

On a Theorem of Rooney Concerning the Spectrum of the Singular Integral Operator

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The authors have recently described the spectrum and the essential spectrum of the singular integral operator on spaces with general Muckenhoupt weights. In this note we show how these results imply a sharpened version of a theorem by Rooney on the spectrum of the singular integral operator on spaces with a "weakly" perturbed power weight.

Key words: *Singular integral operators, Wiener–Hopf operators, spaces with Muckenhoupt weight*

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1. Introduction. A famous theorem of Hunt, Muckenhoupt, and Wheeden [3] describes all the weights v on \mathbb{R} for which the singular integral operator S on \mathbb{R} , given by

$$(Sf)(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty} (t-x)^{-1} f(t) dt \quad (\bar{x} \in \mathbb{R})$$

is bounded on $L^p(\mathbb{R}, v)$ ($1 < p < \infty$). Because for such weights $S^2 = I$, the spectrum of S on $L^p(\mathbb{R}, v)$ is the doubleton $\{-1, 1\}$. The much more interesting problem of identifying the spectrum on $L^p(\mathbb{R}_+, v)$ of the compression S_+ of S to the positive half-line $\mathbb{R}_+ = (0, \infty)$,

$$(S_+f)(x) = \frac{1}{\pi i} \int_0^{\infty} (t-x)^{-1} f(t) dt \quad (x \in \mathbb{R}_+),$$

has attracted many people for a long time, including Widom [9], Duduchava [2], Schneider [7], Rooney [6], Roch and Silbermann [4], and the authors [1,8].

The operator S_+ is the simplest example of a Wiener–Hopf integral operator with a piecewise continuous symbol: the symbol of S_+ is $-\operatorname{sgn} \xi$, and it has two jumps, namely at the origin and at infinity. It is well known in Wiener–Hopf theory that for determining the spectrum of a Wiener–Hopf operator it suffices to describe the essential spectrum of the operator (i.e. its spectrum modulo compact operators) and to establish an index formula for Fredholmian Wiener–Hopf operators. For operators with piecewise continuous symbols, and so in particular for S_+ , the latter problem can be solved in a standard way once only the essential spectrum is known.

Widom [9], Duduchava [2], and Schneider [7] studied S_+ on $L^p(\mathbb{R}_+, v)$ for certain special classes of Hunt–Muckenhoupt–Wheeden weights v and arrived at the conclusion that the essential spectrum of S_+ is the union of two (possibly coinciding) circular arcs with the endpoints ± 1 whose shape is determined by the value of p and the behaviour of the weight v at the origin and at infinity.

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The case of general Hunt–Muckenhoupt–Wheeden weights was disposed of only recently in [1,8], where we showed that the essential spectrum of S_+ is the union of two so-called horns; a horn is a closed subset of \mathbb{C} bounded by two circular arcs joining -1 and $+1$.

However, exactly as it is by no means a triviality to check whether a given weight satisfies the Hunt–Muckenhoupt–Wheeden condition, it is in general no easy matter to describe the concrete shape of the horns constituting the essential spectrum of S_+ . It is not difficult to show that the two horns degenerate to circular arcs in the Widom–Duduchava–Schneider cases, and we were also able to produce Hunt–Muckenhoupt–Wheeden weights for which the horns do not degenerate to circular arcs (see [1,8]). An interesting intermediate case is the weights considered by Rooney [5,6]. These weights are of the form $w(x) = x^{\mu-1/p}v(\log x)$, where $\mu \in (0, 1)$ and v is an Hunt–Muckenhoupt–Wheeden weight. Although these weights involve besides a power of x a “proper Hunt–Muckenhoupt–Wheeden portion”, Rooney [6] showed that the spectrum of S_+ on $L^p(\mathbb{R}_+, w)$ is a subset of a certain circular arc.

The purpose of this note is to prove that in Rooney’s case the spectrum of S_+ is actually all of the circular arc and secondly, to clarify why in the case of Rooney weights the two horns collapse to a single circular arc.

