

Weakly Singular Hammerstein-Volterra Operators in Orlicz and Hölder Spaces

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Acting, boundedness, and compactness conditions for nonlinear Hammerstein-Volterra operators are given either between two Orlicz spaces, or from an Orlicz space into a (generalized) Hölder space. Particular emphasis is put on weakly singular kernels. This leads to (local) solvability results for Hammerstein-Volterra equations of second kind.

Key words: *Hammerstein-Volterra operators, weakly singular kernels, strong nonlinearities, fractional derivatives, Orlicz spaces, Hölder spaces*

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0. Introduction

As is well known, the study of initial value problems for systems of ordinary differential equations leads to Hammerstein-Volterra equations of the form

$$x(t) = \int_0^t k(t, s)f(s, x(s)) ds \quad (t \geq 0), \quad (1)$$

where $k: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}^{N \times N}$ is a matrix-valued kernel function and $f: \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a given nonlinearity which satisfies a Carathéodory condition. In nonlinear analysis one usually considers (1) as fixed point equation for the *Hammerstein-Volterra operator* $H = VF$ which may be written as composition of the linear *Volterra operator*

$$Vy(t) = \int_0^t k(t, s)y(s) ds \quad (2)$$

and the nonlinear *Nemytskij operator*

$$Fx(s) = f(s, x(s)). \quad (3)$$

In the classical setting, these operators are usually studied either in the space C of continuous functions (e.g., if the nonlinearity f is continuous in both variables), or in the space L_p ($1 \leq p < \infty$) of p -integrable functions (e.g., if the function f has polynomial growth). However, if the nonlinearity f exhibits a non-polynomial (e.g., exponential) growth in the second variable, it is a useful device to study equation (1) in some Orlicz space which is "manufactured" according to the data k and f .

The main purpose of the present paper is to study equation (1) in appropriate Orlicz spaces. If the Volterra operator (2) is replaced by the *Fredholm operator*

$$Ky(t) = \int_0^T k(t, s)y(s) ds, \tag{4}$$

the corresponding fixed point equation $x = KFx$ has been widely considered in the literature. On the other hand, there are only very few results, as far as we know, taking into account the specific features of the Volterra operator (2). For example, it is well known that, in case of a *weakly singular kernel function*

$$k(t, s) = \frac{h(t, s)}{|t - s|^\lambda} \tag{5}$$

(h bounded and sufficiently regular), the operator V maps the Lebesgue space L_p into the Hölder space C^α with $\alpha = 1 - \lambda - 1/p$. In the first section below we show that V has a similar “smoothing property” from Orlicz spaces into certain generalized Hölder spaces, or into the space C of continuous functions. Afterwards, we establish some boundedness for the operator V between Orlicz spaces which build on Riordan’s generalization of the classical Marcinkiewicz interpolation theorem. Combining these results with well known mapping properties of the Nemytskij operator (3), we are lead to local existence theorems for the Hammerstein-Volterra equation (1). In this connection, particular emphasis is put again on kernel functions of the potential type (5).

1. Equations with Hölder continuous kernels

Let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a *Young function*, i.e. Φ is increasing, even, convex, and continuous with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. The *Orlicz space* $L_\Phi = L_\Phi([0, T], \mathbb{R}^N)$ consists, by definition, of all (classes of) measurable functions $x : [0, T] \rightarrow \mathbb{R}^N$ for which the (Luxemburg) norm

$$\|x\|_\Phi = \inf \left\{ \alpha > 0 : \int_0^T \Phi \left(\frac{|x(t)|}{\alpha} \right) dt \leq 1 \right\}$$

is finite (see, e.g., [9, 17]). The particular choice $\Phi(u) = \Phi_p(u) = \frac{1}{p}|u|^p$ ($1 \leq p < \infty$) leads, of course, to the Lebesgue space $L_p = L_p([0, T], \mathbb{R}^N)$.

Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a *Hölder function*, i.e. ϕ is increasing and continuous with $\phi(0) = 0$. The (generalized) *Hölder space* $C^\phi = C^\phi([0, T], \mathbb{R}^N)$ consists, by definition, of all continuous functions $x : [0, T] \rightarrow \mathbb{R}^N$ satisfying an estimate

$$|x(t) - x(\tau)| \leq L\phi(|t - \tau|)$$

for some $L > 0$. Equipped with the norm

$$\|x\|_\phi = \|x\|_C + [x]_\phi,$$

where

$$\|x\|_C = \max_{0 \leq t \leq T} |x(t)| \quad \text{and} \quad [x]_\phi = \sup_{t \neq \tau} \frac{|x(t) - x(\tau)|}{\phi(|t - \tau|)},$$

the space C^ϕ becomes a Banach space. The particular choice $\phi(t) = t^\alpha$ ($0 < \alpha \leq 1$) leads, of course, to the classical Hölder space $C^\alpha = C^\alpha([0, T], \mathbb{R}^N)$.

Orlicz spaces occur, loosely speaking, whenever one has to deal with differential equations involving strong nonlinearities; a typical application may be found in [7]. Moreover, the so-called *Orlicz-Sobolev spaces* $W^k L_\Phi$ (containing functions whose k -th order distributional derivatives belong to L_Φ) may be imbedded into generalized Hölder spaces C^ϕ in rather the same way as classical Sobolev spaces W_p^k into classical Hölder spaces C^α (see, e.g., [1, 11]).

