

On cyclic CAT(0) domains of discontinuity

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Abstract. Let X be a CAT(0) space, and G a discrete cyclic group of isometries of X . We investigate the domain of discontinuity for the action of G on the boundary ∂X .

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1. Introduction

The idea of a domain of discontinuity was first investigated in the setting of Kleinian groups, discrete groups of isometries of hyperbolic m -space, \mathbb{H}^m . Using the Poincaré ball model of \mathbb{H}^m , where \mathbb{H}^m is the interior of the open unit m -ball, we see that the visual boundary of \mathbb{H}^m (see Section 2 for definition) is the unit sphere \mathbb{S}^{m-1} , and we see that isometries of \mathbb{H}^m extend to homeomorphisms of the closed m -ball, $\mathbb{H}^m \cup \mathbb{S}^{m-1}$.

Let G be a discrete group of isometries of \mathbb{H}^m , and (g_i) a sequence of distinct elements of G . Fixing a point $a \in \mathbb{H}^m$, using compactness and passing to a subsequence, we may assume that $g_i(a) \rightarrow p \in \mathbb{S}^{m-1}$ and $g_i^{-1}(a) \rightarrow n \in \mathbb{S}^{m-1}$. It can be shown that $g_i(x) \rightarrow p$ uniformly on compact subsets of $\mathbb{S}^{m-1} - \{n\}$. Since any sequence of elements of G has such a subsequence, we say that G acts as a convergence group on \mathbb{S}^{m-1} .

The set of all such p is the limit set of the Kleinian group G ,

$$\Lambda G = \{p \in \mathbb{S}^{m-1} : \text{there exists } (g_i) \subset G \text{ with } g_i(x) \rightarrow p \text{ for some } x \in \mathbb{H}^m\}$$

and the domain of discontinuity of G , $\Omega G = \mathbb{S}^{m-1} - \Lambda G$. Clearly ΛG is G -invariant and closed, and therefore ΩG is G -invariant and open. Let $p \in \Lambda G$ and $(g_i) \subset G$ with $g_i(x) \rightarrow p$ and $g_i^{-1}(x) \rightarrow n \in \Lambda G$ for some (any) $x \in \mathbb{H}^m$. Then for any $a \in \Omega G$, $g_i(a) \rightarrow p$. Thus if $\Omega G \neq \emptyset$, then p is a limit point of ΩG , and so ΩG is dense in \mathbb{S}^{m-1} .

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We thus arrive at the following result: If G is a discrete group of isometries of hyperbolic m -space, \mathbb{H}^m , then $\partial\mathbb{H}^m = \mathbb{S}^{m-1}$ is a disjoint union of the limit set of G , ΛG , and the domain of discontinuity of G , ΩG , where ΩG is open and either dense or empty. Since G acts as a convergence group on \mathbb{S}^{m-1} , the action of G on ΩG is *properly discontinuous* (for any compact $C \subset \Omega G$, $\{g \in G : g(C) \cap C \neq \emptyset\}$ is finite).

This result holds whenever G acts as a discrete convergence group on a compact Hausdorff space, in particular when G is a properly discontinuous group of isometries of a proper δ -hyperbolic space [9].

In this paper we consider the case where the negative curvature (δ -hyperbolic) condition is relaxed to a non-positive curvature (CAT(0)) condition. This result fails in general, as [10] gives an example of a discrete group of isometries of a CAT(0) space whose limit set contains a nonempty open subset of the boundary, but is not the whole boundary.

The conjecture seems to be that if a cyclic subgroup acts “nicely” on a CAT(0) space and the action on the boundary is not virtually trivial, then there should be an open dense subset of the boundary on which the cyclic subgroup acts properly discontinuously.

The conjecture is realized when the Tits diameter of the boundary is large:

Theorem 4. *Let X be a complete CAT(0) space with Tits radius of ∂X more than 3π and $|\partial X| > 2$. If h is a hyperbolic isometry of X , then $\langle h \rangle$ acts properly discontinuously on the open subset $\Omega = \partial X - [\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})]$. If $[\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})] \subset \Lambda X$, then Ω is dense in ∂X .*

Herein $\bar{B}_T(x, r)$ is the closed ball in the Tits metric about x of radius r and ΛX is the limits set of the isometry group of X .

Theorem 6. *Let X be a proper CAT(0) space with Tits radius more than 3π , and h a parabolic isometry of X . Then there exists $m \in \partial X$, a fixed point of h , such that $\langle h \rangle$ acts properly discontinuously on the open dense subset $\Omega = \partial X - \bar{B}_T(m, \pi)$ of ∂X . If $\bar{B}_T(m, \pi) \subset \Lambda X$, then Ω is dense in ∂X .*

When the Tits radius is small, we get the following weaker (but much more difficult) result.

Main Theorem. *Let g be a hyperbolic isometry of the cocompact proper CAT(0) space X . If $\Omega = \partial X - S_g \neq \emptyset$, then Ω is a dense open subset of ∂X , and $\langle g \rangle$ acts discretely on Ω .*

Herein S_g is the union of $\{g^\pm\}$ and all Tits geodesics of length π (if any) between g^- and g^+ in ∂X .

The case where $\partial X = S_g$ will not be addressed in this note.

In the process of proving this main theorem, we obtain the following result which is of independent interest (perhaps more so than our original problem). The version given here is slightly weaker than what we prove, but is easier to state.

