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Location of the Complex Zeros of Bessel Functions and Lommel Polynomials

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Some inequalities for the complex zeros of Bessel functions and the zeros of Lommel polynomials, which improve previously known results, are presented.

Key words: *Complex zeros, Bessel functions, Lommel polynomials*
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1. Introduction

Many results about the zeros of the Bessel function J_μ of the first kind and of order $\mu = \nu + i\tau$ are concerning with the case where μ is real ($\tau = 0$) and in particular with the case $\mu = \nu > -1$. A little is known for the case where μ is complex ($\tau \neq 0$). For $\tau \neq 0$, an important result is that the function J_μ cannot have real zeros [1]. Also from the results in [1] it is known that when $\nu > -1$ and $\tau > 0$ ($\tau < 0$) the real and imaginary parts of any zero $\rho = \rho_1 + i\rho_2$ have the same sign (different sign). More over, in [6] was proved that, for $\nu \geq 0$ and $\tau > 0$, the zeros of J_μ lie in the first and third quadrant of the z-plane and for $\nu \geq 0$ and $\tau < 0$ they lie in the second and fourth quadrant.

In this work we first refine and discuss some inequalities, which follow easily by the method of [1 - 3]. These inequalities are the following:

$$|\rho_1| > j_{\nu,1}, \quad \nu > -1 \tag{1.1}$$

$$|\rho_2| > |\tau|, \tag{1.2}$$

where $j_{\nu,1}$ is the first positive zero of the function J_ν, which for $\nu > -1$ has been extensively studied. Also, we prove the inequality

$$\tau\rho_1/\rho_2 > 1 + \nu \tag{1.3}$$

for any real ν and real $\tau \neq 0$, which incorporates many results found previously [1, 6].

We use a similar method and study the complex zeros $\lambda = \lambda_1 + i\lambda_2$ of the polynomial $P_{n+1,\mu}$ of degree n, defined by the following recurrence relation:

$$P_{n+1,\mu}(x) + P_{n-1,\mu}(x) = 2(n+\mu)xP_{n,\mu}(x), \quad P_{0,\mu}(x) = 0, \quad P_{1,\mu}(x) = 1, \tag{1.4}$$

for $\mu = \nu + i\tau$, $\tau \neq 0$. The polynomials $P_{n+1,\mu}$ are the same as the *Lommel polynomials* $R_{n,\mu+1}$ defined by

$$R_{n+1,\mu}(x) + R_{n-1,\mu}(x) = 2(n+\mu)xR_{n,\mu}(x), \quad R_{-1,\mu}(x) = 0, \quad R_{0,\mu}(x) = 1. \tag{1.5}$$

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For any real ν and $\tau \neq 0$, it is proved that $\lambda_2 \neq 0$ and the inequalities

$$1+\nu < (-\tau)\lambda_1/\lambda_2 < N+\nu \quad \text{and} \quad |\lambda| < cos(\pi/(N+1))/|\tau| \tag{1.6}$$

hold (Theorem 4.1). In the case $\nu > -1$, $\tau \neq 0$ the inequality

$$|\lambda_1| < 1/j_{\nu,1} \tag{1.7}$$

is found, where $j_{\nu,1}$ is the first positive zero of the Bessel function J_ν.

We compare all the results in this paper with those of H.J. Runckel found in [8] with a different method.

2. Preliminaries

In this section, we explain the method and some results we shall use in the next section. Consider an abstract separable Hilbert space H with orthonormal basis $\{e_n\}_{n\geq 1}$. Denote by V the shift operator with respect to that basis ($Ve_n = e_{n+1}$) and by V^* the adjoint of V ($V^*e_1 = 0$, $V^*e_n = e_{n-1}$, for $n > 1$). The operator $T_0 = V + V^*$ is selfadjoint with purely continuous spectrum covering the entire interval [-2, 2]. In particular $\|T_0\| = 2$. For completeness, we give below a simple proof of this well-known result. Suppose that $\lambda = 2cos\theta = e^{i\theta} + e^{-i\theta}$, $0 \leq \theta \leq 2\pi$ is a regular point of $V + V^*$ or belongs to the point spectrum of $V + V^*$. In the first case, there exists an $x_1 \neq 0$ being the unique solution of the inhomogeneous equation $(V + V^* - 2cos\theta)x_1 = e_1$. In the second case, there exists an $x_2 \neq 0$ being the solution of the homogeneous equation $(V + V^* - 2cos\theta)x_2 = 0$. Since $V^*e_1 = 0$ and $V^*V = I$ in both cases there must exists an $x \neq 0$ such that

