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Strong Approximation of Spherical Functions by Cesàro Means

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The paper deals with the approximation of spherical functions by Cesàro means. The strong approximation order of the Cesàro means on sets of full measure is established.

Key words: *Strong approximation, Cesàro means, spherical harmonic expansions*
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§1. Introduction

Let Σ_d be the unit sphere (with center at the origin) in the $(d+1)$-dimensional Euclidean space $R^{(d+1)}$. By $L^p(\Sigma_d)$, $1 \le p < \infty$, we denote the space of (the equivalence classes of) p-th integrable functions on Σ_d for which the norm

$$\|f\|_p := \left\{ \int_{\Sigma_d} | f(x) |^p \, dx \right\}^{\frac{1}{p}}$$

is finite. A function $f \in L^1(\Sigma_d)$ can be expanded in a series of surface spherical harmonics; i.e.,

$$f(x) \sim \sum_{k=0}^{\infty} Y_k(f;x),$$

where

$$Y_k(f;x) := \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_{\Sigma_d} P_k^\lambda(xy) f(y)\, dy, \qquad k \in \mathrm{N}_0,$$

P_k^λ, $\lambda = \frac{1}{2}(d-1)$, being the ultraspherical (or Gegenbauer) polynomials. The Cesàro means of order δ of the harmonic series of f are defined by

$$C_n^\delta(f;x) := \sum_{k=0}^{n} \frac{A_{n-k}^\delta}{A_n^\delta} Y_k(f;x); \quad n \in \mathrm{N}_0, \tag{1.1}$$

where $\delta > -1$ and $A_n^\delta := \binom{n+\delta}{\delta} = \frac{\Gamma(n+\delta+1)}{\Gamma(\delta+1)\Gamma(n+1)}$. It is well-known that, for some appropriate index δ, $C_n^\delta(f;x)$ converge to $f(x)$ almost everywhere on Σ_d and in norm; we refer the reader to Bonami and Clerc [1], Sogge [3] for details. We also know

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that, for $\delta = 0$, $C_n^\delta(f;\cdot)$ is the usual n-th partial sum of the series of the surface spherical harmonics of f. Up to now, it is not known whether, for $f \in L^2(\Sigma_d)$, $C_n^0(f;x)$ converges to $f(x)$ almost everywhere or not; therefore the index δ of the Cesàro means is restricted to be positive in our arguments. Considering strong convergence, the index can be extended to be negative; similarly for strong approximation. Here we want to study strong summability and strong approximation by Cesàro means on sets of full measure. To be more specific, we want to study the validity of the formula

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} | C_k^\delta(f;x) - f(x) |^2 = 0 \qquad a.e., \tag{1.2}$$

and to estimate its convergence rate. Let $\beta > 0$, and let $f \in L^2(\Sigma_d)$. If there exists a function $g \in L^2(\Sigma_d)$ such that $Y_k(g;\cdot) = k^\beta Y_k(f;\cdot)(k \in \mathrm{N}_0)$, we call g the *Riesz derivative* of f of order β and write $f^{\{\beta\}} = g$ (we use this notion in analogy to the one in the theory of Fourier transforms, see [5, Chapter V]). We see that $f^{\{\beta\}}$ is uniquely determined by f, and define the Riesz space $L^{2,\beta}(\Sigma_d)$ in $L^2(\Sigma_d)$ to be

$$L^{2,\beta}(\Sigma_d) := \{f \in L^2(\Sigma_d);\ \|f\|_{2,\beta} < \infty\},$$

where

$$\|f\|_{2,\beta} := \|f^{\{\beta\}}\|_2 = \left\{\sum_{k=0}^{\infty} k^{2\beta}\,\|Y_k(f)\|_2^2\right\}^{\frac{1}{2}}.$$

$L^{2,\beta}(\Sigma_d)$ is a complete linear subspace of $L^2(\Sigma_d)$ under the norm $\|\cdot\|_{2,\beta}$, and continuously imbedded in $L^2(\Sigma_d)$.

