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Identifiability of the Transmissivity Coefficient in an Elliptic Boundary Value Problem

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Abstract. We deal with a coefficient inverse problem describing the filtration of ground water in a region $\Omega \subset R^n$, $n \geq 2$. Introducing a weak formulation of the problem, discretization and regularization methods can be constructed in a natural way. These methods converge to the normal solution of the problem, i.e. to a transmissivity coefficient of a minimal $L^2(\Omega)$-norm. Thus a question about L^2-identifiability (identifiability among functions of the class $L^2(\Omega)$) of the transmissivity coefficient arises. Our purpose is to describe subregions of Ω where the transmissivity coefficient is really L^2-identifiable or even L^1-identifiable. Thereby we succeed introducing physically realistic conditions on the data of the problem, e.g. piecewise smooth surfaces in Ω are allowed where the data of the inverse problem may have discontinuities. With some natural changes, our results about the L^1-identifiability extend known results about the identifiability among more smooth functions given by G. R. Richter [4], C. Chicone and J. Gerlach [1], and K. Kunisch [3].

Keywords: *Inverse problems, ground water filtration problems, identifiability*

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1. Inverse problems

1.1 Boundary value problem formulation. Let $\Omega \in R^n$ $(n \geq 2)$ be an open region with a piecewise smooth boundary $\partial\Omega$; we denote by ν the outer unit normal to $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be a relatively open set having a piecewise smooth boundary on $\partial\Omega$. We shall deal with the following inverse problem:

Find a coefficient $a \in L^2(\Omega)$ such that

$$-\operatorname{div}(a(x)\nabla u(x)) = f(x) \quad (x \in \Omega), \qquad (a(x)\nabla u(x)) \cdot \nu(x) = g(x) \quad (x \in \Gamma) \qquad (1.1)$$

where $u \in W^{1,\infty}(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ are given functions. Physically, u can be interpreted as the piezometrical head of the ground water in Ω; the function f characterizes the sources and sinks in Ω and the function g characterizes the inflow and outflow through $\Gamma \subset \partial\Omega$. The filtration (transmissivity) coefficient a is, physically, positive and piecewise smooth with possible discontinuities of the first kind on some surfaces in Ω.

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We do not exclude the case of $\Gamma = \emptyset$. The boundary condition is omitted in (1.1) in this case.

Conditions (1.1) can be understood in the sense of distributions. We prefer to deal with the weak formulation of the problem.

1.2 Weak formulation. Let us provisionally assume that the functions a and u are smooth (e.g. $a \in H^1(\Omega)$ and $u \in W^{2,\infty}(\Omega)$). Multiplying the first equation of (1.1) by $w \in H^1(\Omega)$, integrating by parts and using the second equation of (1.1) we obtain

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = \int_\Omega fw \, dx + \int_\Gamma gw \, dS + \int_{\partial\Omega\setminus\Gamma} (a\nabla u) \cdot \nu w \, dS.$$

Introduce the subspace

$$H^1(\Omega, \Gamma) = \{w \in H^1(\Omega) : w(x) = 0 \text{ for } x \in \partial\Omega \setminus \Gamma\} \subseteq H^1(\Omega).$$

We obtain the following weak formulation of the inverse problem (1.1):

Given u find $a \in L^2(\Omega)$ such that

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = \int_\Omega fw \, dx + \int_\Gamma gw \, dS \quad \text{for all } w \in H^1(\Omega, \Gamma). \tag{1.2}$$

The same formulation can be obtained in case of a piecewise smooth function u. Problem (1.2) makes sense for $u \in W^{1,\infty}(\Omega)$ and $a \in L^2(\Omega)$. We assume throughout that $u \in W^{1,\infty}(\Omega)$.

1.3 Operator equation formulation. Let us denote by G the space of gradients of functions $w \in H^{1,\infty}(\Omega, \Gamma)$:

$$G = G(\Omega, \Gamma) = \{\nabla w : w \in H^1(\Omega, \Gamma)\} \subset (L^2(\Omega))^n.$$

Let Q_G denote the orthoprojector in $(L^2(\Omega))^n$ corresponding to G. We observe that problem (1.2) is equivalent to the equation

$$Ta = \nabla\psi \tag{1.3}$$

where the operator $T = T_u \in \mathcal{L}(L^2(\Omega), \Gamma)$ is defined via the formula

$$Ta = Q_G(a\nabla u) \qquad (a \in L^2(\Omega)) \tag{1.4}$$

and $\psi = \psi_{f,g}$ is a solution to the direct problem

$$\begin{gathered} -\nabla\psi(x) = f(x) \quad (x \in \Omega), \\ \nabla\psi(x) \cdot \nu(x) = g(x) \quad (x \in \Gamma), \qquad \psi(x) = 0 \quad (x \in \partial\Omega \setminus \Gamma). \end{gathered} \tag{1.5}$$

Indeed, ψ satisfies the variational equality

$$\int_\Omega \nabla\psi \cdot \nabla w \, dx = \int_\Omega fw \, dx + \int_\Gamma gw \, dS \qquad \text{for all } w \in H^1(\Omega, \Gamma), \tag{1.6}$$

thus (1.2) takes the form

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = \int_\Omega \nabla\psi \cdot \nabla w \, dx \qquad \text{for all } w \in H^1(\Omega, \Gamma),$$

and this is equivalent to (1.3) since $Q_G \nabla\psi = \nabla\psi$.

In case $\Gamma \neq \partial\Omega$, problem (1.5) is uniquely solvable. In case $\Gamma = \partial\Omega$, problem (1.5) is solvable if $\int_\Omega f \, dx + \int_{\partial\Omega} g \, dS = 0$; this condition is also necessary for the solvability of the inverse problem (1.2) if $\Gamma = \partial\Omega$.

1.4 Ill-posedness of the inverse problem. The operator $T \in \mathcal{L}(L^2(\Omega), \Gamma)$ has a very simple adjoint operator $T^* \in \mathcal{L}(G, L^2(\Omega))$:

$$T^* \nabla w = \nabla u \cdot \nabla w \qquad (\nabla w \in G). \tag{1.7}$$

It is easy to see that the range $R(T^*) \subset L^2(\Omega)$ is non-closed in $L^2(\Omega)$ even if $|\nabla u| \geq c_0 > 0$ in Ω (here our case $n \geq 2$ essentially differs from the case $n = 1$). It is also clear that T^* is non-compact. Consequently, $T \in \mathcal{L}(L^2(\Omega), \Gamma)$ has a non-closed range $R(T) \subset G$ and is non-compact, too. Thus (1.3) is an ill-posed problem with a non-compact operator. This circumstance essentially influences the construction of discretization and regularization schemes for problem (1.1).

