Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1993), 677 - 682

On the Nonlinear Tricomi Problem

A. SCHOLL

Let $G \subset \mathbb{R}^2$ be a domain, bounded for $y > 0$ by a curve Γ_0 joining two given points and for $y < 0$ by the characteristics Γ_1 and Γ_2 of the following equation under consideration. In this paper the nonlinear Tricomi problem

$$T[u] := yu_{xx} + u_{yy} = f(x, y, u) \quad \text{in } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1$$

is considered. With Schauder's fixed point theorem, the existence of generalized solutions and some regularity results are obtained for a great class of nonlinearities. Here, the right-hand side f is not supposed to satisfy some Lipschitz condition. Furthermore, for $f = u|u|^\rho$ $(\rho > -1/2)$, an existence theorem is proved.

Key words: *Tricomi problems*
AMS subject classification: 35M

1. Introduction

Consider the nonlinear Tricomi problem

$$T[u] := yu_{xx} + u_{yy} = f(x, y, u) \quad \text{in } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1, \tag{1}$$

where G is a domain in $\mathbb{R}^2$, bounded for $y > 0$ by a curve Γ_0 joining the points $(-1, 0)$ and $(0, 0)$ and for $y < 0$ by the characteristics

$$\Gamma_1 : \ x - \tfrac{2}{3}(-y)^{3/2} = -1 \qquad \text{and} \qquad \Gamma_2 : \ x + \tfrac{2}{3}(-y)^{3/2} = 0$$

of equation (1). The intersection point of these characteristics is denoted by $C(-\frac{1}{2}, y_c)$. Using known results for the linear problem two questions are examined in this paper:
- Existence of solutions of the nonlinear problem (1)
- Regularity results for solutions of the nonlinear problem (1).

In Section 2 some notations are introduced and known results for the linear problem are cited, as far as they are needed in the upcoming sections. These results can be obtained from the estimates (3) and (4) (see, e.g., [2, 4, 5]).

With the well-known Leray-Schauder linearization trick and the application of a fixed point theorem, existence results for problem (1) are proved in Section 3. The existence theorem obtained here includes the existence results pointed out in [3] for the nonlinear Tricomi problem. But in contrast to [3] no Lipschitz condition is imposed on the right-hand side f of (1), so that the class of nonlinearities regarded here is much wider.

A. Scholl: Universität Karlsruhe, Math. Institut I, Kaiserstr. 12, D - 76135 Karlsruhe

ISSN 0232-2064 / $ 2.50 © Heldermann Verlag Berlin

In Section 4 the regularity of these solutions is shown under some restrictions to the right-hand side f of (1). Up to now, very few is known about the regularity of solutions of the Tricomi problem. For the linear problem some first developments have recently been carried out in [8]. This result for the linear problem together with the linearization trick makes it possible to prove the regularity of the solutions of the nonlinear problem obtained in Section 3.

Unfortunately the existence results pointed out in Section 3 do not hold, if the right-hand side f of (1) is some power of u. This problem is therefore discussed in Section 5. There, a more general existence result than that given in [10] is proved.

2. The linear problem

Let U and V be the function spaces

$$\begin{aligned} U &= \left\{u \in C^\infty(\overline{G}) : u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1\right\} \\ V &= \left\{v \in C^\infty(\overline{G}) : v = 0 \quad \text{on } \Gamma_0 \cup \Gamma_2\right\}. \end{aligned}$$

Note that V determines the adjoint boundary conditions to problem (1) with respect to the scalar product in $L^2(G)$. The bilinear form $B : W_2^1(bd) \times W_2^1(bd^+) \to I\!R$ is given by

$$B[u,v] := (Tu, v)_{L^2(G)} = -\iint_G (yu_x v_x + u_y v_y)\, d(x,y)$$

with $W_2^1(bd) := \overline{U}^{\|\cdot\|_{1,2}}$ and $W_2^1(bd^+) := \overline{V}^{\|\cdot\|_{1,2}}$. Here, $\|\cdot\|_{1,2}$ denotes the norm in the Sobolev space $W_2^1(G)$.

Definition 1: Let $f \in L^2(G)$. A function $u \in W_2^1(bd)$ is called a *generalized solution* of the linear problem

$$yu_{xx} + u_{yy} = f(x,y) \quad \text{in } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1 \tag{2}$$

if $B[u,v] = (f,v)_{L^2(G)}$ for all $v \in W_2^1(bd^+)$.

