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A Real Inversion Formula for the Laplace Transform

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Let f be the Laplace transform of a square integrable function F and set

$$F_N(t) = \int_0^\infty f(s)e^{-st} P_N(st)\,ds \qquad (N = 0,1,2,\ldots)$$

for the polynomials

$$P_N(\xi) = \sum_{0\le\nu\le n\le N} \frac{(-1)^{\nu+1}(2n)!}{(n+1)!\nu!(n-\nu)!(n+\nu)!}\xi^{n+\nu} \times \left\{\frac{2n+1}{n+\nu+1}\xi^2 - \left(\frac{2n+1}{n+\nu+1} + 3n+1\right)\xi + n(n+\nu+1)\right\}.$$

Then it is proved that the sequence $\{F_N\}_{N=0}^\infty$ converges to F in the sense that

$$\lim_{N\to\infty} \int_0^\infty |F(t) - F_N(t)|^2\,dt = 0.$$

Furthermore, a general formula for this result is established.

Key words: *Bergman-Selberg spaces, real inversion formulas, Laplace transform, reproducing kernels, reproducing kernel Hilbert spaces*

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1. Introduction and result

For any $q > 0$, we let L_q^2 be the class of all square integrable functions with respect to the measure $t^{1-2q}\,dt$ on the half line $(0,\infty)$. Then we consider the Laplace transform

$$[\mathcal{L}F](z) = \int_0^\infty F(t)e^{-zt}\,dt \qquad (z \in R^+ = \{\mathcal{R}z > 0\}$$

for $F \in L_q^2$. In [2, §7], it was shown that the image of L_q^2 under the Laplace transform $\mathcal{L}$ coincides with the reproducing kernel Hilbert space H_q (Bergman-Selberg space) admitting the reproducing kernel $K_q(z,\overline{u}) = \Gamma(2q)/(z+\overline{u})^{2q}$ and $\mathcal{L}$ is an isometry of

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of the space L_q^2 onto the space H_q. For $q > \frac{1}{2}$, the Hilbert space H_q consists of all functions f analytic in R^+ with finite norms

$$\|f\|_q^2 = \frac{4^{q-1}}{\pi\Gamma(2q-1)} \iint_{R^+} |f(z)|^2 x^{2q-2}\, dx dy \qquad (z = x + iy)$$

and

$$H_{\frac{1}{2}} = \left\{ f : f \text{ analytic in } R^+,\ \|f\|_{\frac{1}{2}}^2 = \frac{1}{2\pi} \sup_{x>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2\, dy < \infty \right\}.$$

The inverse of the Laplace transform $\mathcal{L}$ is, in general, given by complex forms. The observation in many fields of sciences however gives us, intuitively, real data $[\mathcal{L}F](x)$ only, and so it is important to establish its inversion formula in terms of real data $[\mathcal{L}F](x)$. Such a formula was given for $L^1[(0,\infty), dt]$-functions F by R. P. Boas and D. V. Widder about fifty years ago (see [7, p. 386]). By use of the representations of H_q-norms on the positive real line in [5], we shall establish in the next theorem the natural inversion formula of the Laplace transform $\mathcal{L}$ on the space L_q^2 in terms of real data $[\mathcal{L}F](x)$ in the framework of the Hilbert space L_q^2.

Theorem. *For any fixed number $q > 0$ and for any function $F \in L_q^2$, put $f = \mathcal{L}F$. Then the inversion formula*

$$F(t) = s - \lim_{N\to\infty} \int_0^\infty f(x) e^{-xt} P_{N,q}(xt)\, dx \qquad (t > 0)$$

is valid, where the limit is taken in the space L_q^2 and the polynomials $P_{N,q}$ are given by the formula

$$P_{N,q}(\xi) = \sum_{0\le\nu\le n\le N} \frac{(-1)^{\nu+1}\Gamma(2n+2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)} \xi^{n+\nu+2q-1}$$
$$\times \left\{ \frac{2(n+q)}{n+\nu+2q}\xi^2 - \left(\frac{2(n+q)}{n+\nu+2q} + 3n + 2q \right) \xi + n(n+\nu+2q) \right\}.$$

