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A Further Result on Analyticity of some Kernels

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Abstract. Let $\mathcal{H}_n$ $(n \in \mathbb{N})$ be the Hermite functions. The object is to prove that the series $\sum \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is an analytical function if the sequence (λ_n) is such that $\sup_{n\in\mathbb{N}} R^{\sqrt{n}}|\lambda_n| < +\infty$ for some constant $R > 1$. This answers completely in the affirmative as a consequence the question treated in [3]. The method given here also gives an alternative proof of a theorem proved in that earlier paper.

Keywords: *Hermite functions, Hermite polynomials, analytic functions*

AMS subject classification: 26E05, 26B99, 45C05, 47B10

1. Introduction

In this section we will formulate the result and give some preliminaries. Let $\mathcal{H}_m$ be the Hermite functions as defined in [4: p. 261], let $R > 1$ be a constant and set

$$\Gamma_R^{1/2} = \left\{ (\lambda_n)_{n\in\mathbb{N}} \,\middle|\, \sup_n R^{\sqrt{n}}|\lambda_n| < +\infty \right\}.$$

The object of this paper is to prove the following theorem and settle completely a question in the affirmative raised by A.L. Brown and mentioned already in [3].

Theorem: *Let* $(\lambda_n)_{n\in\mathbb{N}} \in \Gamma_R^{1/2}$. *Then the series* $\sum_{n=1}^{\infty} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ *is an analytic function in* $\mathbb{R}^2$.

Before we proceed to prove this theorem, we give some preliminaries. The following definitions of Hermite functions and operators $\tau_\pm$ are taken from [4: p. 261]. The mth *Hermite function* $\mathcal{H}_m$ (m a non-negative integer) is defined as $\mathcal{H}_m(x) = H_m(x)e^{-\pi x^2}$ where

$$H_m(x) = \frac{1}{C_m} e^{2\pi x^2} \frac{d^m}{dx^m} e^{-2\pi x^2} \qquad \text{with} \quad C_m = (-1)^m \sqrt{m!}\, 2^{m-1/4} \pi^{m/2}.$$

The following recurrence formula is well-known and can be easily proved using the definition of Hermite functions:

$$\sqrt{m}\mathcal{H}_m = 2\sqrt{\pi}x\mathcal{H}_{m-1} - \sqrt{(m-1)}\mathcal{H}_{m-2} \qquad \text{for all} \quad m \geq 2.$$

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Define the operators τ_+ and τ_- from the space C^1 of all once continuously differentiable functions to the space C of continuous functions as

$$\tau_+\varphi = \frac{d\varphi}{dx} + 2\pi x\varphi \qquad \text{and} \qquad \tau_-\varphi = -\frac{d\varphi}{dx} + 2\pi x\varphi.$$

It is easily seen that

$$\tau_-(\tau_+\varphi) = -\frac{d^2\varphi}{dx^2} + (4\pi^2x^2 - 2\pi)\varphi \qquad \text{for all} \quad \varphi \in C^2$$

where C^2 means the space of all twice differentiable functions. Using the recurrence formula, it can be easily seen that

$$\tau_+\mathcal{H}_m = 2\sqrt{\pi m}\mathcal{H}_{m-1} \qquad \text{and} \qquad \tau_-\mathcal{H}_m = 2\sqrt{\pi(m+1)}\mathcal{H}_{m+1}.$$

Therefore, $\tau_-(\tau_+\mathcal{H}_m) = 4\pi m\mathcal{H}_m$. Consider in $\mathbb{R}^2$ the operator

$$L = -\Delta + 4\pi^2(x^2+y^2) - 4\pi$$

where Δ is the Laplacian. It is easy to see that

$$L(\mathcal{H}_n(x)\mathcal{H}_n(y)) = 8\pi n\ \mathcal{H}_n(x)\mathcal{H}_n(y). \tag{1.1}$$

Definition 1: A sequence $(\mu_n)_{n\in\mathbb{N}}$ is said to be *rapidly decreasing* if $\sup_{n\in\mathbb{N}} n^k|\mu_n| < +\infty$ for all $k \in \mathbb{N}$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the *Schwartz space* of functions on $\mathbb{R}^n$. If the sequence (μ_n) is rapidly decreasing, then it is proved in [4: p. 262] that $\sum_{n=1}^{\infty}\mu_n\mathcal{H}_n \in \mathcal{S}(\mathbb{R})$. It can be easily proved by adapting the argument there that if the sequence (μ_n) is rapidly decreasing, then $\sum_{n=1}^{\infty}\mu_n\mathcal{H}_n(x)\mathcal{H}_n(y) \in \mathcal{S}(\mathbb{R}^2)$ and that, for $m \to \infty$,

$$\sum_{n=1}^{m}\mu_n\mathcal{H}_n(x)\mathcal{H}_n(y) \longrightarrow \sum_{n=1}^{\infty}\mu_n\mathcal{H}_n(x)\mathcal{H}_n(y)$$

in the topology of $\mathcal{S}(\mathbb{R}^2)$.

