frac{\theta(\phi) + \underline{S}}{2k} + \mathcal{O}_0(k^{-1}) \tag{3.2}$$

where $\mathcal{O}_0(k^{-1})$ denotes k^{-1} times a function $\varepsilon(k)$ on X with $\varepsilon(k) \to 0$ in $C^l(X,\mathbb{R})$ as $k \to 0$.

Proof. By definition

$$\sigma_k(1)^*\omega(\phi) - \omega(\phi) = \sqrt{-1}\,\partial\bar{\partial}\psi_k(\phi),$$

so

$$\sigma_1(1/k)^*\omega(\phi) - \omega(\phi) = \sqrt{-1}\,\partial\bar{\partial}\psi_k(\phi),$$

where $\sigma_1(1/k)$ is equal to $\exp\bigl(-\frac{1}{4k}V^*\bigr)$. Dividing by $1/k$, and letting k go to infinity, we get

$$\mathcal{L}_{-\frac{1}{4}V^*}\omega(\phi) = \sqrt{-1}\,\partial\bar{\partial}\lim_{k\to\infty} k\psi_k(\phi).$$

Then by Cartan's formula,

$$\mathcal{L}_{-\frac{1}{4}V^*}\omega(\phi) = -\tfrac{1}{4}d\omega_\phi(V^*,\cdot) = -\tfrac{1}{4}dg_\phi(V^*, J\cdot),$$

and by definition of holomorphy potentials,

$$\mathcal{L}_{-\frac{1}{4}V^*}\omega(\phi) = -\tfrac{1}{4}dd^c\theta(\phi) = \tfrac{1}{2}\sqrt{-1}\,\partial\bar{\partial}\theta(\phi),$$

thus

$$\lim_{k\to\infty} k\psi_k(\phi) = \frac{\theta(\phi)+c}{2}$$

for some constant c. By the normalization (2.6) of the function $\psi_k(\phi)$ we deduce

$$\frac{N_k}{k^n} = \int_X \exp\left(\psi_{\sigma_k,\phi}\right)d\mu_\phi = \int_X \left(1 + \frac{\theta(\phi)+c}{2k} + \mathcal{O}(k^{-2})\right)d\mu_\phi.$$

Recall that we choose $\theta(\phi)$ normalized to have mean value zero. Using formula (2.2) to expand $N_k = \dim(H^0(X, L^k))$, we conclude that $c = \underline{S}$. □

From the above computations we also deduce the following:

Proposition 3.4. *Let $\phi \in \mathcal{H}$ be a k^{th} σ_k-balanced metric. Then ϕ is a critical point of $\mathcal{L}_k^\sigma$.*

Proof. By equation (2.4) of σ_k-balanced metrics and by definition (2.5) of $\psi_k(\phi)$ we deduce

$$\rho_k(\phi) = C\exp(\psi_k(\phi))$$

for some constant C. Integrating over X and using the expansions (2.2) and (3.2) we deduce

$$\rho_k(\phi) = k^n\exp(\psi_k(\phi)).$$

The result follows from the computation of the differential of $\mathcal{L}_k^\sigma$ (equation (3.1)). □

A direct computation implies the similar result for Z_k^σ (see Corollary 3.9).

3.2. Comparison of Z_k^σ and $\mathcal{L}_k^\sigma$

The aim of this section is to show that $Z_k^\sigma \circ Hilb_k$ and $\mathcal{L}_k^\sigma$ converge to the same functional. We will need the following two lemmas:

Lemma 3.5. *The second derivative of I_k^σ along a path $\phi_s \in \mathcal{H}^G$ is equal to*

$$\frac{d^2}{ds^2} I_k^\sigma(\phi_s) = k^n \int_X \left(\phi'' - \frac{1}{2}|d\phi'|^2\right)(k + \Delta_{\phi_s}) e^{\psi_k(\phi_s)}\, d\mu_{\phi_s}.$$

Proof. The proof of this result is given in the Appendix, Section 4.2. □

Lemma 3.6. *Let $\phi \in \mathcal{H}^G$. Then there exists an integer k_0, depending on ϕ, such that for each $k \geq k_0$, the functional I_k^σ is concave along the path*

$$[0,1] \to \mathcal{H}^G, \qquad s \mapsto \phi + \frac{s}{k}\log(\rho_k(\phi)).$$

Proof. By Lemma 3.5, the second derivative of I_k^σ along the path $\phi_k(s) = \phi + \frac{s}{k}\log(\rho_k(\phi))$ is

$$k^n \int_X \left(\phi_k'' - \frac{1}{2}|d\phi_k'|^2\right)(k + \Delta_{\phi_k(s)}) e^{\psi_k(\phi_k(s))}\, d\mu_{\phi_k(s)}.$$

