Zeitschrift für Analysis
und ihre Anwendungen
Volume 12 (1993), 13 - 26
Heldermann Verlag Berlin

Some New Classes in Topological Sequence Spaces Related to L_r-Spaces and an Inclusion Theorem for K(X)-Spaces

J. BOOS and T. LEIGER

The aim of the present paper is to get inclusion theorems for K(X)–spaces, that is, sequence spaces over any Fréchet space X endowed with a K–topology (e.g. domains of operator valued matrices). Since Kalton's closed graph theorem is an essential tool to get inclusion theorems in the case that X equals the set of all complex numbers and since domains of operator valued matrices are not necessarily separable FK(X)–spaces we can no longer make use of FK–space theory. Therefore, it is necessary to develop new ideas to get inclusion theorems. For this we introduce two new classes of K(X)–spaces and prove a closed graph theorem for inclusion maps. One of them is closely related to the class of L_r–spaces introduced by Jinghui Qiu and to the closed graph theorem of J. Qiu, the other is connected with a well–known result of K. Zeller in summability theory. As an immediate corollary of the inclusion theorem proved in this paper we get a generalization of a theorem of Mazur–Orlicz type due to the authors.

Keywords: *Topological sequence spaces (over an F–space), summability in abstract structures, (operator valued) matrix maps, inclusion theorems, consistency of (operator valued) matrix maps, closed graph theorems*

AMS subject classification: 46A45, 46A30, 40H05 and 40J05

1. Notations and preliminaries

For a given dual pair (E,F) we denote by $\sigma(E,F)$ and $\tau(E,F)$ the weak topology and the Mackey topology of the duality (E,F), respectively. If (X,τ_X) is a given locally convex space, then X^* and X' denotes the algebraic dual of X and the topological dual of (X,τ_X), respectively. Furthermore, if S is a linear subspace of X^*, then we use the following notations:

$$\overset{\sqcap}{S} := \left\{ g \in X^* \mid \exists\, (g_n) \text{ in } S : g_n \longrightarrow g \; (\sigma(X^*,X)) \right\},$$

$$\overset{\sqcup}{S} := \bigcap \left\{ V < X^* \mid S \subset V = \overset{\sqcap}{V} \right\} \quad \text{(where '$<$' stands for 'is a linear subspace of');}$$

in particular, $\overset{\sqcup}{S}$ is the smallest linear subspace of X^* containing S and being sequentially closed in $(X^*,\sigma(X^*,X))$. Note, $\overset{\sqcup}{S} \subset X'$ if $S \subset X'$ and if (X,τ_X) is barrelled on account of the Banach–Steinhaus Theorem. A locally convex space X is said to be *weakly compactly generated* (denoted *WCG–space*) if there exists an absolutely convex weakly compact total subset of X and it is called *sub WCG–space* if it is topologically isomorphic to a linear subspace of a WCG–space (see [8]).

If X is any vector space, then $\omega(X)$ denotes the *set of all sequences $x = (x_k)$ in X*. A subspace of $\omega(X)$ is called *sequence space (over X)*. Throughout the whole paper we assume

J. Boos: FernUniversität - Gesamthochschule, FB Math., Lützowstr. 125, D - 5800 Hagen
T. Leiger: Universität Tartu, Lehrstuhl für Math. Analysis, EE Tartu, Estland

ISSN 0232-2064

that (X,τ_X) and (Y,τ_Y) are (locally convex) Fréchet spaces (F–spaces) and, as usual, $B(X,Y)$ denotes the set of all continuous linear maps from X to Y. In the sequel we use the notations $m(X)$, $c(X)$ and $\varphi(X)$ for the set of all sequences being bounded in X, convergent in X and finite, respectively. In the classical case, that is $X := \mathbb{K}$ ($\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$) we write ω, m, c, φ instead of $\omega(X),\ldots,\varphi(X)$. Furthermore we use the notation bv for the set of all sequences in $\mathbb{K}$ having bounded total variation. On $c(X)$ we consider the mapping $\lim : c(X) \longrightarrow X$, $(x_k) \longrightarrow \lim_k x_k$ where the limit is taken in (X,τ_X).

Now we remember the notion of FK(X)–spaces (see [11], [1] and [5]). To that we consider on $\omega(X)$ the product topology τ_ω which is the topology of coordinatewise convergence. A locally convex space (E,τ) is called *K(X)–space* if E is a sequence space over X and the inclusion map $i : (E,\tau) \to (\omega(X),\tau_\omega)$ is continuous. If in addition (E,τ) is an F–space, then it is called *FK(X)–space*. Furthermore, a normable FK(X)–space is called *BK(X)–space*. For example, $(\omega(X),\tau_\omega)$ and $m(X)$ and $c(X)$ provided with a suitable topology τ_∞ are FK(X)–spaces. If X is a Banach space, then $m(X)$ and $c(X)$ together with the obvious supremum norm are BK(X)–spaces. We remark that the limit function $\lim : (c(X),\tau_\infty) \longrightarrow (X,\tau_X)$ is continuous (see [5, Example 2.3(c)]). In [5] distinguished subspaces of FK(X)–spaces are examined by the authors.

We recall the notion of weak sectional convergence. Let (E,τ_E) be a K(X)–space containing $\varphi(X)$. Then we consider the embedding map

$$e_i : X \longrightarrow E \ , \ a \longrightarrow (0,\ldots,a,0,\ldots)$$

where a stands in the i–th position, and for each $x = (x_k) \in \omega(X)$ and $n \in \mathbb{N}$ the sequence

$$x^{[n]} := (x_1,\ldots,x_n,0,\ldots) = \sum_{k=1}^{n} e_k(x_k)$$

which is called *n–th section of x*. Then

$$\begin{aligned} W_E &:= \left\{ x \in E \,\middle|\, x^{[n]} \longrightarrow x \ (\sigma(E,E')) \right\} \\ &= \left\{ x = (x_k) \in E \,\middle|\, \forall f \in E' : f(x^{[n]}) = \sum_{k=1}^{n} f(e_k(x_k)) \longrightarrow f(x) \right\} \end{aligned}$$

is the set of all members of E being weakly sectionally convergent. For a sequence space G over X we define the *β–dual of G* by

$$G^\beta := \left\{ (A_k) \in \omega(X') \,\middle|\, \forall (x_k) \in G : \sum_k A_k(x_k) \text{ converges} \right\}$$

and the space

$$\mathcal{M}(G) := \left\{ (y_k) \in \omega \,\middle|\, \forall (x_k) \in G : (y_k x_k) \in G \right\}$$

of all *(scalar) factor sequences of G*.