Moreover, the series

$$\sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [x f'(x)]|^2 x^{2n+2q-1}\, dx$$

converges and the truncation error is estimated by the inequality

$$\left\| F(t) - \int_0^\infty f(x) e^{-xt} P_{N,q}(xt)\, dx \right\|_{L_q^2}^2$$
$$\le \sum_{n=N+1}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [x f'(x)]|^2 x^{2n+2q-1}\, dx.$$

Note that, even if $q = \frac{1}{2}$, our polynomial $P_{N,\frac{1}{2}}$ is different from the one of R. P. Boas and D. V. Widder.

2. Preliminaries

In order to prove Theorem, we prepare three lemmas.

Lemma 1. *For any fixed $q > 0$, let the function f be a member of the space H_q and set, for any non-negative integer N,*

$$f_N(z) = \sum_{n=0}^{N} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial_\xi^n[\xi f'(\xi)]\overline{\partial_\xi^n[\xi\partial_\xi K_q(\xi,\overline{z})]}\xi^{2n+2q-1}\,d\xi$$

for $z \in R^+$. Then, the function f_N belongs to the space H_q, and the sequence $\{f_N\}_{N=0}^\infty$ converges to f in H_q.

Proof. Recall first the following representation of the norm in the space H_q:

$$\|f\|_q^2 = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n[xf'(x)]|^2 x^{2n+2q-1}\,dx \tag{1}$$

(see [5]). From the reproducing property of $K_q(\cdot,\overline{z})$, we have the expressions

$$\begin{aligned} K_q(z,\overline{u}) &= \left(K_q(\cdot,\overline{u}), K_q(\cdot,\overline{z})\right)_q \\ &= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial_\xi^n[\xi\partial_\xi K_q(\xi,\overline{u})]\overline{\partial_\xi^n[\xi\partial_\xi K_q(\xi,\overline{z})]}\xi^{2n+2q-1}\,d\xi \end{aligned}$$

and

$$\begin{aligned} f(z) &= \left(f(\cdot), K_q(\cdot,\overline{z})\right)_q \\ &= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial_\xi^n[\xi f'(\xi)]\overline{\partial_\xi^n[\xi\partial_\xi K_q(\xi,\overline{z})]}\xi^{2n+2q-1}\,d\xi, \end{aligned}$$

where $(\cdot,\cdot)_q$ denotes the inner product in H_q. Hence, we see by [6, p. 170] (see also [4, p. 96]) that f_N is a member in H_q and

$$\begin{aligned} &\|f - f_N\|_q^2 \\ &= \left\| \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial_\xi^n[\xi f'(\xi)]\overline{\partial_\xi^n[\xi\partial_\xi K_q(\xi,\overline{z})]}\xi^{2n+2q-1}\,d\xi \right\|_q^2 \\ &\le \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_\xi^n[\xi f'(\xi)]|^2 \xi^{2n+2q-1}\,d\xi. \end{aligned} \tag{2}$$

Therefore, our claim is true ∎

Lemma 2. *For any fixed $q > 0$, let the function f be a member of the space H_q and set, for any non-negative integer N.*

$$F_N(t) = \sum_{n=0}^{N} \frac{t^{2q-1}}{n!\Gamma(n+2q+1)} \int_0^\infty \partial_x^n[xf'(x)]\partial_x^n[x\partial_x(e^{-tx})]x^{2n+2q-1}\,dx$$

for $t \in (0,\infty)$. Then, the function F_N belongs to the space L_q^2, and furthermore, for the functions f_N defined in Lemma 1, $\mathcal{L}F_N = f_N$.