As $\phi_k' = \frac{1}{k}\log(\rho_k(\phi))$ and $\phi_k'' = 0$, this expression simplifies:

$$\frac{d^2}{ds^2} I_k^\sigma(\phi_k(s)) = -k^n \int_X \frac{1}{2}\left|d\frac{1}{k}\log(\rho_k(\phi))\right|^2 (k + \Delta_{\phi_k(s)}) e^{\psi_k(\phi_k(s))}\, d\mu_{\phi_k(s)}.$$

We compute the leading term in the above expression as k goes to infinity. To simplify notation, let $T_k(\phi) = FS_k \circ Hilb_k(\phi)$. Note that $\omega_{\phi_1} = \omega_{T_k(\phi)}$. From (2.1),

$$\omega_{\phi_0} - \omega_{\phi_1} = \mathcal{O}(k^{-2}).$$

Thus we have the estimates

$$\Delta_{\phi_k(s)} = \Delta_\phi + \mathcal{O}(k^{-1}), \qquad d\mu_{\phi_k(s)} = d\mu_\phi + \mathcal{O}(k^{-1}), \qquad \psi_k(\phi_k(s)) = \psi_k(\phi) + \mathcal{O}(k^{-1}).$$

Hence

$$\frac{d^2}{ds^2} I_k^\sigma(\phi_k(s)) = -k^n \int_X \frac{1}{2}\left|d\frac{1}{k}\log(\rho_k(\phi))\right|^2 (k + \Delta_\phi) e^{\psi_k(\phi)}\, d\mu_\phi + \mathcal{O}(k^{n-4}).$$

From this we deduce that the leading term as k tends to infinity is

$$-\frac{k^{n-3}}{8} \int_X |dS(\phi)|^2\, d\mu_\phi < 0,$$

where once again we used the expansions of the Bergman kernel and of $\psi_k(\phi)$ from Lemma 3.3. □

Now we can prove the main result of this section:

Proposition 3.7. *For each potential $\phi \in \mathcal{H}^G$, we have*

$$\lim_{k\to\infty} k^{-n}(\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)) = 0,$$

Proof. By definition,

$$k^{-n}(\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)) = -k^{-n}\big(I_k^\sigma(T_k(\phi)) - I_k^\sigma(\phi) - k^n \log(k^n)\operatorname{Vol}(X)\big)$$

where $T_k = FS_k \circ Hilb_k$. From Lemma 3.6, for k large enough, the functional I_k^σ is concave along the path

$$s \mapsto \phi + \frac{s}{k}\log(\rho_k(\phi))$$

going from ϕ to $T_k(\phi)$ in $\mathcal{H}^G$. Thus

$$(\delta I_k^\sigma)_\phi\left(\frac{1}{k}\log\rho_k(\phi)\right) \geq I_k^\sigma(T_k(\phi)) - I_k^\sigma(\phi) \geq (\delta I_k^\sigma)_{T_k(\phi)}\left(\frac{1}{k}\log\rho_k(\phi)\right). \tag{3.3}$$

We deduce from the definitions that

$$k^{-n}(\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)) \geq -k^{-n}(\delta I_k^\sigma)_\phi\left(\frac{1}{k}\log\rho_k(\phi)\right) + \log(k^n)\operatorname{Vol}(X), \tag{3.4}$$

$$-k^{-n}(\delta I_k^\sigma)_{T_k(\phi)}\left(\frac{1}{k}\log\rho_k(\phi)\right) + \log(k^n)\operatorname{Vol}(X) \geq k^{-n}(\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)), \tag{3.5}$$

and it remains to show that the right hand side of (3.4) and the left hand side of (3.5) tend to zero. First,

$$\begin{aligned}
&k^{-n}(\delta I_k^\sigma)_\phi\left(\frac{1}{k}\log\rho_k(\phi)\right) - \log(k^n)\operatorname{Vol}(X)\\
&\quad= \int_X \left(\frac{1}{k}\log(\rho_k(\phi))\right)(k+\Delta_\phi)e^{\psi_k(\phi)}\,d\mu_\phi - \operatorname{Vol}(X)\log(k^n)\\
&\quad= \int_X \left(\log(k^n) + \frac{S(\phi)}{2k} + \mathcal{O}(k^{-2})\right)\left(1+\frac{\Delta_\phi}{k}\right)\left(1+\frac{\theta(\phi)+\underline{S}}{2k} + \mathcal{O}_0(k^{-1})\right)d\mu_\phi\\
&\qquad- \operatorname{Vol}(X)\log(k^n)
\end{aligned}$$

by the expansion of the Bergman kernel and Lemma 3.3. It follows that

$$\begin{aligned}
&k^{-n}(\delta I_k^\sigma)_\phi\left(\frac{1}{k}\log\rho_k(\phi)\right) - \log(k^n)\operatorname{Vol}(X)\\
&\qquad= \operatorname{Vol}(X)\log(k^n) + \mathcal{O}(k^{-1}) - \operatorname{Vol}(X)\log(k^n) \to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Note that we did not make use of the fact that the derivative δI_k^σ was evaluated at ϕ, so the above argument extends to the last term of the inequality (3.5), evaluated at $T_k(\phi)$, which thus tends to zero as well. □