Since $(\lambda_n)_{n\in\mathbb{N}} \in \Gamma_R^{1/2}$, this sequence is easily seen to be rapidly decreasing. Hence, for $m \to \infty$,

$$\sum_{n=1}^{m}\lambda_n\mathcal{H}_n(x)\mathcal{H}_n(y) \longrightarrow f = \sum_{n=1}^{\infty}\lambda_n\mathcal{H}_n(x)\mathcal{H}_n(y) \in \mathcal{S}(\mathbb{R}^2)$$

and, for $m \to \infty$,

$$\sum_{n=1}^{m}\lambda_n\mathcal{H}_n(x)\mathcal{H}_n(y) \longrightarrow f \qquad \text{in} \quad \mathcal{S}(\mathbb{R}^2). \tag{1.2}$$

In an obvious manner we have the representation

$$f = \lambda_1\mathcal{H}_1(x)\mathcal{H}_1(y) + g \qquad \text{where} \quad g = \sum_{n=2}^{\infty}\lambda_n\mathcal{H}_n(x)\mathcal{H}_n(y).$$

Since $\mathcal{H}_n$ is analytic for all $n \in \mathbb{N}$, to prove that f is analytic, it is sufficient to prove that g is analytic. This is what we shall show in the next section.

2. Proof of the Theorem

We shall prove now that the function $g = \sum_{n=2}^{\infty} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is analytic. For this we make use of the following result due to T. Kotaké and M.S. Narasimhan (see [2: Theorem 3.8.9]) which we recall without proof.

Let $\Omega \subset \mathbb{R}^n$ be non-trivial and open and $L : C^\infty(\Omega) \to C^\infty(\Omega)$ be an elliptic operator of order m with analytic coefficients. If $f \in C^\infty(\Omega)$ and if for any relatively compact open subset $\Omega' \subset \Omega$ there exists a constant $M > 0$ such that $\|L^r f\|_2^{\Omega'} \leq M^{r+1}(rm!)$ for all $r \in \mathbb{N}$ where $\|L^r f\|_2^{\Omega'}$ stands for $(\int_{\Omega'} |L^r f|^2 dx)^{1/2}$, then f is analytic in Ω.

Proof of the Theorem: We have the representation $g = f - \lambda_1 \mathcal{H}_1(x)\mathcal{H}_1(y)$. Hence $g \in \mathcal{S}(\mathbb{R}^2)$ as $f \in \mathcal{S}(\mathbb{R}^2)$ and $\mathcal{H}_1(x)\mathcal{H}_1(y) \in \mathcal{S}(\mathbb{R}^2)$. Therefore $g \in C^\infty(\mathbb{R}^2)$. We shall deduce the analyticity of the function g by using the above result of T. Kotaké and M.S. Narasimhan, taking for L the second-order operator $\Delta - 4\pi^2(x^2 + y^2) + 4\pi$. Since L is of order 2, we have to prove that there exists a positive $M \in \mathbb{R}$ such that $\|L^r g\|_2^{\Omega'} \leq M^{r+1}(2r!)$ for all $r \in \mathbb{N}$. We shall prove that there exists a positive real number M such that $\|L^r g\|_2 \leq M^{r+1}(2r!)$ for all $r \in \mathbb{N}$ where $\|L^r g\|_2$ stands for $(\int_{\mathbb{R}^2} |L^r g|^2 dx)^{1/2}$.

Let us estimate $\|L^r g\|_2^2$. From (1.2), $\sum_{n=2}^{m} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y) \longrightarrow g$ in the topology of $\mathcal{S}(\mathbb{R}^2)$ as $m \to \infty$. Since $L : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R}^2)$ is a continuous linear operator, we have

$$L\left(\sum_{n=2}^{m} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)\right) \longrightarrow Lg \qquad \text{in} \quad \mathcal{S}(\mathbb{R}^2).$$