Let $A = (A_{nk})$ be an infinite matrix with $A_{nk} \in B(X,Y)$ for $k,n \in \mathbb{N}$. Then

$$\begin{aligned} \omega(Y)_A &:= \left\{ (x_k) \in \omega(X) \,\middle|\, \forall n \in \mathbb{N} : \sum_k A_{nk}(x_k) \text{ converges in } (Y,\tau_Y) \right\}, \\ \omega(Y)_{A_w} &:= \left\{ (x_k) \in \omega(X) \,\middle|\, \forall n \in \mathbb{N} : \sum_k A_{nk}(x_k) \text{ converges in } (Y,\sigma(Y,Y')) \right\} \end{aligned}$$

are called *application domain of* A and *weak application domain of* A, respectively. Therefore the matrix map

$$A : \omega(Y)_A \longrightarrow \omega(Y)\,,\, x = (x_k) \longrightarrow Ax := \Big(\sum_k A_{nk}(x_k)\Big)_n$$

and the weak matrix map

$$A_w : \omega(Y)_{A_w} \longrightarrow \omega(Y)\,,\, x = (x_k) \longrightarrow A_w x := \Big(\sigma(Y,Y')\text{-}\sum_k A_{nk}(x_k)\Big)_n$$

are well-defined. If E is any sequence space over Y, then the *domain of* A and the *weak domain of* A *(with respect to* E*)* are defined by

$$E_A := \left\{ x \in \omega(Y)_A \;\middle|\; Ax \in E \right\} \quad \text{and} \quad E_{A_w} := \left\{ x \in \omega(Y)_{A_w} \;\middle|\; A_w x \in E \right\},$$

respectively. In the special case $E := c(Y)$ we call it simply *domain*. For the limit function corresponding to the domain $c(Y)_A$ of A we use the notation $\lim_A := \lim \circ A$. From [1] and [5] it is known that $c(Y)_A$ is an FK(X)–space and $\lim_A : c(Y)_A \longrightarrow Y$ is a continuous linear map. For the distinguished subset $W_{c(Y)_A}$ of the FK(X)–space $c(Y)_A$ we write W_A. In case of $Y := \mathbb{K}$ we write c_A instead of $c(Y)_A$.

2. Questioning

The authors proved in [4] that in the classical case (that is $X = \mathbb{K} = Y$) the implication

$$M \cap W_E \subset F \implies M \cap W_E \subset W_F \tag{1}$$

holds for every separable FK–space F, for every FK–space E containing the set φ of all finite sequences, and for each sequence space M having suitable factor sequences. For this, they first showed that (1) is true in the special case that F is the domain c_B of any complex valued matrix B. Then the general case was an immediate corollary of the following inclusion theorem of G. Bennett and N. J. Kalton.

Proposition 2.1 (G. Bennett and N. J. Kalton [2, Theorem 5]). *If* $\varphi \subset G < \omega$, *then the following statements are equivalent:*

(a) $(G^\beta, \sigma(G^\beta, G))$ *is sequentially complete.*

(b) *The inclusion map* $i : (G, \tau(G, G^\beta)) \longrightarrow (F, \tau_F)\,,\, x \longrightarrow x$ *is continuous for each separable FK-space* F *with* $G \subset F$.

(c) *The implication* $G \subset F \implies G \subset W_F$ *holds for each separable FK–space* F.

(d) *The implication* $G \subset c_B \implies G \subset W_B$ *holds for each (infinite) matrix* B.

We remark that in the proof of this inclusion theorem the assumption of the separability of F is decisive since G. Bennett and N. J. Kalton used Kalton's closed graph theorem.

Now, it is obvious to ask whether (1) remains true in the more general setting of sequence spaces over an F–space and, in particular, for separable FK(X)–spaces. With the aim of providing a positive answer to this question the authors proved in [6], on the basis of the proof of the corresponding classical result, that (1) remains even true if F is the domain of an operator valued matrix. To reformulate this theorem in detail we need the definition of a special class $\mathcal{E}^*$ of '(scalar) factor sequences' and the 'gliding humps property' of sequence spaces (over $\mathbb{K}$), see Definitions 2.1 and 2.2 in [6].

Definition 2.2 (see [4]). Let $y = (y_k) \in \omega$, then, by definition, $y \in \mathcal{E}^*$ if

$$(y_k - y_{k+1}) \in c_0 \quad \text{and} \quad y_k \geq 0 \ \ (k \in \mathbb{N})$$

and if there exist two index sequences (k_j) and (k_j^*) with the following properties $(j, \mu \in \mathbb{N})$:

$$k_j^* < k_j < k_{j+1}^*, \quad y_k = \begin{cases} 0 & \text{if } k_{2\mu-1} < k \leq k_{2\mu}^* \\ 1 & \text{if } k_{2\mu} < k \leq k_{2\mu+1}^*, \end{cases}$$

$$y_k \leq y_n \quad \text{if} \quad k_{2\mu}^* < k \leq n \leq k_{2\mu} \qquad \text{and} \qquad y_n \leq y_k \quad \text{if} \quad k_{2\mu+1}^* < k \leq n \leq k_{2\mu+1}.$$

Definition 2.3 (see [12] and [4]). Let V be a sequence space containing φ. Assume that for each index sequence (p_n) and for each sequence $(y^{(j)})$ in ω satisfying $y_k^{(j)} = 0$ for $k \notin [p_j, p_{j+1}]$ and $(y^{(j)})$ bounded in bv, there exists a subsequence $(y^{(q_j)})$ of $(y^{(j)})$ such that the pointwise sum $\sum_j y^{(q_j)}$ is an element of V. Then V will be said to have the *gliding humps property*.