Corollary 16. *Let X be a proper CAT(0) space with almost extendable geodesics and finite dimensional Tits boundary, and G a group of isometries of X . Suppose there is a convex subset W of X satisfying the following conditions:*

- $W = Y \times Z$ where $Y \cong \mathbb{E}^m$.
- ∂W contains an open subset U of ∂X .
- There exists a geodesic ray $R \subset W$ with $R(\infty) \in U$, and a compact $C \subset X$ with $R \subset \bigcup_{g \in G} g(C)$.

Then every point of ∂Y is a spherical suspension point of ∂X .

Herein a is a spherical suspension point of ∂X if there exists $b \in \partial X$ and $W \subset \partial X$ such that $\partial X = \{a, b\} * W$, where $*$ is the spherical join. The author wishes to thank M. Kapovich, G. Levitt and K. Ruane for helpful conversations, and the referee and M. Kapovich for suggestions of improvement.

2. Definitions and basic results

We refer the reader to [5] or [1] for more details of the following.

Definition. For X a geodesic metric space and $\Delta(a, b, c)$ a geodesic triangle in X with vertices $a, b, c \in X$ there is a comparison triangle $\bar{\Delta} = \Delta(\bar{a}, \bar{b}, \bar{c}) \subset \mathbb{E}^2$ with $d(a, b) = d(\bar{a}, \bar{b})$, $d(a, c) = d(\bar{a}, \bar{c})$ and $d(b, c) = d(\bar{b}, \bar{c})$. We define the comparison angle $\bar{\angle}_a(b, c) = \angle_{\bar{a}}(\bar{b}, \bar{c})$.

Each point $z \in \Delta(a, b, c)$ has a unique comparison point, $\bar{z} \in \bar{\Delta}$. We say that the triangle $\Delta(a, b, c)$ is CAT(0) if for any $y, z \in \Delta(a, b, c)$ with comparison points $\bar{y}, \bar{z} \in \bar{\Delta}$, $d(y, z) \leq d(\bar{y}, \bar{z})$. The space X is said to be CAT(0) if every geodesic triangle in X is CAT(0).

If X is CAT(0), notice that for any geodesics $\alpha: [0, r] \rightarrow X$ and $\beta: [0, s] \rightarrow X$ with $\alpha(0) = \beta(0) = a$, the function

$$\theta(r, s) = \bar{\angle}_{r(0)}(\alpha(r), \beta(s))$$

is an increasing function of r, s . Thus $\lim_{r,s \rightarrow 0} \theta(r, s)$ exists and we call this limit $\angle_a(\alpha(r), \beta(s))$. It follows that for any $a, b, c \in X$, a CAT(0) space,

$$\angle_a(b, c) \leq \bar{\angle}_a(b, c).$$

For the following discussion, we assume that X is a complete CAT(0) space. The (visual) boundary, ∂X , is the set of equivalence classes of rays, where rays are

equivalent if they are within finite Hausdorff distance from each other. Given a ray R and a point $x \in X$, there is a ray S emanating from x with $R \sim S$. Fixing a base point $\mathbf{0} \in X$, we define the visual topology on $\bar{X} = X \cup \partial X$ by taking the basic open sets of $x \in X$ to be the open metric balls about x . For $y \in \partial X$, and R a ray from $\mathbf{0}$ representing y , we construct basic open sets $U(R, n, \varepsilon)$ where $n, \varepsilon > 0$. We say $z \in U(R, n, \varepsilon)$ if the unit speed geodesic, $S: [0, d(\mathbf{0}, z)] \rightarrow \bar{X}$, from $\mathbf{0}$ to z satisfies $d(R(n), S(n)) < \varepsilon$. These sets form a basis for a regular topology on \bar{X} and ∂X . For any $x \in X$ and $u, v \in \partial X$, we can define $\angle_x(u, v)$ and $\bar{\angle}_x(u, v)$ by parameterizing the rays $[x, u)$ and $[x, v)$ by $t \in [0, \infty)$ and taking the limit of \angle_x as $t \rightarrow 0$ and $t \rightarrow \infty$ respectively.

For $u, v \in \partial X$, we define $\angle(u, v) = \sup_{p \in X} \angle_p(u, v)$. It follows from [5] that $\angle(u, v) = \bar{\angle}_p(u, v)$ for any $p \in X$. Notice that isometries of X preserve the angle between points of ∂X . The angle defines a path metric, d_T on the set ∂X , called the Tits metric, whose topology is finer than the visual topology of ∂X . Also $\angle(a, b)$ and $d_T(a, b)$ are equal whenever either of them is less than π . For any $u \in \partial X$, we define $B_T(u, \varepsilon) = \{v \in \partial X : d_T(u, v) < \varepsilon\}$ and $\bar{B}_T(u, \varepsilon) = \{v \in \partial X : d_T(u, v) \leq \varepsilon\}$.

The set ∂X with the Tits metric is called the Tits boundary of X , denoted TX . Isometries of X extend to isometries of TX

The identity function $TX \rightarrow \partial X$ is continuous, but the identity function $\partial X \rightarrow TX$ is only lower semi-continuous. That is for any sequences $(u_n), (v_n) \subset \partial X$ with $u_n \rightarrow u$ and $v_n \rightarrow v$ in ∂X , then

$$\underline{\lim} d_T(u_n, v_n) \geq d_T(u, v).$$

Lemma 1 ([7], p. 175, Proposition 3.4.3). *Let X be a complete CAT(0) space, $v \in \partial X$, and $H \subset X$ a horoball centered at v . Then $\bar{H} \cap \partial X = \bar{B}_T(v, \frac{\pi}{2})$.*

Proof. We first show that $\bar{H} \cap \partial X \subseteq \bar{B}_T(v, \frac{\pi}{2})$. Suppose that $w \in \bar{H} \cap \partial X$ with $\angle(v, w) > \frac{\pi}{2}$. Choose a point $y \in X$ with $\angle_y(v, w) > \frac{\pi}{2}$.