2. Discretization and regularization (a survey)

2.1 Discretization. A natural way to discretize the inverse problem (1.1) is to apply finite element approximations to the weak formulation (1.2) of the problem. Introduce finite-dimensional subspaces $S_h \subset H^1(\Omega, \Gamma)$ depending on a discretization parameter $h > 0$; we assume that S_h is complete in $H^1(\Omega, \Gamma)$ as $h \to 0$, i.e. for every $w \in H^1(\Omega, \Gamma)$, there exist $w_h \in S_h$ such that $w_h \to w$ in $H^1(\Omega)$ as $h \to 0$. We introduce the following discrete version of problem (1.2):

Find $a_h \in L^2(\Omega)$ of minimal $L^2(\Omega)$-norm such that

$$\int_\Omega a_h \nabla u \cdot \nabla w_h \, dx = \int_\Omega f w_h \, dx + \int_\Gamma g w_h \, dS \quad \text{for all } w_h \in S_h. \tag{2.1}$$

Problem (2.1) has never more than one solution. If problem (1.2) is solvable, then problem (2.1) is solvable, too, and the solutions satisfy the relation $a_h = P_{h,u} a$ where $P_{h,u}$ is the orthoprojector in $L^2(\Omega)$ corresponding to the subspace $\{a_h \in L^2(\Omega) : a_h = \nabla u \cdot \nabla v_h, v_h \in S_h\}$; a consequence is that $a_h \to a_0$ in $L^2(\Omega)$ as $h \to 0$ where $a_0 \in L^2(\Omega)$ is the normal solution (the solution of minimal $L^2(\Omega)$-norm) of problem (1.2). Conversely, if problem (1.2) is non-solvable in $L^2(\Omega)$, but problem (2.1) is solvable, then $\|a_h\|_{L^2(\Omega)} \to \infty$ as $h \to 0$.

Choosing a basis $w_j = w_{j,h}$ $(j = 1, \ldots, l = l_h)$ of S_h, problem (2.1) can be reformulated as follows:

Find

$$a_h = \sum_{j=1}^{l} c_j \nabla u \cdot \nabla w_j \tag{2.2}$$

solving the system of linear equations

$$Ac = d \tag{2.3}$$

where c is an l-vector with components c_j, d is an l-vector with components

$$d_i = \int_\Omega f w_i \, dx + \int_\Gamma g w_i \, dS \qquad (i = 1, \ldots, l), \tag{2.4}$$

and $A = (a_{ij})$ is an $l \times l$-matrix with elements

$$a_{ij} = \int_\Omega (\nabla u \cdot \nabla w_j)(\nabla u \cdot \nabla w_i) \, dx \qquad (i, j = 1, \ldots, l). \tag{2.5}$$

2.2 Regularization. Consider the case where, instead of exact data denoted here by u_0, f_0, and g_0 we have polluted data $u = u_\eta \in W^{1,\infty}(\Omega)$, $f = f_\delta \in L^2(\Omega)$ and $g = g_\delta \in L^2(\Gamma)$ at our disposal. Then numerical difficulties should be expected especially for fine grids, and a precedent regularization of problem (2.1) is needed. Tikhonov regularization yields the following numerical scheme (cf. (2.2) - (2.3)):

$$a_{\alpha,h} = \sum_{j=1}^{l} c_{j,\alpha} \nabla u \cdot \nabla w_j, \qquad (\alpha B + A) c_\alpha = d.$$

Here d is an l-vector with components d_i defined in (2.4), c_α is an l-vector with components $c_{j,\alpha}$ $(j = 1, \ldots, l)$, $A = (a_{ij})$ and $B = (b_{ij})$ are $l \times l$-matrices with elements a_{ij} defined in (2.5) and $b_{ij} = \int_\Omega \nabla w_j \cdot \nabla w_i \, dx$ $(i, j = 1, \ldots, l)$. A suitable value of regularization parameter $\alpha > 0$ depends on the errror level of the data. Assume that

$$\|\nabla \psi_\delta - \nabla \psi_0\|_{(L^2(\Omega))^n} \le \delta, \tag{2.6}$$

$$\sup_{x \in \Omega} |\nabla u_\eta(x) - \nabla u_0(x)| \le \eta \tag{2.7}$$

where ψ_0 and ψ_δ are the solutions to the direct problem (1.5) with right-hand terms f_0, g_0 and f_δ, g_δ, respectively, and δ, η are small positive numbers. Then an a priori choice $\alpha = \alpha(h, \delta, \eta)$ such that

$$\alpha(h, \delta, \eta) \to 0, \quad (\delta^2 + \eta^2)/\alpha(h, \delta, \eta) \to 0 \quad \text{as } h, \delta, \eta \to 0$$

guarantees the convergence $a_{\alpha(h,\delta,\eta),h} \to a_0$ in $L^2(\Omega)$-norm as $h, \delta, \eta \to 0$ where a_0 is the normal solution to problem (1.2) corresponding to exact data u_0, f_0, g_0 (we assume that problem (1.2) with the exact data is solvable in $L^2(\Omega)$). The same result holds if $\alpha = \alpha(h, \delta, \eta)$ is chosen, according to the residual principle, so that

$$\delta + \langle A c_\alpha, c_\alpha \rangle^{1/2} \eta \le \langle A c_\alpha - d, B^{-1}(A c_\alpha - d) \rangle^{1/2} \le \beta(\delta + \langle A c_\alpha, c_\alpha \rangle^{1/2} \eta)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in R^l and $\beta \ge 1$ is a constant not depending on h, δ, η.

The convergence results concerning the a priori parameter choice remains valid if (2.7) is replaced by the conditions

$$a_0 \in L^\infty(\Omega), \quad \sup_{x\in\Omega} |u_\eta(x)| \le c, \quad \|\nabla u_\eta - \nabla u_0\|_{(L^2(\Omega))^n} \le \eta$$

where the constant c does not depend on η.