We can now state the main results.

Theorem 1: *Let $f \in L^{2,\beta}(\Sigma_d), 0 \le \beta \le 1$, and let $\delta > 0$. For almost all x in Σ_d*

$$|C_n^\delta(f;x) - f(x)| = \begin{cases} o_x(\frac{1}{n^\beta}) & \text{if } 0 \le \beta < 1 \\ O_x(\frac{1}{n}) & \text{if } \beta = 1. \end{cases} \tag{1.3}$$

Theorem 2: *Let $f \in L^{2,\beta}(\Sigma_d), 0 \le \beta \le 1$, and let $\delta > -\frac{1}{2}$. For almost all x in Σ_d*

$$\frac{1}{n}\sum_{k=0}^{n} | C_k^\delta(f;x) - f(x) |^2 = \begin{cases} O_x(\frac{1}{n^{2\beta}}) & \text{if } 0 \le \beta < \frac{1}{2} \\ O_x(\frac{\log n}{n}) & \text{if } \beta = \frac{1}{2} \\ O_x(\frac{1}{n}) & \text{if } \beta > \frac{1}{2}. \end{cases} \tag{1.4}$$

Remark: Replacing the Cesàro means of f by the Bochner-Riesz means $S_R^\delta(f;\cdot)$, defined by $S_R^\delta(f;\cdot) := \sum_{k<R}(1 - \frac{k^2}{R^2})^\delta Y_k(f;\cdot)$, we have analoguous results.

§2. Auxiliary lemmas

We begin with the strong summability of C_n^δ . For $f \in L^1(\Sigma_d)$ let

$$C^\delta(f;x) = \sup_{n\geq 1}\left\{\frac{1}{n}\sum_{k=0}^{n} \mid C_k^\delta(f;x)\mid^2\right\}^{\frac{1}{2}}.$$

Following arguments similar to those in [1, p.237–239], we can deduce that for $\delta = \sigma + i\tau$ and for any $f \in L^2(\Sigma_d)$

$$\|C^\delta(f)\|_2 \leq \text{const}_\delta\, e^{c\tau^2}\|f\|_2, \quad \sigma > -\frac{1}{2},\ 0 < c \leq \pi, \tag{2.1}$$

and that for any $f \in L^p(\Sigma_d)$, $1 < p \leq 2$,

$$\|C^\delta(f)\|_p \leq \text{const}_{\delta,p}\, e^{c\tau^2}\|f\|_p, \quad \sigma > \frac{1}{2}(d-1),\ 0 < c \leq \pi. \tag{2.2}$$

Here and in the following, $\text{const}_{\delta,p,\ldots}$ denotes a constant depending only on the listed subindices. By linearizing the operator C^δ and by applying Stein's interpolation theorem, we get from (2.1) and (2.2) the following

Proposition: *Let $\delta > d(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$. For $f \in L^p(\Sigma_d), 1 < p \leq 2$,*

$$\|C^\delta(f)\|_p \leq \text{const}_{\delta,p}\,\|f\|_p \tag{2.3}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}|C_k^\delta(f;x) - f(x)|^2 = 0 \quad a.e. \tag{2.4}$$

Proof: It follows from straightforward modifications of the proof of the corresponding result in [4]; for the sake of completeness, let us give just a sketch. Let $n = n(u)\,(u \in \Sigma_d)$ be any step function taking positive integral values and let $\{\varphi_k(u)\}$ be any sequence of measurable functions defined on Σ_d which satisfy the condition

$$\frac{1}{n}\sum_{k=0}^{n}\varphi_k^2(u) \leq 1, \quad \forall u \in \Sigma_d \quad \text{and} \quad \forall n \in \mathbf{N}.$$

Keeping the functions $n(u)$ and $\varphi_k(u)$ momentarily fixed, we define by L_δ the linear operators

$$L_\delta(f;\,\cdot) = \frac{1}{n}\sum_{k=0}^{n}C_k^\delta(f;\,\cdot)\varphi_k.$$

