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Quadrature and Collocation Methods for the Double Layer Potential on Polygons

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This paper is concerned with approximation methods for Neumann's integral equation on curves with corners. Necessary and sufficient conditions for the stability of the piecewise constant ε - collocation and for the quadrature method, using the rectangular rule, are given.

Key words: *Neumann integral equation, polygonal domains, quadrature and collocation methods*
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0. Introduction

We consider for $f \in L^2(\Gamma) \cap \mathbf{R}(\Gamma)$ the second kind integral equation

$$A_\Gamma u := (I - K)u = f \text{ on } \Gamma \tag{1}$$

where

$$Ku(x) = -\frac{1}{\pi} \int_\Gamma u(y) \frac{\partial}{\partial n_y} \log |x - y| ds_y \tag{2}$$

and $\partial / \partial n$ denotes the normal derivative with respect to the outer normal n which exists except at the corners of the polygon Γ, consisting of straight line segments Γ^j. The double layer potential (2) can be rewritten as

$$Ku(x) = -\frac{1}{\pi} \int_\Gamma u(y) d\theta_x(y) \tag{3}$$

where $\theta_x(y)$ denotes the angle between $y - x$ and some fixed direction.

Many boundary value problems in physics and engineering can be reduced to the equation (1) where u is the unknown solution. For the numerical solution of (1) spline approximation methods are widely used, especially collocation and quadrature schemes. For Γ being a smooth closed curve a fairly complete error analysis of collocation methods for (1) using smooth splines has been established (see [1, 5, 10, 11]). For Γ being a polygon, convergence of point collocation for (1) with piecewise linear trial functions is shown in [4] by rewriting the collocation scheme as a Petrov-Galerkin scheme with delta-distributions in the break points as test functions.

In the following we prove convergence for the collocation method of (1) with piecewise constant trial functions by first analysing a quadrature scheme. Our analysis follows closely and uses heavily the analysis by Prössdorf and Rathsfeld [9] which prove convergence of collocation and quadrature schemes for singular integral equations with Cauchy kernel on closed, piecewise smooth curves.

For the collocation method with piecewise constants on the grid $\Delta_n = \{y_1, \ldots, y_n\}$ we need a finite set of collocation points $\{\tau_k^{(n)}, k = 0, \ldots, n-1\} \subset \Gamma$ where $\tau_k^{(n)} \notin \Delta_n$

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and approximate the exact solution u of (1) by the piecewise constant functions u_n on Δ_n satisfying

$$(I-K)u_n(\tau_k^{(n)}) = f(\tau_k^{(n)}),\ k=0,\dots,n-1. \tag{4}$$

In order to solve this system one has to compute $(K\,u_n)(\tau_k^{(n)})$ which in the case of a curved polygon Γ has to be done by the use of quadrature rules. This leads to quadrature schemes which are another numerical method to solve (1) approximately.

Our quadrature and collocation methods both replace the equation $A_\Gamma u = f$ by a discrete operator equation $A_n u_n = f_n$ where A_n is an approximate operator of A acting in the space X_n of piecewise constant functions on a quasiuniform mesh and $f_n \in X_n$ is an interpolation of f. Such a numerical method is called *stable* if A_n is invertible for n sufficiently large and $sup\|A_n^{-1}\| < \infty$. If the method is stable, f Riemann integrable, and A_n converges strongly to A, then the approximate solutions u_n converge to u (see [8]). Thus, the crucial point is the proof of the stability of the scheme. This is done by showing stability of a corresponding model problem on an angle, making use of a localization principle by Gohberg and Krupnik. Following Prössdorf and Rathsfeld we apply Mellin techniques from Costabel and Stephan [3] to handle the model problem.

We now introduce some notation used below :

Π — unit circle $\{z \in \mathbb{C} : |z| = 1\}$
$\mathbf{R}(\Gamma)$ — class of bounded Riemann integrable functions on Γ
$PC(\Gamma)$ — class of piecewise continuous functions on Γ
l^2 — Hilbert space of sequences $\{\xi_n\}_{n=0}^{\infty}$, $\xi_n \in \mathbb{C}$
$\tilde{l}^2$ — Hilbert space of sequences $\{\xi_n\}_{n=-\infty}^{\infty}$, $\xi_n \in \mathbb{C}$
X — an abstract Banach space
X_n — linear space of column vectors of length n with entries from X
$X_{n\times n}$ — linear space of $n \times n$ matrices with entries from X
$T(a)$ — Toeplitz operator generated by $a \in PC(\Pi)$
$\mathcal{L}(X)$ — Banach space of continuous linear operators on X.

