

## Quasi-isometric embeddings into diffeomorphism groups

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**Abstract.** Let  $M$  be a smooth compact connected oriented manifold of dimension at least two endowed with a volume form. Assuming certain conditions on the fundamental group  $\pi_1(M)$  we construct quasi-isometric embeddings of either free Abelian or direct products of non-Abelian free groups into the group of volume preserving diffeomorphisms of  $M$  equipped with the  $L^p$  metric induced by a Riemannian metric on  $M$ .

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### 1. Introduction

**1.A. The  $L^p$ -metric.** Let  $M$  be a compact connected and oriented Riemannian manifold and let  $\text{Diff}(M, \mu)$  denote the group of  $C^r$ -diffeomorphisms of  $M$  acting by the identity on a neighborhood of the boundary and preserving the volume form  $\mu$  induced by the metric. Unless otherwise stated we assume that  $\text{Diff}(M, \mu)$  is equipped with the  $C^k$ -topology for some fixed  $1 \leq k \leq r \leq \infty$ .

In the present paper we study the geometry of the identity component  $\text{Diff}_0(M, \mu)$  of the above group endowed with the right invariant  $L^p$ -metric. It is defined as follows. Let

$$\mathcal{L}_p\{g_t\} := \int_0^1 dt \left( \int_M |\dot{g}_t(x)|^p \mu \right)^{\frac{1}{p}}$$

be the  $L^p$ -length of a smooth isotopy  $\{g_t\}_{t \in [0,1]} \subset \text{Diff}_0(M, \mu)$ , where  $|\dot{g}_t(x)|$  denotes the length of the tangent vector  $\dot{g}_t(x) \in T_x M$  induced by the Riemannian metric. Observe that this length is right-invariant, that is,  $\mathcal{L}_p\{g_t \circ f\} = \mathcal{L}_p\{g_t\}$  for any  $f \in \text{Diff}(M, \mu)$ . It defines a right-invariant metric on  $\text{Diff}_0(M, \mu)$  by

$$\mathbf{d}_p(g_0, g_1) := \inf_{g_t} \mathcal{L}_p\{g_t\},$$

where the infimum is taken over all paths from  $g_0$  to  $g_1$ .

If  $p = 2$  then the group  $\text{Diff}_0(M, \mu)$  is in fact equipped with a Riemannian metric inducing the above  $L^2$ -length. The geodesics of this metric are the solutions of the

equations of the flow of an incompressible fluid [1], which makes the  $p = 2$  case the most interesting. It is known that if  $M$  is a simply connected Riemannian manifold of dimension at least three then the  $L^2$ -diameter of the group  $\text{Diff}_0(M, \mu)$  is finite [18]. On the other hand Eliashberg and Ratiu [10] proved that this diameter is infinite for surfaces and for manifolds with positive first Betti number. See Arnol'd–Khesin [2] and Khesin–Wendt [14], Section 3.6, for a detailed discussion.

**Remark 1.1.** The property of preserving the volume is essential to prove the right invariance of the above metric. One can define the  $L^p$ -metric by defining first a norm of a diffeomorphism  $g$  by

$$\|g\|_p := \inf_{g_t} \mathcal{L}_p\{g_t\},$$

where the infimum is taken over all isotopies from the identity to  $g$ . Then the metric  $\mathbf{d}_p(g, h) := \|gh^{-1}\|_p$  is right invariant by definition. However, in order to prove the triangle inequality it is necessary to use the property of preserving the volume.

**1.B. The main result.** A map  $\psi: (X_1, d_1) \rightarrow (X_2, d_2)$  between metric spaces is called *large scale Lipschitz* [17], Remark 1.9, if there exist constants  $A, B \geq 0$  such that

$$d_2(\psi(x), \psi(y)) \leq A \cdot d_1(x, y) + B.$$

Let  $m \in M$  be a reference point. Let  $\text{ev}_m: \text{Diff}_0(M, \mu) \rightarrow M$  be the evaluation map defined by  $\text{ev}_m(f) := f(m)$  and let  $G_\mu \subset \pi_1(M)$  be the image of the homomorphism induced by  $\text{ev}_m$ . It is easy to prove that  $G_\mu$  is contained in the center of  $\pi_1(M)$ . The subgroup  $G_\mu$  is called the *Gottlieb group* associated with the volume form  $\mu$  because the groups of similar origin were first studied by Gottlieb in [11].

Let  $\text{Diff}(M, \mu, m) \subset \text{Diff}_0(M, \mu)$  be the isotropy of the reference point  $m$ . Let us define a homomorphism

$$\Phi: \text{Diff}(M, \mu, m) \rightarrow \pi_1(M)/G_\mu$$

as follows. Let  $g \in \text{Diff}(M, \mu, m)$  and let  $\{g_t\} \subset \text{Diff}_0(M, \mu)$  be a smooth isotopy from the identity to  $g$ . The value  $\Phi(g)$  is represented by the loop  $\{g_t(m)\}$  and it is straightforward to show that  $\Phi$  is well defined.