By Schwarz's inequality $|L_\delta(f;\,\cdot)| \leq C^\delta(f;\,\cdot)$. Moreover there is not difficulty to verify that, for any p, $\|C^\delta(f)\|_p = \sup\|L_\delta(f)\|_p$, here the supremum is taken over all functions $n(u)$ and $\varphi_k(u)$ of the type described above. We now define an analytic family of operators $\{T_z\}_{z\in\mathbf{C}}$ by

$$T_z(f;\,\cdot) := L_{\delta(z)}(f;\,\cdot), \qquad z\in\mathbf{C},$$

where $\delta(z) = (\varepsilon_0 - \frac{1}{2})(1-z) + (\frac{1}{2}(d-1)+\varepsilon_1)z\,(\varepsilon_0,\varepsilon_1 > 0)$. By (2.1) and (2.2) we have for $\tau = [-(\varepsilon_0-\frac{1}{2}) + (\frac{1}{2}(d-1)+\varepsilon_1)]y$,

$$\|T_{iy}(f)\|_2 = \|L_{\delta(iy)}(f)\|_2 \le \|C^{\delta(iy)}(f)\|_2 \le \mathrm{const}_{\varepsilon_0} e^{c\tau^2}\|f\|_2, \quad 0<c\le\pi,$$

and

$$\|T_{1+iy}(f)\|_{p_1} \le \|C^{\delta(1+iy)}(f)\|_{p_1} \le \mathrm{const}_{\varepsilon_1} e^{c\tau^2}\|f\|_{p_1}, \quad 1<p_1<2,\ 0<c\le\pi.$$

It is important to notice that $\mathrm{const}_{\varepsilon_0}$ and $\mathrm{const}_{\varepsilon_1}$ do not depend on $n(u)$ and $\varphi_k(u)$. Let $0<t<1$, $\frac{1}{p} = \frac{1}{2}(1-t) + \frac{1}{p'}t$, and $\frac{1}{p}+\frac{1}{p'} = 1$. Applying Stein's interpolation theorem (see [4]), $\|T_t(f)\|_p \le \mathrm{const}_t\|f\|_p$. Again const_t does not depend on $n(u)$ and $\varphi_k(u)$. Finally, $\|C^{\delta(t)}(f)\|_p \le \mathrm{const}_t\|f\|_p$. It is clear that $\delta(t)$ is a continuous function of p_1, ε_0, and ε_1. Thus, by continuity, we can always realize any $\delta(t)$ satisfying $\delta(t) > \frac{d-1}{2}(\frac{2}{p}-1) - \frac{1}{p'}$ by choosing $p_1>1, \varepsilon_0>0$, and $\varepsilon_1>0$ appropriately. ■

The proposition makes our results meaningful. For the proof of the theorem we introduce the auxiliary maximal functions

$$\begin{aligned}
M^\delta_\beta(f;x) &:= \sup_{n\ge 0} n^\beta\,|C^\delta_n(f;x) - f(x)| \\
N^\delta_\beta(f;x) &:= \sup_{n\ge 0} n^\beta\,|C^{\delta+1}_n(f;x) - C^\delta_n(f;x)| \\
g^\delta_\beta(f;x) &:= \Big\{\sum_{n=0}^\infty n^{2\beta-1}\,|C^{\delta+1}_n(f;x) - C^\delta_n(f;x)|^2\Big\}^{\frac{1}{2}}.
\end{aligned}$$

The first two ones are maximal functions, while the last one is a Littlewood-Paley function. We want to study the boundedness of these functions; to do so we need some extra-ordinary identities of the Cesàro means. Throughout this paper, the indeterminant expression 0^0 will be understood to be equal to 1 whenever it comes up.