The goal of this survey is to motivate the concepts of L^2-identifiability of the transmissivity coefficient which will be studied in the following sections. We refer to [5, 6] for a more detailed exposition of discretization and regularization methods for problem (1.2), including iterative regularization, to [7] for the general theory of regularization, to [2–4, 8] for other methods (without a regularization) to solve an inverse problem of the type (1.1).

3. L^1-identifiability in the case of smooth u

3.1 Introduction. The discretization and regularization methods considered in Section 2 converge to the normal solution of problem (1.2). A question about the uniqueness (identifiability) of the transmissivity coefficient among the functions of the class $L^2(\Omega)$ acutely arises. For sufficiently smooth data, the identifiability of the transmissivity coefficient among smooth functions is sufficiently fully analysed by G. R. Richter [4], C. Chicone and J. Gerlach [1], and K. Kunisch [3]. Here, imposing only physically realistic assumptions, we concentrate on identifiability within the class of $L^2(\Omega)$-functions or more generally $L^1(\Omega)$-functions.

We slightly generalize our inverse problem (cf. (1.2)): having data $u \in W^{1,\infty}(\Omega)$, $f \in L^1(\Omega)$, $g \in L^1(\Omega)$ at our disposal, we look for a function $a \in L^1(\Omega)$ such that

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = \int_\Omega fw \, dx + \int_\Gamma gw \, dS \qquad \text{for all } w \in W^{1,\infty}(\Omega, \Gamma) \tag{3.1}$$

where $W^{1,\infty}(\Omega,\Gamma) = \{w \in W^{1,\infty}(\Omega) : w(x) = 0 \text{ for } \partial\Omega \setminus \Gamma\}$. Let us recall that $\partial\Omega$, the boundary of an open bounded region $\Omega \subset R^n$, is assumed to be piecewise smooth and $\Gamma \subset \partial\Omega$ is a relatively open subset of $\partial\Omega$ with a piecewise smooth (relative) boundary on $\partial\Omega$.

We say that the transmissivity coefficient a is *L^1-identifiable from problem* (3.1) *on a subregion* $\Omega' \subseteq \Omega$ if, for any solutions $a_1, a_2 \in L^1(\Omega)$ to problem (3.1), $a_1(x) = a_2(x)$ for a.e. $x \in \Omega'$. Our goal is to describe subregions $\Omega' \subseteq \Omega$ where a is L^1-identifiable. It is clear that L^1-identifiability of a from (3.1) on Ω' implies L^2-identifiability of a from (1.2) on the same set Ω'.

It is clear also that a is L^1-identifiable from (3.1) on $\Omega' \subseteq \Omega$ if and only if any solution $a \in L^1(\Omega)$ to the homogeneous problem

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = 0 \qquad \text{for all } w \in W^{1,\infty}(\Omega, \Gamma) \tag{3.2}$$

vanishes almost everywhere on Ω'. Thus, for L^1-identifiability, assumptions on $u \in W^{1,\infty}(\Omega)$ are deciding. Instead of $f \in L^1(\Omega), g \in L^1(\Gamma)$ we could assume that f, g

define linear continuous functionals on $W^{1,\infty}(\Omega, \Gamma)$. In this Section 3 we consider the case where ∇u is continuous; in Section 4 we shall treat the case where ∇u may have jumps.

3.2 Flow curves. More precisely, we assume here that

$$u \in W^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega_\epsilon) \qquad \text{for all } \epsilon > 0 \tag{3.3}$$

where Ω_ϵ consists of all points $x \in \Omega$ such that the distance from x to a nearest non-smoothness point of $\partial\Omega$ exceeds ϵ; if $\partial\Omega$ is smooth, then (3.3) means that $u \in W^{2,\infty}(\Omega)$. Introduce a flow curve $x = \varphi(t, y)$ through a point $y \in \Omega$ as the maximal solution (the solution on the maximal time interval) to the Cauchy problem

$$dx/dt = -\nabla u(x), \qquad x(0) = y. \tag{3.4}$$

Physically, the ground water flows along those curves but the speed depends on ∇u and the transmissivity coefficient (*Darcy's law*).

Due to (3.3), ∇u is bounded and locally Lipschitz continuous on Ω, with possible singularities of Lipschitz coefficient only as x tends to a non-smoothness point of $\partial\Omega$. Therefore, problem (3.4) is uniquely solvable and $\varphi(t, y)$ is defined on a finite or infinite time interval (t_y^-, t_y^+); if t_y^- or t_y^+ is finite, then $\varphi(t, y)$ tends to a point on $\partial\Omega$ as $t \downarrow t_y^-$ or $t \uparrow t_y^+$, respectively. If $\nabla u(y) \neq 0$, then $\nabla u(\varphi(t, y)) \neq 0$ for all $t \in (t_y^-, t_y^+)$ and $u(\varphi(t, y))$ is strictly decreasing:

$$du(\varphi(t,y))/dt = \nabla u(\varphi(t,y)) \cdot d\varphi(t,y)/dt = -|\nabla u(\varphi(t,y))|^2 < 0 \quad (t \in (t_y^-, t_y^+)).$$

A corollary is that problem (3.4) allows no periodic solutions.

Introduce further the following subsets of Ω:

$$\Omega_C = \{y \in \Omega : \nabla u(y) = 0\}$$

$$\Omega^+ = \{y \in \Omega : \nabla u(y) \neq 0,\ t_y^+ = +\infty\}, \quad \Omega^- = \{y \in \Omega : \nabla u(y) \neq 0,\ t_y^- = -\infty\}$$

$$\Omega_\Gamma^+ = \left\{ y \in \Omega \;\middle|\; \begin{array}{l} \nabla u(y) \neq 0,\ t_y^+ < +\infty,\ \varphi(t,y) \text{ transversely (non-tangen-} \\ \text{tially) reaches a smoothness point of } \Gamma \subseteq \partial\Omega \text{ as } t \uparrow t_y^+ \end{array} \right\}$$

$$\Omega_\Gamma^- = \left\{ y \in \Omega \;\middle|\; \begin{array}{l} \nabla u(y) \neq 0,\ t_y^- > -\infty,\ \varphi(t,y) \text{ transversely (non-tangen-} \\ \text{tially) reaches a smoothness point of } \Gamma \subseteq \partial\Omega \text{ as } t \downarrow t_y^- \end{array} \right\}.$$

Since $\Gamma \subset \partial\Omega$ is relatively open, Ω_Γ^+ and Ω_Γ^- are open subsets of Ω. The interior of Ω^+ and Ω^- will be denoted by $\operatorname{int}\Omega^+$ and $\operatorname{int}\Omega^-$, respectively.