Now, we recall the main result of [6] which generalizes Theorem 1 of [4].

Theorem 2.4. *Let M be a sequence space over X containing $\varphi(X)$ such that $\mathcal{E}^* \subset \mathcal{M}(M)$ or such that $\mathcal{M}(M)$ has the gliding humps property. Then the implication*

$$M \cap W_E \subset c(Y)_B \implies M \cap W_E \subset W_B$$

holds for every FK(X)-space E containing $\varphi(X)$ and each matrix $B = (B_{nk})$ with $B_{nk} \in B(X,Y)$.

Using Theorem 2.4 we may prove the validity of (1) in case of separable FK(X)–spaces but the obtained theorem would not contain Theorem 2.4 since domains of operator valued matrices are not necessarily separable FK(X)–spaces as the following simple example shows: the domain $c(m)_I$ of the identity matrix I (together with its FK(m)–topology) is not separable as the BK–space m is not separable.

The main idea of the present paper is to find a class of K(X)–spaces containing both the separable FK(X)–spaces and the domains of operator valued matrices and such that the inclusion theorem presented in Proposition 2.1 remains true in the case of general sequence spaces and if F is any member of this class. In the next section we will motivate and present such a class.

3. L_r–K(X)–spaces and some related classes of K(X)–spaces

Kalton's closed graph theorem [9] says that every closed linear map $T : E \longrightarrow F$ is continuous whereby F is a B_r–complete locally convex space and E is a Mackey space such that

$(E', \sigma(E', E))$ is sequentially complete. Introducing the notion of L_r-spaces, Jinghui Qiu generalized in [10] Kalton's closed graph theorem and he proved that the class of all L_r-spaces is the maximal class of range spaces F in this result (see [10, Theorem 3]). A locally convex space (E, τ) is called *L_r-space* if $\overset{\sqcup}{S} \cap E' = E'$ for each $\sigma(E', E)$-dense subspace S of E'. We recall Qiu's closed graph theorem and we will refine Qiu's proof since some details of his proof seem to be incorrect.

Proposition 3.1 (see [10, Theorem 1]) . *Let (E, τ) be a Mackey space such that the dual $(E', \sigma(E', E))$ is sequentially complete, (F, τ_F) be an L_r-space and $T : (E, \tau(E, E')) \longrightarrow (F, \tau_F)$ be a linear map with closed graph. Then T is continuous.*

Proof. Let $D_T^* := \left\{ f \in F^* \mid f \circ T \in E' \right\}$ and $D_T := D_T^* \cap F'$. Then D_T is $\sigma(F', F)$-dense in F' since T is closed. The continuity of T is proved if we can show $D_T = F'$. To that end we will prove $D_T^* = \overset{\sqcap}{D_T^*}$ thus $D_T = \overset{\sqcup}{D_T} \cap F'$ which implies $D_T = F'$ since (F, τ_F) is an L_r-space. Let $f \in F^*$ be given and (f_n) be a sequence in D_T^* such that $f_n \longrightarrow f \ (\sigma(F^*, F))$. In particular, $f_n \circ T \in E'$ and there exists $g \in E^*$ such that $f_n \circ T \longrightarrow g \ (\sigma(E^*, E))$. Thus $(f_n \circ T)$ is a $\sigma(E', E)$-Cauchy sequence which converges to g since $(E', \sigma(E', E))$ is sequentially complete, therefore $g = f \circ T \in E'$, that is $f \in D_T^*$. Altogether, $D_T^* = \overset{\sqcap}{D_T^*}$ is proved ∎

Remark 3.2 . J. Qiu proved $\overset{\sqcap}{D_T} \cap F' = F'$ and derived from this fact that D_T is sequentially closed in $(F^*, \sigma(F^*, F))$ which fails in general: Let $(E, \tau) := (\varphi, \tau(\varphi, \omega))$, $(F, \tau_F) := (\varphi, \tau(\varphi, \varphi))$ and let $T : \varphi \longrightarrow \varphi$ be the inclusion map. Then obviously $\overset{\sqcap}{D_T} \cap F' = F' = \varphi$ and $\overset{\sqcup}{D_T} = \overset{\sqcup}{\varphi} = \omega$ in $(F^*, \sigma(F^*, F))$.

Furthermore, Jinghui Qiu proved in Theorem 2 of [10] that every separable B_r-complete space is an L_r-space. Thus Proposition 3.1 generalizes Kalton's closed graph theorem. As every separable locally convex space is a subWCG–space (see [8]) these considerations are contained in the following result.

Theorem 3.3 . *Every B_r-complete subWCG–space is an L_r-space.*

Proof. Using a result of R. J. Hunter and J. Lloyd [8, Proposition 3·11] the proof is quite similar to the proof for separable spaces. Let E be a B_r-complete subWCG–space and S be a $\sigma(E', E)$-dense subspace of E'. We'll prove $\overset{\sqcup}{S} \cap E' = E'$. Note, $\overset{\sqcup}{S} \cap E'$ is a $\sigma(E', E)$-dense subspace of E'. Let U be a neighborhood of 0 in E and U° be the polar of U in E'. If we can show that $(\overset{\sqcup}{S} \cap E') \cap U^\circ$ $(= \overset{\sqcup}{S} \cap U^\circ)$ is $\sigma(E', E)$-closed, then we are done on account of the B_r-completeness of E. Let f be an fixed element in the $\sigma(E', E)$-closure of $\overset{\sqcup}{S} \cap U^\circ$ in E'. Then, because of [8, Proposition 3·11] we may choose a sequence (f_n) in $\overset{\sqcup}{S} \cap U^\circ$ being $\sigma(E', E)$-convergent to f. From the definition of $\overset{\sqcup}{S}$ and the fact that U° is $\sigma(E', E)$-closed in E' we conclude $f \in \overset{\sqcup}{S} \cap U^\circ$ ∎

In Example 3.13 we will prove that the domain of an operator valued matrix is not necessarily an L_r-space. Therefore, the class of L_r-K(X)-spaces is not the desired class described at the end of Section 2. The way out is the consideration of topological sequence spaces closely related to L_r-spaces. The definition of these spaces is motivated by the following theorem.