Let  $B(m, r) \subset M$  be a ball of radius  $r > 0$  centered at a reference point  $m \in M$ . Let  $\text{Diff}(M, \mu, B(m, r))$  denote the subgroup of  $\text{Diff}_0(M, \mu)$  consisting of diffeomorphisms preserving the ball  $B(m, r)$  pointwise. The metric on the group  $\text{Diff}(M, \mu, B(m, r))$  is induced from the  $L^p$ -metric on  $\text{Diff}_0(M, \mu)$ . The following is our main technical result which is proven in Section 2.

**Theorem 1.2.** *Let  $M$  be a compact connected and oriented Riemannian manifold. For all small enough  $r > 0$  the homomorphism*

$$\Phi: \text{Diff}(M, \mu, B(m, r)) \rightarrow \pi_1(M)/G_\mu$$

*is surjective and large scale Lipschitz with respect to the  $L^p$ -metric on the group  $\text{Diff}(M, \mu, B(m, r))$  and the word metric on  $\pi_1(M)/G_\mu$ .*

Recall that the word norm on a group  $\Gamma$  generated by a symmetric finite set  $S \subset \Gamma$  is defined by

$$|\gamma|_S := \min\{k \in \mathbb{N} \mid \gamma = s_1 \dots s_k \text{ where } s_i \in S\}.$$

The word metric is defined by  $d_S(\gamma_1, \gamma_2) := |\gamma_1(\gamma_2)^{-1}|_S$ . It is right-invariant and it depends on the choice of a finite generating set up to a quasi-isometry (defined below) [5], Example 8.17.

**1.C. Applications.** Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be two metric spaces. A function  $f : X_1 \rightarrow X_2$  is a *quasi-isometric embedding* if there exist two constants  $A \geq 1$  and  $B \geq 0$  such that

$$\frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B. \tag{1}$$

In the case when  $(X_1, d_1)$  and  $(X_2, d_2)$  are metric groups, we require  $f$  to be an *injective homomorphism*. We say that  $f$  is a *quasi-isometry* if, in addition to (1), there exists a constant  $C \geq 0$  such that for every  $u \in X_2$  there exists  $x \in X_1$  with the property

$$d_2(u, f(x)) \leq C.$$

Let  $\Gamma$  be a group equipped with the word metric associated with a finite generating set  $S \subset \Gamma$ . An element  $\gamma$  is called *undistorted* in  $\Gamma$  if there exists a positive constant  $C > 0$  such that

$$|\gamma^n|_S \geq C \cdot n.$$

Otherwise,  $\gamma$  is called *distorted*. These properties do not depend on the choice of a finite generating set.

**Theorem 1.3.** *Let  $M$  be a compact connected and oriented Riemannian manifold of dimension at least two. If  $\pi_1(M)/G_\mu$  contains an undistorted element then  $(\text{Diff}_0(M, \mu), \mathbf{d}_p)$  contains quasi-isometrically embedded free Abelian group of an arbitrary rank. In particular, the metric group  $(\text{Diff}_0(M, \mu), \mathbf{d}_p)$  has infinite diameter.*

This result generalizes a theorem of Eliashberg and Ratiu [10] where they prove the infiniteness of the diameter under the assumption that the first Betti number of  $M$  is positive and the center of the fundamental group is trivial. Notice that there exist compact oriented manifolds with the fundamental group isomorphic to an arbitrary finitely presented group. Such a group can be chosen to have finite abelianization (hence the first Betti number is equal to zero) and usually groups have undistorted elements. For example,  $\text{SL}(2, \mathbb{Z})$  has finite abelianization. Moreover, if  $g \in \text{SL}(2, \mathbb{Z})$  has an eigenvalue  $\lambda$  such that  $|\lambda| \neq 1$  then the cyclic group generated by  $g$  is undistorted.

In fact, no finitely presented group with all its elements distorted is known. On the other hand, Osin [16] constructs an example of an infinite finitely generated group with exactly two conjugacy classes. In particular, such a group does not have undistorted elements.

**Theorem 1.4.** *Let  $M$  be a compact connected and oriented Riemannian manifold of dimension at least three. If  $\pi_1(M)/G_\mu$  contains quasi-isometrically embedded non-Abelian free group then  $(\text{Diff}_0(M, \mu), \mathbf{d}_p)$  contains quasi-isometrically embedded direct product of any finite number of free groups of arbitrary ranks.*

Essentially, the idea of the proof is to appropriately embed disjoint copies of the figure eight into  $M$  and suitably apply Theorem 1.2. Thus the situation is a bit different for surfaces, where we get the following slightly weaker statement.