3.3 Main results and comments. If Ω_C, the set of critical points of u, has a positive Lebesgue measure, then the function $a \in L^\infty(\Omega)$ defined by $a(x) = 1$ if $\nabla u(x) = 0$ and $a(x) = 0$ elsewhere in Ω satisfies the homogeneous problem (3.2) but does not vanish a.e. on Ω_C. Thus, a cannot be L^1-identifiable from (3.1) on Ω_C if $\operatorname{meas}\Omega_C > 0$.

Theorem 1. *Under condition* (3.3), *the transmissivity coefficient* a *is* L^1*-identifiable from problem* (3.1) *on the sets* $\operatorname{int}\Omega^+$, $\operatorname{int}\Omega^-$ *and* Ω_Γ^+, Ω_Γ^-; *on* $\operatorname{int}\Omega^+$ *and* $\operatorname{int}\Omega^-$, L^1*-identifiability holds even if* $\Gamma = \emptyset$.

The proof of Theorem 1 is given in Subsection 3.4.

Figure 1 illustrates a case where Ω^+ and Ω^- cover Ω except for the isolated critical points of u. In this case we can identify a putting $\Gamma = \emptyset$. Figure 2 illustrates a case where a boundary condition on a part Γ of $\partial\Omega$ is necessary to identify a all over Ω.

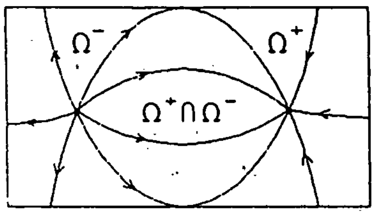

Fig. 1

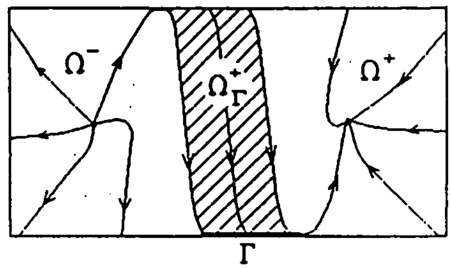

Fig. 2

A result of C. Chicone and J. Gerlach [1] says the following: if $u \in C^2(\tilde{\Omega})$ where $\tilde{\Omega}$ is open and contains $\bar{\Omega}$ (the closure of Ω), then a is C^1-identifiable (identifiable among functions a of the class $C^1(\bar{\Omega})$) from problem (1.1) with $\Gamma = \emptyset$ on the closure of the set $\operatorname{int}\Omega^+ \cup \operatorname{int}\Omega^-$. A result of G.R. Richter [4] can be interpreted as C-identifiability on the closures of Ω^+_Γ and Ω^-_Γ (the smoothness conditions and a priori assumptions on a are not explicitly formulated but a must be differentiable at least in the direction of ∇u). Theorem 1 extends these results to the case where no a priori smoothness of a is assumed.

Remark 1. Theorem 1 fails if assumption (3.3) is replaced by $u \in W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)$, $p < \infty$ (see a counter-example in Subsection 5.1). Thus, for L^1-identifiability, the conditions of Theorem 1 are rather close to the necessary ones. For the L^2-identifiability of a on $\operatorname{int}\Omega^\pm$ and $\Omega^\pm_\Gamma$ the conditions of Theorem 1 are only sufficient and seem to be far from the necessity — examining the examples one can conjecture that here (3.3) may be replaced by $u \in C^1(\bar{\Omega}) \cap H^2(\Omega)$. Unfortunately, our proof method does not work in this case since the flow curves may be non-uniquely determined from (3.4) if (3.3) fails.

Remark 2. Under condition $u \in W^{2,\infty}(\Omega)$, it is easy to see that $\operatorname{meas}\partial\Omega^\pm_\Gamma = 0$, therefore the L^1-identifiability result of Theorem 1 can be extended from Ω^+_Γ and Ω^-_Γ to their closures. On the other hand, for some (rather exotic) functions $u \in W^{2,\infty}(\Omega)$ and even $u \in C^2(\bar{\Omega})$, the sets $\partial(\operatorname{int}\Omega^\pm)$ may be of positive measure, and the result of Theorem 1 about the L^1-identifiability of a on $\operatorname{int}\Omega^+$ and $\operatorname{int}\Omega^-$ cannot be extended to the closures of those sets (see a counter-example in Subsection 5.2). Here a difference between the results about L^1-identifiability and C-identifiability appears.

Remark 3. If $u \in W^{2,\infty}(\Omega)$ and $u(x) = 0$ for $x \in \partial\Omega\backslash\Gamma$, then

$$\overline{\Omega^+} \cup \overline{\Omega^-} \cup \overline{\Omega^+_\Gamma} \cup \overline{\Omega^-_\Gamma} \supset \Omega\backslash\Omega_C. \tag{3.5}$$

The proof is outlined in Subsection 3.5.

3.4 Proof of Theorem 1. We have to show that any solution $a \in L^1(\Omega)$ to the homogeneous problem (3.2) vanishes on sets $\operatorname{int}\Omega^{\mp}$ and $\Omega_\Gamma^{\mp}$.

(i) We first prove that $a(x) = 0$ for a.e. $x \in \Omega_\Gamma^+$. It suffices to show that

$$\int_{U(x^0,\epsilon)} a(x)\,dx = 0 \qquad (0 < \epsilon < \epsilon_0 = \epsilon_0(x^0)) \tag{3.6}$$

where $x^0 \in \Omega_\Gamma^+$ is arbitrary fixed and $U(x^0,\epsilon)$ are small neighbourhoods of x^0 constructed as follows. Introduce the set $N(x^0,\epsilon) = \{y \in \Omega : u(y) = u(x^0), |y - x^0| < \epsilon\}$ and put (see Figure 3)

$$U(x^0,\epsilon) = \{x \in \Omega : x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ -\epsilon < t < \epsilon\}$$
$$V(x^0,\epsilon) = \{x \in \Omega : x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ -\epsilon < t < t_y^+\}.$$