3.3. The metrics $Hilb_k(\omega^*)$ are almost σ-balanced

Lemma 3.8. *The functional Z_k^σ is convex along geodesics in $\mathcal{B}_k^G$.*

Proof. We follow [10, proof of Proposition 1] (also [22, Lemma 3.1]). Here we abbreviate the subscript k. Take a geodesic $\{H(s)\}_{s\in\mathbb{R}}$ in $\mathcal{B}^G$. By choosing an appropriate orthonormal basis $\{\tau_\alpha\}$ of $H(0)$, one can represent $H(s)$ by

$$H(s) = \mathrm{diag}(e^{2\lambda_\alpha s}), \qquad \lambda_\alpha \in \mathbb{R},\ \sum_\alpha \lambda_\alpha = 0, \tag{3.6}$$

with respect to the basis $\{\tau_\alpha\}$. We denote the associated one-parameter subgroup of $SL(H^0(M, L))$ by $\varrho(s)$. We define the Kähler potential $\phi_s = FS(H(s))$ by

$$\phi_s = \log\Big(\sum_\alpha |\varrho(s)\cdot\tau_\alpha|^2 / \sum_\beta |\tau_\beta|^2\Big).$$

First of all, we will show the first variation of Z^σ along ϕ_s. From (4.3), we have

$$\begin{aligned}\frac{dZ^\sigma}{ds}(s) &= \int_X \phi_s'(1+\Delta_{FS(H(s))})e^{\psi_s}\, d\mu_{FS(H(s))}\\ &= \int_X \Big(\phi_s' e^{\psi_s} + \frac{d}{ds}e^{\psi_s}\Big)\, d\mu_{FS(H(s))}\\ &= \int_X \Big\{ \frac{\sum_\alpha 2\lambda_\alpha|\varrho(s)\cdot\tau_\alpha|^2}{\sum_\beta|\varrho(s)\cdot\tau_\beta|^2}\,\frac{\sum_\gamma|\varrho(s)\cdot\sigma^*\tau_\gamma|^2}{\sum_\beta|\tau_\beta|^2}\\ &\qquad + \frac{d}{ds}\Big(\frac{\sum_\gamma|\varrho(s)\cdot\sigma^*\tau_\gamma|^2}{\sum_\beta|\varrho(s)\cdot\tau_\beta|^2}\Big)\Big\}\, d\mu_{FS(H(s))}\\ &= \int_X \frac{\sum_\alpha 2\widetilde{\lambda}_\alpha|\varrho(s)\cdot\sigma^*\tau_\alpha|^2}{\sum_\beta|\varrho(s)\cdot\tau_\beta|^2}\, d\mu_{FS(H(s))},\end{aligned} \tag{3.7}$$

where ψ_s denotes $\psi_{\sigma,FS(H(s))}$. In (3.7), $H(s)$ is represented by

$$H(s) = \mathrm{diag}(e^{2\widetilde{\lambda}_\alpha s}), \qquad \lambda_\alpha \in \mathbb{R},\ \sum_\alpha \widetilde{\lambda}_\alpha = 0,$$

with respect to the basis $\{\sigma^*\tau_\alpha\}$. Let

$$\varphi_s' := \frac{\sum_\alpha 2\widetilde{\lambda}_\alpha|\varrho(s)\cdot(\sigma^*\tau_\alpha)|^2}{\sum_\beta|\varrho(s)\cdot\tau_\beta|^2}.$$

Then

$$\frac{d^2Z^\sigma}{ds^2}(0) = \int_X \{\varphi_0'' - (\nabla\varphi_0', \nabla\phi_0')\}\, d\mu_{FS(H(0))}. \tag{3.8}$$

Here we denote the connection of type $(1, 0)$ by ∇. Following [10], it is sufficient to prove that the integrand of (3.8) is equal to

$$\sum_\alpha |(\nabla\phi_0', \nabla(\sigma^*\tau_\alpha)) - (2\widetilde{\lambda}_\alpha - \phi_0')(\sigma^*\tau_\alpha)|^2_{FS(H(0))} \tag{3.9}$$

pointwise on X. Expanding out (3.9) yields

$$\sum_\alpha |(\nabla\phi_0', \nabla(\sigma^*\tau_\alpha))|^2_{FS(H(0))} - 2\sum_\alpha (2\widetilde{\lambda}_\alpha - \phi_0')((\nabla\phi_0', \nabla(\sigma^*\tau_\alpha)), \sigma^*\tau_\alpha) + \sum_\alpha (2\widetilde{\lambda}_\alpha - \phi_0')^2 |\sigma^*\tau_\alpha|^2_{FS(H(0))}. \tag{3.10}$$