2. Hunt–Muckenhoupt–Wheeden weights. Let v be a non-negative function on \mathbb{R} which does not vanish identically. The function v is said to belong to A_p ($1 < p < \infty$) and is then called a *Hunt–Muckenhoupt–Wheeden weight* if v and v^{-1} are locally in $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ ($1/p + 1/q = 1$), respectively, and if

$$\sup_{-\infty < a < b < \infty} (b-a)^{-1} \left(\int_a^b v(x)^p dx \right)^{1/p} \left(\int_a^b v(x)^{-q} dx \right)^{1/q} < \infty.$$

It was shown in [3] that S is a bounded operator on $L^p(\mathbb{R}, v)$ with the norm

$$\|f\| := \left(\int_{-\infty}^{\infty} |f(x)v(x)|^p dx \right)^{1/p}$$

if and only if $v \in A_p$. In case $v \in A_p$, the compression S_+ of S is clearly bounded on the space $L^p(\mathbb{R}_+, v) := L^p(\mathbb{R}_+, v|_{\mathbb{R}_+})$.

Theorem 2.1 [1]: *If $v \in A_p$, then each of the sets*

$$I_0(p, v) := \{ \alpha \in \mathbb{R} : |x|^\alpha |x - i|^{-\alpha} v(x) \in A_p \},$$

$$I_\infty(p, v) := \{ \alpha \in \mathbb{R} : |x - i|^{-\alpha} v(x) \in A_p \}$$

is an open interval of a length not greater than 1 which contains the origin, i.e., for $\xi = 0$, ∞ we have

$$I_\xi(p, v) := (-\nu_\xi^-(p, v), 1 - \nu_\xi^+(p, v)) \text{ with } 0 < \nu_\xi^-(p, v) \leq \nu_\xi^+(p, v) < 1.$$

Given a number $\beta \in (0, 1)$, we denote by σ_β the circular arc with the endpoints ± 1 passing through the point $i \cot \pi\beta$. For $0 < \gamma \leq \delta < 1$, we define the horn $\mathcal{H}(\gamma, \delta)$ as the union of all the arcs σ_β such that $\gamma \leq \beta \leq \delta$.

Theorem 2.2 [1]: *If $v \in A_p$, then the essential spectrum of S_+ on $L^p(\mathbb{R}_+, v)$ equals*

$$\mathcal{H}(\nu_0^-(p, v), \nu_0^+(p, v)) \cup \mathcal{H}(1 - \nu_\infty^+(p, v), 1 - \nu_\infty^-(p, v)).$$

3. Rooney weights. In [5,6], a non-negative function w on \mathbb{R}_+ which does not vanish identically is defined to belong to \mathfrak{A}_p ($1 < p < \infty$) if w and w^{-1} are locally in $L^p(\mathbb{R}_+)$ and $L^q(\mathbb{R}_+)$ ($1/p + 1/q = 1$), respectively, and if

$$\sup_{-\infty < a < b < \infty} \left(\log \frac{\beta}{\alpha} \right)^{-1} \left(\int_a^b w(y)^p dy/y \right)^{1/p} \left(\int_a^b w(y)^{-q} dy/y \right)^{1/q} < \infty.$$

It is readily seen that $w \in \mathfrak{A}_p$ if and only if $w(x) = v(\log x)$ for some $v \in A_p$.

Lemma 3.1 [5, p.261]: *If $w \in \mathfrak{A}_p$ and $\mu \in (0, 1)$, then the weight w_μ defined by $w_\mu(x) = |x|^{\mu-1/p}w(|x|)$ for $x \in \mathbb{R}$ belongs to A_p .*

This lemma yields in particular the boundedness of S_+ on $L^p(\mathbb{R}_+, w_\mu)$. The following theorem sharpens the result obtained by Rooney in [6].

Theorem 3.2: *If $w \in \mathfrak{A}_p$ and $\mu \in (0, 1)$, then both the spectrum and the essential spectrum of S_+ on $L^p(\mathbb{R}_+, w_\mu)$ coincide with the circular arc σ_μ .*

Proof: In [6], it was shown that the spectrum of S_+ is a subset of σ_μ . Our Theorem 2.2 implies that the essential spectrum of S_+ is connected, and since the essential spectrum is contained in the spectrum, it follows that both spectra are equal to all of σ_μ , which completes the proof. ■

In what follows we give a more direct proof of Theorem 3.2: we show how this theorem can be derived from the Theorems 2.1 and 2.2 without having recourse to [6].