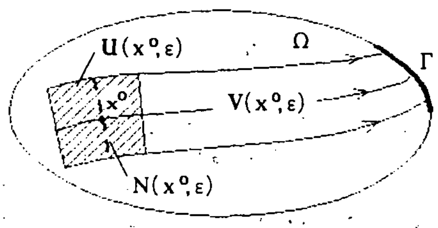

Fig. 3

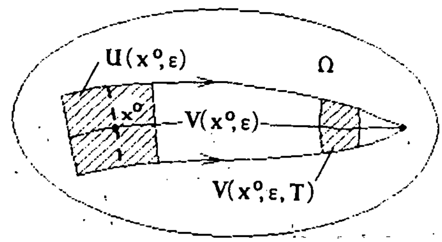

Fig. 4

Since $\nabla u(x^0) \neq 0$ and Γ is open and due to the transversality condition in the definition of Ω_Γ^+, the sets $N(x^0,\epsilon), U(x^0,\epsilon)$ and $V(x^0,\epsilon)$ are well-defined for small $\epsilon > 0$. Further introduce the functions

$$w_\epsilon(x) = \begin{cases} t + \epsilon & \text{for } x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ -\epsilon < t < +\epsilon \\ 2\epsilon & \text{for } x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ +\epsilon \le t < t_y^+ \\ 0 & \text{elsewhere in } \Omega \end{cases}$$

$$e_{\epsilon,\delta}(x) = \begin{cases} b_{\epsilon,\delta}(y) & \text{for } x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ -\epsilon < t < t_y^+ \\ 0 & \text{elsewhere in } \Omega \end{cases}$$

where $b_{\epsilon,\delta} : N(x^0,\epsilon) \to R$ is a smooth function such that $0 \le b_{\epsilon,\delta}(y) \le 1$ and

$$b_{\epsilon,\delta}(y) = \begin{cases} 1 & \text{for } y \in N(x^0,\epsilon),\ |y - x^0| < \epsilon - 2\delta \\ 0 & \text{for } y \in N(x^0,\epsilon),\ |y - x^0| > \epsilon - \delta \end{cases}$$

and $\delta \in (0, \epsilon/2)$ is a parameter. The function $e_{\epsilon,\delta}w_\epsilon$ is continuous and piecewise continuously differentialble on Ω, therefore $e_{\epsilon,\delta}w_\epsilon \in W^{1,\infty}(\Omega)$; the support of $e_{\epsilon,\delta}w_\epsilon$ lies in the closure of $V(x^0,\epsilon)$ which, for sufficiently small $\epsilon > 0$, intersects $\partial\Omega$ on Γ, therefore $e_{\epsilon,\delta}w_\epsilon$ vanishes on $\partial\Omega \setminus \Gamma$ and belongs to $W^{1,\infty}(\Omega,\Gamma)$. From (3.2) we obtain

$$\int_{V(x^0,\epsilon)} a\nabla u \cdot \nabla(e_{\epsilon,\delta}w_\epsilon)\,dx = 0 \qquad (0 < \epsilon < \epsilon_0). \tag{3.7}$$

Since $x = \varphi(t,y)$ is the solution to the Cauchy problem (3.4), we have

$$\nabla u(\varphi(t,y)) \cdot \nabla w_\epsilon(\varphi(t,y)) = -dw_\epsilon(\varphi(t,y))/dt = \begin{cases} -1 & \text{for } -\epsilon < t < +\epsilon \\ 0 & \text{for } +\epsilon < t \end{cases}$$

and

$$\nabla u(\varphi(t,y)) \cdot \nabla e_{\epsilon,\delta}(\varphi(t,y)) = -de_{\epsilon,\delta}(\varphi(t,y))/dt = 0,$$

thus (3.7) takes the form

$$-\int\limits_{U(x^0,\epsilon)} a(x) e_{\epsilon,\delta}(x)\,dx = 0 \qquad (0 < \epsilon < \epsilon_0).$$

Taking the limit $\delta \to 0$ we obtain (3.6).

(ii) Now we prove that $a(x) = 0$ for a.e. $x \in \operatorname{int}\Omega^+$. Again, it suffices to establish equality (3.6) for any fixed $x^0 \in \operatorname{int}\Omega^+$; the construction of $U(x^0,\epsilon)$ is the same as in part (i) of the proof and

$$V(x^0,\epsilon) = \{x \in \Omega : x \in \varphi(t,y),\ y \in N(x^0,\epsilon),\ -\epsilon < t < +\infty\}$$

(see Figure 4). Define the functions

$$w_{\epsilon,T}(x) = \begin{cases} t+\epsilon & \text{for } x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ -\epsilon < t < +\epsilon \\ 2\epsilon & \text{for } x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ +\epsilon \le t \le +T \\ 2\epsilon - (t-T) & \text{for } x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ +T < t < +T + 2\epsilon \\ 0 & \text{elsewhere in } \Omega \end{cases}$$

$$e_{\epsilon,\delta}(x) = \begin{cases} b_{\epsilon,\delta}(y) & \text{for } x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ -\epsilon < t < +\infty \\ 0 & \text{elsewhere in } \Omega \end{cases}$$

where $b_{\epsilon,\delta}$ is the same function as in part (i). This time, $\operatorname{supp}(e_{\epsilon,\delta} w_{\epsilon,T}) \subset \Omega$, thus $e_{\epsilon,\delta} w_{\epsilon,T} \in W^{1,\infty}(\Omega,\Gamma)$ again, and (3.2) yields

$$\int\limits_{V(x^0\epsilon)} a\nabla u \cdot \nabla(e_{\epsilon,\delta} w_{\epsilon,T})\,dx = 0. \tag{3.8}$$

We have again $\nabla u \cdot \nabla e_{\epsilon,\delta} = 0$ and, for $x = \varphi(t,y) \in V(x^0,\epsilon)$,

$$\nabla u(\varphi(t,y)) \cdot \nabla w_{\epsilon,T}(\varphi(t,y)) = \begin{cases} -1 & \text{for } -\epsilon < t < +\epsilon \\ 0 & \text{for } +\epsilon < t < +T \text{ and } t > T + 2\epsilon \\ +1 & \text{for } +T < t < +T + 2\epsilon, \end{cases}$$

thus (3.8) takes the form

$$-\int\limits_{U(x^0,\epsilon)} a e_{\epsilon,\delta}\,dx + \int\limits_{V(x^0,\epsilon,T)} a e_{\epsilon,\delta}\,dx = 0$$

where

$$V(x^0,\epsilon,T) = \{x \in V(x^0,\epsilon) : x = \varphi(t,y),\ y \in N(x^0,\epsilon),\ T < t < T + 2\epsilon\}.$$

Taking the limit $\delta \to 0$ we obtain

$$-\int\limits_{U(x^0,\epsilon)} a\,dx + \int\limits_{V(x^0,\epsilon,T)} a\,dx = 0 \qquad (0 < \epsilon < \epsilon_0).$$

Now we obtain (3.6) since meas $V(x^0,\epsilon,T) \to 0$ as $T \to \infty$. To see the last relation, note that, for $k = 1,2,\dots$, the sets $V(x^0,\epsilon,k)$ are disjoint for $\epsilon < 1/2$ and therefore $\sum_k \operatorname{meas} V(x^0,\epsilon,k) < \operatorname{meas}\Omega$ and meas $V(x^0,\epsilon,k) \to \infty$ as $k \to \infty$.