Theorem 3.4 . *Let E be a K(X)-space. Then $\varphi(X')$ and therefore $E^\beta \cap E'$ is $\sigma(E', E)$-dense in E'.*

Proof. This statement is an immediate corollary of the fact that the polar of $\varphi(X')$ equals $\{0\}$ ∎

Definition 3.5 . Let (E, τ_E) be a K(X)–space.

(a) E *has φ-sequentially dense dual* [or *β-sequentially dense dual*] if $\varphi(X')$ [or $E^\beta \cap E'$] is sequentially $\sigma(E', E)$–dense in E'.

(b) E is called *L_φ-space* if $\overline{\varphi(X')} \cap E' = E'$.

Remarks 3.6 . **(a)** On account of Theorem 3.4 and $\varphi(X') \subset E^\beta \subset \overline{\varphi(X')}$ we get the following relations for K(X)–spaces containing $\varphi(X)$:

(i) If E has φ–sequentially dense dual, then E has β–sequentially dense dual.

(ii) If E has β–sequentially dense dual, then E is an L_φ–space.

(iii) If E is an L_r–space, then E is an L_φ–space.

In Examples 3.12 — 3.14 we will learn that the inversion of each of these implications fails in general.

(b) A K(X)–space is an L_φ–space if and only if it fulfils $\overline{E^\beta \cap E'} \cap E' = E'$. (This is an immediate corollary of $\varphi(X') \subset E^\beta \subset \overline{\varphi(X')}$ and of the definition of an L_φ–space.)

(c) If E is any separable (or even subWCG–) FK(X)–space, then E is an L_r–space thus an L_φ–space.

(d) Any subspace of an L_φ–K(X)–space (E, τ) containing $\varphi(X)$ is an L_φ–K(X)–space, and E remains an L_φ–space if we replace τ by any weaker K(X)–topology.

There is a characterization of spaces with β–sequentially dense dual which is closely related to a result of K. Zeller [13] saying that each continuous linear functional on a matrix domain may be represented by the limit functional of a suitable matrix. Furthermore, we get also a similar characterization for K(X)–spaces with φ–sequentially dense dual.

Theorem 3.7 . *Let E be a K(X)-space and let $E^\beta \subset E'$ (for example if E is an FK(X)-space). Then the following statements are equivalent:*

(a) *E has β-sequentially dense dual.*

(b) *For each $f \in E'$ there exists a sequence (Φ_n) in $\omega(E^\beta \cap E')$ such that $\Phi_n \longrightarrow f \ (\sigma(E', E))$.*

(c) *For each $f \in E'$ there exists a matrix $B = (B_{nk})$ with $B_{nk} \in X'$ such that $E \subset c(\mathbb{K})_B$ and $f = \lim_B |_E$.*

If $E^\beta \not\subset E'$, then (a) $\Longleftrightarrow$ (b) $\Longrightarrow$ (c).

Theorem 3.8 . *Let E be a $K(X)$-space. Then the following statements are equivalent:*

(a) *E has φ-sequentially dense dual.*

(b) *For each $f \in E'$ there exists a sequence (Φ_n) in $\omega(\varphi(X'))$ such that $\Phi_n \longrightarrow f \ (\sigma(E', E))$.*

(c) *For each $f \in E'$ there exists a row finite matrix $B = (B_{nk})$ with $B_{nk} \in X'$ such that $E \subset c(\mathbb{K})_B$ and $f = \lim_B |_E$.*

The proofs are straightforward and therefore we omit them.

Using the idea of the proof of the mentioned result of K. Zeller [13] we may prove that domains of operator valued matrices and of row finite operator valued matrices are FK(X)-spaces with β-sequentially dense dual and φ-sequentially dense dual, respectively.

Theorem 3.9 . *Let $A = (A_{nk})$ be a matrix with $A_{nk} \in B(X,Y)$. Then $c(Y)_A$ has β-sequentially dense dual. If, in addition, A is row finite, then $c(Y)_A$ has φ-sequentially dense dual.*

Proof. For a proof of the first statement we apply Theorem 3.7'(a)$\Longleftrightarrow$(c)'. Let $F := c(Y)_A$ and $f \in c(Y)'_A$. Then f has a representation

$$f(x) = \mu(\lim{}_A x) + \sum_n t_n \left(\sum_k A_{nk}(x_k) \right) + \sum_k h_k(x_k) \qquad (x \in F) \tag{2}$$

where $(t_n) \in \ell(Y')$, $\mu \in Y'$ and $(h_k) \in F^\beta$ are suitably chosen (see [1] and [5, Theorem 2.15]). If the matrix $D = (D_{nk})$ is defined by

$$D_{nk} := \begin{cases} \mu & \text{for } k = n \\ t_k & \text{for } k < n \\ 0 & \text{for } k > n, \end{cases}$$

then we obtain

$$D_n(y) = \mu(y_n) + \sum_{k=1}^{n-1} t_k(y_k) \qquad (y \in c(Y))$$

for the n-th row functional of D. Furthermore we put $C := DA$, that is

$$C_{ni} = \mu \circ A_{ni} + \sum_{k=1}^{n-1} t_k \circ A_{ki} \qquad (n, i \in \mathbb{N}).$$

Thereby we get

$$C_n(x) = \mu\big(A_n(x)\big) + \sum_{k=1}^{n-1} t_k(A_k(x)) \qquad (x \in F)$$

(A_k denotes the k-th row functional of A) and thus

$$\lim{}_C x = \mu(\lim{}_A x) + \sum_k t_k(A_k(x)) \qquad (x \in F).$$

So we proved $F \subset c(Y)_C$. Finally, we define the matrix $B = (B_{nk})$ by

$$B_{nk} := \begin{cases} h_k & \text{for } n = 1 \\ C_{n-1,k} + h_k & \text{for } n > 1 \end{cases} \qquad (n, k \in \mathbb{N}).$$

Obviously B has the desired properties and the first statement is proved.