In this and the following sections we are interested in mapping and boundedness properties of the *weakly singular Volterra operator*

$$Vy(t) = \int_0^t \frac{h(t,s)}{|t-s|^\lambda} y(s) ds, \quad (6)$$

where h is some continuous matrix-valued function on the triangular domain $\Delta = \{(t,s) : 0 \leq s \leq t \leq T\}$, and $0 < \lambda < 1$. Let Φ be a fixed Young function and denote by $\bar{\Phi}$ the *conjugate Young function* [9]

$$\bar{\Phi}(u) = \sup_{v \geq 0} \{|u|v - \Phi(v)\}; \quad (7)$$

for instance, $\bar{\Phi}_p = \Phi_{\bar{p}}$ with $\frac{1}{p} + \frac{1}{\bar{p}} = 1$ for $p > 1$. Put

$$f(t) = \int_0^t \bar{\Phi}(s^{-\lambda}) ds \quad (8)$$

and

$$\psi(t) = \inf \{\alpha > 0 : f(\alpha^{1/\lambda} t) \leq \alpha^{1/\lambda}\}. \quad (9)$$

Then $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a well-defined Hölder function. In fact, the derivative of the function $f_t(\rho) = \rho - f(\rho t)$ satisfies

$$\frac{df_t(\rho)}{d\rho} = 1 - t f'(\rho t) = 1 - t \bar{\Phi}(\rho^{-\lambda} t^{-\lambda}),$$

and hence is positive for ρ large enough (recall that $\bar{\Phi}(u) \rightarrow 0$ as $u \rightarrow 0$). Consequently, $f(\rho t) \leq \rho$ for sufficiently large ρ , and thus the function (9) is well-defined. For $0 \leq t \leq \tau$ we have

$$f(\psi(\tau)^{1/\lambda} t) \leq f(\psi(\tau)^{1/\lambda} \tau) \leq \psi(\tau)^{1/\lambda},$$

since f is increasing; this shows that we may put $\alpha = \psi(\tau)$ in (9), and hence $\psi(t) \leq \psi(\tau)$. Finally, the continuity of (9) follows from the continuity and concavity of (8), and the equality $\psi(0) = 0$ follows from $f(0) = 0$.

Theorem 1. *Suppose that $h : \Delta \rightarrow \mathbb{R}^{N \times N}$ is continuous and*

$$[h]_\xi = \sup_{(t,s) \neq (\tau,\sigma)} \frac{|h(t,s) - h(\tau,\sigma)|}{\xi(|t-\tau| + |s-\sigma|)} < \infty$$

for some Hölder function ξ . Suppose that the function f defined in (8) is finite, and put $\phi(t) = \max \{\psi(t), \xi(t)\}$ with ψ given by (9). Then the weakly singular Volterra operator (6) is bounded from the Orlicz space L_Φ into the Hölder space C^ϕ and satisfies

$$\|V|L_\Phi \rightarrow C^\phi\| \leq 2(1 + \phi(T)) \max \{2, \psi(T)\} \|h\|_\xi. \quad (10)$$

Proof. For $0 \leq t \leq \tau \leq T$ fixed and $\alpha > 0$, consider the integrals

$$I_1(\alpha) = \int_0^t \tilde{\Phi} \left(\frac{|h(\tau, s) - h(t, s)|}{\alpha|\tau - s|^\lambda} \right) ds, \tag{11}$$

$$I_2(\alpha) = \int_0^t \tilde{\Phi} \left(\left| \frac{1}{\alpha|\tau - s|^\lambda} - \frac{1}{\alpha|t - s|^\lambda} \right| |h(t, s)| \right) ds \tag{12}$$

$$I_3(\alpha) = \int_t^\tau \tilde{\Phi} \left(\frac{|h(\tau, s)|}{\alpha|\tau - s|^\lambda} \right) ds, \tag{13}$$

and let

$$a_j = \inf \{ \alpha > 0 : I_j(\alpha) \leq 1 \} \quad (j = 1, 2, 3).$$

We derive upper estimates for a_1, a_2 and a_3 . Let first $C = [h]_{\xi\xi}(|\tau - t|)$. Assuming $C > 0$ (without loss of generality) and substituting $s \mapsto \tau - (C/\alpha)^{1/\lambda}s$ leads to the estimate

$$I_1(\alpha) \leq \int_0^t \tilde{\Phi} \left(\frac{C}{\alpha|\tau - s|^\lambda} \right) ds = \left(\frac{C}{\alpha} \right)^{1/\lambda} \int_{(\tau-t)(\alpha/C)^{1/\lambda}}^{\tau(\alpha/C)^{1/\lambda}} \tilde{\Phi}(s^{-\lambda}) ds,$$

hence

$$\begin{aligned} I_1(\alpha) &\leq \left(\frac{C}{\alpha} \right)^{1/\lambda} \left[f \left(\tau \left(\frac{\alpha}{C} \right) \right)^{1/\lambda} - f \left((\tau - t) \left(\frac{\alpha}{C} \right)^{1/\lambda} \right) \right] \\ &\leq \left(\frac{C}{\alpha} \right)^{1/\lambda} f \left(T \left(\frac{\alpha}{C} \right)^{1/\lambda} \right). \end{aligned}$$