Lemma 1: *Let $\delta > -\frac{1}{2}$. For $f\in L^{2,\beta}(\Sigma_d)$, $0\le\beta<1$,*

$$\|g^\delta_\beta(f)\|_2 \le \mathrm{const}_{\delta,\beta}\,\|f\|_{2,\beta}. \tag{2.5}$$

Proof: We have

$$C^{\delta+1}_n(f;x) - C^\delta_n(f;x) = \frac{1}{n+\delta+1}\sum_{k=0}^n \frac{A^\delta_{n-k}}{A^\delta_n}\,k\,Y_k(f;x).$$

It follows from the orthogonality of the projection operators $Y_k (k \in \mathbf{N}_0)$ that

$$\begin{aligned}\|g_\beta^\delta(f)\|_2^2 &= \sum_{n=0}^{\infty} n^{2\beta-1} \frac{1}{(n+\delta+1)^2} \int_{\Sigma_d} \Big| \sum_{k=0}^{n} \frac{A_{n-k}^\delta}{A_n^\delta}\, k\, Y_k(f;x) \Big|^2 dx \\ &= \sum_{n=0}^{\infty} n^{2\beta-1} \frac{1}{(n+\delta+1)^2} \sum_{k=0}^{n} \Big(\frac{A_{n-k}^\delta}{A_n^\delta}\Big)^2 k^2 \|Y_k(f)\|_2^2 \\ &= \sum_{k=0}^{\infty} k^{2\beta} \|Y_k(f)\|_2^2\, k^{2-2\beta} \sum_{n=k}^{\infty} \frac{n^{2\beta-1}}{(n+\delta+1)^2} \Big(\frac{A_{n-k}^\delta}{A_n^\delta}\Big)^2 \\ &\le \mathrm{const}_{\delta,\beta}\, \|f\|_{2,\beta}^2.\end{aligned}$$

In the last estimate we used the fact that, for $0 \le \beta < 1$,

$$\sum_{n=k}^{\infty} \frac{n^{2\beta-1}}{(n+\delta+1)^2} \Big(\frac{A_{n-k}^\delta}{A_n^\delta}\Big)^2 \le \mathrm{const}_{\delta,\beta}\, k^{2\beta-2}. \qquad \blacksquare$$

Lemma 2: *Let $\delta > 0$. For $f \in L^{2,\beta}(\Sigma_d), 0 \le \beta \le 1$,*

$$\|N_\beta^\delta(f)\|_2 \le \mathrm{const}_{\delta,\beta}\, \|f\|_{2,\beta}. \tag{2.6}$$

Proof: We first notice that

$$C_n^{\delta+1}(f;x) - C_n^\delta(f;x) = \frac{1}{2A_n^{\delta+1}} \sum_{k=0}^{n} A_{n-k}^{\frac{\delta-1}{2}} A_k^{\frac{\delta+1}{2}} \Big(C_k^{\frac{\delta+1}{2}}(f;x) - C_k^{\frac{\delta-1}{2}}(f;x)\Big). \tag{2.7}$$

In fact, for $\gamma > 0$ and $\alpha > -1$, we have

$$C_n^{\gamma+\alpha+1}(f;x) - C_n^{\gamma+\alpha}(f;x) = \frac{1}{\gamma+\alpha+1} \frac{1}{A_n^{\gamma+\alpha+1}} \sum_{k=0}^{n} k\, A_{n-k}^{\gamma+\alpha} Y_k(f;x). \tag{2.8}$$

Since

$$A_{n-k}^{\gamma+\alpha} = \sum_{j=0}^{n-k} A_j^{\gamma-1} A_{n-k-j}^{\alpha}, \tag{2.9}$$

$$\begin{aligned}\sum_{k=0}^{n} k\, A_{n-k}^{\gamma+\alpha} Y_k(f;x) &= \sum_{k=0}^{n} k \sum_{j=0}^{n-k} A_j^{\gamma-1} A_{n-k-j}^{\alpha} Y_k(f;x) \\ &= \sum_{j=0}^{n} \sum_{k=0}^{n-k} k\, A_j^{\gamma-1} A_{n-k-j}^{\alpha}\, Y_k(f;x) \\ &= \sum_{j=0}^{n} A_j^{\gamma-1} A_{n-j}^{\alpha+1} (\alpha+1) \Big(C_{n-j}^{\alpha+1}(f;x) - C_{n-j}^{\alpha}(f;x)\Big) \\ &= \sum_{k=0}^{n} A_{n-k}^{\gamma-1} A_k^{\alpha+1} (\alpha+1) \Big(C_k^{\alpha+1}(f;x) - C_k^{\alpha}(f;x)\Big).\end{aligned} \tag{2.10}$$