Let $R: [0, \infty) \rightarrow X$ be the geodesic ray from y to v , and $b_R: X \rightarrow \mathbb{R}$ be the Buseman function associated to R , $b_R(x) = \lim_{t \rightarrow \infty} [d(x, R(t)) - t]$. Note the $b_R(y) = 0$ and $b_R(R(t)) = -t$ for $t > 0$. By [5], Exercise 8.23 (1), p. 272, any horoball based at v is within finite distance of any other horoball based at v . It follows that all horoballs based at v will have the same limit points in the boundary. Thus we may assume that $H = b_R^{-1}([-1, -\infty))$.

Since $w \in \bar{H}$, there exists $\hat{w} \in H$ such that $\angle_y(v, \hat{w}) > \frac{\pi}{2}$.

We recall [5], Exercise 8.23 (4), p. 273. Choose a geodesic line $\bar{R}: \mathbb{R} \rightarrow \mathbb{E}^2$. For each $x \in X$ and $n \in \mathbb{N}$ consider a comparison triangle $\bar{\Delta}(x, R(0), R(n))$ with $\bar{R}(i)$ the comparison point for $R(i)$ ($i = 0, n$) and $x_n \in \mathbb{E}^2$ the comparison point for x . Choose $r_n \in \mathbb{R}$ with $\bar{R}(r_n) = \pi_{\bar{R}}(x_n)$ (where $\pi_{\bar{R}}$ is orthogonal projection to the line \bar{R}). Then $r_n \rightarrow r$ where $b_R(x) = -r$. Since $b_R(\hat{w}) \leq -1$, for $n \gg 0$, $r_n > 0$. It follows that $\angle_{\bar{R}(0)}(\bar{R}(n), \hat{w}_n) < \frac{\pi}{2}$. However $\frac{\pi}{2} < \angle_y(v, \hat{w}) = \angle_{R(0)}(R(n), \hat{w}) \leq$

$\bar{Z}_{R(0)}(R(n), \hat{w}) = \angle_{\bar{R}(0)}(\bar{R}(n), \hat{w}_n) < \frac{\pi}{2}$ which is a contradiction. Thus $\bar{H} \cap \partial X \subseteq \bar{B}_T(v, \frac{\pi}{2})$.

We now show that $\bar{H} \cap \partial X \supseteq \bar{B}_T(v, \frac{\pi}{2})$. Since \bar{H} is closed in \bar{X} , it suffices to show that $B_T(v, \frac{\pi}{2}) \subset \bar{H}$. Choose $R: [0, \infty) \rightarrow X$ unit speed geodesic ray to v , and with associated Buseman function $b_R(x) = \lim_{t \rightarrow \infty} [d(x, R(t)) - t]$. We may assume that $H = b_R^{-1}([0, -\infty))$. Let $y = R(0)$. We note the $b_R(y) = 0$ and $b_R(R(t)) = -t$ for $t > 0$.

Let $w \in \partial X$ with $\angle(w, v) = \theta < \pi$, and $S: [0, \infty) \rightarrow X$ be the unit speed geodesic from y to w . We will show that $S([0, \infty)) \subset H$. Since the function b_R is convex and $b_R \circ S(0) = 0$, it suffices to show that $b_R \circ S(s) \leq 0$ for all $s \gg 0$. By [1], Proposition II 4.2, for $s \gg 0$ $\angle_{S(s)}(y, v) > \frac{\pi}{2}$. Fix such $s \gg 0$ and let $x = S(s)$, so $\angle_x(y, v) > \frac{\pi}{2}$.

It suffices to show that $b_R(x) \leq 0$. Since $\lim_{t \rightarrow \infty} \angle_x(y, R(t)) = \angle_x(y, v)$, for $t \gg 0$, $\angle_x(y, R(t)) > \frac{\pi}{2}$. Since $\bar{Z}_x(y, R(t)) \geq \angle_x(y, R(t))$, then for $t \gg 0$, $\bar{Z}_x(y, R(t)) > \frac{\pi}{2}$ implying that $d(x, R(t)) < d(y, R(t)) = t$. Thus $[d(x, R(t)) - t] < 0$ for all $t \gg 0$ and so $b_R(x) \leq 0$ as required. \square

3. Large Tits radius

For $A \subset \partial X$, we define

$$\text{radius}_T(A) = \inf\{r : A \subset B_T(u, r) \text{ for some } u \in \partial X\}.$$

If g is an infinite order isometry of X , and $\langle g \rangle$ is proper, then g is either hyperbolic or parabolic. When g is hyperbolic, it acts by translation on a line (called an axis of g) in X with endpoints g^+ (in the direction of translation) and g^- (see [5], Theorem 6.8, p. 231).

Recall that a hyperbolic isometry h of X is said to be rank 1 if h has an axis L which does not bound a half flat. From [1] we see that $d_T(h^+, \alpha) = \infty$ for all $\alpha \neq h^+$, so if X has a rank 1 isometry, then $\text{radius}_T(\partial X) = \infty$.

Recall a result from [16].

Theorem 2 ([16]). *Let X be a complete CAT(0) space and (g_i) a sequence of isometries of X with the property that $g_i(x) \rightarrow p \in \partial X$ and $g_i^{-1}(x) \rightarrow n \in \partial X$ for any $x \in X$. Then for any $\theta \in [0, \pi]$ and any compact set $K \subset \partial X - \bar{B}_T(n, \theta)$, $g_n(K) \rightarrow \bar{B}_T(p, \pi - \theta)$ (in the sense that for any open $U \supset \bar{B}_T(p, \pi - \theta)$, $g_n(K) \subset U$ for all n sufficiently large).*

This theorem is stated in [16] with stronger hypotheses, but they were not used in the proof. We will refer to this result as π -convergence. For a given hyperbolic element h with axis L , we set $g_i = h^i$, $n = L(-\infty) = h^-$ and $p = L(\infty) = h^+$.