Lemma 3.3: *Let $w \in \mathfrak{A}_p$ and $\mu \in (0, 1)$. Then*

$$I_0(p, w_\mu) = (-\mu, 1 - \mu), \quad I_\infty(p, w_\mu) = (-1 + \mu, \mu).$$

Proof: Given a weight v on \mathbb{R} , define a weight ρ on the complex unit circle \mathbb{T} by

$$\rho(t) = v(i(t + 1)/(t - 1))|t - 1|^{1-2/p} \quad (t \in \mathbb{T}).$$

One can show (see e.g. [1, proof of Theorem 2.10]) that $v \in A_p$ if and only if ρ satisfies the Hunt-Muckenhoupt-Wheeden condition $A_p(\mathbb{T})$ on the unit circle, i.e. if and only if we have

$$\sup_I \frac{1}{|I|} \left(\int_I \rho^p dm \right)^{1/p} \left(\int_I \rho^{-q} dm \right)^{1/q} < \infty, \tag{1}$$

where the supremum is taken over all subarcs I of \mathbb{T} whose arc length $|I|$ is less than any prescribed $\delta > 0$ and dm refers to arc measure on \mathbb{T} . From Lemma 3.1 with $\mu - 1/p =: \beta$ we so infer that $|t + 1|^\beta |t - 1|^{-\beta} |t - 1|^{1-2/p} w(|t + 1|/|t - 1|)$ is in $A_p(\mathbb{T})$ for all $\beta \in (-1/p, 1/q)$. Taking $\delta > 0$ sufficiently small, so that no I in (1) contains both -1 and 1 , we see that $|t + 1|^\beta |t - 1|^{1-2/p} w(|t + 1|/|t - 1|)$ and $|t - 1|^{-\gamma} |t - 1|^{1-2/p} w(|t + 1|/|t - 1|)$ are also in $A_p(\mathbb{T})$ for all $\beta, \gamma \in (-1/p, 1/q)$. Hence once more using (1) with sufficiently small δ , we obtain

$$|t + 1|^\beta |t - 1|^{-\gamma} |t - 1|^{1-2/p} w \left(\frac{|t + 1|}{|t - 1|} \right) \in A_p(\mathbb{T}) \quad \forall \beta, \gamma \in \left(-\frac{1}{p}, \frac{1}{q} \right). \tag{2}$$

We now have

$$\begin{aligned} I_0(p, w_\mu) &= \{ \alpha \in \mathbb{R} : |x|^\alpha |x - i|^{-\alpha} |x|^{\mu-1/p} w(|x|) \in A_p \} \\ &= \left\{ \alpha \in \mathbb{R} : |t+1|^{\alpha+\mu-1/p} |t-1|^{-\mu+1/p} |t-1|^{1-2/p} w \left(\frac{|t+1|}{|t-1|} \right) \in A_p(\mathbb{T}) \right\}, \end{aligned}$$

and so (2) gives that $(-\mu, 1-\mu)$ is a subset of $I_0(p, w_\mu)$. Analogously,

$$\begin{aligned} I_\infty(p, w_\mu) &= \{ \alpha \in \mathbb{R} : |x - i|^{-\alpha} |x|^{\mu-1/p} w(|x|) \in A_p \} \\ &= \left\{ \alpha \in \mathbb{R} : |t+1|^{\mu-1/p} |t-1|^{\alpha-\mu+1/p} |t-1|^{1-2/p} w \left(\frac{|t+1|}{|t-1|} \right) \in A_p(\mathbb{T}) \right\}, \end{aligned}$$

and (2) shows anew that $(-1+\mu, \mu)$ is contained in $I_\infty(p, w_\mu)$. Since, by Theorem 2.1, the lengths of $I_0(p, w_\mu)$ and $I_\infty(p, w_\mu)$ are not greater than 1, we arrive at the assertion. ■

Second proof of Theorem 3.2: Lemma 3.3 gives $\nu_0^\pm(p, w_\mu) = \mu$, $\nu_\infty^\pm(p, w_\mu) = 1 - \mu$, and hence the essential spectrum of S_+ on $L^p(\mathbb{R}_+, w_\mu)$ is $\mathcal{H}(\mu, \mu) \cup \mathcal{H}(\mu, \mu) = \sigma_\mu$ due to Theorem 2.2. Because the winding number of the curve obtained by tracing out σ_μ from 1 to -1 and then back from -1 to 1 is zero with respect to every point in $\mathbb{C} \setminus \sigma_\mu$, it follows that σ_μ is also the spectrum of S_+ . ■

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