(iii) For Ω_Γ^- and int Ω^- the proof is similar as for Ω_Γ^+ and int Ω^+ in parts (i) and (ii), respectively. The proof of Theorem 1 is completed.

3.5 Proof of Remark 3. Assume that (3.5) does not hold: for a point $x^0 \in \Omega\backslash\Omega_C$, we have $x^0 \notin \overline{\Omega^+} \cup \overline{\Omega^-} \cup \overline{\Omega_\Gamma^+} \cup \overline{\Omega_\Gamma^-}$. Since $\Omega\backslash\Omega_C$ is open, there exists a number $r > 0$ such that

$$B(x^0,r) \subset \Omega\backslash\Omega_C, \qquad B(x^0,r) \cap \Omega^\pm = \emptyset, \qquad B(x^0,r) \cup \Omega_\Gamma^\pm = \emptyset. \tag{3.9}$$

The first two relations in (3.9) mean that, for any $y \in B(x^0,r)$, we have $t_y^- > -\infty$, $t_y^+ < +\infty$; let us denote $z_y^\pm = \lim_{t\to t_y^\pm} \varphi(t,y) \in \partial\Omega$. We assert that at least one of the points z_y^+, z_y^- belongs to Γ. Indeed, if z_y^+, $z_y^- \in \partial\Omega\backslash\Gamma$, then, according to the condition of Remark 3, $u(z_y^-) = u(z_y^+) = 0$. Due to the mean value theorem, there exists $\tilde{t} \in (t_y^-, t_y^+)$ such that $(d/dt)u(\varphi(\tilde{t},y)) = 0$. Using (3.4) we find that $\varphi(\tilde{t},y) \in \Omega$ is a critical point of u:

$$0 = \frac{d}{dt}u(\varphi(\tilde{t},y)) = \nabla u(\varphi(\tilde{t},y)) \cdot \frac{d}{dt}\varphi(\tilde{t},y) = -|\nabla u(\varphi(\tilde{t},y))|^2.$$

But this is impossible since a critical point can be attained by $\varphi(t,y)$ only asymptotically as $t \to \pm\infty$. Thus, for any $y \in B(x^0,r)$, z_y^- or z_y^+ belongs to Γ. Due to the last equality (3.9), Γ is non-smooth at $z_y^\pm$ or $\varphi(t,y)$ reaches $z_y^\pm$ tangentially. Both types of points $z_y^\pm \in \Gamma$ can constitute on Γ only manifolds of lower dimensions than $n-1$ as y varies in Ω. Hence some of the flow curves $\varphi = \varphi(t,y), y \in B(x,r)$, reach common points on $\Gamma \subset \partial\Omega$ in a finite time. This contradicts the assumption $u \in W^{2,\infty}(\Omega)$ and proves the remark.

3.6 The case of the Dirichlet problem. We briefly turn to the inverse problem of type (1.1) but with homogeneous Dirichlet boundary condition:

Find $a \in L^1(\Omega)$ such that

$$-\operatorname{div}(a(x)\nabla u(x)) = f(x) \quad (x \in \Omega) \qquad \text{and} \qquad u(x) = 0 \quad (x \in \partial\Omega). \tag{3.10}$$

The weak formulation of this problem is given in the following way:

Find $a \in L^1(\Omega)$ such that

$$\int\limits_\Omega a\nabla u \cdot \nabla w\,dx = \int\limits_\Omega fw\,dx \qquad \text{for all } w \in W_0^{1,\infty}(\Omega), \tag{3.11}$$

where $W_0^{1,\infty}(\Omega) = \{w \in W^{1,\infty}(\Omega) : w(x) = 0 \text{ for } x \in \partial\Omega\}$. Thus problem (3.11) can be viewed as problem (3.1) with $\Gamma = \emptyset$ and Theorem 1 can be applied: under condition (3.3),

a is L^1-identifiable from (3.11) on $\operatorname{int}\Omega^+$ and $\operatorname{int}\Omega^-$. Thereby, the boundary condition $u(x) = 0$ for $x \in \partial\Omega$ implies the equality

$$\Omega^+ \cup \Omega^- = \Omega\backslash\Omega_C. \tag{3.12}$$

Indeed, the inclusion $\Omega^+ \cup \Omega^- \subset \Omega\backslash\Omega_C$ is trivial and the inclusion $\Omega\backslash\Omega_C \subset \Omega^+ \cup \Omega^-$ means that, for any $y \in \Omega$ with $\nabla u(y) \neq 0$, we have $t_y^- = -\infty$ or $t_y^+ = +\infty$. If t_y^- and t_y^+ both are finite, then $u(z_y^-) = u(z_y^+) = 0$ for $z_y^\pm = \lim_{t\to t_y^\pm} \varphi(t,y) \in \partial\Omega$. Repeating an argument from the proof of Remark 3, we obtain a contradiction.

4. L^1-identifiability in the case of piecewise smooth u

4.1 Transversality condition for ∇u. Now consider the case where u remains continuous on Ω but ∇u may have discontinuities on piecewise smooth surfaces M_i $(i = 1, \dots, m)$ in Ω. Physically, M_i are the surfaces between different types of soil. Denote $M = \cup_{i=1}^m M_i$. We assume that

$$u \in W^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega_{\epsilon,M}) \quad \text{for all } \epsilon > 0 \tag{4.1}$$

where $\Omega_{\epsilon,M}$ consists of all points $x \in \Omega\backslash M$ such that the distance from x to a nearest non-smoothness point of $\partial\Omega$ and M_i $(i = 1, \dots, m)$ as well to a nearest intersection point of a pair of surfaces $\partial\Omega, M_i$ $(i = 1, \dots, m)$ exceeds ϵ. Further, we introduce the following *consistency condition*:

$$\left.\begin{array}{l}\text{There is a strictly positive piecewise-smooth function } a_{\text{test}},\\ \text{with possible jumps on } M, \text{ such that } \operatorname{div}(a_{\text{test}}\nabla u) \in L^1(\Omega)\end{array}\right\} \tag{4.2}$$

where the derivatives are understood in the sense of distributions. Usually, the "physical" solution of the inverse problem (1.1) meets this requirement.