The following theorem is due to Ballmann.

Theorem 3 (Ballmann). *If h is a rank 1 isometry of the complete CAT(0) space X , then the group $\langle h \rangle$ generated by h , acts properly discontinuously on the open set $\partial X - \{h^\pm\}$ which is dense if it is non-empty.*

Proof. Let K be a compact subset of $\partial X - \{h^\pm\}$. The Tits distance from K to $n = h^-$ will be infinite. Thus, by π -convergence, $h^i(K) \rightarrow p = h^+$ and $h^{-i}(K) \rightarrow n = h^-$. Thus $\{i \in \mathbb{Z} : h^i(K) \cap K \neq \emptyset\}$ is finite and the action of $\langle h \rangle$ on $\partial X - \{h^\pm\}$ is properly discontinuous. Clearly $\partial X - \{h^\pm\}$ is an open subset and for any $a \in \partial X - \{h^\pm\}$, $h^i(a) \rightarrow h^+$ and $h^{-i}(a) \rightarrow h^-$ so $\partial X - \{h^\pm\}$ is dense in ∂X . \square

Definition. For X , a complete CAT(0) space, we define the limit set $\Lambda X \subset \partial X$ to be the set $\{p \in \partial X : \text{there exist } (g_i) \text{ isometries of } X \text{ with } g_i(x) \rightarrow p \text{ and } g_i^{-1}(x) \rightarrow n \text{ for some } n \in \partial X \text{ and for all } x \in X\}$.

The following is a slight generalization of a result of Karlsson [12].

Theorem 4. *Let X be a complete CAT(0) space with $\text{radius}_T(\partial X) > 3\pi$ and $|\partial X| > 2$. If h is a hyperbolic isometry of X , then $\langle h \rangle$ acts properly discontinuously on the open subset $\Omega = \partial X - [\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})]$. If $[\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})] \subset \Lambda X$, then Ω is dense in ∂X .*

Proof. By Theorem 3, we may assume that h is not rank 1. Thus some axis of h bounds a half-flat, which corresponds to a Tits geodesic of length π from h^+ to h^- , so $d_T(h^+, h^-) = \pi$.

If there is any point of $w \in \partial X$ which is isolated in the Tits metric, ($d_T(w, v) = \infty$ for all $v \neq w$) then, by π -convergence, the orbit of w under $\langle h \rangle$ is infinite and each element of the orbit will be isolated as well.

By the triangle inequality, for any $q \in \partial X$, there exist $u, v \in \partial X$ such that $d_T(q, u), d_T(q, v) \geq \pi$ and $d_T(u, v) > 2\pi$. It follows from π -convergence that for any $w \in \Lambda X$ and any neighborhood W of w in ∂X , the Tits diameter $\text{diam}_T(W) > 2\pi$. Since

$$\text{diam}_T[\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})] \leq 2\pi,$$

$W \not\subset \bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})$, so $W \cap \Omega \neq \emptyset$. Thus Ω is dense in ΛX . Since closed Tits balls are closed in ∂X , Ω is open.

For any compact $K \subset \Omega$, by π -convergence $h^i(K) \rightarrow \bar{B}_T(h^+, \frac{\pi}{2})$ and $h^{-i}(K) \rightarrow \bar{B}_T(h^-, \frac{\pi}{2})$. Using this, we can show that $\{i \in \mathbb{Z} : h^i(K) \cap K \neq \emptyset\}$ is finite and so the action of $\langle h \rangle$ on Ω is properly discontinuous. \square

Example 5. Notice that Ω need not be dense in ∂X if we remove the condition that $[\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})] \subset \Lambda X$. You start with the half plane $\{(x, y) : y \geq 0\}$ where h is unit translation in the first coordinate. Now you attach a line at the origin and let g act by translation in that line. You construct the CAT(0) space X by translating this picture by g and h . This gives you an action of $\langle g, h \rangle$ on the

CAT(0) space X . The isometry g is a rank 1 hyperbolic element, so X is rank 1, but $\Omega = \partial X - [\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})]$ is not dense in the boundary, in fact its closure hits $[\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})]$ only in h^\pm .

Definition. Recall that a metric space is *proper* if closed metric balls are compact, and a metric space is *cocompact* if the quotient of the space by the isometry group is compact.

Theorem 6. *Let X be a proper CAT(0) space with $\text{radius}_T(\partial X) > 3\pi$, and h a parabolic isometry of X . There exists $m \in \partial X$ fixed point of h such that $\langle h \rangle$ acts properly discontinuously on the open dense subset $\Omega = \partial X - \bar{B}_T(m, \pi)$ of ∂X . If $\bar{B}_T(m, \pi) \subset \Lambda X$, then Ω is dense in ∂X .*

Proof. Clearly Ω is non-empty and open. By [5], Proposition 8.25, p. 275, there exists $m \in \partial X$ such that h leaves invariant each horoball centered at m , which implies that h fixes m . Suppose that $\langle h \rangle$ does not act properly discontinuously on Ω . Then there exists K , a compact subset of Ω , and a strictly increasing sequence $(i_j) \subset \mathbb{N}$ such that $K \cap h^{i_j}(K) \neq \emptyset$ for all $j \in \mathbb{N}$.