Proof. We first prove that, for any n, the function g_n defined by

$$g_n(t) = t^{q-\frac{1}{2}} \int_0^\infty \partial_x^n[xf'(x)]\partial_x^n[x\partial_x(e^{-tx})]x^{2n+2q-1}\,dx$$

belongs to the space $L^2[(0,\infty),dt]$. By the Leibniz rule,

$$\partial_x^n[x\partial_x(e^{-tx})]t^{q-\frac{1}{2}} = (-1)^n t^{n+q-\frac{1}{2}}(n-tx)e^{-tx},$$

and we have

$$\begin{aligned} g_n(t) = (-1)^n n t^{n+q-\frac{1}{2}} \int_0^\infty \partial_x^n[xf'(x)]e^{-tx}x^{2n+2q-1}\,dx \\ -(-1)^n t^{n+q+\frac{1}{2}} \int_0^\infty \partial_x^n[xf'(x)]e^{-tx}x^{2n+2q}\,dx. \end{aligned}$$

Moreover, the expression (1) implies that the functions defined by

$$\partial_x^n[xf'(x)]x^{2n+2q-1} \qquad \text{and} \qquad \partial_x^n[xf'(x)]x^{2n+2q}$$

are contained in the spaces L_{n+q}^2 and L_{n+q+1}^2, respectively. Hence the function g_n is the restriction of a member in the set

$$\left\{\tau^{n+q-\frac{1}{2}}h_1(\tau) + \tau^{n+q+\frac{1}{2}}h_2(\tau) : h_1 \in H_{n+q} \text{ and } h_2 \in H_{n+q+1}\ (\tau = t + i\hat{t})\right\}$$

to the half-axis $(0,\infty)$, and it is represented by

$$g_n(t) = t^{n+q-\frac{1}{2}}\hat{h}_1(t) + t^{n+q+\frac{1}{2}}\hat{h}_2(t)$$

for some functions $\hat{h}_1 \in H_{n+q}$ and $\hat{h}_2 \in H_{n+q+1}$. If $n = 0$, we have $g_n(t) = t^{q+\frac{1}{2}}h_3(t)$ for some function $h_3 \in H_{q+1}$. Furthermore, for $n \neq 0$ we have the representation

$$g_n(t) = t^{n+q-\frac{1}{2}}k_1'(t) + t^{n+q+\frac{1}{2}}k_2'(t)$$

for some functions $k_1 \in H_{n+q-1}$ and $k_2 \in H_{n+q}$ (see [3]). Hence, from (1) we get the relations

$$\int_0^\infty |t^{n+q-\frac{1}{2}}k_1'(t)|^2 dt = \int_0^\infty |tk_1'(t)|^2 t^{2n+2q-3} dt < \infty$$

and

$$\int_0^\infty |t^{n+q+\frac{1}{2}}k_2'(t)|^2 dt = \int_0^\infty |tk_2'(t)|^2 t^{2n+2q-1} dt < \infty,$$

and so the function g_n $(n \neq 0)$ belongs to the space $L^2[(0,\infty),dt]$. Likewise, the function g_0 is also a member of the space $L^2[(0,\infty),dt]$. By virtue of the isometry $s(t) \longmapsto$

$s(t)t^{q-\frac{1}{2}}$ of the space $L^2[(0,\infty),dt]$ onto the space L^2_q, we conclude that the function F_N belongs to the space L^2_q.

Next, in order to prove that $\mathcal{L}F_N = f_N$, we examine, for a fixed number $\xi > 0$, the integrability of the functions

$$\varphi(x,t;n,\xi) = \partial_x^n\{xf'(x)\}\partial_x^n\{x\partial_x(e^{-tx})\}e^{-\xi t}t^{2q-1}x^{2n+2q-1} \qquad (n=0,1,2,\ldots)$$

with respect to the Lebesgue measure on the set $(0,\infty)\times(0,\infty)$. We first have the estimate