**Theorem 1.5.** *Let  $\Sigma_{g,k}$  be a compact connected and oriented surface of genus  $g$  with  $k$  boundary components. Then  $(\text{Diff}_0(\Sigma_{g,k}, \mu), \mathbf{d}_p)$  contains quasi-isometrically embedded direct product of  $2g + k - 2$  copies of finitely generated non-Abelian free groups of arbitrary ranks.*

**Remark 1.6.** Let  $\mathbb{D}^2$  be a unit Euclidean disc in  $\mathbb{R}^2$ , i.e. in our notation  $\mathbb{D}^2$  is diffeomorphic to  $\Sigma_{0,1}$ . In [3] Benaim and Gambaudo showed, using a different method, that the group  $(\text{Diff}(\mathbb{D}^2, \mu), \mathbf{d}_p)$  contains quasi-isometrically embedded finitely generated free or free Abelian group of arbitrary rank. Crisp and Wiest proved the same fact for planar right-angled Artin groups [7]. We also would like to mention that M. Kapovich proved that any right-angled Artin group embeds into the group of Hamiltonian diffeomorphisms of any symplectic manifold  $(M, \omega)$ , see [12]. However, it is not known whether the embedding he constructs in [12] is quasi-isometric with respect to  $L^p$ -metric.

## 1.D. Examples

**Example 1.7.** Since the Artin pure braid group  $\mathbf{P}_3$  on three strands is isomorphic to  $\mathbb{F}_2 \times \mathbb{Z}$  (see the proof of Theorem 1.16 in [13]) it embeds quasi-isometrically into  $(\text{Diff}_0(M, \mu), \mathbf{d}_p)$ , where the manifold  $M$  is as in Theorem 1.4.

**Example 1.8.** If a subgroup  $\Gamma$  of a direct product of  $n$  free groups has finitely generated homology up to degree  $n$  then  $\Gamma$  contains a finite index subgroup isomorphic to a direct product of at most  $n$  free groups [6]. On the other hand, there are examples of finitely presented subgroups of a direct product of free groups which have more complicated finiteness properties. More concretely, the kernel  $\Gamma$  of the homomorphism  $(\mathbb{F}_2)^n \rightarrow \mathbb{Z}$  sending each generator to one is of type  $\text{FP}_n$  but not  $\text{FP}_{n+1}$ . These examples are known as *Bieri–Stallings groups*, see Bestvina–Brady [4], Example 6.3.

Suppose that  $\Gamma \rightarrow \mathbb{F}_2 \times \cdots \times \mathbb{F}_2$  is the inclusion of a finitely presented Bieri–Stallings group. It follows from the proof of Theorem 11.7 of Dison’s thesis [9] that

this inclusion is a quasi-isometric embedding. Indeed, since the quotient is isomorphic to  $\mathbb{Z}$ , its isoperimetric function is linear and hence, by Dison’s Lemma 9.5 in [9] the distortion function of the above inclusion is linear. A more straightforward proof can be found in Bridson–Haefliger [5], Exercise 5.12 (3).

Recall that if  $\psi: H \rightarrow G$  is an injective homomorphism of metric groups then its distortion function  $\Delta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is defined by

$$\Delta(r) := \max\{d_H(1, h) \mid d_G(1, \psi(h)) < r\}.$$

Observe that the distortion function is linear if and only if  $\psi$  is a quasi-isometric embedding.

**Remark 1.9.** Let  $M = \mathbb{T}^n$  be the  $n$ -dimensional torus equipped with a volume form  $\mu$ . Neither the result of Eliashberg and Ratiu nor our Theorem 1.3 apply to  $\text{Diff}(\mathbb{T}^n, \mu)$  because  $G_\mu = \pi_1(\mathbb{T}^n)$  and the whole group is central.

**Example 1.10.** Let  $M \subset \mathbb{R}^3$  be a closed domain with free non-Abelian fundamental group. The geometry of  $(\text{Diff}_0(M, \mu), \mathbf{d}_p)$  models the behavior of an incompressible fluid filling a tank of the shape of  $M$ . Theorem 1.4 describes a large scale complexity of mixing such a fluid.

**1.E. The symplectic case.** If  $(M, \omega)$  is a symplectic manifold, then the group  $\text{Diff}_0(M, \mu)$  in all the results above can be replaced either by the group  $\text{Symp}_0(M, \omega)$  of symplectic diffeomorphisms isotopic to the identity, or by the group  $\text{Ham}(M, \omega)$  of Hamiltonian diffeomorphisms, see Remark 2.2 in the proof of Theorem 1.2.