Fix $x \in X$. Passing to a subsequence, we may assume that $h^{i_j}(x) \rightarrow p \in \partial X$ and $h^{-i_j}(x) \rightarrow n \in \partial X$. Since h leaves the horosphere S centered at m passing through x invariant, then $n, p \in \partial S \subset \bar{B}_T(m, \frac{\pi}{2})$ by Lemma 1. Since ∂X is normal, there exists $U \subset \partial X$ open with $\bar{B}_T(m, \pi) \subset U$ and $U \cap K = \emptyset$. Since $\bar{B}_T(n, \frac{\pi}{2}), \bar{B}_T(p, \frac{\pi}{2}) \subset \bar{B}_T(m, \pi) \subset U$, by π -convergence $h^{i_j}(K) \subset U$ for all $j \gg 0$ which implies $h^{i_j}(K) \cap K = \emptyset$ for all $j \gg 0$ contradicting the choice of (i_j) . Thus $\langle h \rangle$ acts properly on Ω .

By the triangle inequality, for any $q \in \partial X$, there exists $u, v \in \partial X$ such that $d_T(q, u), d_T(q, v) \geq \pi$ and $d_T(u, v) > 2\pi$. It follows from π -convergence that for any $w \in \Lambda X$ and any neighborhood W of w in ∂X , the Tits diameter $\text{diam}_T(W) > 2\pi$.

Since $\text{diam}_T(\bar{B}_T(m, \pi)) \leq 2\pi$, $W \not\subset \bar{B}_T(m, \pi)$ so $W \cap \Omega \neq \emptyset$. Thus Ω is dense in ΛX . □

4. Small Tits radius

We now assume that X is a proper CAT(0) space.

Definition. We say that $a, b \in \partial X$ are antipodes if $d_T(a, b) \geq \pi$. The suspension of the antipodes a and b , S_a^b , is the union of a and b together with of all Tits geodesics from a to b of length π (if any). Notice that $S_a^b = S_b^a$.

Let $a \in \partial X$ and (g_i) a sequence of X . We say that (g_i) *pulls from* $n \in \partial X$ if there is some unit speed geodesic ray $R: [0, \infty) \rightarrow X$ representing n , and a sequence

$(s_i) \subset [0, \infty)$ with $s_i \rightarrow \infty$ such that the sequence $(g_i(R(s_i)))$ is bounded. Clearly this is independent of the ray chosen. Passing to a subsequence, we may assume that $g_i(R(s_i)) \rightarrow b$, and that the sequence of rays $(g_i(R))$ (each reparametrized) converges uniformly on compact subsets to a geodesic line L with $g_i(-i) \rightarrow L(-\infty)$, and $L(0) = b$. Notice that for some (and so any) $x \in X$, $g_i(x) \rightarrow L(-\infty)$ and $g_i^{-1}(x) \rightarrow n$. We say a group of isometries *pulls* from $n \in \partial X$, if there is a sequence in G which pulls from n . Notice that if there is a compact $C \subset X$ such that $R([0, \infty)) \subset \bigcup_{g \in G} g(C)$, then G pulls from $n = R(\infty)$.

For any $x \in \partial X$, passing to a subsequence, we may assume that $g_i(x) \rightarrow \hat{x} \in \partial X$. Thus for any compact set $C \subset TX$ we can define $f: C \rightarrow \partial X$ by $f(x) = \hat{x}$. In fact we can define f whenever C has a countable dense subset as a subset of TX . By [5], $f: C \rightarrow TX$ is a contraction (Lipschitz with constant one). Notice that $f(n) = L(-\infty)$.

Lemma 7. *In the above setting for any $a \in \bar{B}_T(n, \pi)$, $f: [n, a] \rightarrow TX$ is an isometric embedding of $[n, a]$ into $S_{L(-\infty)}^{L(\infty)}$, where $[n, a]$ is a Tits geodesic from n to a .*

Proof. Because f is a contraction, if $d_T(n, a) = d_T(f(n), f(a))$ then $f: [n, a] \rightarrow TX$ is an isometry. Since $f(n) = L(-\infty)$, it suffices to show that $d_T(f(n), f(a)) + d_T(f(a), L(\infty)) = \pi$.

Let $\theta = d_T(n, a)$, so by π -convergence $\pi - \theta \geq d_T(f(a), L(\infty))$. Since f is Lipschitz with constant one, $d_T(f(n), f(a)) \leq \theta$. However,

$$\begin{aligned} \pi &= \theta + (\pi - \theta) \geq d_T(f(n), f(a)) + d_T(f(a), L(\infty)) \\ &\geq d_T(L(-\infty), L(\infty)) \geq \pi. \end{aligned} \quad \square$$

Example 8. The function f defined above need not be an embedding on $B_T(n, \pi)$. Consider the half flat $Y = \mathbb{R} \times [0, \infty)$. For each $n \in \mathbb{Z}$ glue the quarter flat $X_n = [n, \infty) \times [0, \infty)$ by identifying $(a, b) \in Y$ with $(c, d) \in X_n$ if $a = c, b = d$, and $\ln(a + 1 - n) \geq b$. Let Z be the resulting space. ∂Z is broom attached at ∞ in the \mathbb{R} factor of Y . Notice that \mathbb{Z} acts on Z ($1 \in \mathbb{Z}$ sends X_n to X_{n+1}). Letting g_i be subtraction by i , we have $f(\partial Z) = \partial Y$.

Definition. Let g be a hyperbolic isometry of the CAT(0) space X . We define $S_g = S_g^{g^+}$, that is, the suspension of the endpoints of g .