$$V^*(V + V^* - e^{i\theta} - e^{-i\theta})x = 0 \text{ or } (V^* - e^{i\theta})(V^* - e^{-i\theta})x = 0.$$

The last equation means that either $e^{i\theta}$ or $e^{-i\theta}$ is an eigenvalue of V^*, which is impossible, because it is easy to see that all points on the unit disc belong to the continuous spectrum of the operator V^* (as well also of V).

Another operator which is used in the method, we follow, is the diagonal unbounded operator $C_0 : e_n \to ne_n$, $n \geq 1$. It is defined on the basis $\{e_n\}$ as before and can be extended to a linear manifold $D(C_0)$ which is dence in H [5]. In [1] it has been proved that for every $\mu \neq -n$, $n \geq 1$, real or complex, the value ρ is a zero of the Bessel function J_μ, if and only if, it is an eigenvalue of the generalized eigenvalue problem

$$(C_0 + \mu)f = \rho T_0 f/2, \; f \neq 0. \tag{2.1}$$

The assumption $\mu \neq -n$, $n \geq 1$ is not a restriction to the problem, because the functions J_n and J_{-n} have the same zeros ($J_n = (-1)^n J_{-n}$). In the case where $\mu = \nu$ is real and $\nu > -1$ the operator $C_0 + \nu$ is positive definite, in fact $(C_0 f, f) \geq \|f\|^2$ and

$$((C_0 + \nu)f, f) \geq (1+\nu)\|f\|^2, \; f \in D(C_0), \tag{2.2}$$

where by $(\cdot,\cdot)$ we mean the scalar product in H. The inverse of $C_0+\nu$, the operator L_ν: $e_n \to (1/(n+\nu))e_n$, is positive, in the sense $(L_\nu f,\ f) > 0,\ f \in H$. Its square root $L_\nu^{1/2}$ exists and is equal to $(C_0+\nu)^{-1/2}$. Thus we can set in (2.1) $f = L_\nu^{1/2}x$ and transform the eigenvalue problem (2.1) into the regular eigenvalue problem

$$S_\nu x = 2x/\rho, \quad x \neq 0, \tag{2.3}$$

where

$$S_\nu = L_\nu^{1/2} T_0 L_\nu^{1/2} \tag{2.4}$$

is a selfadjoint and compact operator. We can easily see that if $2/\rho$ is an eigenvalue of S_ν corresponding to the eigenvector x, then $-2/\rho$ is also an eigenvalue of S_ν corresponding to the eigenvector Ux, where U is the diagonal operator $Ue_n = (-1)^n e_n,\ n \geq 1$. Also, it is easy to see that the eigenvalues of S_ν are simple, because the eigenvectors are uniquely determined from (2.3), up to a factor $(x, e_1) = \alpha \neq 0$. Thus the eigenvalues of the operator S_ν are $\pm 2/j_{\nu,n},\ n \geq 1$, where $j_{\nu,n}$ are the positive zeros of the function $J_\nu,\ \nu > -1$. Moreover the maximal principle for compact and self-adjoint operators yields

$$\|S_\nu\| = 2/j_{\nu,1}. \tag{2.5}$$

From (2.1) for $f = L_\nu^{1/2}x$ we can find $2((C_0+\mu)f, f) = \rho(T_0 f, f)$. By setting here $\mu = \nu + i\tau$ and $\rho = \rho_1 + i\rho_2$ and by comparing real and imaginary parts we find

$$2((C_0+\nu)f, f) = \rho_1(T_0 f, f) \quad \text{and} \quad 2\tau(f, f) = \rho_2(T_0 f, f). \tag{2.6}$$