## 2. Proofs

**2.A. An abstract lemma.** Let  $(G, d_G)$  be a metric group with the identity element  $1_G$ . For  $g \in G$  we set

$$\|g\|_G := d_G(1_G, g) \quad \text{and} \quad \text{diam}(G) := \sup_{g \in G} \|g\|_G.$$

**Lemma 2.1.** *Let  $(G, d_G), (H, d_H), (K, d_K)$  be three metric groups, such that  $H$  is finitely generated and  $d_H$  is a word metric w.r.t. some finite generating set  $S$ . Suppose that  $\Phi: G \rightarrow K$  is a large scale Lipschitz homomorphism. Let  $\Psi: H \rightarrow G$  be a homomorphism, such that  $\Phi\Psi: H \rightarrow K$  is a quasi-isometric embedding. Then  $\Psi$  is a quasi-isometric embedding.*

*Proof.* The homomorphism  $\Psi$  is injective, because  $\Phi\Psi$  is injective. Let  $h \in H$ . The homomorphism  $\Phi\Psi$  is a quasi-isometric embedding, hence there exist two constants  $A_1 \geq 0$  and  $B_1 \geq 0$  such that

$$A_1 \|h\|_H - B_1 \leq \|\Phi\Psi(h)\|_K \tag{2}$$

The homomorphism  $\Phi$  is a large scale Lipschitz, hence there exist two constants  $A_2 > 0$  and  $B_2 > 0$  such that

$$\|\Phi\Psi(h)\|_K \leq A_2\|\Psi(h)\|_G + B_2. \quad (3)$$

Combining inequalities (2) and (3) we get

$$\frac{A_1}{A_2}\|h\|_H - \left(\frac{B_1 + B_2}{A_2}\right) \leq \|\Psi(h)\|_G. \quad (4)$$

Let  $k$  be such that  $\|h\|_H = k$ . The group  $H$  is finitely generated, hence  $h = h_1 \dots h_k$ , where  $h_i \in S$  for each  $1 \leq i \leq k$ . Denote by

$$M_\Psi := \max\{\|\Psi(h_1)\|_G, \dots, \|\Psi(h_k)\|_G\}.$$

It follows that

$$\|\Psi(h)\|_G = \|\Psi(h_1) \dots \Psi(h_k)\|_G \leq \sum_{i=1}^k \|\Psi(h_i)\|_G \leq M_\Psi \cdot k = M_\Psi \|h\|_H. \quad (5)$$

Inequalities (4) and (5) conclude the proof of the lemma.  $\square$

In the proofs below we make use of the fact that the metric on the groups  $\text{Diff}(M, \mu, B(m, r))$  and  $\text{Diff}(M, \mu, \sqcup_i B(m_i, r))$  (this group is defined in the proof of Theorem 1.3) is induced from the  $L^p$ -metric on the whole group  $\text{Diff}(M, \mu)$ .

**2.B. Proof of Theorem 1.2.** In the first part we prove the surjectivity of the homomorphism  $\Phi: \text{Diff}(M, \mu, B(m, r)) \rightarrow \pi_1(M)/G_\mu$ .

Let  $S' := \{\gamma_1, \dots, \gamma_k\} \subset \pi_1(M)$  be a symmetric generating set of the fundamental group of  $M$  such that each representative  $\gamma_i$  is a simple closed curve. It follows from the tubular neighborhood theorem that for each  $\gamma_i$  there exists  $r_i > 0$ , the standard  $(n-1)$ -dimensional ball  $B_{r_i}^{n-1} \subset \mathbb{R}^n$  of radius  $r_i > 0$ , and a volume-preserving embedding

$$\text{emb}_i: B_{r_i}^{n-1} \times \mathbb{S}^1 \hookrightarrow M,$$

such that  $\text{emb}_i|_{\{0\} \times \mathbb{S}^1}$  is the curve  $\gamma_i$  and  $r_i = d_M(\gamma_i, \text{emb}_i|_{\partial B_{r_i}^{n-1} \times \mathbb{S}^1})$ . The volume form on the product  $B_{r_i}^{n-1} \times \mathbb{S}^1$  is the standard Euclidean volume and  $d_M$  denotes the distance on  $M$  induced by the Riemannian metric.

Let  $r = \min_{1 \leq i \leq k} r_i$ . Then  $\text{emb}_i: B_r^{n-1} \times \mathbb{S}^1 \hookrightarrow M$  is volume-preserving for each  $i$  and  $\text{emb}_i|_{\{0\} \times \mathbb{S}^1} = \gamma_i$ . It is straightforward to construct a smooth isotopy of volume-preserving diffeomorphisms

$$g_t: B_r^{n-1} \times \mathbb{S}^1 \rightarrow B_r^{n-1} \times \mathbb{S}^1$$

between  $g_0 = \text{Id}$  and  $g_1$  such that:

- For each  $t \in [0, 1]$  the diffeomorphism  $g_t$  equals to the identity in the neighborhood of  $\partial B_r^{n-1} \times \mathbb{S}^1$ , and the time-one map  $g_1$  is equal to the identity on  $B_{r'}^{n-1} \times \mathbb{S}^1$ , where  $0 < r' < r$ .
- Each diffeomorphism  $g_t$  preserves the foliation of  $B_r^{n-1} \times \mathbb{S}^1$  by the circles  $\{x\} \times \mathbb{S}^1$  and for every  $x \in B_{r'}^{n-1}$  the restriction  $g_t : \{x\} \times \mathbb{S}^1 \rightarrow \{x\} \times \mathbb{S}^1$  is the rotation by  $2\pi t$ . It follows that the time-one map  $g_1$  equals the identity on  $B_{r'}^{n-1} \times \mathbb{S}^1$ .

We identify  $B_r^{n-1} \times \mathbb{S}^1$  with its image with respect to the embedding  $\text{emb}_i$ . Then we extend each isotopy  $g_t$  by the identity on  $M \setminus B_r^{n-1} \times \mathbb{S}^1$  obtaining smooth isotopies  $g_{t,i} \in \text{Diff}_0(M, \mu)$ . This shows that every representative  $\gamma_i$  of a generator of the fundamental group of  $M$  arises as simple closed curve  $\{g_{t,i}(m)\}$  and hence the homomorphism

$$\Phi: \text{Diff}(M, \mu, B(m, r)) \rightarrow \pi_1(M)/G_\mu$$

is surjective.

**Remark 2.2.** Notice that if  $M$  is a symplectic manifold then the above isotopies can be constructed to be Hamiltonian.

Let  $\Pi: \pi_1(M) \rightarrow \pi_1(M)/G_\mu$  be the projection homomorphism. Consequently  $S := \Pi(S')$  is a finite generating set for the quotient  $\pi_1(M)/G_\mu$ . Let  $\Pi_M: M_\bullet \rightarrow M$  be the Riemannian covering associated with  $\Pi$ . This means that the metric on  $M_\bullet$  is induced from the Riemannian metric on  $M$ . The corresponding distance will be denoted by  $d_\bullet$ .

Now we shall prove that  $\Phi$  is a large scale Lipschitz map. That is, we show that there exist positive constants  $A$  and  $B$  independent of  $g$  such that

$$A \cdot \|g\|_p + B \geq \|\Phi(g)\|_S,$$

where  $\|g\|_p := \mathbf{d}_p(g, \text{Id})$  is the  $L^p$ -norm of the diffeomorphism  $g$ .

Let  $g \in \text{Diff}(M, \mu, B(m, r))$  and let  $\{g_t\}_{t \in [0,1]} \in \text{Diff}_0(M, \mu)$  be an isotopy from the identity to  $g$ . It follows from the Hölder inequality that  $\|g\|_p \geq C_p \cdot \|g\|_1$ , where  $C_p$  is some positive constant independent of  $g$ . Hence it is enough to prove the statement for  $p = 1$ .

Let  $m_\bullet \in \Pi_M^{-1}(m)$ , and let  $\{g_{\bullet,t}(m_\bullet)\}$  be the lift of  $\{g_t(m)\}$  starting at the point  $m_\bullet$ . The manifold  $M$  is compact, hence by the Švarc–Milnor lemma [5], [15], the inclusion of the orbit of  $m_\bullet$  with respect to the deck transformation group  $\pi_1(M)/G_\mu$  defines a quasi-isometry

$$\pi_1(M)/G_\mu \stackrel{\text{q.i.}}{\simeq} (M_\bullet, d_\bullet).$$

In particular, it means that there exist positive constants  $A', B'$  such that

$$d_\bullet(m_\bullet, g_{\bullet,1}(m_\bullet)) \geq A' \|\Phi(g)\|_S - B'. \tag{6}$$

Let  $x \in B(m, r) \subset M$ . We claim that the length of the flow-line  $g_t(x)$  is bounded by the distance  $d_\bullet(m_\bullet, g_{\bullet,1}(m_\bullet))$  up to the diameter of the ball. To see this, consider the lift of each flow-line  $g_t(x)$  starting at a ball of radius  $r$  in  $M_\bullet$  centered at  $m_\bullet$  and observe that such a lift ends in a ball of radius  $r$  centered at  $g_{\bullet,1}(m_\bullet)$ . Indeed, let  $\alpha : [0, 1] \rightarrow B(m, r)$  be a path between  $x$  and  $m$ . Then the map

$$H : [0, 1] \times [0, 1] \rightarrow M,$$

defined by  $H(t, s) = g_t(\alpha(s))$ , is a homotopy from  $\{g_t(m)\}$  to  $\{g_t(x)\}$ . Lifting this homotopy shows that the lift of  $\{g_t(x)\}$  ends at the ball  $B(g_{\bullet,1}(m_\bullet), r)$ . Finally, we obtain that