To prove the second statement we adapt the proof of the first statement. For this, we assume A to be row finite. Then we may choose a representation (2) of f with the additional property $(h_k) \in \varphi(X)$. Therefore the matrix B constructed above is row finite and the second statement follows from Theorem 3.8'(a)$\Longleftrightarrow$(c)' ∎

In the classical case it is known that the domain E_A with respect to a separable FK–space is a separable FK–space. Therefore, it is obvious to ask whether in the general case of operator valued matrices the domain E_A with respect to an L_φ–FK(Y)–space E is an L_φ–space. (Trivially, it is an FK(X)–space.) We are able to prove this supposition in case of an FK(Y)–space E with β–sequentially dense dual.

Theorem 3.10 . *Let $A = (A_{nk})$ be a matrix with $A_{nk} \in B(X,Y)$ and E be an FK(Y)–space with β–sequentially dense dual. Then the FK(X)–space E_A is an L_φ–space.*

Proof. Let $f \in E_A'$. We may choose $g \in E'$ and $\alpha \in \omega(Y)_A^\beta$ with $f = g \circ A + \alpha$ and, because E^β is sequentially dense in $(E', \sigma(E', E))$, a matrix $B = (B_{nk})$ with $B_{nk} \in Y'$ $(n, k \in \mathbb{N})$, $c_B \supset E$ and $g = \lim_B|_E$. Therefore, for each $x = (x_i) \in E_A$ we get

$$f(x) = \lim\nolimits_B Ax + \alpha x = \lim_n \sum_k B_{nk} \left(\sum_i A_{ki}(x_i) \right) + \sum_i \alpha_i(x_i). \tag{3}$$

For all $n, r, i \in \mathbb{N}$ we define $C_{ri}^{(n)}$ by $C_{ri}^{(n)} = \sum_{k=1}^{r} B_{nk} \circ A_{ki}$. Obviously we get

$$C^{(n,r)} := \left(C_{ri}^{(n)} \right)_{i \in \mathbb{N}} \in E_A^\beta \qquad (n, r \in \mathbb{N}).$$

On account of (3) for each $n \in \mathbb{N}$ the linear functional

$$C^{(n)} : E_A \longrightarrow \mathbb{K},\ x = (x_i) \longrightarrow \lim_r C^{(n,r)}(x) = \sum_k B_{nk} \left(\sum_i A_{ki}(x_i) \right)$$

is well–defined and we obtain $C^{(n,r)} \xrightarrow{r \to \infty} C^{(n)}$ in $(E_A', \sigma(E_A', E_A))$ for each $n \in \mathbb{N}$; in particular,

$$C^{(n)} \in \overset{\sqcup}{E_A^\beta} \quad \text{and therefore} \quad C^{(n)} + \alpha \in \overset{\sqcup}{E_A^\beta} \qquad (n \in \mathbb{N}).$$

Altogether, we get from (3) the statement $C^{(n)} + \alpha \xrightarrow{n \to \infty} f$ in $(E_A', \sigma(E_A', E_A))$ and thus $f \in \overset{\sqcup}{E_A^\beta}$, that is $\overset{\sqcup}{E_A^\beta} = E_A'$ ∎

We complete Theorem 3.10 with the following

Remarks 3.11 . **(a)** We don't know whether Theorem 3.10 remains true if we (only) assume that E is an L_φ–FK(Y)–space. If E is a separable FK(Y)–space (therefore an L_φ–space), then the FK(X)–space E_A is separable too and hence it is an L_φ–space.

(b) From Example 3.14 we will get that, in general, the domain E_A in Theorem 3.10 need not have β–sequentially dense dual.

Now, we are going to give the counterexamples promised in Remark 3.6 (a).

Example 3.12 . P. Erdös and G. Piranian [7, Theorem 1] gave an example of a regular real valued (row infinite) matrix A such that there exists no row finite regular matrix B with $c_A \subset c_B$. (Thereby, a matrix A is called regular if $c \subset c_A$ and $\lim Ax = \lim x$ for each $x \in c$.) Thus, if we put $E := c_A$, then because of Theorem 3.9 the FK–space E *has* β*-sequentially dense dual* but, combining Theorem 3.8(c) in case of $f := \lim_A$ and the argument of Erdös and Piranian, we obtain that the FK–space E does not have φ–sequentially dense dual.

Example 3.13 . (a) The BK–space $(m, \|\ \|_\infty)$ is not an L_φ–space thus not an L_r–space. To prove this statement we consider the FK–space $E := \omega$ and put $G := m \cap W_E = m$. From the Mazur–Orlicz Theorem 2.4 and the Inclusion Theorem 2.1 we know that $(G^\beta, \sigma(G^\beta, G))$ is sequentially complete (which is a well-known result). Thus $\overset{\sqcup}{\varphi} = G^\beta = \ell_1$ implying that $(m, \|\ \|_\infty)$ is no L_φ–space thus no L_r–space on account of Remark 3.6(a).

(b) The BK(m)–space $c(m)$ has φ–sequentially dense dual, but it is not an L_r–space. The first statement is an immediate corollary of the second statement in Theorem 3.9 because $c(m)$ is the domain of the identity matrix. Furthermore, the BK(m)–space $c(m)$ is not an L_r–space as the BK–space $(m, \|\ \|_\infty)$ may be embedded by e_i isometrically isomorphically in the BK(m)–space $c(m)$ (see [10, Theorem 2]).