Theorem 9. *The group $\langle g \rangle$ acts discretely on the open set $\Omega = \partial X - S_g$.*

Proof. It suffices to show that for any point $a \in \partial X$, the limit points of the $\{g^i(a) : i \in \mathbb{N}\}$ are in S_g . Let b be a limit point of $\{g^i(a) : i \in \mathbb{N}\}$. Choose an increasing sequence $(i_j) \subset \mathbb{N}$ such that $g^{i_j}(a) \rightarrow b$.

Let L be an axis of g , then the sequence (g^{i_j}) pulls from $n = L(-\infty) = g^-$, leaving L invariant. If $d_T(a, n) \geq \pi$, then by π -convergence $b = L(\infty) = g^+$.

If $d_T(a, n) < \pi$, then we construct the function $f : [n, a] \rightarrow TX$ as above and so $f(a) = b$.

By Lemma 7, f embeds $[n, a]$ into $S_{L(-\infty)}^{L(\infty)} = S_g$. Thus $b \in S_g$. □

Definition. A bounded metric space X is almost a product (K) of Y and Z if there is a convex subset $W \cong Y \times Z$ and $X \subset \text{Nbh}(W, K)$.

Definition. For $C > 0$, we say that X has almost extendable (C) geodesics if for any $x, y \in X$ there exists $z \in B(y, C)$ such that the geodesic segment $[x, z]$ extends to a ray from x . If X has almost extendable geodesics (C) for some C , we say X has almost extendable geodesics.

By [11], if X is cocompact then it has almost extendable geodesics.

Lemma 10. *Let X be a proper CAT(0) space X with almost extendable geodesics. Given there is a convex subset W of X where $W = Y \times Z$, and a non-empty open subset $V \subset \partial X$ with $V \subset \partial W$. If the isometries of X pull from a point of V , then X is almost a product of \hat{Y} and \hat{Z} , convex subsets of X . Additionally, if Y is cocompact, then $\hat{Y} \cong Y$.*

Proof. Let C be the almost extendability constant. We may assume that Y and Z are (convex) subsets of W . Fix a base point $w \in W$. Let $\alpha \in V$ be a point which the isometries of X pulls from. Let $R : [0, \infty) \rightarrow W$ be a geodesic ray with $R(0) = w$ and $R(\infty) = \alpha$. There exist $N \in \mathbb{N}, \varepsilon > 0$ such that $U(\alpha, N, \varepsilon) \subset V$.

By definition of pulling from α , there is a sequence of isometries (g_i) and $t_i \rightarrow \infty$ such that $g_i(R_{t_i}) \rightarrow L$ uniformly on compact subset where R_{t_i} is the ray R reparametrize to have domain $[-t_i, \infty)$. Using the fact that X is proper and passing to a subsequence, there are unbounded sequences $(y_i) \subset Y, (z_i) \subset Z$ so that $g_i(w_i) \rightarrow \hat{w} \in X$ where $w_i = (y_i, z_i)$. Using properness and passing to a subsequence, we have that $g_i(W) \rightarrow \hat{W}$, a convex subset of X . Let Y_i be the copy of Y in W which contains z_i , and Z_i the copy of Z in W which contains y_i , so that $Y_i \cap Z_i = \{w_i\}$. Passing to a subsequence, we may assume that $g_i(Y_i) \rightarrow \hat{Y}$, a convex subset of \hat{W} and $g_i(Z_i) \rightarrow \hat{Z}$, a convex subset of \hat{W} . Now let $\hat{a}, \hat{b} \in \hat{W}$. Thus there are sequences $(a_i), (b_i) \subset W$ with $g_i(a_i) \rightarrow \hat{a}$ and $g_i(b_i) \rightarrow \hat{b}$. For each i , let $a_i^Y = \pi_{Y_i}(a_i)$, the projection of a_i in Y_i . Define b_i^Y, a_i^Z and b_i^Z similarly. For each i , W is the metric product of Y_i and Z_i , i.e.,

$$d(a_i, b_i)^2 = d(a_i^Y, b_i^Y)^2 + d(a_i^Z, b_i^Z)^2.$$

The sequence $(g_i(a_i^Y))$ will clearly converge to the closest point projection $\pi_{\hat{Y}}(\hat{a})$. Similarly $g_i(a_i^Z) \rightarrow \pi_{\hat{Z}}(\hat{a}), g_i(b_i^Y) \rightarrow \pi_{\hat{Y}}(\hat{b}),$ and $g_i(b_i^Z) \rightarrow \pi_{\hat{Z}}(\hat{b})$. It follows that

$$d(\hat{a}, \hat{b})^2 = d(\pi_{\hat{Y}}(\hat{a}), \pi_{\hat{Y}}(\hat{b}))^2 + d(\pi_{\hat{Z}}(\hat{a}), \pi_{\hat{Z}}(\hat{b}))^2,$$

so $\widehat{W} = \widehat{Y} \times \widehat{Z}$.

When Y admits a cocompact action, we can arrange it so that for some $y \in Y$, and h_i isometry of Y , $g_i(h_i(y))$ is bounded and so there is an isometric embedding $h: Y \rightarrow X$ such that $h(Y) = \widehat{Y}$, and similarly for Z .

Now we claim that $X \subset \text{Nbh}(\widehat{W}, C)$. Suppose not, then let $x \in X$ with $d(x, \widehat{W}) > C$. Notice that $g_i^{-1}(\overline{B}(x, C)) \rightarrow \alpha$. Thus for $i \gg 0$ $g_i^{-1}(\overline{B}(x, C)) \subset \tilde{U}(\alpha, N, \varepsilon)$.