$$\text{Length}(g_t(x)) := \int_0^1 |\dot{g}_t(x)| dt \geq d_\bullet(m_\bullet, g_{\bullet,1}(m_\bullet)) - 2r. \tag{7}$$

as claimed. Combining inequalities (6) and (7) we get that

$$\text{Length}(g_t(x)) \geq A' \|\Phi(g)\|_S - (B' + 2r)$$

for every  $x \in B(m, r)$ . Hence by Fubini theorem and the above inequality we have

$$\begin{aligned} \mathcal{L}_1(\{g_t\}) &= \int_0^1 dt \left( \int_M |\dot{g}_t(x)| \mu \right) \\ &= \int_M \mu \left( \int_0^1 |\dot{g}_t(x)| dt \right) \\ &\geq \text{vol}(B(m, r)) \cdot \min_{x \in B(m, r)} \text{Length}(g_t(x)) \\ &\geq \text{vol}(B(m, r)) \cdot A' \|\Phi(g)\|_S - \text{vol}(B(m, r)) \cdot (B' + 2r). \end{aligned}$$

Since the above inequalities hold for any isotopy  $\{g_t\}_{t \in [0,1]}$  between the identity and  $g$ , we obtain that

$$\|\Phi(g)\|_S \leq A \cdot \|g\|_p + B,$$

where  $A = (C_p \cdot \text{vol}(B(m, r))^{-1} \cdot A'$  and  $B = \frac{B'+2r}{A'}$  and this concludes the proof. □

**2.C. Proof of Theorem 1.3.** Let  $n \in \mathbb{N}$  be a positive integer. Recall that we need to prove that there exists a quasi-isometric embedding of a free Abelian group of rank  $n$  into  $\text{Diff}_0(M, \mu)$ .

Assume first that the dimension of  $M$  is at least three. Let  $m_1, \dots, m_n$  be distinct points in the interior of  $M$  and let  $r > 0$  be such that the balls  $B(m_i, r)$  of radius  $r$  centered at  $m_i$  are pairwise disjoint. Let  $\gamma_{i,j}$  be simple closed curves representing the generators of  $\pi_1(M, m_i)$ . We also assume that  $\gamma_{i_1, j_1}$  is disjoint from  $\gamma_{i_2, j_2}$  whenever  $i_1 \neq i_2$ . We choose  $r$  small enough such that the tubular neighborhood of radius  $r$  of the above generators are disjoint.

We denote by  $G_i \subset \pi_1(M, m_i)$  be the corresponding Gottlieb group. The groups  $\pi_1(M, m_i)/G_i$  are pairwise isomorphic. Let  $\gamma_i \in \pi_1(M, m_i)/G_i$  be an undistorted element which exists according to the hypothesis. Let

$$h: \mathbb{Z}^n \rightarrow \pi_1(M, m_1)/G_1 \times \cdots \times \pi_1(M, m_n)/G_n$$

be the homomorphism defined by

$$h(k_1, \dots, k_n) := (\gamma_1^{k_1}, \dots, \gamma_n^{k_n}).$$

It immediately follows from the fact that each  $\gamma_i$  is undistorted that  $h$  is a quasi-isometric embedding. Let  $\text{Diff}(M, \mu, \sqcup_i B(m_i, r)) \subset \text{Diff}_0(M, \mu)$  be the subgroup consisting of diffeomorphisms preserving the disjoint union of balls  $B(m_i, r)$  point-wise. Let

$$\Phi_i: \text{Diff}(M, \mu, B(m_i, r)) \rightarrow \pi_1(M, m_i)/G_i$$

be the homomorphism defined in Theorem 1.2. Consider a homomorphism

$$\tilde{\Phi}: \text{Diff}(M, \mu, \sqcup_i B(m_i, r)) \rightarrow \pi_1(M, m_1)/G_1 \times \cdots \times \pi_1(M, m_n)/G_n$$

which is the composition of the (diagonal) inclusion

$$\iota: \text{Diff}(M, \mu, \sqcup_i B(m_i, r)) \hookrightarrow \prod_i \text{Diff}(M, \mu, B(m_i, r))$$

followed by the product homomorphism

$$\prod_i \Phi_i: \prod_i \text{Diff}(M, \mu, B(m_i, r)) \rightarrow \prod_i \pi_1(M, m_i)/G_i.$$

Since the inclusion  $\iota$  is an isometric embedding and the  $\prod_i \Phi_i$  is large scale Lipschitz, according to Theorem 1.2, we obtain that  $\tilde{\Phi}$  is a large scale Lipschitz homomorphism.