From (1.1), therefore,

$$-\sum_{n=2}^{m} \lambda_n (8\pi n)\mathcal{H}_n(x)\mathcal{H}_n(y) \longrightarrow Lg \qquad \text{in} \quad \mathcal{S}(\mathbb{R}^2)$$

follows. By induction, there follows that

$$(-1)^r \sum_{n=2}^{m} \lambda_n (8\pi n)^r \mathcal{H}_n(x)\mathcal{H}_n(y) \longrightarrow L^r g \qquad \text{in} \quad \mathcal{S}(\mathbb{R}^2)$$

for all $r \in \mathbb{N}$. Hence

$$(-1)^r \sum_{n=2}^{m} \lambda_n (8\pi n)^r \mathcal{H}_n(x)\mathcal{H}_n(y) \longrightarrow L^r g \qquad \text{in} \quad L^2(\mathbb{R}^2)$$

and

$$\left\|\sum_{n=2}^{m} \lambda_n (8\pi n)^r \mathcal{H}_n(x)\mathcal{H}_n(y)\right\|_2^2 \longrightarrow \|L^r g\|_2^2 .$$

Since $\{\mathcal{H}_n(x)\mathcal{H}_n(y)\}_{n\in I\!N}$ is an orthonormal system in $L^2(I\!R^2)$,

$$\left\|\sum_{n=2}^{m}\lambda_n(8\pi n)^r\mathcal{H}_n(x)\mathcal{H}_n(y)\right\|_2^2 = \sum_{n=2}^{m}(8\pi n)^{2r}|\lambda_n|^2 = (8\pi)^{2r}\sum_{n=2}^{m}n^{2r}|\lambda_n|^2.$$

Therefore, we have

$$\|L^r g\|_2^2 = (8\pi)^{2r}\sum_{n=2}^{\infty}n^{2r}|\lambda_n|^2. \tag{2.1}$$

Since $(\lambda_n)\in\Gamma_R^{1/2}$, there exists a positive constant C such that $R^{\sqrt{n}}|\lambda_n|\le C$ for all $n\in I\!N$. Hence there holds $|\lambda_n|^2\le C^2/R^{2\sqrt{n}}$ for all $n\in I\!N$. Therefore, from (2.1) we obtain

$$\|L^r g\|_2^2 \le (8\pi)^{2r}C^2\sum_{n=2}^{\infty}n^{2r}R^{-2\sqrt{n}}. \tag{2.2}$$

We shall now estimate $\sum_{n=2}^{\infty}n^{2r}R^{-2\sqrt{n}}$ using the theory of Γ-function. Let $k = 2\log R$. Consider the integral

$$\int_0^\infty e^{-\sqrt{t}}t^{2r}dt \ge \int_1^\infty e^{-k\sqrt{t}}t^{2r}dt = \sum_{n=2}^{\infty}\int_{n-1}^{n} e^{-k\sqrt{t}}t^{2r}dt.$$

As $n\ge 2$, there is $n-1\ge\frac{1}{2}n$. Hence we have

$$\int_{n-1}^{n} e^{-k\sqrt{t}}t^{2r}dt \ge e^{-k\sqrt{n}}\left(\frac{n}{2}\right)^{2r}.$$

From this it follows that

$$\int_0^\infty e^{-k\sqrt{t}}t^{2r}dt \ge \sum_{n=2}^{\infty}e^{-k\sqrt{n}}\frac{n^{2r}}{2^{2r}} = \frac{1}{2^{2r}}\sum_{n=2}^{\infty}n^{2r}R^{-2\sqrt{n}}.$$

Therefore, there holds

$$\sum_{n=2}^{\infty}n^{2r}R^{-2\sqrt{n}} \le 2^{2r}\int_1^\infty e^{-k\sqrt{t}}t^{2r}dt. \tag{2.3}$$

By the substitution $k\sqrt{t}=y$ it is easily seen that

$$\int_1^\infty e^{-k\sqrt{t}}t^{2r}dt = \frac{2}{k^{4r+2}}\Gamma(4r+2) \qquad \text{where} \quad \Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$$

for $n > 0$. Hence, from (2.2) and (2.3) we see that

$$\|L^r g\|_2^2 \le C^2(8\pi)^{2r}\frac{2^{2r+1}}{k^{4r+2}}\Gamma(4r+2) = C^2(8\pi)^{2r}\frac{2^{2r+1}}{k^{4r+2}}(4r+1)!$$

Hence, we have

$$\|L^r g\|_2 \le C(8\pi)^r \frac{2^{r+1/2}}{k^{2r+1}}(4r+1)!^{1/2}. \tag{2.4}$$

By the Stirling formula, $n! \sim e^{-n}n^{n+1/2}\sqrt{2\pi}$. Hence, there holds

$$(4r+1)! \sim e^{-(4r+1)}(4r+1)^{4r+1+1/2}\sqrt{2\pi}$$

and, therefore,

$$(4r+1)!^{1/2} \sim e^{-(2r+1/2)}(4r+1)^{2r+3/4}\sqrt{2\pi}^{1/2}.$$

Further, we have $(2r)! \sim e^{-2r}(2r)^{2r+1/2}\sqrt{2\pi}$. Hence, there exists a positive constant D such that, for all $r \in I\!N$,