From the above, we get the following results:

1. *If $\tau \neq 0$, then $|\rho_2| > |\tau|$.* This means that if $\tau \neq 0$, then the function J_μ cannot have real zeros. The inequality $|\rho_2| > |\tau|$ follows from the second relation of (2.6) using Schwartz's inequality $|(T_0 f, f)| < \|T_0 f\|\|f\| \leq \|T_0\|\|f\|^2$ and the relation $\|T_0\| = 2$. The equality in Schwartz's inequality is excluded, because otherwise $T_0 f = kf$, which is imposible because T_0 has no eigenvalues. That justifies the strict inequality $|\rho_2| > |\tau|$.

2. *If $\nu > -1$ and $\tau \neq 0$, then $\rho_1 \neq 0$.* This follows from the first relation of (2.6) and the relation (2.2) and means that, for $\nu > -1$ and $\tau \neq 0$, the function J_μ cannot have purely imaginary zeros.

3. *If ν and $\tau \neq 0$ are real, then $\tau\rho_1/\rho_2 > 1+\nu$.* This follows from (2.6) and the inequality (2.2).

From $\tau\rho_1/\rho_2 > 1+\nu$ we find that if $\nu+1 > 0$, then

$$\rho_2/\rho_1 < \tau/(1+\nu) \ \text{ for } \ \tau > 0 \ \text{ or } \ \rho_2/\rho_1 > \tau/(1+\nu) \ \text{ for } \ \tau < 0 \tag{2.7}$$

and if $\nu+1 < 0$, then

$$\rho_2/\rho_1 > \tau/(1+\nu) \ \text{ for } \ \tau > 0 \ \text{ or } \ \rho_2/\rho_1 < \tau/(1+\nu) \ \text{ for } \ \tau < 0. \tag{2.8}$$

From (2.7) and (2.8) we obtain

$$\text{if } \tau/(\nu+1) > 0, \text{ then } \rho_2/\rho_1 < \tau/(\nu+1)$$

$$\text{if } \tau/(\nu+1) < 0, \text{ then } \rho_2/\rho_1 > \tau/(\nu+1).$$

Relation $\tau\rho_1/\rho_2 > 1+\nu$ means that if $\nu > -1$ and $\tau > 0$ ($\tau < 0$), then ρ_1 and ρ_2 have the same (different) sign. This means that the complex zeros of J_μ for $\nu > -1$ and $\tau > 0$ ($\tau < 0$) can only lie in the first and third (second and fourth) quadrant of the (ρ_1, ρ_2)-plane.

3. Proof of inequality (1.1)

Setting $f = L_\nu^{1/2}x$ in (2.1) we find

$$(C_0+\mu)L_\nu^{1/2}x = \rho T_0 L_\nu^{1/2}x/2 \quad \text{or} \quad (C_0+\nu)^{1/2}x + i\tau L_\nu^{1/2}x = \rho T_0 L_\nu^{1/2}x/2$$

and

$$x + i\tau L_\nu x = \rho S_\nu x/2. \tag{3.1}$$

For $\rho = \rho_1 + i\rho_2$ and $\|x\| = 1$, we find

$$1 + i\tau(L_\nu x, x) = \rho_1(S_\nu x, x)/2 + i\rho_2(S_\nu x, x)/2. \tag{3.2}$$

Note that $(L_\nu x, x)$ and $(S_\nu x, x)$ are real, because the operators involved are self-adjoint. Comparing real and imaginary parts in (3.2), we obtain

$$1 = \rho_1(S_\nu x, x)/2 \quad \text{and} \quad \tau(L_\nu x, x) = \rho_2(S_\nu x, x)/2 = \rho_2/\rho_1. \tag{3.3}$$