Consequently, $f(T(\alpha/C)^{1/\lambda}) \leq (\alpha/C)^{1/\lambda}$ implies $I_1(\alpha) \leq 1$, hence

$$a_1 \leq \psi(T)[h]_{\xi\xi}(|\tau - t|). \tag{14}$$

Now let $D = \|h\|_C$. Assuming $D > 0$ (without loss of generality) and substituting $s \mapsto t - (D/\alpha)^{1/\lambda}s$ and $s \mapsto \tau - (D/\alpha)^{1/\lambda}s$, respectively, leads to the estimate

$$\begin{aligned} I_2(\alpha) &\leq \int_0^t \tilde{\Phi} \left(\frac{D}{\alpha(t-s)^\lambda} - \frac{D}{\alpha(\tau-s)^\lambda} \right) ds \\ &\leq \int_0^t \tilde{\Phi} \left(\frac{D}{\alpha(t-s)^\lambda} \right) ds - \int_0^t \tilde{\Phi} \left(\frac{D}{\alpha(\tau-s)^\lambda} \right) ds \\ &= \left(\frac{D}{\alpha} \right)^{1/\lambda} \int_0^{t(\alpha/D)^{1/\lambda}} \tilde{\Phi}(s^{-\lambda}) ds - \left(\frac{D}{\alpha} \right)^{1/\lambda} \int_{(\tau-t)(\alpha/D)^{1/\lambda}}^{t(\alpha/D)^{1/\lambda}} \tilde{\Phi}(s^{-\lambda}) ds \\ &= \left(\frac{D}{\alpha} \right)^{1/\lambda} \left[f \left(t \left(\frac{\alpha}{D} \right)^{1/\lambda} \right) - f \left(\tau \left(\frac{\alpha}{D} \right)^{1/\lambda} \right) + f \left((\tau - t) \left(\frac{\alpha}{D} \right)^{1/\lambda} \right) \right] \\ &\leq \left(\frac{D}{\alpha} \right)^{1/\lambda} f \left((\tau - t) \left(\frac{\alpha}{D} \right)^{1/\lambda} \right), \end{aligned}$$

where we have used the convexity of $\tilde{\Phi}$ and the monotonicity of f . Consequently, $f((\tau - t)(\alpha/D)^{1/\lambda}) \leq (\alpha/D)^{1/\lambda}$ implies that $I_2(\alpha) \leq 1$, hence

$$a_2 \leq \psi(|\tau - t|)\|h\|_C. \tag{15}$$

An analogous reasoning leads to the estimate

$$I_3(\alpha) \leq \left(\frac{D}{\alpha}\right)^{1/\lambda} f\left((\tau-t)\left(\frac{\alpha}{D}\right)^{1/\lambda}\right),$$

hence

$$a_3 \leq \psi(|\tau-t|)\|h\|_C. \quad (16)$$

Now fix $x \in L_\Phi$ and $0 \leq t \leq \tau \leq T$ as above. For $0 \leq s \leq T$, put

$$\begin{aligned} h_1(s) &= \begin{cases} \frac{|h(\tau,s)-h(t,s)|}{|\tau-s|^\lambda} & \text{if } s < t \\ 0 & \text{otherwise} \end{cases} \\ h_2(s) &= \begin{cases} \left|\frac{1}{|\tau-s|^\lambda} - \frac{1}{|t-s|^\lambda}\right| |h(t,s)| & \text{if } s < t \\ 0 & \text{otherwise} \end{cases} \\ h_3(s) &= \begin{cases} \frac{|h(\tau,s)|}{|\tau-s|^\lambda} & \text{if } t \leq s < \tau \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

A comparison with (11) - (13) shows that $h_j \in L_\Phi$ with $\|h_j\|_\Phi = a_j$ ($j = 1, 2, 3$). Writing $h(\tau, s) = a$, $|\tau - s|^{-\lambda} = b$, $h(t, s) = c$, and $|t - s|^{-\lambda} = d$, for short, we get

$$\begin{aligned} |Vx(\tau) - Vx(t)| &= \left| \int_0^t (ab - cd)x(s) ds + \int_t^\tau (ab)x(s) ds \right| \\ &\leq \int_0^t (|a - c| |b| + |b - d| |c|) |x(s)| ds + \int_t^\tau |ab| |x(s)| ds \\ &= \int_0^T [h_1(s) + h_2(s) + h_3(s)] |x(s)| ds \\ &\leq 2(a_1 + a_2 + a_3) \|x\|_\Phi. \end{aligned}$$

Combining this with (14) - (16) yields

$$|Vx(\tau) - Vx(t)| \leq 2(2\|h\|_C + \psi(T)\{h\}_\xi) \|x\|_\Phi \phi(|\tau - t|), \quad (17)$$

hence

$$[Vx]_\Phi \leq 2(2\|h\|_C + \psi(T)\{h\}_\xi) \|x\|_\Phi. \quad (18)$$

Observing now that

$$|Vx(t)| = |Vx(t) - Vx(0)| \leq \phi(T)[Vx]_\Phi$$

we finally get

$$\|Vx\|_C \leq 2(2\phi(T)\|h\|_C + \phi(T)\psi(T)\{h\}_\xi) \|x\|_\Phi$$

which together with (18) proves (10) ■

2. Equations with continuous kernels

Theorem 1 shows that the weakly singular Volterra operator (6) maps the Orlicz space L_Φ into an appropriate Hölder space C^ϕ , provided the kernel function h is Hölder continuous on the triangular domain Δ . If we merely require h to be continuous on Δ , it is not

surprising that we end up in the space $C = C([0, T], \mathbb{R}^N)$. On the other hand, as an additional gift we get then (see the following theorem) the compactness of the operator (6).