The second term of (3.10) is equal to

$$\begin{aligned} -2\sum_\alpha (2\widetilde{\lambda}_\alpha - \phi_0')(\nabla\phi_0', (\sigma^*\tau_\alpha, \nabla(\sigma^*\tau_\alpha))) &= -2\sum_\alpha (2\widetilde{\lambda}_\alpha - \phi_0')(\nabla\phi_0', \nabla(|\sigma^*\tau_\alpha|^2_{FS(H(0))})) \\ &= -2(\nabla\phi_0', \nabla\varphi_0') + 2\phi_0'(\nabla\phi_0', \nabla e^{\psi_0}) \\ &= -2(\nabla\phi_0', \nabla\varphi_0') + 2\phi_0'\psi_0' e^{\psi_0} \\ &= -2(\nabla\phi_0', \nabla\varphi_0') + 2\phi_0'(\varphi_0' - \phi_0' e^{\psi_0}). \end{aligned} \tag{3.11}$$

Above, we use (4.3) of the Appendix and

$$\psi_0' e^{\psi_0} = \frac{d}{ds}\bigg|_{s=0} e^{\psi_s} = \varphi_0' - \phi_0' e^{\psi_0}.$$

The third term of (3.10) is equal to

$$\sum_\alpha 4\widetilde{\lambda}_\alpha^2 |\sigma^*\tau_\alpha|^2_{FS(H(0))} - 2\varphi_0'\phi_0' + (\phi_0')^2 e^{\psi_0}. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10) shows that (3.10) is equal to

$$\sum_\alpha |(\nabla\phi_0', \nabla(\sigma^*\tau_\alpha))|^2_{FS(H(0))} - 2(\nabla\phi_0', \nabla\varphi_0') - (\phi_0')^2 e^{\psi_0} + \sum_\alpha 4\widetilde{\lambda}_\alpha^2 |\sigma^*\tau_\alpha|^2_{FS(H(0))}.$$

Since

$$\varphi_0'' = \sum_\alpha 4\widetilde{\lambda}_\alpha^2 |\sigma^*\tau_\alpha|^2_{FS(H(0))} - \varphi_0'\phi_0',$$

it remains to prove that

$$\sum_\alpha |(\nabla\phi_0', \nabla(\sigma^*\tau_\alpha))|^2_{FS(H(0))} = (\nabla\phi_0', \nabla\varphi_0') + (\phi_0')^2 e^{\psi_0} - \varphi_0'\phi_0'. \tag{3.13}$$

In the computation in (3.11), we found

$$-(\phi_0')^2 e^{\psi_0} + \varphi_0'\phi_0' = (\nabla\phi_0', \phi_0'\nabla e^{\psi_0}).$$

Hence, (3.13) is equivalent to

$$\sum_\alpha |(\nabla\phi_0', \nabla(\sigma^*\tau_\alpha))|^2_{FS(H(0))} = (\nabla\varphi_0', \nabla\phi_0') - (\nabla\phi_0', \phi_0'\nabla e^{\psi_0}). \tag{3.14}$$

This follows from the definition of the restriction $\omega_{FS(H(0))}$ of the Fubini–Study metric. To see (3.14), recall that the Fubini–Study metric is given by

$$\frac{\sum_i dz^i \wedge d\bar{z}^i}{1+\sum |z^k|^2} - \frac{(\sum \bar{z}^i dz^i) \wedge (\sum z^j d\bar{z}^j)}{1+\sum |z^k|^2}$$

in the coordinate chart $U_0 = \{(1, z^2, \dots, z^N) \in \mathbb{CP}^{N-1}\}$. Then

$$|(\nabla\phi_0', \nabla\tau_\alpha)|^2_{FS(H(0))} = \frac{(\lambda_\alpha^2 + (\phi_0')^2 - 2\phi_0'\lambda_\alpha)|\tau_\alpha|^2}{\sum_\beta |\tau_\beta|^2}, \tag{3.15}$$

$$(\nabla\phi_0', \nabla|\tau_\alpha|^2_{FS(H(0))}) = \frac{(\lambda_\alpha - \phi_0')|\tau_\alpha|^2}{\sum_\beta |\tau_\beta|^2}, \tag{3.16}$$

$$(\nabla\phi_0', \phi_0'\nabla|\tau_\alpha|^2_{FS(H(0))}) = \frac{\phi_0'\lambda_\alpha|\tau_\alpha|^2 - (\phi_0')^2|\tau_\alpha|^2}{\sum_\beta |\tau_\beta|^2}. \tag{3.17}$$

We get (3.14) by substituting (3.15) into the left hand side of (3.14), and (3.16) and (3.17) into the right hand side of (3.14). □

The following corollary is fundamental for understanding the idea of this paper, although we do not use as it stands in the proof of the main theorem.