Example 3.14 . Now, we give an example of a domain E_A where E is an FK–space with β–sequentially dense dual, E_A is an L_φ–space which fails to have β–sequentially dense dual. In case of the matrices $M = (m_{nk})$ with

$$m_{nk} := \begin{cases} 2^{-p} & \text{if } k = 2^{n-1}(2p-1) \quad (p, n \in \mathbb{N}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$A := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

P. Erdös and G. Piranian [7, Theorem 1 and 5] proved the following statements:

(a) If C is any regular matrix such that $c_M \subset c_C$, then C is not row finite.

(b) If B is any regular matrix with $c_A \subset c_B$, then B is row finite and fulfils

$$\exists\, k_0 \in \mathbb{N} \quad \forall\, k > k_0 \quad \forall\, n \in \mathbb{N} : \ b_{n,2k+1} = b_{n,2k+2}. \tag{4}$$

We put $E := c_M$ and $F := E_A = (c_M)_A$. The FK–space E has β–sequentially dense dual (Theorem 3.9) and the FK–space F is an L_φ–space (Theorem 3.10). The map

$$T : F \longrightarrow c,\ x = (x_k) \longrightarrow Tx := \left(\sum_k m_{nk} \sum_i a_{ki} x_i \right)_{n \in \mathbb{N}}$$

is well-defined, linear and continuous (with respect to the FK–topologies). Furthermore, we have $\lim_T := \lim \circ T \in F'$ and $\lim_T x = \lim x$ for each $x \in c$. We are going to prove that there does not exist any matrix B with

$$F \subset c_B \quad \text{and} \quad \lim_B|_F = \lim_T. \tag{5}$$

(Thus, on account of Theorem 3.7 the FK–space F fails to have β–sequentially dense dual.) We assume that B is a matrix satisfying (5). It is regular and because of $c_A \subset F$ and the above statement (b) it is row finite and fulfils condition (4). We consider the matrix $B^* = (b^*_{nk})$ with $b^*_{nk} := 2\,b_{n,2k-1}$ $(n,k \in \mathbb{N})$. As B also B^* is row finite and regular. Therefore, on account of statement (a) there exists a $y \in c_M \setminus c_{B^*}$. Putting $x := (2y_1, 0, 2y_2, 0, \ldots)$ we get $Ax = y$, thus $x \in F$. On the other hand,

$$y \notin c_{B^*} \quad \text{and} \quad \sum_k b_{nk} x_k = \sum_k b^*_{nk} y_k \quad (n \in \mathbb{N})$$

implies $x \notin c_B$ and alltogether $x \in F \setminus c_B$. Therefore, the FK–space F fails to have β–sequentially dense dual.

Collecting the results of the present section, we state that the class of all L_φ–K(X)–spaces contains both the separable FK(X)–spaces (see Remark 3.6) and the domains of operator valued matrices (see Theorem 3.9).

4. A general inclusion theorem for K(X)–spaces

For the proof of the aspired inclusion theorem we need that matrix maps between suitable K(X)– and K(Y)–spaces have closed graph. Furthermore, we need the continuity of the inclusion map in case of special K(X)–spaces.

Lemma 4.1 . *Let X and Y be F-spaces and G be a sequence space over X containing $\varphi(X)$.*

(a) *If $B_k \in B(X,Y)$ for each $k \in \mathbb{N}$ and $\sum_k B_k(x_k)$ converges for each $x = (x_k) \in G$, then*

$$s : (G, \tau(G, G^\beta)) \longrightarrow (Y, \tau_Y)\,,\ (x_k) \longrightarrow \sum_k B_k(x_k)$$

is continuous. This statement remains true if we replace $\sum_k B_k(x_k)$ by $\sigma(Y,Y')$-$\sum_k B_k(x_k)$.

(b) *If F is a $K(Y)$-space, then each matrix map $A : (G, \tau(G, G^\beta)) \longrightarrow (F, \tau_F)$ has closed graph. This is also true in case of weak matrix maps A_w.*

Proof. The statements in (b) are immediate corollaries of the statements in (a) and the property of F to be a K(Y)–space. For a proof of the (weak) continuity of s it is sufficient to show that for any $f \in Y'$ there exists a $g \in G^\beta$ such that $g = f \circ s$. It is easy to verify that $g := (f \circ B_k)_k$ is the desired member of G^β. (Note, we have identified G^β and the dual of $(G, \tau(G, G^\beta))$ in an obvious way.) ∎

Theorem 4.2 . *Let F be an L_φ-K(X)-space, let G and H be subspaces of F and G^*, respectively, such that $(H, \sigma(H,G))$ is sequentially complete and $(G, \tau(G,H))$ is a K(X)-space. Then the inclusion map $i : (G, \tau(G,H)) \longrightarrow (F, \tau_F)$, $x \longrightarrow x$ is continuous.*

Proof. The proof is quite similar to that of the closed graph theorem of J. Qiu (see Proposition 3.1). Let $i' : F' \longrightarrow G^*$ be the transpose of i and $D_i := (i')^{-1}(H) = \left\{ f \in F' \mid f \circ i \in H \right\}$.