Theorem 2. *Suppose that $h : \Delta \rightarrow \mathbb{R}^{N \times N}$ is continuous and the function f defined in (8) is finite, and let ψ be given by (9). Then the weakly singular Volterra operator (6) is compact from the Orlicz space L_Φ into the space C and satisfies*

$$\|V|L_\Phi \rightarrow C\| \leq 2\psi(T)\|h\|_C. \tag{19}$$

Proof. We modify the proof of Theorem 1. To this end, for $0 \leq t \leq T$ fixed and $\alpha > 0$, consider the integral

$$I_4(\alpha) = \int_0^t \tilde{\Phi} \left(\frac{|h(t, s)|}{\alpha|t - s|^\lambda} \right) ds,$$

and let

$$a_4 = \inf \{ \alpha > 0 : I_4(\alpha) \leq 1 \}.$$

As in the proof of Theorem 1, a straightforward calculation shows that

$$a_4 \leq \psi(T)\|h\|_C,$$

hence

$$|Vx(t)| \leq 2a_4\|x\|_\Phi \leq 2\psi(T)\|h\|_C\|x\|_\Phi,$$

by the Hölder inequality. This proves (19). To see that V is compact, let $\|x\|_\Phi \leq 1$. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|h(\tau, s) - h(t, s)| < \varepsilon$ and $\psi(|\tau - t|) < \varepsilon$ for $|\tau - t| < \delta$. For $I_1(\alpha)$ as in (11) we get then, after substituting $s \mapsto \tau - (\varepsilon/\alpha)^{1/\lambda}s$, that $I_1(\alpha) \leq \varepsilon\psi(T)$. Reasoning as in the proof of Theorem 1 we conclude that

$$|Vx(\tau) - Vx(t)| \leq (2\|h\|_C + \psi(T))\varepsilon.$$

This shows that the set $\{Vx : \|x\|_\Phi \leq 1\}$ is equicontinuous, and thus the assertion of V follows from the Arzelà-Ascoli compactness criterion. ■

We illustrate Theorem 1 and Theorem 2 by two typical examples.

Example 1. Let $\Phi(u) = \Phi_p(u) = \frac{1}{p}|u|^p$ ($1 < p < \infty$). A trivial calculation shows that the function (8) is finite in this case if and only if $\lambda p < p - 1$, and $f(t) \sim t^{1-\lambda p/(p-1)}$ hence $\psi(t) \sim t^{1-\lambda-1/p}$. In particular, for $h(t, s) \equiv 1$ we get the classical result that the Abel operator

$$Ax(t) = \int_0^t \frac{x(s)}{|t - s|^\lambda} ds \tag{20}$$

maps the Lebesgue space L_p into the Hölder space C^α with $\alpha = 1 - \lambda - 1/p$ (see, e.g., [6, 8]).

Example 2. Let $\Phi(u) = e^{|u|} - |u| - 1$. In this case the properties of the Orlicz space L_Φ are essentially different from those of the Lebesgue space L_p , since Φ does not satisfy a Δ_2 condition (see [9, 17]). The conjugate Young function (7) is here

$$\tilde{\Phi}(u) = (1 + |u|)\log(1 + |u|) - |u|.$$

We show that, for this choice of Φ , the function

$$f(t) = \int_0^t (1 + s^{-\lambda}) \log(1 + s^{-\lambda}) ds - \int_0^t s^{-\lambda} ds \tag{21}$$

is finite for any $\lambda \in (0, 1)$. The second integral in (21) is trivially finite; it is the first integral which requires a more careful analysis. Now, integrating

$$I_\epsilon = \int_\epsilon^t (1 + s^{-\lambda}) \log(1 + s^{-\lambda}) ds$$

by parts yields

$$I_\epsilon = \left[\left(s + \frac{1}{1-\lambda} s^{1-\lambda} \right) \log(1 + s^{-\lambda}) \right]_\epsilon^t - \int_\epsilon^t \left(s + \frac{1}{1-\lambda} s^{1-\lambda} \right) \frac{-\lambda s^{-\lambda-1}}{1 + s^{-\lambda}} ds.$$

The first term is bounded for ϵ near 0, while the integrand in the second term may be majorized by the integrable function $\lambda(1+s^{-\lambda}/(1-\lambda))$. Consequently, I_ϵ remains bounded, as $\epsilon \downarrow 0$, and so we are done.

From Theorem 1 and Theorem 2 we conclude that the Volterra operator (6) acts from L_Φ into C if the kernel function h is continuous, and from L_Φ into C^α ($0 < \alpha < 1$ appropriate) if h belongs to C^α . As a typical example, we may again consider the Abel operator (20).

3. Equations with bounded kernels

Now we further weaken the regularity assumption on the kernel function (5) by simply requiring that h is essentially bounded on Δ . As a consequence, we obtain mapping and boundedness properties for the Volterra operator (6) from one Orlicz space L_Φ into another Orlicz space L_Ψ . A basic tool will be Riordan's generalization [18] of the classical Marcinkiewicz interpolation theorem [12].