$$\begin{aligned}\frac{((4r+1)!)^{1/2}}{(2r)!} &\le D\frac{(4r+1)^{2r+3/4}}{(2r)^{2r+1/2}}\\ &\le D\frac{(8r)^{2r+3/4}}{(2r)^{2r+1/2}} = D4^{2r}2^{7/4}r^{1/4} \le D2^{7/4}4^{2r+2r} = D2^{7/4}4^{4r}\end{aligned}$$

as $r^{1/4} \le 4^{2r}$ for all $r > 0$. Hence, from (2.4),

$$\begin{aligned}\|L^r g\|_2 &\le DC2^{7/4}(8\pi)^r\frac{2^{r+1/2}}{k^{2r+1}}4^{4r}(2r!)\\ &= 2^{7/4+1/2}\frac{DC}{k}\left(\frac{2\cdot 8\pi\cdot 4^4}{k^2}\right)^r(2r!) \le M_1^r(2r!)\end{aligned}$$

for some positive number $M_1 \in I\!R$ independent of r. Sinc $M_1^r \le (M_1+1)^r \le (M_1+1)^{r+1}$, putting $M = M_1 + 1$, we have $\|L^r g\|_2 \le M^{r+1}(2r!)$ for all $n \in I\!N$ ∎

Corollary 1: *Let the sequence $(\lambda_n)_{n\in I\!N}$ be such that there exists a positive constant C and a number $\rho > 1$ such that*

$$|\lambda_n| \le \frac{C}{\rho^n} \quad \textit{for all } n \in I\!N. \tag{2.5}$$

Then $\sum_{n=1}^{\infty}\lambda_n\mathcal{H}_n(x)\mathcal{H}_n(y)$ is an analytic function.

Proof: This follows immediately from the above theorem by noting that if the sequence $(\lambda_n)_{n\in I\!N}$ satisfies condition (2.5), $(\lambda_n)_{n\in I\!N} \in \Gamma_R^{1/2}$ for all $R > 1$ ∎

Remark: In [3] the above corollary was proved by a different method for all sequences $(\lambda_n)_{n\in I\!N}$ that satisfy condition (2.5) for some $\rho > 2$.

Corollary 2: *Let the sequence $(\lambda_n)_{n\in I\!N}$ be such that, for some $\varepsilon \in (0, \frac{1}{2})$ and for some real $R > 1$, there holds*

$$\sup_{n\in I\!N} R^{n^{1-\varepsilon}}|\lambda_n| < +\infty. \tag{2.6}$$

Then $\sum_{n=1}^{\infty}\lambda_n\mathcal{H}_n(x)\mathcal{H}_n(y)$ is an analytic function.

Proof: Note that if condition (2.6) is satisfied, then the sequence $(\lambda_n)_{n\in I\!N}$ belongs to the class $\Gamma_R^{1/2}$. Hence the result follows immediately from the above theorem ∎

3. Application to integral operators

In this section, we shall apply our Theorem to the study of kernels of integral operators in relation to their eigenvalues.

Proposition: *Let $T: L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ be a self-adjoint integral operator given by a kernel belonging to the space $L^2(\mathbb{R}^2)$ such that its eigenvalues are in the class $\Gamma_R^{1/2}$ for some constant $R > 1$. Then T is unitarily equivalent to an integral operator T_G given by a kernel G which is analytic and belongs to $\mathcal{S}(\mathbb{R}^2)$.*

Proof: Let the eigenvalues of T be (λ_n). By assumption $(\lambda_n) \in \Gamma_R^{1/2}$ for some constant $R > 1$. Hence by the Theorem, $G(x,y) = \sum_{n=1}^{\infty} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is an analytic function belonging to the space $\mathcal{S}(\mathbb{R}^2)$. Let T_G be the integral operator given by G. Then T_G has the eigenvalues λ_n. Now the result follows from the fact that if two compact symmetric operators have the same eigenvalues, then they are unitarily equivalent ∎

S. Ganapathiraman in his thesis considers integral operators K on $L^2(I)$ where I is a closed bounded interval $[a,b]$, induced by kernels K which are analytic in a neighbourhood of $I \times I$. He proves (see also [1: Theorem 3.2]) that the eigenvalues of K belong to the space Γ^- which is defined as the set of all sequences (λ_n) such that the sequence $R^{n^{1-\varepsilon}}|\lambda_n|$ is bounded for all constants $R > 0$ and all $\varepsilon \in (0,1)$. Obviously, $\Gamma^- \subset \Gamma_R^{1/2}$ for any R. Hence, if $(\lambda_n) \in \Gamma^-$, then the function $\sum \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is analytic. This answers in the affirmative the question asked by A.L. Brown, in the course of my lectures on these problems.

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