Inequality (1.1) follows from (3.3) and (2.5). The strict inequality follows from the strict inequality in Schwatz's inequality:

$$|(S_\nu x, x)| < \|S_\nu x\|\|x\| \le \|S_\nu\| = 2/j_{\nu,1}. \tag{3.4}$$

In fact, equality in (3.4) is excluded, because otherwise we must have $S_\nu x = kx$, for some eigenvalue k of S_ν, or $(S_\nu x, x) = k$. But, from (3.3), $k = 2/\rho_1$ and from (3.1)

$$x + i\tau L_\nu x = (\rho_1/2)(2/\rho_1)x + i(\rho_2/2)(2/\rho_1)x \quad \text{or} \quad \tau L_\nu x = (\rho_2/\rho_1)x.$$

This means that x is an eigenvector of L_ν, i.e., $x = e_n$ for some n. That is impossible. We stress the fact that equality in (1.1) is possible only in the case $\tau = 0$.

Remark 3.1: The inequality $|\rho_2| > |\tau|$, $\nu > -1$ means that for $\mu = \nu + i\tau$ and $\nu > -1$ the region $\{z \in C : \ |Imz| \le |\tau|\}$ is zero-free. This result has been proved, with another method, in [8]. Also the inequalities

$$\rho_2/\rho_1 < \tau/(1+\nu), \quad \tau/(1+\nu) > 0 \quad \text{and} \quad \rho_2/\rho_1 > \tau/(1+\nu), \quad \tau/(1+\nu) < 0$$

can be obtained from the results of [8, Corollary 1]. Instead of the inequality (1.1), in [8] there has been proved the inequality

$$\rho_1^2 \geq (1+\nu)(2+\nu), \quad \tau/(\nu+1) > 0. \tag{3.5}$$

We note that inequality (1.1) is better than inequality (3.5) for every $\mu = \nu + i\tau$, $\tau \neq 0$ and $\nu > -1$. In fact for $-1 < \nu < 0$ this statement follows from the inequality (see[7]) $j_{\nu,1}^2 > 4(1+\nu)(2+\nu)^{1/2}$ and for $\nu > 0$ it follows from the inequality (see[3]) $j_{\nu,1}^2 \geq j_{0,1}^2 + \nu^2 + 2\nu(j_{0,1}^2 + 4)^{1/2}$.

4. The complex zeros of the polynomials (1.4)

We consinder the Lommel polynomials $P_{n,\mu}$ defined by (1.4). We know from [4] that the zeros λ of the polynomial $P_{N+1,\mu}$ of degree N are the eigenvalues of the generalized eigenvalue problem

$$T_0 f = 2\lambda(C_0 + \mu)f, \quad f \neq 0, \tag{4.1}$$

where T_0 and C_0 are the same operators as in Section 2, but they are defined here in an N-dimensional Hilbert space H_N with orthonormal basis $\{e_1, e_2, ..., e_N\}$. Precisely $T_0 = V + V^*$, where V is the truncated shift ($Ve_n = e_{n+1}$ for $n < N$, $Ve_N = 0$) and V^* is the adjoint of V ($V^* e_1 = 0$, $V^* e_n = e_{n-1}$ for $n = 2, 3, ..., N$). In this case [4]

$$\|T_0\|_{H_N} = 2cos(\pi/(N+1)), \tag{4.2}$$

so we get $\|T_0\| = 2$ for $N \to \infty$. The operator C_0 is the diagonal operator ($C_0 e_n = ne_n$ for $n = 1, 2, ..., N$). So, for any $f = \sum_{n=1}^{N}(f, e_n)e_n$ in H_N we have the relation

$$(C_0 f, f) = \sum_{n=1}^{N} n|(f, e_n)|^2, \tag{4.3}$$

from which the inequalities

$$\|f\|^2 \leq (C_0 f, f) \leq N\|f\|^2 \tag{4.4}$$

follow imediately. In the case that f is an eigenvector of the problem (4.1) strict inequalities can be easily proved in (4.4), i.e.

$$1 < (C_0 f, f) < N, \quad \|f\| = 1. \tag{4.5}$$

Taking the scalar product by $f \neq 0$ in (4.1) and comparing real and imaginary parts for $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \nu + i\tau$ we obtain

$$\lambda_2(C_0 f, f) + (\lambda_2\nu + \lambda_1\tau)\|f\|^2 = 0 \tag{4.6}$$

$$(T_0 f, f) = 2\lambda_1(C_0 f, f) + 2(\lambda_1\nu - \lambda_2\tau)\|f\|^2. \tag{4.7}$$

We have the following results.