Recall that a linear operator A is called of weak type (p, q) ($1 \leq p, q < \infty$) if A is bounded as an operator from the Lebesgue space L_p into the Marcinkiewicz space M_q (see, e.g., [4, 5, 10]). More explicitly, A is of weak type (p, q) if

$$\text{mes}(D(Ax; h)) \leq \left(\frac{c}{h} \|x\|_p \right)^q \quad (h > 0) \tag{22}$$

for all $x \in L_p$ and some $c > 0$, where $D(y; h)$ denotes the Lebesgue set of all $t \in [0, T]$ such that $|y(t)| > h$. Since $L_q \subseteq M_q$, every bounded linear operator from L_p into L_q is of course of weak type (p, q) ; the converse is not true. (To see this, consider $A = V$ as in (2) with $k(t, s) = 1/t$ for $p = q = 1$.)

Theorem 3 [18]: *Suppose that A is both of weak type (p_1, q_1) and (p_2, q_2) , where $q_1 > q_2, p_1 \neq p_2, p_1 \leq q_1$, and $p_2 \leq q_2$. Let*

$$\alpha = \frac{q_2/p_2 - q_1/p_1}{q_2 - q_1} \quad \text{and} \quad \beta = \frac{q_1^{-1} - q_2^{-1}}{p_1^{-1} - p_2^{-1}}.$$

Assume that Ψ is a Young function such that

$$\int_t^\infty s^{-q_1} d\Psi(s) = O(t^{-q_1} \Psi(t)) \quad \text{and} \quad \int_0^t s^{-q_2} d\Psi(s) = O(t^{-q_2} \Psi(t)).$$

Finally, let Φ be a Young function such that

$$\Phi(t) \sim \left[\Psi \left(\frac{t}{\Phi(t)^\alpha} \right) \right]^\beta. \quad (23)$$

Then the operator A is bounded from the Orlicz space L_Φ into the Orlicz space L_Ψ .

We point out that, in case $\Phi(u) = |u|^p$ and $\Psi(u) = |u|^q$, Theorem 3 reduces to the classical Marcinkiewicz interpolation theorem.

Applying Theorem 3 to the Volterra operator (6), one may get various boundedness results for V between Orlicz spaces: it simply suffices to find $p, q \in [1, \infty)$ such that V is of weak type (p, q) . As a sample result, we mention the following.

Theorem 4. *Suppose that $h : \Delta \rightarrow \mathbb{R}^{N \times N}$ is measurable and essentially bounded, and let $0 < \lambda < 1$. Assume that either*

$$\frac{1}{1-\lambda} \leq p < \infty \quad (24)$$

or

$$1 \leq p < \frac{1}{1-\lambda}, \quad 1 \leq q \leq \frac{p}{1-p(1-\lambda)}. \quad (25)$$

Then the weakly singular Volterra operator (6) is of weak type (p, q) .

Proof. It suffices to prove the assertion for the "worst case" $p = 1$ and $q = 1/\lambda$. Let $\|h\|_\infty = \eta < \infty$; we show that (22) holds with $c = \eta/(1-\lambda)$. In fact, given $x \in L_1$ and $h > 0$, for $D = D(Vx; h)$ we have

$$\begin{aligned} h \operatorname{mes}(D) &= \int_D h \, ds \leq \int_D |Vx(s)| \, ds \leq \eta \int_D \int_0^T \frac{\chi_{[0,s]}(\sigma)}{|s-\sigma|^\lambda} |x(\sigma)| \, d\sigma \, ds \\ &= \eta \int_0^T \left(\int_D \frac{\chi_{[\sigma,T]}(s)}{|s-\sigma|^\lambda} \, ds \right) |x(\sigma)| \, d\sigma \leq \eta \int_0^T \frac{\operatorname{mes}(D)^{1-\lambda}}{1-\lambda} |x(\sigma)| \, d\sigma \\ &= \frac{\eta \|x\|_1}{1-\lambda} \operatorname{mes}(D)^{1-\lambda}, \end{aligned}$$

hence $\operatorname{mes}(D)^\lambda \leq \frac{\eta \|x\|_1}{1-\lambda} \frac{1}{h}$ as claimed ■

We make some comments on Theorem 4. First of all, we point out that (at least in case $h(t, s) \equiv 1$) the conditions (24) and (25) are sharp in the following sense:

If the operator (20) is of weak type (p, q) for some $\lambda \in (0, 1)$, then (24) or (25) hold.

Second, the conditions (24) and (25) are similar to classical boundedness conditions for convolution operators (e.g., the Abel operator (20)) on Lebesgue spaces. For instance, it is well-known ([8], see also [22]) that the operator (20) is bounded from L_p into L_q for

$$1 \leq p \leq \frac{1}{1-\lambda} \quad \text{and} \quad q = \frac{p}{1-p(1-\lambda-\varepsilon)}, \quad (26)$$

where $\varepsilon > 0$ and, at least for $p = 1$ or $p = 1/(1-\lambda)$, one must not take $\varepsilon = 0$ in (26). In [6: Theorem 4.1.1] the authors claim that the estimate

$$\|Vx\|_q \leq T^\varepsilon \left(1 + \frac{\lambda}{\varepsilon} \right)^{\lambda+\varepsilon} \|x\|_p$$

holds for p and q as in (26). This is indeed true for $q \geq p$, but false for $q < p$, as the following example shows.