By taking $\alpha = \frac{\delta-1}{2} (> -\frac{1}{2})$ and $\gamma = \delta - \alpha$, hence $\gamma + \alpha = \delta$ and $\gamma = \frac{\delta+1}{2}$, the formulas (2.8) and (2.10) yield (2.7). Next we apply Schwarz's inequality, use the estimate $A_n^\delta = O(n^\delta)$, and obtain

$$\begin{aligned}|C_n^{\delta+1}(f;x) - C_n^\delta(f;x)| &\leq \frac{1}{2A_n^{\delta+1}} \Big\{ \sum_{k=0}^n \left(A_{n-k}^{\frac{\delta-1}{2}} A_k^{\frac{\delta+1}{2}} \right)^2 k^{1-2\beta} \Big\}^{\frac{1}{2}} \\ &\quad \times \Big\{ \sum_{k=0}^n k^{2\beta-1} |C_k^{\frac{\delta+1}{2}}(f;x) - C_k^{\frac{\delta-1}{2}}(f;x)|^2 \Big\}^{\frac{1}{2}} \\ &\leq \mathrm{const}_{\delta,\beta}\, n^{-\delta-1}\, g_\beta^{\frac{\delta-1}{2}}(f;x) \Big\{ \sum_{k=0}^n (n-k)^{\delta-1} k^{\delta+1} k^{1-2\beta} \Big\}^{\frac{1}{2}} \\ &\leq \mathrm{const}_{\delta,\beta}\, n^{-\beta}\, g_\beta^{\frac{\delta-1}{2}}(f;x),\end{aligned}$$

or

$$n^\beta |C_n^{\delta+1}(f;x) - C_n^\delta(f;x)| \leq \mathrm{const}_{\delta,\beta}\, g_\beta^{\frac{\delta-1}{2}}(f;x), \quad a.e.$$

and finally,

$$N_\beta^\delta(f;x) \leq \mathrm{const}_{\delta,\beta}\, g_\beta^{\frac{\delta-1}{2}}(f;x) \quad a.e.$$

It is important to recall that $\frac{\delta-1}{2} > -\frac{1}{2}$. By Lemma 1, estimate (2.6) is proven for $0 \leq \beta < 1$. For $\beta = 1$, we have

$$n|C_n^{\delta+1}(f;x) - C_n^\delta(f;x)| = \left| \frac{n}{n+\delta+1} \sum_{k=0}^n \frac{A_{n-k}^\delta}{A_n^\delta} k\, Y_k(f;x) \right| \leq \sup_{n\geq 1} |C_n^\delta(f^{\{1\}};x)|.$$

Here $f^{\{1\}} \in L^2(\Sigma_d)$ is , again, the 1-st Riesz derivative of f having the spherical harmonic expansion $f^{\{1\}}(\cdot) = \sum_{k=0}^\infty k\, Y_k(f;\cdot)$. It follows from the L^2-boundedness of $\sup_n |C_n^\delta(f;x)|, \delta > 0$, that

$$\|N_1^\delta(f)\|_2 \leq \|\sup_n |C_n^\delta(f^{\{1\}})|\|_2 \leq \mathrm{const}_\delta \|f^{\{1\}}\|_2 = \mathrm{const}_\delta \|f\|_{2,1},$$