Existence and uniqueness of solutions of the linear Tricomi problem have been obtained by many authors. The basic tools are some a priori estimates. The results cited below are essential for the following sections. Their proofs can be found in [2 - 4, 9].

Theorem 1: *Suppose Γ_0 is piecewise of class C^1,*

$$\alpha^1 n_1 + \alpha^2 n_2 \geq 0 \quad on\ \Gamma_0 \qquad and \qquad \tilde{\alpha}^1 n_1 + \tilde{\alpha}^2 n_2 \geq 0 \quad on\ \Gamma_0$$

with $\vec{n} = (n_1, n_2)$ being the outward normal vector and

$$\alpha^1 = -(-y_c)^{1/2} + 2x, \qquad \alpha^2 = \tilde{\alpha}^2 = 1 + y, \qquad \tilde{\alpha}^1 = 2(1+x) + (-y_c)^{1/2}.$$

Then there is a constant $c > 0$ such that the following inequalities hold:

$$\sup_{0 \neq u \in W_2^1(bd)} \frac{|B[u,v]|}{\|u\|_{1,2}} \geq c\|v\|_{L^2(G)} \qquad for\ all\ v \in W_2^1(bd^+) \tag{3}$$

$$\sup_{0 \neq v \in W_2^1(bd^+)} \frac{|B[u,v]|}{\|v\|_{1,2}} \geq c\|u\|_{L^2(G)} \qquad for\ all\ u \in W_2^1(bd). \tag{4}$$

Theorem 2: *Suppose, the conditions of Theorem 1 hold. Then for each right-hand side $f \in L^2(G)$ of (1) there is a unique generalized solution u_f of problem (2) and the inequality*

$$\|u_f\|_{1,2} \leq c^{-1}\|f\|_{L^2(G)} \tag{5}$$

holds with the constant c from Theorem 1.

Theorem 2 immediately leads to

Lemma 1: *Suppose, the conditions of Theorem 1 are satisfied. Then the operator $T^{-1}: L^2(G) \to L^2(G), T^{-1}f = u_f$ exists, is linear and compact.*

Proof: The existence of the operator T^{-1} follows from Theorem 2, and its compactness is a consequence of the compact imbedding of the space $W_2^1(G)$ into the space $L^2(G)$ (see [11, Theorem 7.9]) ∎

Remarks: 1. A further investigation of the operator T^{-1} is given in [9]. It is based on a priori estimates similar to (3) and (4). 2. Lemma 1 implies the equality $B[T^{-1}f, v] = (f, v)_{L^2(G)}$ for all $v \in W_2^1(bd^+)$.

With the help of the estimates (3), (4) and the existence and uniqueness of generalized solutions of the linear Tricomi problem the following regularity results are obtained in [8].

Theorem 3: *Suppose, the conditions of Theorem 1 are fulfilled. In addition, let Γ_0 be a C^{j+2}-curve satisfying the equations*

$$-\sigma^2(x+1) + y = 0 \quad (\sigma \neq 0) \qquad \text{and} \qquad \tau^2 x + y = 0 \quad (\tau \neq 0)$$

in some neighbourhoods of the points $(-1, 0)$ and $(0, 0)$, respectively. Suppose furthermore, there is an integer $j \geq 2$ such that for the right-hand side f of (1)

$$f \in W_2^{j-1}(G) \qquad \text{and} \qquad f = \frac{\partial f}{\partial n} = \ldots = \frac{\partial^{j-2} f}{\partial n^{j-2}} = 0 \quad \text{on } \Gamma_1,$$

with $\frac{\partial^k f}{\partial n^k}$ denoting the k-times derivative of f on Γ_1 in outer normal direction. Then the generalized solution u_f of problem (2) satisfies

$$u_f \in W_2^1(bd) \cap W_2^j(G) \cap W_2^{j+2}(G_+ \cup \Gamma_0) \quad \text{and} \quad \frac{\partial u_f}{\partial n} = \ldots = \frac{\partial^{j-1} u_f}{\partial n^{j-1}} = 0 \quad \text{on } \Gamma_1.$$

3. An existence theorem for the nonlinear Tricomi problem

Before proving the announced existence theorem, the concept of generalized solutions for the nonlinear problem will be introduced in an evident manner.