Lemma 1. *Let $x^0 \in M_i$ be in the smooth part of M_i, and let $\nu_i(x^0)$ denote a unit normal to M_i at this point. Then, under conditions* (4.1) *and* (4.2),

$$\lim_{\substack{(x-x^0)\cdot\nu_i(x^0)>0\\ x\to x^0}} a_{\text{test}}(x)\nabla u(x)\cdot\nu_i(x^0) = \lim_{\substack{(x-x^0)\cdot\nu_i(x^0)<0\\ x\to x^0}} a_{\text{test}}(x)\nabla u(x)\cdot\nu_i(x^0). \tag{4.3}$$

Proof. Let $B = B(x^0, \epsilon)$ be an open ball in R^n centered at x^0 and of radius ϵ such that $M_i \cap B$ is in the smooth part of M_i and B does not intersect $\partial\Omega$ and other $M_j, j \neq i$. For any $w \in \mathcal{D}(B)$, i.e. $w \in C^\infty(B)$ with support in B, we have, according to the definition of distribution derivatives,

$$\int_B \operatorname{div}(a_{\text{test}}\nabla u) w\, dx = -\int_B a_{\text{test}}\nabla u\cdot\nabla w\, dx.$$

On the other hand, since $\operatorname{div}(a_{\text{test}}\nabla u) \in L^1(\Omega)$, we can devide the integral over B into the subsets B^+ and B^- on different sides of M_i, and integrating by parts we obtain

$$\begin{aligned}\int_B \operatorname{div}(a_{\text{test}}\nabla u) w\, dx &= \int_{B^+} \operatorname{div}(a_{\text{test}}\nabla u) w\, dx + \int_{B^-} \operatorname{div}(a_{\text{test}}\nabla u) w\, dx\\ &= -\int_B a_{\text{test}}\nabla u\cdot\nabla w\, dx + \int_{M_i\cap B} (a_{\text{test}}^+ - a_{\text{test}}^-)\nabla u\cdot\nu_i(x) w\, dS\end{aligned}$$

where a^+_{test} and a^-_{test} are the limit values of a_{test} on M_i from different sides. Thus,

$$\int_{M_i \cap B} (a^+_{\text{test}} - a^-_{\text{test}}) \nabla u \cdot \nu_i(x) w \, dS = 0 \qquad \text{for all } w \in \mathcal{D}(B),$$

and (4.3) follows.

Note that Lemma 1 holds without the positiveness assumption of a_{test}. The positiveness of a_{test} is needed when the flow curves are considered.

4.2 Flow curves. The following assertion is a direct corollary from (4.1) - (4.3): If a flow curve $x = \varphi(t, x)$, in a finite time moment, transversely reaches a smoothness point of M_i, then this flow curve passes M_i transversely to the other side of M_i and continues there. We can define sets $\Omega^+, \Omega^-, \Omega^+_\Gamma, \Omega^-_\Gamma$ as in Subsection 3.2 adding a requirement about the transversal cuttings of M, e.g.,

$$\Omega^+ = \left\{ y \in \Omega \;\middle|\; \begin{array}{l} \nabla u(y) \neq 0,\ t^+_y = +\infty \text{ and, for } 0 \le t < \infty,\ \varphi(t,y) \text{ cuts } M \text{ not} \\ \text{more than finite times whereby every cutting is transversal} \\ \text{and takes place at a smoothness point of an } M_i,\ 1 \le i \le m \end{array} \right\}$$

$$\Omega^-_\Gamma = \left\{ y \in \Omega \;\middle|\; \begin{array}{l} \nabla u(y) \neq 0,\ t^-_y > -\infty,\ \varphi(t,y) \text{ transversely reaches a smooth-} \\ \text{ness point of } \Gamma \subseteq \partial\Omega \text{ as } t \downarrow t^-_y \text{ and, for } t^-_y < t \le 0, \text{ cuts } M \\ \text{not more than finite times whereby every cutting is transver-} \\ \text{sal and takes place at a smoothness point of an } M_i,\ 1 \le i \le m \end{array} \right\}.$$

4.3 Extension of the main results. The proof of the following assertion is analogous to the proof of Theorem 1.

Theorem 2. *Under conditions* (4.1) *and* (4.2), *the transmissivity coefficient a is L^1-identifiable from problem* (3.1) *on the sets* $\operatorname{int}\Omega^+$, $\operatorname{int}\Omega^-$ *and* $\Omega^+_\Gamma, \Omega^-_\Gamma$ *specified in Subsection* 4.2; *on* $\operatorname{int}\Omega^+$ *and* Ω^- *the L^1-identifiability holds even if* $\Gamma = \emptyset$.

In Subsection 5.3 we present an example which clarifies the role of the consistency condition (4.2).

5. Counter-examples to L^1-identifiability

5.1 Counter-example in case $u \in W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)$, $p < \infty$. Let

$$\Omega = \{x = (x_1, x_2) \in R^n : -1 < x_1 < 1,\ 0 < x_2 < 1\}, \quad u(x) = |x_1|^\alpha \quad (1 < \alpha < 2).$$

Then $u \in C^1(\bar\Omega) \cap W^{2,p}(\Omega)$ $(p < 1/(2-\alpha))$, $\Omega^+ = \{x \in \Omega : x_1 \neq 0\}$ – all flow curves reach the critical line $x_1 = 0$ in a finite time and stop here. Putting $\Gamma = \emptyset$ or $\Gamma = \{x \in \partial\Omega : x_2 = 0 \text{ or } x_2 = 1\}$, a hypothetical extension of Theorem 1 to the case $u \in W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)$ says that a is L^1-identifiable from (3.1) on Ω^+. But this assertion is false: $a = |x_1|^{1-\alpha} \operatorname{sign} x_1$ is a solution to the homogeneous problem (3.2) belonging to $L^q(\Omega), q < 1/(\alpha - 1)$, and non-vanishing in any point of Ω^+. Indeed,

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = \alpha \int_\Omega (\partial w/\partial x_1) \, dx = 0 \qquad \text{for all } w \in W^{1,\infty}(\Omega, \Gamma),$$

while a function $w \in W^{1,\infty}(\Omega, \Gamma)$ vanishes for $x_1 = -1$ and $x_1 = +1$.