As an inclusion map between K(X)–spaces i has closed graph, that is $\overline{D_i}^{\sigma(F',F)} = F'$ (see [9, Lemma 2·1]). For a proof of the (weak) continuity of i it is sufficient to prove $D_i = F'$. To that end let $f \in F^*$ and a sequence (f_n) in F^* with $f_n \circ i \in H$ and $f_n \longrightarrow f$ in $(F^*, \sigma(F^*, F))$ be given. Then there exists $g \in G^*$ such that $f_n \circ i \longrightarrow g$ in $(G^*, \sigma(G^*, G))$. In particular, $(f_n \circ i)$ is a $\sigma(H,G)$–Cauchy sequence, that is $g \in H$ as $(H, \sigma(H,G))$ is sequentially complete. Thus $g = f \circ i$ which proves that $D_i^* = \{f \in F^* \mid f \circ i \in H\}$ is sequentially closed in $(F^*, \sigma(F^*, F))$. In particular, this implies $\underline{D_i} \cap F' = D_i$. Since F and $(G, \tau(G,H))$ are K(X)–spaces we get $\varphi(X') \subset H \cap F'$ $(= D_i)$. Using that F is an L_φ–K(X)–space we get $F' = \underline{\varphi(X')} \cap F' \subset \underline{D_i} \cap F' = D_i$ ∎

Remark 4.3. Similar to the argument in Theorem 3 of [10] we may prove that in Theorem 4.2 L_φ–space is the best assumption for the range space in the following sense: Let (F, τ_F) be a K(X)–space. If the inclusion map $i : (G, \tau(G,H)) \longrightarrow (F, \tau_F)$, $x \longrightarrow x$ is continuous for each subspace G of F and H of G^* such that $(H, \sigma(H,G))$ is sequentially complete and $(G, \underline{\tau(G,H)})$ is a K(X)–space, then (F, τ_F) is an L_φ–space. (As a hint for the proof we remark $G^\beta \subset \underline{\varphi(X')}$.)

Now we are in a position to formulate and prove the announced inclusion theorem.

Theorem 4.4. *If G is a sequence space over X containing $\varphi(X)$, then the following statements are equivalent:*

(a) $(G^\beta, \sigma(G^\beta, G))$ *is sequentially complete.*

(b*) *Each weak matrix map $A_w : (G, \tau(G, G^\beta)) \longrightarrow (F, \tau_F), x \longrightarrow A_w x$ is continuous whenever (F, τ_F) is an L_φ–K(Y)–space, where Y is any F–space.*

(b) *Each matrix map $A : (G, \tau(G, G^\beta)) \longrightarrow (F, \tau_F), x \longrightarrow Ax$ is continuous whenever (F, τ_F) is an L_φ–K(Y)–space, where Y is any F–space.*

(c) *The inclusion map $i : (G, \tau(G, G^\beta)) \longrightarrow (F, \tau_F), x \longrightarrow x$ is continuous for each L_φ–K(X)–space (F, τ_F) containing G.*

(d) *The implication $G \subset F \implies G \subset W_F$ holds for each L_φ–K(X)–space F.*

(e) *The implication $G \subset c(Y)_B \implies G \subset W_B$ is true for each matrix $B = (B_{nk})$ with $B_{nk} \in B(X,Y)$ whereby Y is any F–space.*

(f) *The implication $G \subset c_B \implies G \subset W_B$ is valid for each matrix $B = (B_{nk})$ with $B_{nk} \in X'$.*

Proof. First of all, we prove the implications '(a) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (a)', then '(a) ⇒ (b*) ⇒ (b) ⇒ (c)'.

'(a) ⇒ (c)' is Theorem 4.2 in case of $H := G^\beta$. (Note that $(G, \tau(G, G^\beta))$ is a K(X)–space.)

'(c) ⇒ (d)' is obviously true since the continuity of i implies the weak continuity and $x^{[n]} \longrightarrow x$ in $(G, \sigma(G, G^\beta))$ for each $x \in G$.

'(d) $\Rightarrow$ (e)' is trivially fulfilled as every domain is an L_φ-K(X)-space (see Theorem 3.9).

'(e) $\Rightarrow$ (f)' since (f) is contained in (e).

'(f) $\Rightarrow$ (a)': Let (Φ_n) with $\Phi_n = (\Phi_{nk})$ be a Cauchy sequence in $(G^\beta, \sigma(G^\beta, G))$. Then the functional valued matrix $B = (\Phi_{nk})$ fulfills $G \subset c_B$ and furthermore for each $x \in G$ we get on account of (f)

$$\lim_n < \Phi_n, x > = \lim_B x = \sum_k (\lim_B \circ e_k)(x_k) = \sum_k \varphi_k(x_k) = < \Phi, x > \qquad (x \in G) \qquad (6)$$

where $\varphi_k := \lim_B \circ e_k$ $(\in X')$ and $\Phi := (\varphi_k) \in G^\beta$. By (6) the sequence (Φ_n) is $\sigma(G^\beta, G)$-convergent in G^β and altogether we proved (a).

'(a) $\Rightarrow$ (b*)': Let $(G, \sigma(G, G^\beta))$ be sequentially complete, F be an L_φ-K(Y)-space and $A_w : (G, \tau(G, G^\beta)) \longrightarrow (F, \tau_F)$ be a weak matrix map. We have to prove the continuity of A_w. Let $D_{A_w} := \left\{ f \in F' \mid f \circ A_w \in G^\beta (= G') \right\}$. Since A_w is a closed map (see Lemma 4.1), D_{A_w} is dense in $(F', \sigma(F', F))$. Since $(G^\beta, \sigma(G^\beta, G))$ is sequentially complete, D_{A_w} is $\sigma(F', F)$-sequentially closed (see [9, Lemma 2.2]). On the other hand $\varphi(X') \cap F' = F'$ since F is an L_φ-K(Y)-space. Therefore the *weak* continuity of A_w is proved if we can show $\varphi(X') \subset D_{A_w}$ because this implies $D_{A_w} = F'$ (that is $\Psi \circ A_w \in G' = G^\beta$ for all $\Psi \in F'$). Let $\Psi = (\Psi_n) \in \varphi(Y')$ and let $f \in F^*_{A_w}$ be defined by

$$f(x) := (\Psi \circ A_w)(x) = \sum_{n=1}^{\nu} \Psi_n \left(\sigma(Y, Y')\text{-}\sum_k A_{nk}(x_k) \right) \qquad (x \in F_{A_w} \supset G).$$

Then for each $x \in F_{A_w}$ we get

$$f(x) = \sum_{n=1}^{\nu} \sum_k \Psi_n(A_{nk}(x_k)) = \sum_k \sum_{n=1}^{\nu} \Psi_n(A_{nk}(x_k)) = \sum_k B_k(x_k)$$

where $B_k := \sum_{n=1}^{\nu} \Psi_n \circ A_{nk}$ for each $k \in \mathbb{N}$. Obviously, $(B_k) \in G^\beta$, that is $f = \Psi \circ A_w \in G^\beta$. Therefore we have shown $\Psi \in D_{A_w}$.