Corollary 3.9. *If $FS_k(H_k)$ is a σ_k-balanced metric for some $H_k \in \mathcal{B}_k^G$, then H_k is a minimum point of Z_k^σ on $\mathcal{B}_k^G$.*

Proof. Since H_k is a σ_k-balanced metric, $\{c(\sigma^*\tau_\alpha)\}_\alpha$ is an orthonormal basis with respect to $T(H_k)$ for some $c > 0$. From (3.7), H_k is a critical point of Z_k^σ on $\mathcal{B}_k^G$. From Lemma 3.8, this is an absolute minimum point of Z_k^σ. □

Proposition 3.10. *Let $\phi \in \mathcal{H}^G$. Then there are functions $\varepsilon_\phi(k)$ such that*

$$k^{-n}(Z_k^\sigma \circ Hilb_k(\phi)) \geq k^{-n}(Z_k^\sigma \circ Hilb_k(\phi^*)) + \varepsilon_\phi(k)$$

and $\lim_{k\to\infty} \varepsilon_\phi(k) = 0$ *in* $C^l(X, \mathbb{R})$ *for* $l \gg 1$.

Proof. We follow Li's proof of [16, Lemma 3.3], adapted to our more general setting. Below, C will stand for a constant depending on ϕ, ϕ^* and the volume of the polarized manifold (X, L), but independent of k. The precise value of this constant might change but it will not be important for us.

Set $H_k^* = Hilb_k(\phi^*)$ and $H_k = Hilb_k(\phi)$. We choose an orthonormal basis $\{\tau_\alpha^{(k)}\}$ of H_k^* in which H_k^* is represented by the identity and

$$H_k = \mathrm{diag}(e^{2\lambda_\alpha^{(k)}}).$$

Then evaluating H_k on the orthonormal vectors $e^{\lambda_\alpha^{(k)}}\tau_\alpha^{(k)}$ gives

$$e^{-2\lambda_\alpha^{(k)}} = \int_X |\tau_\alpha^{(k)}|^2_{h_0^k}\, d\mu_0. \tag{3.18}$$

Comparing the metrics we have the existence of $C > 0$ such that

$$C^{-k} h^k_{\phi^*} \leq h^k_0 \leq C^k h^k_{\phi^*},$$

from which we deduce with (3.18) the following estimate:

$$|\lambda^{(k)}_\alpha| \leq Ck. \tag{3.19}$$

Consider the one-parameter subgroup

$$s \mapsto H_k(s) = \mathrm{diag}(e^{2s\lambda^{(k)}_\alpha})$$

of $\mathcal{B}^G_k$. It is a geodesic that goes from H^*_k to H_k in $\mathcal{B}^G_k$, thus by Lemma 3.8,

$$k^{-n}(Z^\sigma_k(H_k) - Z^\sigma_k(H^*_k)) \geq k^{-n} f'_k(0) \quad \text{with} \quad f_k(s) = Z^\sigma_k(H_k(s)).$$

We need to show that $\lim_{k\to\infty} k^{-n} f'_k(0) = 0$. By a straightforward computation,

$$k^{-n} f'_k(0) = 2k^{-n} \sum_\alpha \lambda^{(k)}_\alpha - \frac{2}{k} \int_X \frac{\rho^\lambda_k}{\rho_k}(k + \Delta) e^{\psi_k}\, d\mu$$

where $\rho^\lambda_k = \sum_\alpha \lambda^{(k)}_\alpha |\tau^{(k)}_\alpha|^2_{h^k_0}$ and the quantities ρ_k, Δ, ψ_k and $d\mu$ are computed with respect to the extremal metric ω_{ϕ^*}. Then

$$2^{-1}k^{-n} f'_k(0) = k^{-n} \sum_\alpha \lambda^{(k)}_\alpha - \int_X \frac{\rho^\lambda_k}{\rho_k} e^{\psi_k}\, d\mu - \frac{1}{k} \int_X \frac{\rho^\lambda_k}{\rho_k} \Delta e^{\psi_k}\, d\mu. \tag{3.20}$$

We first show that the last term in of (3.20) tends to zero. First note that $|\rho^\lambda_k/\rho_k| \leq Ck$ from (3.19), thus

$$\left| \frac{1}{k} \int_X \frac{\rho^\lambda_k}{\rho_k} \Delta e^{\psi_k}\, d\mu \right| \leq C \int_X |\Delta e^{\psi_k}|\, d\mu,$$

and using Lemma 3.3 we deduce that this term goes to zero as k tends to infinity.