$$\begin{aligned}|\varphi(x,t;n,\xi)| &= |\partial_x^n\{xf'(x)\}|\,|\partial_x^n\{x\partial_x(e^{-tx})\}|\,e^{-\xi t}t^{2q-1}x^{2n+2q-1}\\ &= |\partial_x^n\{xf'(x)\}|\,|(-t)^n xe^{-tx} + n(-t)^{n-1}e^{-tx}|\,t^{2q}e^{-\xi t}x^{2n+2q-1}\\ &\le |\partial_x^n\{xf'(x)\}|\left\{t^{n+2q}xe^{-(x+\xi)t} + nt^{n+2q-1}e^{-(x+\xi)t}\right\}x^{2n+2q-1}.\end{aligned}$$

Therefore, since the functions defined by

$$x\int_0^\infty t^{n+2q}e^{-(x+\xi)t}\,dt = \Gamma(n+2q+1)x(x+\xi)^{-(n+2q+1)}$$

and

$$n\int_0^\infty t^{n+2q-1}e^{-(x+\xi)t}\,dt = n\Gamma(n+2q)(x+\xi)^{-(n+2q)}$$

belong to the space $L^2[(0,\infty),x^{2n+2q-1}dx]$, we see by the Schwarz inequality that the function $\varphi(x,t;n,\xi)$ is integrable for all n. By the Fubini theorem, the following sequence of equalities is therefore valid:

$$\begin{aligned}&\int_0^\infty F_N(t)e^{-\xi t}\,dt\\ &= \sum_{n=0}^N \frac{1}{n!\Gamma(n+2q+1)}\\ &\quad\times\int_0^\infty\left[\int_0^\infty \partial_x^n\{xf'(x)\}\partial_x^n\{x\partial_x(e^{-tx})\}x^{2n+2q-1}\,dx\right]t^{2q-1}e^{-\xi t}\,dt\\ &= \sum_{n=0}^N \frac{1}{n!\Gamma(n+2q+1)}\\ &\quad\times\int_0^\infty \partial_x^n[xf'(x)]\left[\int_0^\infty \partial_x^n\{x\partial_x(e^{-tx})\}e^{-\xi t}t^{2q-1}\,dt\right]x^{2n+2q-1}\,dx\\ &= \sum_{n=0}^N \frac{1}{n!\Gamma(n+2q+1)}\\ &\quad\times\int_0^\infty \partial_x^n[xf'(x)]\partial_x^n\left[x\partial_x\int_0^\infty e^{-tx}e^{-\xi t}t^{2q-1}\,dt\right]x^{2n+2q-1}\,dx\\ &= \sum_{n=0}^N \frac{1}{n!\Gamma(n+2q+1)}\\ &\quad\times\int_0^\infty \partial_x^n[xf'(x)]\partial_x^n[x\partial_x K_q(x,\xi)]x^{2n+2q-1}\,dx = f_N(\xi).\end{aligned}$$

Thus the assertions of the lemma are proved ∎

Lemma 3. *For any fixed $q > 0$, let the function f be a member of the space H_q. Then the following statements are true.*

(i) If $n \geq 1$ and $0 \leq m \leq n-1$, then $\partial_x^m[xf'(x)]x^{n+m+2q} = o(1)$ as $x \longrightarrow 0+$.

(ii) $f(x)x^q = O(1)$ as $x \longrightarrow 0+$.

Proof. By the Leibniz rule, we have the equality

$$\partial_x^m[xf'(x)] = x\partial_x^{m+1}f(x) + m\partial_x^m f(x).$$

We also see that the function ∂_x^{m+1} belongs to the space H_{q+m+1} (see [3]), and from the Schwarz inequality the following estimate is valid:

$$\begin{aligned} \left|\partial_x^{m+1}f(x)\right| &= \left|\left(\partial_\xi^{m+1}f(\xi), K_{q+m+1}(\xi,x)\right)_{q+m+1}\right| \\ &\leq \left\|\partial_x^{m+1}f\right\|_{q+m+1} K_{q+m+1}(x,x)^{\frac{1}{2}} \\ &= \left\|\partial_x^{m+1}f\right\|_{q+m+1} \Gamma(2q+2m+2)^{\frac{1}{2}} 2^{-(q+m+1)} x^{-(q+m+1)}. \end{aligned}$$