By almost extendable geodesics(C) there is a point $v_i \in g_i^{-1}(\overline{B}(x, C))$ such that v_i is on a ray $S: [0, \infty) \rightarrow X$ from our base point w . Notice that for $i \gg 0$, $d(x, g_i(W)) > C$, so in that case $d(v_i, W) > 0$, so $S([0, \infty)) \not\subset W$.

Since the (unit speed geodesic) rays based at w give unique representatives of ∂X and every element of ∂W is represented by a ray in W from w , this implies that $S(\infty) \notin \partial W$, but $S(\infty) \in U(\alpha, N, \varepsilon) \subset \partial W$, which is a contradiction. \square

Corollary 11. *Let X be a proper CAT(0) space with almost extendable geodesics and W a convex subset with cocompact stabilizer in the isometry group of X . If some nonempty open subset of ∂X is in ∂W , then X lies in a uniform neighborhood of W , so $\partial X = \partial W$.*

Proof. In the proof of Lemma 10, we may choose the sequence (g_i) in the stabilizer of W , so $\widehat{W} = W$. \square

Unfortunately, the previous result raises more questions than it answers

Question. If X and Y are cocompact proper CAT(0) spaces and $U \subset \partial X, V \subset \partial Y$ non-empty open subsets with $U \cong V$, is $\partial X \cong \partial Y$? (Here \cong is a homeomorphism in cone topology and an isometry in the Tits metric.)

What if additionally we have a group G acting geometrically on both X and Y ? The answer is clearly yes if X or Y is rank 1 and the isomorphism $U \cong V$ is G -equivariant.

See [6] and [15] for related questions.

We now generalize part of a result of Lytchak to our setting.

Theorem 12. [14] *Let Z be a geodesically complete finite dimensional CAT(1) space. Then Z has a unique decomposition $Z = S^n * G_1 * \dots * G_k * Y_1 * \dots * Y_m$ where G_j is a thick irreducible building and Y_j is an irreducible (via spherical join) non-building.*

In our setting, the Tits boundary TX is not geodesically complete. Nonetheless, we obtain the following result: If Z is a finite dimensional CAT(1) space then there is a unique decomposition $Z = S^n * Y$ where Y doesn't have a sphere as a spherical join factor.

Lemma 13. *If Y and Z are CAT(0) spaces and F is a flat sector in $Y \times Z$, then $\pi_Y(F)$ is a flat sector in A .*

Proof. Since a flat sector is the nested union of flat triangles (and this is sufficient), it suffices to show that the projection of a flat triangle is a flat triangle.

Let abc be a triangle in a CAT(0) space. Consider the following condition: For $t \in (0, 1)$ if $e \in [a, b]$ and $f \in [a, c]$ with $d(a, e) = td(a, b)$ and $d(a, f) = td(a, c)$ then $d(e, f) = td(b, c)$. Notice that if abc is flat, then this condition is satisfied by similar triangles. If on the other hand this condition is satisfied, then by similar triangles, the Euclidian comparison angle $\bar{Z}_a(b, c) = \bar{Z}_a(e, f)$. It follows that $\angle_a(b, c) = \bar{Z}_a(b, c)$ and so the triangle abc is flat.

Thus we may assume that abc is a triangle in satisfying this condition. For $t \in (0, 1)$ choose e and f as above. Let a_1, b_1, c_1, e_1, f_1 be the projections of a, b, c, e, f respectively into Y and a_2, b_2, c_2, e_2, f_2 the projections of a, b, c, e, f respectively into Z . By similar Euclidean triangles $d(a_i, e_i) = td(a_i, b_i)$ and $d(a_i, f_i) = td(a_i, c_i)$ for $i = 1, 2$.

By hypothesis $d(e, f) = td(b, c)$, so $d(e, f)^2 = t^2d(b, c)^2$. Thus,

$$d(e_1, f_1)^2 + d(e_2, f_2)^2 = t^2d(b_1, c_1)^2 + t^2d(b_2, c_2)^2.$$

By the CAT(0) inequality applied to the triangles $a_i b_i c_i, i = 1, 2, d(e_i, f_i) \leq td(b_i, c_i)$ for $i = 1, 2$. It follows that $d(e_i, f_i) = td(b_i, c_i)$ for $i = 1, 2$, and so the triangles $a_i b_i c_i$ are flat for $i = 1, 2$. \square

Definition ([5], Corollary 5.11, p. 62). For Y and Z , metric spaces of diameter $\leq \pi$, we define $Y * Z$ to be the quotient of the space $Y \times Z \times [0, \frac{\pi}{2}]$ by the identifications $(y, z, 0) = (y, \hat{z}, 0)$ and $(y, z, \frac{\pi}{2}) = (\hat{y}, z, \frac{\pi}{2})$. We denote the class (y, z, θ) by $y \cos \theta + z \sin \theta$, so $(y, z, 0) = y$ and $(y, z, \frac{\pi}{2}) = z$, and so we have $Y, Z \subset Y * Z$. For $u = y \cos \theta + z \sin \theta, u' = y' \cos \theta' + z' \sin \theta' \in Y * Z$ we define $d(u, u')$ by

$$\cos(d(u, u')) = \cos \theta \cos \theta' \cos(d(y, y')) + \sin \theta \sin \theta' \cos(d(z, z')).$$

Clearly for $\theta \neq 0, \frac{\pi}{2}, d(u, u') = \pi$ if and only if $d(y, y') = \pi = d(z, z')$ and $\theta + \theta' = \frac{\pi}{2}$.