1. Quadrature methods on an angle

We are interested in quadrature methods for approximating the solution of (1) on polygons. We shall give local conditions which are necessary and sufficient for the stability of the methods. For simplicity we consider only the case of Γ being an infinite angle $\Gamma_\omega = \overline{\mathbf{R}^+} \cup e^{i\omega}\overline{\mathbf{R}^+}$ with opening $0 < \omega < 2\pi$. The general case of a polygon Γ follows then by localization arguments. We fix $n \in \mathbb{N}, 0 < \epsilon, \delta < 1$ and for $k \in \mathbb{Z}$ choose the quadrature points $t_k^{(n)}$ as follows. Following Prössdorf and Rathsfeld [9] we introduce

$$t_k^{(n)} = \begin{cases} \frac{k+\delta}{n} & \text{for } k \geq 0 \\ -\frac{k+\delta}{n}e^{i\omega} & \text{for } k < 0 \end{cases} \quad \text{and} \quad \tau_k^{(n)} = \begin{cases} \frac{k+\epsilon}{n} & \text{for } k \geq 0 \\ -\frac{k+\epsilon}{n}e^{i\omega} & \text{for } k < 0 \end{cases} \tag{5}$$

Then using the *rectangular rule* as the *quadrature formula* we obtain for a discretization of (1) the system ($k \in \mathbb{Z}$)

$$\xi_k^{(n)} + \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{j=0}^{\infty} \frac{\xi_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}}\frac{1}{n} + \sum_{j=-\infty}^{-1} \frac{\xi_j^{(n)}}{t_j^{(n)} - \tau_k^{(n)}}\frac{-e^{i\omega}}{n}\right\} = f(\tau_k^{(n)}). \tag{6}$$

If there exists a solution $(\xi_k^{(n)})_{k\in\mathbb{Z}}$, $\xi_k^{(n)} \in \mathbb{R}$, then we obtain an approximation u_n for the solution $u \in L^2(\Gamma_\omega)$ of (1) $Au = f, f \in \mathbf{R}(\Gamma_\omega) \cap L^2(\Gamma_\omega)$, by setting

$$u_n = \sum_{k\in\mathbb{Z}} \xi_k^{(n)} \chi_k^{(n)} \tag{7}$$

where

$$\begin{aligned} \chi_k^{(n)}(t) &= \begin{cases} 1, & \text{if } \frac{k}{n} \le t \le \frac{k+1}{n} \\ 0, & \text{elsewhere} \end{cases} && (k = 0, 1, 2, \ldots) \\ \chi_k^{(n)}(t) &= \begin{cases} 1, & \text{if } \frac{k}{n} \le -e^{-i\omega}t \le \frac{k+1}{n} \\ 0, & \text{elsewhere} \end{cases} && (k = -1, -2, \ldots) \end{aligned} \tag{8}$$

Let A_n denote the matrix of the system (6), $A_n\xi_k^{(n)} = f(\tau_k^{(n)})$. We define the interpolation projection T_n by

$$T_n y = \sum_{k\in\mathbb{Z}} y(\tau_k^{(n)})\chi_k^{(n)} \quad (y \in \mathbf{R}(\Gamma)) \tag{9}$$

and denote the orthogonal projection onto $imT_n \cap L^2(\Gamma_\omega)$ by L_n. In the following we identify the continuous linear operators on imL_n with their matrices corresponding to the base $\{\chi_k^{(n)},\ k \in \mathbb{Z}\}$. Due to

$$\left\| \sum_{k\in\mathbb{Z}} \xi_k \chi_k^{(n)} \right\|_{L^2(\Gamma_\omega)} = n^{-1/2} \|\{\xi_k\}_{k\in\mathbb{Z}}\|_{\tilde{l}^2} \tag{10}$$

these matrices are considered to be operators in $\tilde{l}^2$. In particular, since the matrix $A_n \in \mathscr{L}(\tilde{l}^2)$ is independent of n, the sequence $\{A_n\}$ $(A_n \in \mathscr{L}(imL_n))$ is stable if and only if A_1 is invertible.

Theorem 1 : *The operator $A_1 \in \mathscr{L}(\tilde{l}^2)$ is invertible for all $0 < \omega < 2\pi$.*

To prove this we need some results on Toeplitz operators which are due to Gohberg and Krupnik [6, 7]. Let $\mathfrak{A} \subset \mathscr{L}(l^2)$ denote the smallest algebra containing all Toeplitz operators $T(a)$ with $a \in PC(\Pi)$. Then $\mathfrak{A}_{n\times n} \subset \mathscr{L}(l^2)_{n\times n}$ is an algebra of continuous operators in l^2_n. There exists a multiplicative linear mapping $\mathfrak{A}_{n\times n} \ni B \to \mathscr{A}_B$ into the algebra of bounded $n \times n$ - matrix functions over $\Pi \times [0,1]$. The symbol $\mathscr{A}_B$ of $B = (B_{k,j})_{k,j=1}^n$, $B_{k,j} \in \mathfrak{A}$, is equal to $(\mathscr{A}_{B_{k,j}})_{k,j=1}^n$ and the symbol $\mathscr{A}_{T(a)}$ with $a \in PC(\Pi)$ is given by $\mathscr{A}_{T(a)}(\tau,\mu) = \mu a(\tau+0) + (1-\mu)a(\tau-0)$, where $(\tau,\mu) \in \Pi \times [0,1]$. Furthermore, $B \in \mathfrak{A}_{n\times n}$ is a Fredholm operator if and only if $\det \mathscr{A}_B(\tau,\mu) \ne 0$ for all $\tau \in \Pi$ and $0 \le \mu \le 1$.