Definition 2: A function $u \in W_2^1(bd)$ is called a *generalized solution* of the nonlinear Tricomi problem

$$T[u] = yu_{xx} + u_{yy} = f(x,y,u) \quad \text{in } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1, \tag{6}$$

if

$$B[u,v] = -\iint_G (yu_x v_x + u_y v_y)\, d(x,y) = (g_u, v)_{L^2(G)} \quad \text{for all } v \in W_2^1(bd^+)$$

with $g_u(x,y) := f(x,y,u(x,y))$.

Theorem 4: *Suppose, the right-hand side f of (1) satisfies the Carathéodory condition (see [7, p.22]), the curve Γ_0 satisfies the conditions of Theorem 1, and for each $w \in L^2(G)$ the function g_w defined by $g_w(x,y) = f(x,y,w(x,y))$ belongs to $L^2(G)$. Then problem (6) has a generalized solution.*

Proof: According to Theorems 2.1 and 2.2 in [7], the operator $F : L^2(G) \to L^2(G), F[w] = f(\cdot,\cdot,w)$ is bounded and continuous. For fixed $w \in L^2(G)$ Theorem 2 guarantees a uniquely determined generalized solution $u_w \in W_2^1(bd) \subset L^2(G)$ of the problem

$$yu_{xx} + u_{yy} = f(x,y,w) \quad \text{in } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1.$$

Thus an operator $S : L^2(G) \longrightarrow L^2(G)$, $S[w] = u_w$ is defined. Clearly, $S = T^{-1}F$, with T^{-1} being the operator mentioned in Lemma 1. Therefore S is continuous and its range is a compact subset of $L^2(G)$ [1, Theorem 6.2]. Schauder's fixed point theorem (3.8(3) in [6]) states the existence of a fixed point $u_0 \in L^2(G)$ of S. That means $u_0 = Su_0 = T^{-1} \circ F[u_0]$. Thus $u_0 \in W_2^1(bd)$ and $B[u_0, v] = (f,v)_{L^2(G)}$ for all $v \in W_2^1(bd^+)$ ∎

Remark: In [3] the condition $|f(x,y,u)| \le (A(x,y) + \beta|u|)$ $(A \in L^2(G), \beta \in \mathbb{R})$ is imposed on the right-hand side f of (1). Such an estimate is a consequence of the conditions of Theorem 4 (see Theorem 2.3 in [7]). In contrast to the existence proof given in [3] the explicit knowledge of A and β is not needed here. In addition no Lipschitz condition is required in Theorem 4. So the results obtained here are more general than those pointed out in [3]. But the possibility of getting uniqueness results is lost, because the uniqueness result for the nonlinear problem in [3] is based on a comparison of the Lipschitz constant to the constant c in Theorem 1.

4. Regularity results for the nonlinear Tricomi problem

With the help of Theorem 3, the regularity of solutions of the nonlinear Tricomi problem can be shown.

Theorem 5: *Suppose, the function f satisfies the Carathéodory condition and the curve Γ_0 satisfies the conditions of Theorem 3. Assume, there is an integer $j \ge 2$ such that, for all $k \le j$ and*

$$w \in W_2^1(bd) \cap W_2^{k-1}(G) \quad \textit{with} \quad w = \frac{\partial w}{\partial n} = \ldots = \frac{\partial^{k-2} w}{\partial n^{k-2}} = 0 \quad \textit{on } \Gamma_1,$$

the conditions

$$g_w := f(\cdot,\cdot,w) \in W_2^{k-1}(G) \quad \text{and} \quad g_w = \frac{\partial g_w}{\partial n} = \ldots = \frac{\partial^{k-2} g_w}{\partial n^{k-2}} = 0 \quad \text{on } \Gamma_1$$

hold. Then each generalized solution u of problem (6) satisfies

$$u \in W_2^1(bd) \cap W_2^j(G) \cap W_2^{j+1}(G_+ \cup \Gamma_0), \qquad u = \frac{\partial u}{\partial n} = \ldots = \frac{\partial^{j-1} u}{\partial n^{j-1}} = 0 \quad \text{on } \Gamma_1.$$

Proof: According to Theorem 4, problem (6) has a generalized solution $u \in W_2^1(bd)$. That is $u = 0$ on Γ_1, so that u solves the problem $yu_{xx} + u_{yy} = g_u(x,y)$ on G, $g_u(x,y) = 0$ on Γ_1. Applying Theorem 3 yields

$$u \in W_2^1 \cap W_2^2(G) \cap W_2^3(G_+ \cup \Gamma_0) \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1.$$