which provides the estimate (2.6) for $\beta = 1$. ∎

Remark: Let $\delta > 0$. For $f \in L^{2,1}(\Sigma_d)$,

$$\lim_{n\to\infty} n(f(x) - C_n^\delta(f;x)) = f^{\{1\}}(x) \qquad a.e.$$

Indeed, because of

$$f(x) - C_n^{\delta+1}(f;x) = (\delta+1) \sum_{k=n+1}^\infty \frac{1}{k(k+\delta+1)} C_k^\delta(f^{\{1\}};x),$$

we can verify the equation

$$n(f(x)-C_k^{\delta}(f;x))-f^{\{1\}}(x)=\Big[n(f(x)-C_n^{\delta+1}(f;x))-(\delta+1)f^{\{1\}}(x)\Big]$$
$$-\frac{n}{n+\delta+1}\Big[C_n^{\delta}(f^{\{1\}};x)-f^{\{1\}}(x)\Big]-\frac{\delta+1}{n+\delta+1}f^{\{1\}}(x).$$

We know that the three terms on the right-hand side of the equation above tend to zero almost everywhere as $n\to\infty$.

Lemma 3: *For $f\in L^{2,\beta}(\Sigma_d)$, $0\le\beta\le1$,*

$$\|M_\beta^1(f)\|_2\le \mathrm{const}_\beta\|f\|_{2,\beta}. \tag{2.11}$$

Proof: Set

$$b_{n,k}=\begin{cases}(\frac{k}{n+1})^{1-\beta} & \text{if } 0\le k\le n\\ (\frac{k}{n+1})^{-\beta} & \text{if } n+1\le k<\infty\end{cases}\quad\text{and}\quad \Delta^2 b_{n,k}=b_{n,k}-2b_{n,k+1}+b_{n,k+2}.$$

We define a sequence of linear operators $\{E_n\}$ by $E_n(f;\,\cdot)=\sum_{k=0}^{\infty}b_{n,k}Y_k(f;\,\cdot)$. Then

$$E_n(f;\,\cdot)=\sum_{k=0}^{\infty}(k+1)\Delta^2 b_{n,k}C_k^1(f;\,\cdot). \tag{2.12}$$

Indeed, on the one hand,

$$\sup_n\sum_{k=0}^{\infty}(k+1)|\Delta^2 b_{n,k}|=\sup_n\Big\{\sum_{k=0}^{n}(k+1)|\Delta^2 b_{n,k}|+\sum_{k=n+1}^{\infty}(k+1)|\Delta^2 b_{n,k}|\Big\}$$
$$\le\sup_n 2\Big[1+(n+1)\Big(1-\Big(\frac{n+1}{n+2}\Big)^{\beta}\Big)\Big]\le \mathrm{const}_\beta\,(<\infty);$$

i.e., the sequence of multipliers $\{b_{n,k}\}$ is uniformly quasi-convex. Consequently,

$$\sum_{k=0}^{\infty}(k+1)\Delta^2 b_{n,k}C_k^1(f;\,\cdot)\in L^2(\Sigma_d)\qquad\text{if } f\in L^2(\Sigma_d),$$

and

$$\sup_n\Big|\sum_{k=0}^{\infty}(k+1)\Delta^2 b_{n,k}C_k^1(f;x)\Big|\le \mathrm{const}_\beta\sup_k|C_k^1(f;x)|. \tag{2.13}$$

On the other hand, $Y_m\Big(\sum_{k=0}^{\infty}(k+1)\Delta^2 b_{n,k}C_k^1(f;\,\cdot)\Big)=b_{n,m}Y_m(f;\,\cdot)$ for each $m\ge0$.