Then consider the second term on the right hand side of (3.20). Using the expansions of ψ_k and ρ_k we deduce that

$$\rho_k^{-1} e^{\psi_k} = k^{-n}\left(1 - \frac{S}{2k} + \mathcal{O}(k^{-2})\right)\left(1 + \frac{\theta + \underline{S}}{2k} + \mathcal{O}_0(k^{-1})\right).$$

Here we use our crucial assumption, that ω_{ϕ^*} is extremal, so $S = \theta + \underline{S}$ and thus

$$\rho_k^{-1} e^{\psi_k} = k^{-n}(1 + \mathcal{O}_0(k^{-1})).$$

Then

$$\int_X \frac{\rho^\lambda_k}{\rho_k} e^{\psi_k}\, d\mu = \int_X \frac{\rho^\lambda_k}{k^n}(1 + \mathcal{O}_0(k^{-1}))\, d\mu.$$

As

$$\int_X \frac{\rho_k^\lambda}{k^n}\, d\mu = k^{-n} \sum_\alpha \lambda_\alpha^{(k)},$$

the only remaining term to control at infinity in $k^{-n} f_k'(0)$ is

$$\int_X \frac{\rho_k^\lambda}{k^n} \mathcal{O}_0(k^{-1})\, d\mu.$$

Using (3.19), we obtain

$$\left| \frac{\rho_k^\lambda}{k^n} \mathcal{O}_0(k^{-1}) \right| \leq C k N_k k^{-n} |\mathcal{O}_0(k^{-1})|.$$

By (2.2), $N_k k^{-n}$ is bounded and as $\mathcal{O}_0(k^{-1}) = k^{-1}\epsilon(k)$ with some $\epsilon(k) \to 0$, we conclude that

$$\lim_{k\to\infty} \int_X \frac{\rho_k^\lambda}{k^n} \mathcal{O}_0(k^{-1})\, d\mu = 0 \quad \text{and} \quad \lim_{k\to\infty} k^{-n} f_k'(0) = 0. \qquad \square$$

3.4. *Conclusion: proof of Theorem 1.1*

We show the following stronger theorem, which implies Theorem 1.1 by Remark 3.1:

Theorem 3.11. *Let (X, L) be a polarized manifold that carries extremal metrics representing $c_1(L)$. The modified Mabuchi functional with respect to the G-action induced by the extremal vector field of $c_1(L)$ attains its minimum at extremal metrics.*

Proof. Let $\phi \in \mathcal{H}^G$ and let ϕ^* be the potential of an extremal metric. Since

$$\mathcal{L}_k^\sigma(\phi) = Z_k^\sigma \circ Hilb_k(\phi) + (\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)),$$

by Proposition 3.10 we have

$$\mathcal{L}_k^\sigma(\phi) \geq Z_k^\sigma \circ Hilb_k(\phi^*) + k^n \varepsilon_\phi(k) + (\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)).$$

Then

$$\begin{aligned} \mathcal{L}_k^\sigma(\phi) \geq \mathcal{L}_k^\sigma(\phi^*) &+ (Z_k^\sigma \circ Hilb_k(\phi^*) - \mathcal{L}_k^\sigma(\phi^*)) \\ &+ k^n \varepsilon_\phi(k) + (\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)). \end{aligned} \tag{3.21}$$

To conclude, from Proposition 3.7,

$$k^{-n}(Z_k^\sigma \circ Hilb_k(\phi^*) - \mathcal{L}_k^\sigma(\phi^*)) \to 0 \quad \text{and} \quad k^{-n}(Z_k^\sigma \circ Hilb_k(\phi) - \mathcal{L}_k^\sigma(\phi)) \to 0$$

as k tends to infinity. So does $\varepsilon_\phi(k)$ by construction (see Proposition 3.10). Thus the result follows from Proposition 3.2, after multiplying by k^{-n} and letting $k \to \infty$ in (3.21). $\square$

4. Appendix

We give the proof of the results concerning σ-balanced metrics. We denote by $(\cdot,\cdot)$ any of the following Hermitian pairings:

$$T^*X \times (T^*X \times L) \to L, \quad L \times (T^*X \times L) \to T^*X,$$
$$L \times L \to \mathbb{C}, \qquad T^*X \times T^*X \to \mathbb{C},$$

obtained from $\phi \in \mathcal{H}$ and ω_ϕ. We denote the connection of type $(1,0)$ on the holomorphic tangent bundle $T'X$ by ∇.