Theorem 4.1: *Let $\mu = \nu + i\tau$, $\tau \neq 0$, ν any real number. Then any zero $\lambda = \lambda_1 + i\lambda_2 \neq 0$ of the polynomial $P_{N+1,\mu}$, defined by (1.4), is complex (i.e. $\lambda_2 \neq 0$) and*

$$|\lambda| < cos(\pi/(N+1))/|\tau| \tag{4.8}$$

$$1 + \nu < (-\tau)\lambda_1/\lambda_2 < N + \nu. \tag{4.9}$$

Proof: Assume that $\lambda_2 = 0$. Then from (4.6) we find $\lambda_1\tau = 0$ and since $\tau \neq 0$, $\lambda_1 = 0$, i.e. $\lambda = 0$, contrary to the assumption. Thus $\lambda_2 \neq 0$, so we obtain from (4.6)

$$(C_0 f, f) = -(\nu + \tau\lambda_1/\lambda_2), \quad \|f\| = 1. \tag{4.10}$$

Inequality (4.9) follows imediately from (4.10) and the inequality (4.5). To prove inequality (4.8) we first eliminate $(C_0 f, f)$ from (4.10) and (4.7) and obtain

$$(T_0 f, f) = -2|\lambda|^2\tau/\lambda_2, \quad \|f\| = 1. \tag{4.11}$$

This together with (4.2) yields

$$2|\lambda|^2|\tau|/|\lambda_2| < \|T_0 f\|\|f\| = \|T_0 f\| \leq \|T_0\| = 2cos(\pi/(N+1))$$

and

$$|\lambda|^2 < |\lambda_2| cos(\pi/(N+1))/|\tau|. \tag{4.12}$$

Equality in the Schwarz inequality $|(T_0 f, f)| \leq \|T_0 f\|\|f\|$ is excluded because it implies that $T_0 f = \kappa f$ for some real κ and the eigenvalue equation (4.1) implies that f must be one of the eigenvectors of C_0, i.e. one of the elements $e_1, e_2, ..., e_N$, which is impossible. Since $|\lambda_2| \leq |\lambda|$ we find from (4.12)

$$|\lambda_2| \leq cos(\pi/(N+1))/|\tau| \tag{4.13}$$

and from this, using again (4.12), we obtain the inequality (4.8) ∎

Theorem 4.2: *Let $\mu = \nu + i\tau$, $\tau \neq 0$ and $\nu > -1$. Then every zero $\lambda = \lambda_1 + i\lambda_2 \neq 0$ of the polynomial $P_{N+1,\mu}$ is complex (i.e. $\lambda_2 \neq 0$) with $\lambda_1 \neq 0$ and*

$$|\lambda_1| < 1/j_{\nu,1}, \tag{4.14}$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function J_ν.

Proof: The assertion $\lambda_2 \neq 0$ follows as a particular case of Theorem 4.1 and the assertion $\lambda_1 \neq 0$ follows from (4.9), because $1 + \nu > 0$. Since $\nu > -1$ we set, as in (2.1), $f = L_\nu^{1/2} x$ and transform the problem (4.1) into the following one:

$$S_\nu x = 2\lambda x + 2i\tau\lambda L_\nu x, \quad x \neq 0, \tag{4.15}$$

where the self-adjoint operators $S_\nu = L_\nu^{1/2} T_0 L_\nu^{1/2}$ and L_ν act on the N-dimensional space H_N, spanned by the orthonormal elements $e_1, e_2, ..., e_N$. Scalar product multiplication in (4.15) by x and comparison of the real and imaginary parts leads to