In fact, $Y_m(C_k^1(f;\,\cdot)) = 0$ if $k < m$, and $= (1-\frac{m}{k+1})Y_m(f;\,\cdot)$ if $k \geq m$. Therefore,

$$\begin{aligned} Y_m\left(\sum_{k=0}^{\infty}(k+1)\Delta^2 b_{n,k} C_k^1(f;\,\cdot)\right) &= \sum_{k=m}^{\infty}(k+1)\Delta^2 b_{n,k}\left(1-\frac{m}{k+1}\right)Y_m(f;\,\cdot) \\ &= Y_m(f;\,\cdot)\sum_{k=m}^{\infty}(k+1-m)\Delta^2 b_{n,k} \\ &= Y_m(f;\,\cdot)\sum_{k=0}^{\infty}(k+1)\Delta^2 b_{n,k+m} \\ &= b_{n,m} Y_m(f;\,\cdot), \qquad \forall m \in \mathbf{N}_0. \end{aligned}$$

Using the fact that the sequence of harmonic projections forms a total system, we obtain equation (2.12). If $f \in L^{2,\beta}(\Sigma_d)$, then $(n+1)^\beta(f(\cdot) - C_n^1(f;\,\cdot)) = E_n(f^{\{\beta\}};\,\cdot)$. Furthermore, by estimates (2.12) and (2.13),

$$M_\beta^1(f;\,\cdot) = \sup_n |E_n(f^{\{\beta\}};\,\cdot)| \leq \text{const}_\beta \sup_k |C_k^1(f^{\{\beta\}};\,\cdot)|,$$

and

$$\|M_\beta^1(f)\|_2 \leq \text{const}_\beta \,\|\sup_k |C_k^1(f^{\{\beta\}})|\|_2 \leq \text{const}_\beta \,\|f^{\{\beta\}}\|_2 = \text{const}_\beta \,\|f\|_{2,\beta},$$

which gives the desired estimate. ∎

§3. Proof of the theorems

Having done all necessary preparations, we can present the proof of the theorems by taking, once again, the special properties of the Cesàro means into account. The Banach continuity principle is also used.

Proof of Theorem 1: First we note that, for any $\gamma > 0$,

$$\|M_\beta^{\gamma+1}(f)\|_2 \leq \text{const}_{\gamma,\beta} \,\|f\|_{2,\beta}. \tag{2.14}$$

In fact, we have

$$\begin{aligned} n^\beta |C_n^{\gamma+1}(f;x) - f(x)| &= n^\beta \left|(A_n^{\gamma+1})^{-1}\sum_{k=0}^{n} A_n^1 A_{n-k}^{\gamma-1}(C_k^1(f;x) - f(x))\right| \\ &\leq n^\beta (A_n^{\gamma+1})^{-1}\sum_{k=0}^{n} A_{n-k}^{\gamma-1}\, k^{-\beta}\, M_\beta^1(f;x) \\ &\leq \text{const}_{\beta,\gamma}\, M_\beta^1(f;x). \end{aligned}$$

Therefore $M_\beta^{\gamma+1}(f;x) \leq \text{const}_{\beta,\gamma}\, M_\beta^1(f;x)$, and by Lemma 3,

$$\|M_\beta^{\gamma+1}(f)\|_2 \le \text{const}_\beta \|M_\beta^1(f)\|_2 \le \text{const}_{\beta,\gamma}\|f\|_{2,\beta}.$$

Let $\delta > 0$. For $f \in L^{2,\beta}(\Sigma_d)$, $0 \le \beta \le 1$, we have $M_\beta^\delta(f;x) \le N_\beta^\delta(f;x) + M_\beta^{1+\delta}(f;x)$. Applying Lemma 2 and the estimate (2.14),

$$\|M_\beta^\delta(f)\|_2 \le \|N_\beta^\delta(f)\|_2 + \|M_\beta^{1+\delta}(f)\|_2 \le \text{const}_{\delta,\beta}\,\|f\|_{2,\beta};$$

i.e. $|C_n^\delta(f;x) - f(x)| = O_x(\frac{1}{n^\beta})\,a.e.$, $0 \le \beta \le 1$. Let $0 \le \beta < 1$. For any spherical polynomial g we have $\lim_{n\to\infty} n^\beta |C_n^\delta(g;x) - g(x)| = 0$. Since the spherical polynomials are dense in $L^{2,\beta}(\Sigma_d)$, and M_β^δ is bounded in $L^{2,\beta}(\Sigma_d)$, by the Banach continuity principle (see, e.g., [2]), we finally obtain for $f \in L^{2,\beta}(\Sigma_d)$,

$$|C_n^\delta(f;x) - f(x)| = o_x\Big(\frac{1}{n^\beta}\Big) \quad a.e., \quad 0 \le \beta < 1. \qquad \blacksquare$$