Let  $g_i \in \text{Diff}(M, \mu, B(m_i, r))$  be an element such that  $\Phi_i(g_i) = \gamma_i$  and  $g_i$  is supported in the union of the tubular neighborhoods of the loops representing the generators of  $\pi_1(M, m_i)$  constructed in the beginning of the proof. It follows that the supports of  $g_i$  and  $g_j$  are disjoint if  $i \neq j$ . The existence of  $g_i$  follows from the proof of Theorem 1.2. Let

$$\Psi: \mathbb{Z}^n \rightarrow \text{Diff}(M, \mu, \sqcup_i B(m_i, r)) \subset \text{Diff}_0(M, \mu)$$

be defined by

$$\Psi(k_1, \dots, k_n) := g_1^{k_1} \circ \cdots \circ g_n^{k_n}.$$

It is well defined because  $g_i$  have pair-wise disjoint supports. Recall that we have that

$$h = \tilde{\Phi} \circ \Psi: \mathbb{Z}^n \rightarrow \prod_i \pi_1(M, m_i)/G_i$$

and we know that  $h$  is a quasi-isometric embedding and  $\tilde{\Phi}$  is large scale Lipschitz. Consequently the map  $\Psi$  is a quasi-isometric embedding according to Lemma 2.1.

Let us now consider the two-dimensional case. Let  $M = \Sigma_{g,k}$  be a compact oriented surface of genus  $g$  with  $k$  boundary components. Observe that  $\pi_1(\Sigma_{g,k})/G_\mu$  is trivial if either  $g = 0$  and  $k \leq 2$  or  $g = 1$  and  $k = 0$ . Otherwise it is either free non-Abelian group or the fundamental group of a closed oriented surface. In each case it is straightforward to define an embedding

$$\text{emb}: \mathbb{S}^1 \times [0, 2n] \rightarrow M$$

in such a way that each loop  $\text{emb}(\mathbb{S}^1 \times \{t\})$  represents an undistorted element in  $\pi_1(M, \text{emb}(1, t))$ . Let  $g_i: M \rightarrow M$ , for  $i = 1, \dots, n$  be an area preserving diffeomorphism satisfying each of the following conditions:

- it is supported in  $\text{emb}(\mathbb{S}^1 \times (2i - 2, 2i))$ ;
- it preserves the ball  $\text{emb}(B_i)$ , where  $B_i \subset \mathbb{S}^1 \times [0, 2n]$  is a ball of diameter one centered at  $(1, 2i - 1)$ ;
- it is the time one map of an isotopy from the identity which acts as the full rotation on the loop  $\text{emb}(\mathbb{S}^1 \times \{2i - 1\})$ .

As in the previous part the homomorphism  $\Psi: \mathbb{Z}^n \rightarrow \text{Diff}_0(M, \mu)$  defined by  $\Psi(k_1, \dots, k_n) := g_1^{k_1} \circ \dots \circ g_n^{k_n}$  is the required quasi-isometric embedding. □

**2.D. Proof of Theorem 1.4.** This proof is a modification of the proof of Theorem 1.3 for three dimensional  $M$  where the cyclic group  $\mathbb{Z}$  is replaced by a non-Abelian free group  $\mathbb{F}_2$  on two generators. More precisely, let  $f_i, g_i \in \text{Diff}_0(M, \mu)$  be diffeomorphisms satisfying each of the following conditions (we use here the notation of the proof of Theorem 1.3):

- the support of  $f_i$  and  $g_i$  is contained in the neighborhood of the union of the loops  $\gamma_{i,j}$ ;
- the free non-Abelian group in  $\pi_1(M, m_i)/G_i$  is generated by the images  $\Phi_i(f_i)$  and  $\Phi_i(g_i)$ .

Such diffeomorphisms can be constructed in a similar way as  $g_i$ 's in the proof of Theorem 1.3.

Let  $w \in \mathbb{F}_2$  be a reduced word and given two elements  $f, g \in \text{Diff}(M, \mu)$  let  $w(f, g)$  denote the induced diffeomorphism of  $M$ . Let

$$\Psi: \mathbb{F}_2 \times \dots \times \mathbb{F}_2 \rightarrow \text{Diff}(M, \mu, \sqcup_i B(m_i, r)) \subset \text{Diff}_0(M, \mu)$$

be defined by  $\Psi(w_1, \dots, w_n) := w_1(f_1, g_1) \circ \dots \circ w_n(f_n, g_n)$ . As before  $\Psi$  is a quasi-isometric embedding of a product of free groups on two generators into  $\text{Diff}_0(M, \mu)$ . Since  $\mathbb{F}_2$  contains quasi-isometrically embedded a non-Abelian free group of an arbitrary finite rank [8] the proof is finished. □

**2.E. Proof of Theorem 1.5.** The proof of the two dimensional case of Theorem 1.3 amounts to constructing a number of disjoint simple closed curves representing an undistorted element in the fundamental group of  $M$ . The present proof is analogous in the sense that we need to construct an embedding of the disjoint union of  $2g + k - 2$  copies of the figure eight into  $M$  such that each embedding induces a quasi-isometric embedding  $\mathbb{F}_2 \rightarrow \pi_1(M, m_i)$  for  $i = 1, \dots, 2g + k - 2$ . We leave this straightforward construction as an exercise to the reader.