This counter-example can be modified so that a homogeneous Neumann condition $a\nabla u \cdot \nu = 0$ is given on $\Gamma = \partial\Omega$. The idea is to construct a function $u = \varphi(\arctan(x_2/x_1))$ on an annulus $\Omega = \{x \in R^2 : 1 < |x| < 2\}$.

5.2 Non-L^1-identifiability on the closure of $\operatorname{int}\Omega^+$. Let Ω be the rectangle as in Subsection 5.1. To construct a function $u \in C^2(\bar{\Omega})$, we consider a countable set $\{z_k\}_{k=1}^\infty \subset (0, 1)$ which is dense in [0, 1]. For given $\epsilon > 0$, we recursively construct closed intervals

$$I_1 = [z_1 - \epsilon_1, z_1 + \epsilon_1], \qquad \epsilon_1 < \min\{\epsilon/4, z_1, 1 - z_1\}$$

and, for $k = 2, 3, \ldots$,

$$I_k = [z_{i_k} - \epsilon_k, z_{i_k} + \epsilon_k], \qquad \epsilon_k < \min\left\{e/2^{k+1}, z_{i_k}, 1 - z_{i_k}, \operatorname{dist}(z_{i_k}, \cup_{j=1}^{k-1} I_j)\right\}$$

where z_{i_k} is the first term in the sequence $\{z_k\}$ which is not contained in the set $\cup_{j=1}^{k-1} I_j$. The full set $\cup_{j=1}^\infty I_j$ is dense in the interval [0, 1] since it contains all z_k. On the other hand, its Lebesgue measure on [0, 1] is small: $\operatorname{meas}(\cup_{k=1}^\infty I_k) < \sum_{k=1}^\infty \epsilon/2^k = \epsilon$. Now define

$$u(x) = x_1^2 \sum_{k=1}^\infty u_k(x_2) \qquad (x = (x_1, x_2) \in \Omega),$$

where $u_k \in C^2[0, 1]$ are functions such that $\|u_k\|_{C^2[0,1]} \le 1/k^2$ and $\operatorname{supp} u_k = I_k$ whereby $u_k(z) > 0$ for $z \in \operatorname{int} I_k$ and $u_k'(z) \neq 0$ for $z \in \operatorname{int} I_k$ except the center of the interval. It is clear that $u \in C^2(\bar{\Omega})$. The set of critical points of u is given by the line $x_1 = 0$ and the set $\{x \in \Omega : x_2 \notin \cup_{k=1}^\infty \operatorname{int} I_k\}$. The set Ω^+ consists of the rectangles ($k = 1, 2, \ldots$)

$$\{x \in \Omega : -1 < x_1 < 0, \; z_{i_k} - \epsilon_k < x_2 < z_{i_k} + \epsilon_k\}$$

and

$$\{x \in \Omega : \; 0 < x_1 < 1, \; z_{i_k} - \epsilon_k < x_2 < z_{i_k} + \epsilon_k\};$$

inside of k-th pair of those rectangles, $u(x) = x_1^2 u_k(x_2)$ and flow curves can be examined independently. According to Theorem 1, a is L^1-identifiable from (3.1) on Ω^+ which is open in this example. But a is not L^1-identifiable on the closure of Ω^+ which here coincides with $\bar{\Omega}$, the closure of Ω. Indeed, the homogeneous problem (3.2) has non-trivial solutions, e.g. a function $a \in L^\infty(\Omega)$ defined by $a(x) = 1$ if $\nabla u(x) = 0$ and $a(x) = 0$ if $\nabla u(x) \neq 0$. Note that the Lebesgue measure of $\partial\Omega^+$ as well of Ω_C exceeds $2(1 - \epsilon)$.

5.3 Non-L^1-identifiability in case of failing consistency condition. Consider the square $\Omega = \{x = (x_1, x_2) \in R^n : -1 < x_1, x_2 < +1\}$ which is devided into four triangles $\Omega_1, \ldots, \Omega_4$ by two diagonal straight lines M_1 and M_2 (see Figure 5).

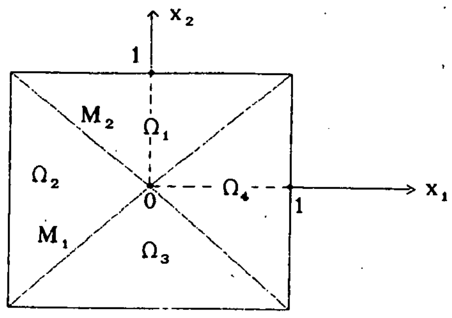

Fig. 5

Define the function $u \in C(\bar{\Omega})$ putting

$$u(x) \quad \text{equals} \quad x_1, -x_2, -x_1, x_2 \quad \text{on } \Omega_1, \dots, \Omega_4, \text{ respectively.}$$

It is clear that $u \in W^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega \backslash M)$, thus assumption (4.1) is fulfilled. On the other hand, the consistency condition (4.2) fails in this example since the limit values of $\nabla u(x) \cdot \nu_i$ from different sides of M_i are of different sign (cf. (4.3)). The flow curves reach M_i in a finite time and cannot be prolonged.

Consider problem (3.1) with $\Gamma = \partial\Omega$. It is interesting that there is no subregion $\Omega' \subseteq \Omega$ where a is L^1-identifiable from the values of u. Indeed, the homogeneous problem (3.2) has a rather large set of solutions — one can check that any function $\alpha \in L(0,1)$ generates a solution $a \in L^1(\Omega)$ to (3.2) via

$$a(x) \quad \text{equals} \quad -\alpha(x_2), \alpha(-x_1), -\alpha(-x_2), \alpha(x_1) \quad \text{on } \Omega_1, \dots, \Omega_4, \text{ respectively.}$$

This example is a modification of an example of K. Ito and K. Kunisch [2] where u satisfies homogeneous Dirichlet condition. In our modification, u satisfies homogeneous Neumann condition $\nabla u \cdot \nu = 0$ on $\partial\Omega$.

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