Example 3. Let $T = 1$, $p = 10$, $q = 1$, $\lambda = 9/10$, and $x(s) \equiv 1$. Then

$$\|Vx\|_1 = 10 \int_0^1 t^{1/10} dt = \frac{100}{11} > 9 \quad \text{but} \quad T^\varepsilon \left(1 + \frac{\lambda}{\varepsilon}\right)^{\lambda+\varepsilon} = \left(\frac{19}{10}\right)^{19/10} < 4.$$

In order to apply Theorem 3 to the operator (6) (or (20)) between specific Orlicz spaces, one has to verify the growth condition (23). Unfortunately, this may be very hard for complicated Young functions. (An example of two Young functions Φ and Ψ satisfying (23) may be found in [17: p. 252].) A different approach to the Abel operator (20) between Orlicz spaces which is based on general properties of convolution operators may be found in [13]. Finally, we remark that there are other papers on the interpolation of Orlicz spaces (e.g. [19]) which may be helpful for obtaining boundedness results for the operator (6) between Orlicz spaces.

4. Hammerstein-Volterra equations in Orlicz spaces

Combining the boundedness and compactness results proved so far for the Volterra operator (6) with well-known boundedness and continuity results for the Nemytskij operator (3), one may obtain various existence theorems for the Hammerstein-Volterra equation (1) with weakly singular kernel function (5).

For the reader's convenience, let us recall the following result on Nemytskij operators between Orlicz spaces ([2], see also [3, 9]).

Theorem 5. *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function. Let Φ and Ψ be two Young functions, and let $r > 0$. Then the Nemytskij operator (3) maps the ball $B_r(L_\Psi) = \{x \in L_\Psi : \|x\|_\Psi \leq r\}$ into the space L_Φ if and only if the growth condition*

$$\Phi \left(\frac{|f(t, u)|}{R} \right) \leq a(t) + b\Psi \left(\frac{|u|}{r} \right) \tag{27}$$

holds for some $a \in L_1$, $b \geq 0$, and $R > 0$. Moreover, in this case the operator (3) is always bounded. Finally, the operator (3) is continuous if the Young function Φ satisfies a Δ_2 condition.

As mentioned before, from Theorem 3 and Theorem 4 one may deduce boundedness conditions for the Volterra operator (6) between two Orlicz spaces L_Φ and L_Ψ . In order to apply classical principles of nonlinear analysis, however, we also need the compactness of the operator (6). Here we recall the following well-known sufficient condition.

Theorem 6 [17]. *Let $k : \Delta \rightarrow \mathbb{R}^{N \times N}$ be a measurable function such that*

$$\int_0^T \int_0^t \Xi \left(\frac{|k(t, s)|}{\alpha} \right) ds dt < \infty \tag{28}$$

for all $\alpha > 0$ and some Young function Ξ . Suppose that there exist $a > 0$ and $u_0 > 0$ such that

$$\Xi(auv) \leq \Phi(u)\tilde{\Psi}(v) \quad (u, v \geq u_0). \tag{29}$$

Then the Volterra operator (6) is compact from L_Φ into L_Ψ and satisfies

$$\|V|L_\Phi \rightarrow L_\Psi\| \leq \Gamma(a, u_0, T)\|k\|_{\Xi},$$

where

$$\Gamma(a, u_0, T) = \frac{2}{a} \left\{ T^2 \Xi(au_0^2) + T\Phi(u_0) + T\tilde{\Psi}(u_0) + 1 \right\}. \tag{30}$$

We remark that (28) simply means that the kernel function k has an *absolutely continuous norm* in the Orlicz space $L_\Xi = L_\Xi([0, T] \times [0, T], \mathbb{R}^{N \times N})$. Equivalently, k may be approximated (in the norm of L_Ξ) by a sequence $(k_n)_n$ of simple kernel functions k_n . This fact was used in [17] to prove an analogous compactness result for the Fredholm operator (4). In fact, the authors claim in the proof of [17: Theorem 6.1.5] that “the corresponding operator $K_n y(t) = \int_0^T k_n(t, s)y(s) ds$ is compact since it has a finite-dimensional range”. However, this is not correct, as may be seen by choosing $k_n(t, s) = \chi_\Delta(t, s)$, say.

Theorem 6 may be combined with Theorem 5 in order to assure the compactness of the Hammerstein-Volterra operator $H = VF$, and thus to obtain solutions of equation (1). We do not want to formulate this as another abstract result, but illustrate this by means of two illuminating examples.

Example 4. Let $h : \Delta \rightarrow \mathbb{R}^{N \times N}$ be measurable and essentially bounded, and take $\lambda = 1/2$. We claim that the corresponding Volterra operator (6) is compact from the Lebesgue space L_2 into the Orlicz space L_Ψ generated by the Young function

$$\Psi(u) = |u|(e^{|u|^\nu} - 1) \quad (0 < \nu < 1).$$

In fact, we may consider the operator VV^* which is generated by the iterated kernel function

$$l(t, s) = \int_s^t k(t, \tau)k(\tau, s) d\tau$$

between the Orlicz spaces $L_{\tilde{\Psi}}$ and L_Ψ . Since

$$|l(t, s)| \leq c_1 + c_2 |\log |t - s||,$$

it is not hard to see that (28) is satisfied for $k = l$, $\Xi = \Psi$, and all $\alpha > 0$. Reasoning now as in [9: §16], we know that VV^* is compact between $L_{\tilde{\Psi}}$ and L_Ψ . Moreover, condition (29) reads now

$$\tilde{\Psi}(auv) \leq u^2 \tilde{\Psi}(v) \quad (u, v \geq u_0). \tag{31}$$

By definition of the Young function Ψ we have $\Psi(u) \sim \Psi(u)^2$; consequently, the conjugate Young function $\tilde{\Psi}$ satisfies a Δ' condition [9]. But this implies (31), since $\Psi(u)$ increases more rapidly than u^2 . By Theorem 6, the corresponding Volterra operator (6) is compact between the spaces L_2 and L_Ψ .