$$(S_\nu x, x) = 2\lambda_1 - 2\tau\lambda_2(L_\nu x, x) \tag{4.16}$$

$$0 = \tau\lambda_1(L_\nu x, x) + \lambda_2, \quad \|x\| = 1. \tag{4.17}$$

Elimination of $(L_\nu x, x)$ from the above gives

$$(S_\nu x, x) = 2|\lambda|^2/\lambda_1, \quad \|x\| = 1. \tag{4.18}$$

Thus

$$2|\lambda|^2/|\lambda_1| = |(S_\nu x, x)| < \|S_\nu x\|_{H_N}\|x\|_{H_N} = \|S_\nu x\|_{H_N} \le \|S_\nu\|_{H_N}. \tag{4.19}$$

Since H_N is a subspace of H, which is spanned by the infinite orthonormal basis $\{e_1, e_2, ...\}$, we have, using relation (2.4),

$$\|S_\nu\|_{H_N} = \sup_{\substack{\|y\|_{H_N}=1 \\ y\in H_N}} |(S_\nu y, y)| \le \sup_{\substack{\|y\|=1 \\ y\in H}} |(S_\nu y, y)| = \|S_\nu\|_H = 2/j_{\nu,1}. \tag{4.20}$$

Now from (4.19) and (4.20) we find

$$|\lambda|^2 < |\lambda_1|/j_{\nu,1} \tag{4.21}$$

and in the same way as in Theorem 4.1 the relation (4.14) follows ∎

Putting together Theorems 4.1 and 4.2 we obtain from (4.8), (4.9) and (4.14)

Theorem 4.3: *Let $\mu = \nu + i\tau$, $\nu > -1$ and $\tau > 0$ $(\tau < 0)$. Then the complex zeros of the polynomial $P_{N+1,\mu}$ of degree N, which is defined by (1.4), lie inside the circle*

$$|\lambda| = \cos(\pi/(N+1))/|\tau|$$

and are restricted in the area:

$$|\lambda_1| < 1/j_{\nu,1}, \quad -\tau/(\nu+1) < \lambda_2/\lambda_1 < -\tau/(N+\nu),$$

$$(-\tau/(\nu+N) < \lambda_2/\lambda_1 < -\tau/(1+\nu)). \tag{4.22}$$

Remark 4.1: If $\nu < -N < -1$, then from (4.9) we obtain the inequalities (4.22) for $\tau > 0$ (or $\tau < 0$). Also, if $-N < \nu < -1$, then from (4.9) we obtain the inequalities

$$-(N+\nu)/\tau < \lambda_1/\lambda_2 < -(1+\nu)/\tau \quad \text{and} -(1+\nu)/\tau < \lambda_1/\lambda_2 < -(N+\nu)/\tau$$

for $\tau > 0$ and $\tau < 0$, respectively.

Remark 4.2: The Lommel polynomials $R_{n,\mu}$ of degree n are defined [9, p.229] by the reccurence relation (1.5). The polynomials $R_{n,\mu+1}$ of degree n defined by (1.5) coincide with the polynomials $P_{n+1,\mu}$ of degree n, defined by (1.4), i.e. $P_{n+1,\mu} = R_{n,\mu+1}$. In [8, Theorem 5 / p.119] there have been studied zero-free regions for the polynomial $R_{N,\mu+1}$. There have been found similar inequalities (not the same), as the inequalities (4.22). Also, in [8] there have been found the inequalities

$$|\lambda| > \begin{cases} Re\lambda + \tau^2/2 & \text{for } \nu \neq -1,\ \tau \in R \\ -\text{Re}\lambda + (1+\nu)(2+\nu)/2 & \text{for } \tau/(\nu+1) > 0\,. \end{cases}$$

The inequality (4.8) and the first one of (4.22) seem to be new.

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