4.1. The definition of I^σ

Proposition 4.1. *$I^\sigma(\phi)$ is independent of the choice of a path from* 0 *to ϕ.*

Proof. Since $I^\sigma(\phi)$ satisfies the cocycle property

$$I^\sigma(\phi_1,\phi_3) = I^\sigma(\phi_1,\phi_2) + I^\sigma(\phi_2,\phi_3)$$

by definition, it is sufficient to prove that $\frac{\partial^2}{\partial s\partial t} I^\sigma(\phi_{0,0},\phi_{t,s})$ is symmetric with respect to s and t for any family of paths

$$\{\Phi = \phi_{t,s} \mid (s,t) \in [0,1]\times[0,1],\ \phi_{0,s} = \phi_{1,s} \equiv 0\}$$

in $\mathcal{H}$. Indeed,

$$\begin{aligned}
&\frac{\partial^2}{\partial s\partial t} I^\sigma(\phi_{0,0},\phi_{t,s}) = \frac{\partial}{\partial s}\int_X \left((1+\Delta_\Phi)\frac{\partial \Phi}{\partial t}\right) e^{\psi_{\sigma,\Phi}}\, d\mu_\Phi \\
&= \int_X \left(\left(\frac{\partial}{\partial s}\Delta_\Phi\right)\frac{\partial\Phi}{\partial t}\right) e^{\psi_{\sigma,\Phi}}\, d\mu_\Phi + \int_X \left((1+\Delta_\Phi)\frac{\partial^2\Phi}{\partial s\partial t}\right) e^{\psi_{\sigma,\Phi}}\, d\mu_\Phi \\
&+ \int_X \left((1+\Delta_\Phi)\frac{\partial\Phi}{\partial t}\right)\left(\frac{\partial e^{\psi_{\sigma,\Phi}}}{\partial s}\right) d\mu_\Phi - \int_X \left((1+\Delta_\Phi)\frac{\partial \Phi}{\partial t}\right) e^{\psi_{\sigma,\Phi}}\left(\Delta_\Phi \frac{\partial\Phi}{\partial s}\right) d\mu_\Phi.
\end{aligned} \tag{4.1}$$

The first term on the right hand side of (4.1) is

$$\int_X \left(\nabla\overline{\nabla}\frac{\partial\Phi}{\partial t}, \nabla\overline{\nabla}\frac{\partial\Phi}{\partial s}\right) e^{\psi_{\sigma,\Phi}}\, d\mu_\Phi,$$

which is symmetric. The second term is obviously symmetric. The third term is

$$\int_X \frac{\partial\Phi}{\partial t}\left(\nabla\psi_{\sigma,\Phi}, \nabla\frac{\partial\Phi}{\partial s}\right) e^{\psi_{\sigma,\Phi}}\, d\mu_\Phi + \int_X \left(\Delta_\Phi\frac{\partial\Phi}{\partial t}\right)\left(\nabla\psi_{\sigma,\Phi}, \nabla\frac{\partial\Phi}{\partial s}\right) e^{\psi_{\sigma,\Phi}}\, d\mu_\Phi. \tag{4.2}$$

Here we use the following equality.

Lemma 4.2.

$$\frac{\partial\psi_{\sigma,\Phi}}{\partial s} = \left(\nabla\psi_{\sigma,\Phi}, \nabla\frac{\partial\Phi}{\partial s}\right). \tag{4.3}$$

Proof. Let v be the gradient vector field of $\frac{\partial\Phi}{\partial s}$, i.e.,

$$v=\mathrm{grad}_{\omega_\Phi}\left(\frac{\partial\Phi}{\partial s}\right)=\sum_{i,j}g^{i\bar{j}}\frac{\partial}{\partial\bar{z}^j}\left(\frac{\partial\Phi}{\partial s}\right)\frac{\partial}{\partial z^i}. \tag{4.4}$$

We have

$$\begin{aligned}\frac{\partial}{\partial s}(\sigma(1)^*\omega_\Phi-\omega_\Phi)=L_v(\sigma(1)^*\omega_\Phi-\omega_\Phi)&=\frac{\sqrt{-1}}{2\pi}d\iota_v\partial\bar{\partial}\psi_{\sigma,\Phi}\\&=\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\right)\end{aligned}$$

where L_v is the Lie derivative along v. Now, there exists some constant c such that

$$\frac{\partial\psi_{\sigma,\Phi}}{\partial s}=\left(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\right)+c. \tag{4.5}$$

Recall that $\int_X\psi_{\sigma,\Phi}\,d\mu_\Phi$ is constant with respect to $s,\ t$ by normalization of $\psi_{\sigma,\Phi}$. Since

$$0=\frac{\partial}{\partial s}\int_X\psi_{\sigma,\Phi}\,d\mu_\Phi=\int_X\left(\frac{\partial\psi_{\sigma,\Phi}}{\partial s}-\left(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\right)\right)d\mu_\Phi,$$

the constant c in (4.5) is zero. Hence, (4.3) is proved. □

The fourth term on the right hand side of (4.1) is

$$-\int_X e^{\psi_{\sigma,\Phi}}\frac{\partial\Phi}{\partial t}\Delta_\Phi\frac{\partial\Phi}{\partial s}\,d\mu_\Phi-\int_X e^{\psi_{\sigma,\Phi}}\Delta_\Phi\frac{\partial\Phi}{\partial t}\Delta_\Phi\frac{\partial\Phi}{\partial s}\,d\mu_\Phi. \tag{4.6}$$