Now, from Theorem 5 we conclude that the growth condition

$$|f(t, u)|^2 \leq a(t) + b\Psi\left(\frac{|u|}{r}\right),$$

where $a \in L_1$, $b \geq 0$, and $r > 0$, implies both the continuity and boundedness of the Nemytskij operator (3) from $B_r(L_\Psi)$ into L_2 . Consequently, the operator $H = VF$ is compact and continuous on the ball $B_r(L_\Psi)$.

To prove the existence of a solution $x \in L_\Psi$ of equation (1), one may proceed in two different ways. On the one hand, if the growth of the nonlinearity f is not too fast, one may find a ball in the space L_Ψ which can be transformed by the operator H into itself and apply Schauder's fixed point principle. On the other hand, if the growth of f is very fast, one may try to find a-priori bounds for the equation $x = \mu Hx$ ($0 < \mu < 1$) and apply Schaefer's continuation method [21]. In any case, the definition of the Young function Ψ shows that the growth of $f(t, u)$ may be faster than any polynomial in u .

In the following example we want to study the "perturbed" equation

$$x(t) = \int_0^t k(t, s)f(s, x(s)) ds + r(t), \quad (32)$$

where $r \in L_\Psi$ is given. We show that (32) is "locally solvable" (i.e. on some subinterval $[0, \tau] \subseteq [0, T]$), provided that r and τ are small in a sense to be made precise.

Example 5. Let Φ, Ψ , and Ξ be three Young functions such that the conditions (28) and (29) hold; in addition, we assume that Φ satisfies a Δ_2 condition. Suppose that f satisfies the growth condition (27). By Theorem 5 and Theorem 6, we know that $F : L_\Psi \rightarrow L_\Phi$ is bounded and continuous, and $V : L_\Phi \rightarrow L_\Psi$ is compact.

Let $\varepsilon > 0$ and $\rho > 0$. We claim that we can find a $\tau \in (0, T]$ such that, for any $r \in L_\Psi$ with $\|r\|_\Psi \leq \rho$, the equation (32) has a solution x on $[0, \tau]$ with $\|x - r\|_\Psi \leq \varepsilon$.

In fact, the boundedness of the operator F implies that there exists $R > 0$ such that $\|Fx\|_\Phi \leq R$ for $\|x\|_\Psi \leq \rho + \varepsilon$. Since the kernel function k has an absolutely continuous norm (see the remark after Theorem 6), for sufficiently small $\tau > 0$ we have

$$\Gamma(a, u_0, \tau) \|\chi_{\Delta(\tau)} k\|_{\Xi} R \leq \varepsilon$$

(see (30) for the definition of Γ), where $\Delta(\tau) = \{(t, s) : 0 \leq s \leq t \leq \tau\}$. Let

$$V_\tau y(t) = \int_0^t \chi_{\Delta(\tau)}(t, s) k(t, s) y(s) ds \quad (0 \leq t \leq \tau), \quad (33)$$

and consider all functions on the interval $[0, \tau]$, rather than $[0, T]$. For $x \in L_\Psi$ with $\|x\|_\Psi \leq \rho + \varepsilon$ we have then

$$\|Hx\|_\Psi = \|V_\tau Fx\|_\Psi \leq \Gamma(a, u_0, \tau) \|\chi_{\Delta(\tau)} k\|_{\Xi} \|Fx\|_\Phi \leq \varepsilon.$$

This shows that the operator H_f defined by $H_f x = Hx + f$ transforms the ball $B_\varepsilon(L_\Psi) + f = \{x \in L_\Psi : \|x - f\|_\Psi \leq \varepsilon\}$ into itself. From Schauder's fixed point principle we conclude that equation (32) has a solution in this ball.

The main trick in getting balls which may be transformed into itself in Example 5 consists in "shrinking" the norm of the operator (33) by choosing τ sufficiently small. The same trick may be used to get existence theorems for the more general *Uryson-Volterra equation*

$$x(t) = \int_0^t g(t, s, x(s)) ds + r(t) \quad (34)$$

in Hölder spaces. The equation (34) has been studied, even for Banach space valued functions, but by means of completely different methods, in [14 - 16].

5. Hammerstein-Volterra equations in Hölder spaces

So far we considered equation (1) in the case when both operators V and F act between Orlicz spaces. In this section we briefly show how to employ Theorem 1 and Theorem 2 in order to establish the existence of a solution of equation (1) in a (generalized) Hölder space.