Using Theorem 3 j-times completes the proof ∎

Remark: Theorems 4 and 5 can not be applied to the problem $yu_{xx} + u_{yy} = u|u|^\rho$ on G, $u = 0$ on $\Gamma_0 \cup \Gamma_1$, because the right-hand side $u|u|^\rho$ is in general not in $L^2(G)$ if $u \in L^2(G)$. This problem will be discussed in the next section. However the above theorems can be applied to the problems

$$\begin{aligned} yu_{xx} + u_{yy} &= \sin u \quad \text{on } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1 \\ yu_{xx} + u_{yy} &= \arctan u \quad \text{on } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1. \end{aligned}$$

5. The problem $T[u] = u|u|^\rho$

Let $w \in L^{2\rho+2}(G)$. Then $w|w|^\rho \in L^2(G)$ and the problem $T[u] = w|w|^\rho$ on G, $u = 0$ on $\Gamma_0 \cup \Gamma_1$ has a unique generalized solution $u_w \in W_2^1(bd)$. The operator

$$F : L^{2\rho+2}(G) \longrightarrow L^2(G), \quad F(w) = w|w|^\rho$$

is continuous and bounded (see [7, Theorems 2.1 and 2.2]) for $\rho \geq -1/2$. The range $R(T^{-1}F) \subset W_2^1(bd)$ is compact in $L^{2\rho+2}(G)$ for $\rho \geq -1/2$ according to the Rellich - Kondrachov theorem (Theorem 6.2 in [1]). The fixed point u obtained by Schauder's fixed point theorem (3.8(3) in [6]) is a generalized solution of the problem

$$T[u] := yu_{xx} + u_{yy} = u\,|u|^\rho \quad \text{in } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1$$

for $\rho \geq -1/2$. So the following theorem holds.

Theorem 6: *Suppose, the curve Γ_0 satisfies the conditions of Theorem 1 and $\rho \geq -1/2$. Then the problem*

$$T[u] := yu_{xx} + u_{yy} = u\,|u|^\rho \quad \text{in } G, \qquad u = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1$$

has a generalized solution $u \in W_2^1(bd)$.

Remarks: **1.** By arguing in the same way, Theorem 6 holds, if the right-hand side is replaced by the more complicated term $f(x,y) + g(x,y)u|u|^{2\rho+2}$ ($f \in L^2(G), g \in L^\infty(G), u \in L^{2\rho+2}(G), \rho \geq -1/2$), because this right-hand side is again in $L^2(G)$. **2.** The methods used in the last section to obtain regularity results can not be applied here, because $(u\,|u|^\rho)_x = u_x\,|u|^\rho\,(1+\rho)$ is not known to be in $L^2(G)$.

References

[1] Adams, R. A.: *Sobolev Spaces*. New York: Academic Press 1975.

[2] Aziz, A. K., Lemmert, R. and M. Schneider: *The existence of generalized solutions for a class of linear and nonlinear equations of mixed type*. Note Mat. **10**/Suppl. 1 (1990), 47 - 64.

[3] Aziz, A. K., Lemmert, R. and M. Schneider: *A finite element method for the nonlinear Tricomi problem*. Numer. Math. **58** (1990), 95 - 108.

[4] Aziz, A. K. and M. Schneider: *The existence of generalized solutions for a class of quasilinear equations of mixed type*. Math. Anal. Appl. **107** (1985), 442 - 445.

[5] Didenko, V. P.: *On the generalized solvability of the Tricomi problem*. Ukr. Math. J. **25** (1973), 10 - 18.

[6] Jeggle, H.: *Nichtlineare Funktionalanalysis*. Stuttgart: Teubner - Verlag 1979.

[7] Krasnoselskij, M. A.: *Topological methods in the theory of nonlinear integral equations*. Oxford: Pergamon Press 1964.

[8] Schneider, M.: *Regularitätsaussagen für Lösungen des Tricomiproblems*. Results in Math. **22** (1992), 749 - 760.

[9] Scholl, A.: *Folgerungen aus a priori Abschätzungen für Differentialgleichungen vom gemischten Typ*. Dissertation. Karlsruhe: Univ. Karlsruhe 1992.

[10] Sun Hesheng : *Tricomi problem for a nonlinear equation of mixed type*. Sci. in China **35**/Series A (1992), 1 - 10.

[11] Wloka, J.: *Partielle Differentialgleichungen*. Stuttgart: Teubner - Verlag 1982.

Received 08.12.1992