The sum of the first term in (4.2) and the first term in (4.6) is

$$-\int_X\frac{\partial\Phi}{\partial t}\left(\Delta_\Phi\frac{\partial\Phi}{\partial s}+\left(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\right)\right)e^{\psi_{\sigma,\Phi}}\,d\mu_\Phi.$$

This is symmetric, because the operator $\Delta_\Phi+(\nabla\psi_{\sigma,\Phi},\nabla)$ is self-adjoint with respect to the weighted volume form $e^{\psi_{\sigma,\Phi}}\,d\mu_\Phi$. The remainder is the second term in (4.2), equal to

$$\begin{aligned}&-\int_X\left(\nabla\overline{\nabla}\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial t}\overline{\nabla}\frac{\partial\Phi}{\partial s}\right)e^{\psi_{\sigma,\Phi}}\,d\mu_\Phi\\&\qquad-\int_X\left(\nabla\frac{\partial\Phi}{\partial t},\nabla\psi_{\sigma,\Phi}\right)\left(\nabla\frac{\partial\Phi}{\partial s},\nabla\psi_{\sigma,\Phi}\right)e^{\psi_{\sigma,\Phi}}\,d\mu_\Phi,\end{aligned}$$

which is symmetric. □

4.2. *Second derivative of I_k^σ*

Proof of Lemma 3.5. We have

$$\begin{aligned}\frac{d^2}{ds^2}I_k^\sigma(\phi_s) &= k^n\frac{d}{ds}\int_X (k+\Delta_\phi)\phi' e^{\psi_{\sigma,\phi}}\,d\mu_\phi\\ &= k^n\int_X(\nabla\overline{\nabla}\phi',\nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi + k^n\int_X(k+\Delta_\phi)\phi'' e^{\psi_{\sigma,\phi}}\,d\mu_\phi\\ &\quad + k^n\int_X((k+\Delta_\phi)\phi')\psi'_{\sigma,\phi}e^{\psi_{\sigma,\phi}}\,d\mu_\phi - k^n\int_X((k+\Delta_\phi)\phi')e^{\psi_{\sigma,\phi}}\Delta_\phi\phi'\,d\mu_\phi. \qquad (4.7)\end{aligned}$$

From (4.3), the third term on the right hand side of (4.7) is equal to

$$k^n\int_X((k+\Delta_\phi)\phi')(\nabla\psi_{\sigma,\phi},\nabla\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi. \qquad (4.8)$$

By integration by parts, the fourth term in (4.7) is equal to

$$\begin{aligned}&-k^{n+1}\int_X|\nabla\phi'|^2e^{\psi_{\sigma,\phi}}\,d\mu_\phi - k^{n+1}\int_X\phi' e^{\psi_{\sigma,\phi}}(\nabla\psi_{\sigma,\phi},\nabla\phi')\,d\mu_\phi\\ &\quad - k^n\int_X(\nabla\Delta_\phi\phi',\nabla\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi - k^n\int_X(\Delta_\phi\phi')(\nabla\psi_{\sigma,\phi},\nabla\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi. \qquad (4.9)\end{aligned}$$

Note that the sum of the second and fourth terms in (4.9) cancels (4.8). The third term in (4.9) is

$$\begin{aligned}&-k^n\int_X(\nabla\overline{\nabla}\phi',\nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi - k^n\int_X(\nabla\overline{\nabla}\phi',\nabla\psi_{\sigma,\phi}\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi\\ &\qquad = -k^n\int_X(\nabla\overline{\nabla}\phi',\nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi - k^n\int_X|\nabla\phi'|^2\Delta_\phi\psi_{\sigma,\phi}e^{\psi_{\sigma,\phi}}\,d\mu_\phi\\ &\qquad\quad + k^n\int_X|\nabla\phi'|^2|\nabla\psi_{\sigma,\phi}|^2e^{\psi_{\sigma,\phi}}\,d\mu_\phi\\ &\qquad = -k^n\int_X(\nabla\overline{\nabla}\phi',\nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}\,d\mu_\phi - k^n\int_X|\nabla\phi'|^2\Delta_\phi e^{\psi_{\sigma,\phi}}\,d\mu_\phi. \qquad (4.10)\end{aligned}$$

Substituting (4.8)–(4.10) into (4.7), we get the desired conclusion. □

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