Likewise, the estimates

$$|\partial_x^m f(x)| \leq \|\partial_x^m f\|_{q+m} \Gamma(2q+2m)^{\frac{1}{2}} 2^{-(q+m)} x^{-(q+m)}$$

and

$$|f(x)| \leq \|f\|_q \Gamma(2q)^{\frac{1}{2}} 2^{-q} x^{-q}$$

are valid. Therefore, our lemma is obtained ∎

3. Proof of Theorem

From Lemma 3, and by integration by parts we have, for any non-negative integer n,

$$\begin{aligned} &\int_0^\infty \partial_x^n[xf'(x)]\partial_x^n[x\partial_x(e^{-tx})]x^{2n+2q-1}dx \\ &= t^n \int_0^\infty xf'(x)\partial_x^n[(n-tx)e^{-tx}x^{2n+2q-1}]\,dx \\ &= -t^n \int_0^\infty f(x)\partial_x[x\partial_x^n\{(n-tx)e^{-tx}x^{2n+2q-1}\}]\,dx. \end{aligned}$$

Meanwhile, for $n \geq 1$ we also have

$$
\begin{aligned}
&- t^n \partial_x \left[x \partial_x^n \left\{ (n - tx) e^{-tx} x^{2n+2q-1} \right\} \right] \\
&= - e^{-tx} t^n \Big[\sum_{\nu=0}^{n} \binom{n}{\nu} (-t)^\nu \left\{ n \partial_x^{n-\nu} x^{2n+2q-1} - t \partial_x^{n-\nu} x^{2n+2q} \right\} \\
&\quad - tx \sum_{\nu=0}^{n} \binom{n}{\nu} (-t)^\nu \left\{ n \partial_x^{n-\nu} x^{2n+2q-1} - t \partial_x^{n-\nu} x^{2n+2q} \right\} \\
&\quad + x \sum_{\nu=0}^{n} \binom{n}{\nu} (-t)^\nu \left\{ n \partial_x^{n-\nu+1} x^{2n+2q-1} - t \partial_x^{n-\nu+1} x^{2n+2q} \right\} \Big] \\
&= e^{-tx} \sum_{\nu=0}^{n} (-1)^{\nu+1} \binom{n}{\nu} (xt)^{n+\nu} \frac{\Gamma(2n+2q)}{\Gamma(n+\nu+2q)} \\
&\quad \times \left\{ \frac{2n+2q}{n+\nu+2q} t^2 x^{2q+1} - \left(\frac{2n+2q}{n+\nu+2q} + 3n + 2q \right) t x^{2q} + n(n+\nu+2q) x^{2q-1} \right\}.
\end{aligned}
$$

Applying Lemma 1 and Lemma 2 to the isometry $\mathcal{L}$, we therefore obtain the inversion formula of $\mathcal{L}$. Also, the inequality (2) gives the estimate of the truncation error ∎

Remark. For any $q > 0$, let the functions F and f be as in Theorem. In [3], we see that the function f' is a member of the space H_{q+1}, and $\|f'\|_{q+1} = \|f\|_q$. Hence, by the inversion formula in [4, p. 85], we have the inversion formula of the Laplace transform $\mathcal{L}$ in the complex form as follows:

$$
F(t) = s - \lim_{n\to\infty} \frac{-4^q t^{2q}}{\pi \Gamma(2q+1)} \iint_{E_n} f'(z) e^{-\bar{z}t} x^{2q} \, dx dy,
$$

where the limit $s - \lim_{n\to\infty}$ is taken in the space L_q^2 and the sequence $\{E_n\}_{n=0}^{\infty}$ is a compact exhaustion of R^+.

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