'(b*) $\Rightarrow$ (b) $\Rightarrow$ (c)' is obviously valid ∎

As an immediate corollary of Inclusion Theorem 4.4 we get the desired generalization of the theorem of Mazur–Orlicz type presented in Theorem 2.4.

Theorem 4.5 . *Let M be a sequence space over X containing $\varphi(X)$ such that $\mathcal{E}^* \subset \mathcal{M}(M)$ or such that $\mathcal{M}(M)$ has the gliding humps property. Then the implication*

$$M \cap W_E \subset F \quad \Longrightarrow \quad M \cap W_E \subset W_F$$

holds for every FK(X)-space E containing $\varphi(X)$ and each L_φ-K(X)-space F.

Proof. If we put $G := M \cap W_E \subset F$, then Theorem 2.4 says that statement (e) and therefore (d) in Theorem 4.4 is true. But (d) is in the present case the statement of the theorem being submitted ∎

We don't know whether we may replace in Theorem 4.4 (b) the matrix map by any linear map with closed graph. But, the following theorem holds.

Theorem 4.6 . *Let G be any sequence space over X containing $\varphi(X)$ and let (F,τ_F) be a K(Y) space and let $T:(G,\tau(G,G^\beta))\longrightarrow(F,\tau_F)$ be a continuous linear operator. Then T is a weak matrix map, that is, there exists a matrix $A=(A_{nk})$ with $A_{nk}\in B(X,Y)$ such that $Tx = A_w x$ for each $x\in G$.*

Proof. Let $T_n := \pi_n \circ T$ ($n\in\mathbb{N}$) where π_n denotes the n-th projection from F to Y. We define the Matrix $A=(A_{nk})$ by $A_{nk} := T_n \circ e_k$ ($n,k\in\mathbb{N}$). Since T_n and e_k are weakly continuous, A_{nk} is weakly continuous and thus continuous, that is $A_{nk}\in B(X,Y)$. Furthermore, we get

$$T(x) = (T_n(x))_n = \left(\sigma(Y,Y')\text{-}\sum_k (T_n\circ e_k)(x_k)\right)_n = A_w x \quad \text{for each } x\in G$$

from the weak continuity of T_n and the AK-property of $(G,\sigma(G,G^\beta))$, that is $x=\sum_k e_k(x_k)$ in $(G,\sigma(G,G^\beta))$ ∎

We complete the paper with some remarks to the relation of the present paper to [6].

Remark 4.7 . The authors gave in Theorem 3.3 of [6] an inclusion theorem connected with weak domains of operator valued matrices. Because the assumptions as well as the statement (a) in Theorem 4.4 of the present paper and in Theorem 3.3 of [6] are identical, we may complete Theorem 4.4 by statements (b) – (e) of Theorem 3.1 of [6].

Example 5.1 and Remark 5.2 of [6] tell us that we cannot deduce the Mazur–Orlicz Theorem 3.3 of [6] from Inclusion Theorem 4.4 if we consider the natural topology on weak domains which makes it to an FK(X)–space (see [3]).

It is obvious to ask for suitable K(X)–topologies on weak domains such that the domain becomes an L_φ–K(X)–space and the limit map becomes continuous because in this case there is a little hope that we can deduce theorems of Mazur–Orlicz type – and consequently consistency theorems in case of weak domains – from Inclusion Theorem 4.4.

The authors wish to thank the referee for helpful hints. In particular, the example in Remark 3.2 is due to the referee.

REFERENCES

[1] BARIC, L. W.: *The chi function in generalized summability.* Studia Math. **39** (1971), 165 – 180.

[2] BENNETT, G. and N. J. KALTON: *Inclusion theorems for K-spaces.* Canad. J. Math. **25** (1973), 511 – 524.

[3] BOOS, J. and P. CASS: *Weak domains of operator valued matrices are FK(X)–spaces.* Analysis (to appear).

[4] BOOS, J. and T. LEIGER: *General theorems of Mazur-Orlicz type.* Studia Math. **92** (1989), 1 – 19.

[5] BOOS, J. and T. LEIGER: *Some distinguished subspaces of domains of operator valued matrices.* Resultate Math. **16** (1989), 199 – 211.

[6] BOOS, J. and T. LEIGER: *Consistency theory for operator valued matrices.* Analysis **11** (1991), 279 – 292.

[7] ERDŐS, P. and G. PIRANIAN: *Convergence fields of row-finite and row-infinite Toeplitz transformations. Proc. Amer. Math. Soc.* 1 (1950), 397 - 401.

[8] HUNTER, R. J. and J. LLOYD: *Weakly compactly generated locally convex spaces.* Math. Proc. Camb. Phil. Soc. **82** (1977), 85 - 98.

[9] KALTON, N. J.: *Some forms of the closed graph theorem.* Proc. Cambridge Phil. Soc. **70** (1971), 401 - 408.

[10] QIU, JINGHUI: *A new class of locally convex spaces and the generalization of Kalton's closed graph theorem.* Acta Math. Scientia **5** (1985), 389 - 397.

[11] RAMANUJAN, M. S.: *Vector sequence spaces and perfect summability matrices of operators in Banach spaces.* Publ. Ramanujan Inst. 1 (1969), 145 - 153.

[12] SNYDER, A. K.: *Consistency theory in semiconservative spaces.* Studia Math. **71** (1982), 1 - 13.

[13] ZELLER, K.: *Allgemeine Eigenschaften von Matrixverfahren.* Math. Z. **53** (1951), 463 - 487.

Received 02.01.1992; in revised form 15.09.1992