Lemma 14. *Let α be a geodesic in $A * B$ where A and B are CAT(1) spaces. Notice that A and B are π -convex subsets of $A * B$. If the image of α misses B , then the projection of α to A is a geodesic.*

Proof. Let $Y = C_0(A)$ and $Z = C_0(B)$ be the Euclidian cones on A and B respectively. By [5], $C_0(A * B) = X \cong Y \times Z$ and so $A * B = \partial(Y \times Z)$. By [5], Proposition 5.15, p. 64, it suffices to show that the projection of α to A is a local Tits geodesic. There is a flat sector $F \subset X$ based at the cone point 0 of X with $\partial F = \alpha$. We can think of Y and Z as being convex subsets of X . By Lemma 13, the projection F_Y of F into Y is a Euclidean sector based at the cone point of Y (which is 0 of course). By the proof of [5], Proposition 5.15, p. 64, the ∂F_Y will be the projection of α to A . It follows that this projection is a Tits geodesic in A . \square

The following can also be done using the main result of [8].

Theorem 15. *If Z is a finite dimensional complete CAT(1) space, then there is a unique decomposition $Z = S * D$, where $S \cong \mathbb{S}^n$ and D does not have a non-empty sphere as a spherical join factor.*

Proof. Suppose that $Z = \mathbb{S}^k * U$ and $Z = \{a, b\} * A$, where $\{a, b\} \not\subset \mathbb{S}^k$. It follows that a is the unique antipode to b in Z and visa versa (so $\{a, b\} \cap \mathbb{S}^k = \emptyset$). By the metric on $Z = \mathbb{S}^k * U$, we have $a_1, b_1 \in \mathbb{S}^k, a_2, b_2 \in U$ and $\theta \in (0, \pi/2]$ with $a = a_1 \cos \theta + a_2 \sin \theta$ and $b = b_1 \sin \theta + b_2 \cos \theta$. It follows that a_2 is the unique antipode of b_2 in U and visa versa (and similarly for a_1 and b_1 in \mathbb{S}^k).

Since $Z = \{a, b\} * A$, it follows that $Z = S_a^b$, and every point of Z is on a Tits geodesic from a to b . Fix $u \in U \subset \mathbb{S}^k * U$; the point u is on a Tits geodesic from a to b which misses \mathbb{S}^k . By Lemma 13, the projection of this Tits geodesic into U is a Tits geodesic from a_2 to b_2 passing through u . It follows that U is the union of all Tits geodesics in U from a_2 to b_2 . This implies by [14], 4.1, that $U = \{a_2, b_2\} * W$, where W is the set of all points of U at Tits distance $\pi/2$ from both a_2 and b_2 . Thus by [5], Proposition 5.15, p. 64, $Z = \mathbb{S}^k * (\{a_2, b_2\} * W) = (\mathbb{S}^k * \{a_2, b_2\}) * W = \mathbb{S}^{k+1} * W$. Since Z is finite dimensional, this process must terminate, and so for some $n, Z = \mathbb{S}^n * Y$, where Y does not have a non-empty sphere as a spherical join factor. Notice by our construction that

$$S = \{a \in Z : Z \text{ is a spherical suspension with suspension point } a\}.$$

Thus this decomposition is canonical. □

Corollary 16. *Suppose that the Tits boundary, TX , is finite dimensional. Under the hypothesis of Lemma 10, if $Y \cong \mathbb{E}^m$ for some m , then $\partial Y \subset S$, the set of suspension points of ∂X .*

Proof. By $\partial X = S * D$ as in Theorem 15. Since \mathbb{E}^n admits a cocompact isometric group action, $\hat{Y} \cong Y \cong \mathbb{E}^n$. $\partial X = \partial(\hat{Y} \times \hat{Z}) = (\partial Y) * (\partial \hat{Z})$. Since $\partial \hat{Y} \cong \partial \mathbb{E}^n = S^{n-1}$, we have $\partial \hat{Y} \subset S$. Let $\alpha \in \partial Y - S$, so $d_T(\alpha, D) < \pi/2$. We may assume $g_i(\alpha) \rightarrow \beta$ preforce with $\beta \in \partial \hat{Y} \subset S$. However $g_i(D) = D$ for all i , so $d_T(\beta, D) \leq d_T(\alpha, D) < \pi/2$, but that is a contradiction since every point of S has distance $\pi/2$ from every point of D . Thus $\partial Y \subset S$. □

Definition. We say a group G acts *discretely* on a space X if every orbit is a discrete subset. This is strictly weaker than saying that a group acts properly discontinuously.

Theorem 17. *Let g be a hyperbolic isometry of the proper CAT(0) space X . Suppose that TX is finite dimensional and the isometry group of X pulls from a dense subset of S_g . If $\Omega = \partial X - S_g \neq \emptyset$, then Ω is a dense open subset of ∂X , and $\langle g \rangle$ acts discretely on Ω .*

Proof. By Theorem 9, $\langle g \rangle$ acts discretely on the open set Ω . We must show that Ω is dense whenever it is non-empty.

Suppose that Ω is not dense. Then there is an open subset V of ∂X with $V \subset \partial S_g$. Recall that $S_g \cong \mathbb{R} \times Z$ for some Z , where the endpoints of \mathbb{R} are g^\pm . Thus by Corollary 16, $g^\pm \in S$, the set of suspension points of ∂X . It follows that $S_g = \partial X$ as required. \square

Main Theorem. *Let g be a hyperbolic isometry of the cocompact proper CAT(0) space X . If $\Omega = \partial X - S_g \neq \emptyset$, then Ω is a dense open subset of ∂X , and $\langle g \rangle$ acts discretely on Ω .*

Proof. By [13], TX is finite dimensional. Since the isometry group of X is cocompact, it pulls from every point of ∂X , and Theorem 17 applies. \square

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