We begin with Theorem 1. First of all, we recall that a Hölder space C^ϕ is *compactly imbedded* into a Hölder space $C^{\hat{\phi}}$ if

$$\phi(t) = o(\hat{\phi}(t)) \quad (t \rightarrow 0); \tag{35}$$

a typical example is of course $\phi(t) = t^\alpha$ and $\hat{\phi}(t) = t^\beta$ with $\alpha > \beta$. Now, Theorem 1 gives a sufficient condition for the operator (6) to be bounded from an Orlicz space L_Φ into a Hölder space C^ϕ . If we require the operator (3) to be bounded and continuous from some Hölder space $C^{\hat{\phi}}$ satisfying (35) into L_Φ , then the compactness of the operator $H = VF$ in the space C^ϕ will be simply a consequence of the compactness of the imbedding $C^\phi \subseteq C^{\hat{\phi}}$.

Conditions for the boundedness and continuity of F from $C^{\hat{\phi}}$ into L_Φ which are both necessary and sufficient are not known; nevertheless, one can give simple sufficient conditions. A very rough condition which ensures the boundedness of F from C (a fortiori, from $C^{\hat{\phi}}$) into L_Φ is that the function f_r , defined by

$$f_r(t) = \sup_{|u| \leq r} |f(t, u)| \tag{36}$$

belongs to L_Φ for any $r > 0$ ([20], see also [3]). Moreover, it is clear that

$$\sup \{ \|Fx\|_\Phi : \|x\|_{\hat{\phi}} \leq r \} \leq \|f_r\|_\Phi \tag{37}$$

in this case. Finally, if the Young function Φ satisfies a Δ_2 condition, F is also continuous from C into L_Φ . This simple observation allows us to obtain the following straightforward existence result in the Hölder space C^ϕ .

Theorem 7. *Suppose that the hypotheses of Theorem 1 are satisfied, where Φ satisfies a Δ_2 condition. Assume, moreover, that the function f_r , defined by (36) belongs to the Orlicz space L_Φ , and let $\hat{\phi}$ be any Hölder function satisfying (35). Finally, suppose that there exists $r > 0$ such that*

$$(1 + \phi(T)) \max \{2, \psi(T)\} \|h\|_\epsilon \|f_r\|_\Phi \leq r.$$

Then equation (1) has a solution $x \in C^\phi$ with $\|x\|_{\hat{\phi}} \leq r$.

Proof. By (10) and (37), the operator $H = VF$ leaves the ball $B_r(C^\phi) = \{x \in C^\phi : \|x\| \leq r\}$ invariant, and hence Schauder's fixed point principle applies ■

Of course, Theorem 2 may be applied in the same way. Since the operator (6) is, under the hypotheses of Theorem 2, even *compact* from L_Φ into C , it is not necessary to imbed C into a larger space, and thus the proof becomes even simpler. We summarize with the following

Theorem 8. *Suppose that the hypotheses of Theorem 2 are satisfied, where Φ satisfies a Δ_2 condition. Assume, moreover, that the function f_r , defined by (36) belongs to the Orlicz space L_Φ . Finally, suppose that there exists $r > 0$ such that*

$$\psi(T) \|h\|_C \|f_r\|_\Phi \leq r.$$

Then equation (1) has a solution $x \in C$ with $\|x\|_C \leq r$.

The following example shows that the operator F may be discontinuous from C into L_Φ if Φ does not satisfy a Δ_2 condition.

Example 6. Let $\Phi(u) = e^{|u|} - |u| - 1$ and $f(t, u) = \log(t + |u|)$. Since $f(t, \cdot)$ is continuous for $t > 0$, f is a Carathéodory function on $[0, 1] \times \mathbb{R}^N$. For $r > 0$ and $0 < t \leq 1$ we have

$$f_r(t) = \max \left\{ \log(t + r), \log \frac{1}{t} \right\} \leq \log(r + 1) + \log \frac{1}{t}.$$

Since

$$\int_0^1 \Phi \left(\frac{1}{2} \log \frac{1}{t} \right) dt = \int_0^1 \left(\frac{1}{\sqrt{t}} - \frac{1}{2} \log \frac{1}{t} - 1 \right) dt < \infty,$$

the function f_r belongs to $L_\Phi = L_\Phi([0, 1], \mathbb{R}^N)$.

Now let $(x_n)_n$ be a sequence in $C = C([0, 1], \mathbb{R}^N)$ with $|x_n(t)| \equiv \frac{1}{n}$. Since

$$\begin{aligned} \int_0^1 \Phi(|Fx_n(t) - F0(t)|) dt &= \int_0^1 \Phi \left(\log \frac{1 + nt}{nt} \right) dt \\ &= \int_0^1 \left(1 + \frac{1}{nt} - \log \frac{1 + nt}{nt} - 1 \right) dt = \infty, \end{aligned}$$

we have $\|Fx_n - F0\|_\Phi \geq 1$, and thus F is discontinuous at zero.

We point out that, if the function f is of the form

$$f(t, u) = g(t)h(u) \quad \text{with } g \in L_\Phi([0, T], \mathbb{R}^{N \times N}), h \in C([0, T], \mathbb{R}^N)$$

or

$$f(t, u) = g(u)h(t) \quad \text{with } g \in C([0, T], \mathbb{R}^{N \times N}), h \in L_\Phi([0, T], \mathbb{R}^N),$$

then F is automatically continuous between C and L_Φ . This shows that our Example 6 cannot be replaced by the (autonomous) function $f = f(u)$ from [9: Example (17.10)].

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