Proof of Theorem 2: If $\delta > 0$, we can deduce the estimate (1.4) directly from the one of (1.3). In order to handle the case $-\frac{1}{2} < \delta \le 0$, we need the help of the Littlewood-Paley function. For $0 \le \beta \le \frac{1}{2}$,

$$\Big\{\frac{n^{2\beta}}{n}\sum_{k=0}^n |C_k^\delta(f;x) - C_k^{\delta+1}(f;x)|^2\Big\}^{\frac{1}{2}} \le \Big\{\sum_{k=0}^n k^{2\beta-1}|C_k^\delta(f;x) - C_k^{\delta+1}(f;x)|^2\Big\}^{\frac{1}{2}} \le g_\beta^\delta(f;x).$$

A combination of (2.5) with the inequality above yields

$$\Big\|\sup_n \Big\{\frac{n^{2\beta}}{n}\sum_{k=0}^n |C_k^\delta(f;\,\cdot) - C_k^{\delta+1}(f;\,\cdot)|^2\Big\}^{\frac{1}{2}}\Big\|_2 \le \|g_\beta^\delta(f)\|_2 \le \text{const}_{\delta,\beta}\|f\|_{2,\beta}.$$

This gives

$$\frac{1}{n}\sum_{k=0}^n |C_k^\delta(f;x) - C_k^{\delta+1}(f;x)|^2 = O_x\Big(\frac{1}{n^{2\beta}}\Big) \qquad a.e.$$

for $f \in L^{2,\beta}(\Sigma_d)$ and $0 \le \beta \le \frac{1}{2}$. If $f \in L^{2,\beta}(\Sigma_d), \beta > \frac{1}{2}$, then $f \in L^{2,\frac{1}{2}}(\Sigma_d)$. We also have

$$\frac{1}{n}\sum_{k=0}^n |C_k^\delta(f;x) - C_k^{\delta+1}(f;x)|^2 = O_x\Big(\frac{1}{n}\Big) \qquad a.e.$$

We summarise the above and obtain

Lemma 4: *Let $0 \le \beta \le 1$. For $f \in L^{2,\beta}(\Sigma_d)$,*

$$\frac{1}{n}\sum_{k=0}^{n} |C_k^\delta(f;x) - C_k^{\delta+1}(f;x)|^2 = \begin{cases} O_x(\frac{1}{n^{2\beta}}) & \text{if } 0 \le \beta < \frac{1}{2} \\ O_x(\frac{1}{n}) & \text{if } \beta \ge \frac{1}{2} \end{cases}$$

almost everywhere.

By this lemma, Theorem 1, and the inequality

$$\frac{1}{n}\sum_{k=0}^{n} |C_k^\delta(f;x) - f(x)|^2 \le \frac{2}{n}\sum_{k=0}^{n} |C_k^\delta(f;x) - C_k^{\delta+1}(f;x)|^2 + \frac{2}{n}\sum_{k=0}^{n} |C_k^{\delta+1}(f;x) - f(x)|^2,$$

we finally get

$$\frac{1}{n}\sum_{k=0}^{n} |C_k^\delta(f;x) - f(x)|^2 = \begin{cases} O_x(\frac{1}{n^{2\beta}}) & \text{if } 0 \le \beta < \frac{1}{2} \\ O_x(\frac{\log n}{n}) & \text{if } \beta = \frac{1}{2} \\ O_x(\frac{1}{n}) & \text{if } \beta > \frac{1}{2} \end{cases}$$

almost everywhere. ■

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