The rest of the proof is similar to the other proofs. That is, we construct relevant diffeomorphisms  $f_i, g_i$  and observe that the map

$$\Psi: \mathbb{F}_2 \times \cdots \times \mathbb{F}_2 \rightarrow \text{Diff}(\Sigma_{g,k}, \mu, \sqcup_i B(m_i, r)) \subset \text{Diff}_0(\Sigma_{g,k}, \mu)$$

defined by  $\Psi(w_1, \dots, w_n) := w_1(f_1, g_1) \circ \dots \circ w_n(f_n, g_n)$  is a quasi-isometric embedding.  $\square$

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## References

- [1] V. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)* **16** (1966), 319–361. [Zbl 0148.45301](#) [MR 0202082](#)
- [2] V. I. Arnold and B. A. Khesin, *Topological methods in hydrodynamics*. Appl. Math. Sci. 125, Springer-Verlag, New York 1998. [Zbl 0902.76001](#) [MR 1612569](#)
- [3] M. Benaim and J.-M. Gambaudo, Metric properties of the group of area preserving diffeomorphisms. *Trans. Amer. Math. Soc.* **353** (2001), 4661–4672. [Zbl 0977.57039](#) [MR 1851187](#)
- [4] M. Bestvina and N. Brady, Morse theory and finiteness properties of groups. *Invent. Math.* **129** (1997), 445–470. [Zbl 0888.20021](#) [MR 1465330](#)
- [5] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Grundlehren Math. Wiss. 319, Springer-Verlag, Berlin 1999. [Zbl 0988.53001](#) [MR 1744486](#)
- [6] M. R. Bridson, J. Howie, C. F. Miller, III, and H. Short, Subgroups of direct products of limit groups. *Ann. of Math. (2)* **170** (2009), 1447–1467. [Zbl 1196.20047](#) [MR 2600879](#)
- [7] J. Crisp and B. Wiest, Quasi-isometrically embedded subgroups of braid and diffeomorphism groups. *Trans. Amer. Math. Soc.* **359** (2007), 5485–5503. [Zbl 1186.20027](#) [MR 2327038](#)
- [8] P. de la Harpe, *Topics in geometric group theory*. Chicago Lectures in Math., The University of Chicago Press, Chicago 2000. [Zbl 0965.20025](#) [MR 1786869](#)

- [9] W. J. Dison, Isoperimetric functions for subdirect products and Bestvina-Brady groups. D.Phil. Thesis, Imperial College, London 2008. [www.maths.bris.ac.uk/~mawjd/thesis.pdf](http://www.maths.bris.ac.uk/~mawjd/thesis.pdf)
- [10] Y. Eliashberg and T. Ratiu, The diameter of the symplectomorphism group is infinite. *Invent. Math.* **103** (1991), 327–340. [Zbl 0725.58006](#) [MR 1085110](#)
- [11] D. H. Gottlieb, A certain subgroup of the fundamental group. *Amer. J. Math.* **87** (1965), 840–856. [Zbl 0148.17106](#) [MR 0189027](#)
- [12] M. Kapovich, RAAGs in Ham. *Geom. Funct. Anal.* **22** (2012), 733–755. [Zbl 1262.53070](#) [MR 2972607](#)
- [13] C. Kassel and V. Turaev, *Braid groups*. Graduate Texts in Math. 247, Springer, New York 2008. [Zbl 1208.20041](#) [MR 2435235](#)
- [14] B. Khesin and R. Wendt, *The geometry of infinite-dimensional groups*. *Ergeb. Math. Grenzgeb.* (3) 51, Springer-Verlag, Berlin 2009. [Zbl 1153.22001](#) [MR 2456522](#)
- [15] J. Milnor, A note on curvature and fundamental group. *J. Differential Geometry* **2** (1968), 1–7. [Zbl 0162.25401](#) [MR 0232311](#)
- [16] D. Osin, Small cancellations over relatively hyperbolic groups and embedding theorems. *Ann. of Math. (2)* **172** (2010), 1–39. [Zbl 1203.20031](#) [MR 2680416](#)
- [17] J. Roe, *Lectures on coarse geometry*. Univ. Lecture Ser. 31, American Mathematical Society, Providence, RI, 2003. [Zbl 1042.53027](#) [MR 2007488](#)
- [18] A. I. Shnirelman, Generalized fluid flows, their approximation and applications. *Geom. Funct. Anal.* **4** (1994), 586–620. [Zbl 0851.76003](#) [MR 1296569](#)

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