By virtue of $l^2 \oplus l^2 = \tilde{l}^2$ we can identify $\mathscr{L}(\tilde{l}^2)$ with $\mathscr{L}(l^2)_{2\times 2}$. In order to prove the assertion of Theorem 1 we show $A_1 \in \mathfrak{A}_{2\times 2}$ and $index\, \mathscr{A}_{A_1} = 0$. First we need the following result by Ratsfeld [10].

Lemma 2 : *Let $z \in \mathbb{C}$, $-1/2 < \mathrm{Re}\, z < 1/2$, $\Lambda^z := ((k+1)^z\delta_{k,j})_{k,j=0}^\infty$ and $a \in PC(\Pi)$. Suppose that there exists $\omega_j \in (0, 2\pi), \omega_0 = 0, \omega_{k+1} = 2\pi$, such that the restriction of a to $\{e^{ix}, \omega_j \le x \le \omega_{j+1}\}$ $(j = 1, \ldots, k)$ is twice differentiable. Then the following assertions hold.*

(i) *The matrix $\Lambda^{-z}T(a)\Lambda^z$ belongs to $\mathfrak{A}$ and $(j = 0, \ldots, k)$*

$$\mathscr{A}_{\Lambda^{-z}T(a)\Lambda^z}(\tau,\mu) = \begin{cases} a(\tau) & \text{if } \tau \ne e^{i\omega_j} \\ \frac{\mu a(\tau+0)+(1-\mu)a(\tau-0)e^{-i2\pi z}}{\mu+(1-\mu)e^{-i2\pi z}} & \text{if } \tau = e^{i\omega_j} \end{cases} \tag{11}$$

(ii) *The function* $z \mapsto \Lambda^{-z}T(a)\Lambda^{z}$ *is continuous on* $\{z, -1/2 < \operatorname{Re} z < 1/2\}$.

Now we are in the position to prove Theorem 1.

Proof of Theorem 1: Firstly, the expression (3) shows that the equation (1) on Γ_ω takes the form

$$Au = \begin{pmatrix} I & -K_{12} \\ -K_{21} & I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{12}$$

where

$$K_{12}u(x) = K_{21}u(x) = \frac{1}{\pi}\int_0^\infty \operatorname{Im} \frac{1}{xe^{i\omega} - y} u(y)dy \tag{13}$$

and $u_1 = u|_{\mathbb{R}^+}$, $u_2 = u|_{e^{i\omega}\mathbb{R}^+}$. Fix mesh width $n = 1$. Then $A_1 \in \mathscr{L}(l^2)_{2\times 2}$ takes the form

$$A_1 = \begin{pmatrix} K_{1,1} & K_{1,2} \\ K_{2,1} & K_{2,2} \end{pmatrix} \tag{14}$$

where $K_{1,1} = K_{2,2} = I$ and

$$K_{2,1} = -\frac{1}{2\pi i}\left(\frac{1}{(j+\delta) + (-k-1+\epsilon)e^{i\omega}} - \frac{1}{(j+\delta)+(-k-1+\epsilon)e^{-i\omega}}\right)_{k,j=0}^{\infty}$$

$$K_{1,2} = -\frac{1}{2\pi i}\left(\frac{1}{(-j-1+\delta) + (k+\epsilon)e^{-i\omega}} - \frac{1}{(-j-1+\delta)+(k+\epsilon)e^{i\omega}}\right)_{k,j=0}^{\infty}.$$

In the following we show that $A_1 \in \mathfrak{A}_{2\times 2}$ is a Fredholm operator by computing its symbol making use of the Mellin transformation and Lemma 2.

For $-1 < \nu < 1$, $\nu \neq 0$, we set

$$f^\nu(e^{i2\pi x}) = \frac{ie^{i\pi\nu(1-2x)}}{\sin \pi\nu},\ 0 \le x < 1. \tag{15}$$

Then computation shows $f^\nu = \sum_{k\in\mathbb{Z}} f_k^\nu t^k$ where $f_k^\nu = -1/i\pi(k+\nu)$. Now let us prove $K_{2,1} \in \mathfrak{A}$. The residue theorem together with the formula [3]

$$\frac{1}{i\pi}\frac{1}{1-xe^{\pm i\omega}} = \frac{1}{2\pi i}\int_{\operatorname{Re} z=1/2} x^{-z}\left\{\frac{-ie^{\mp i(\omega-\pi)z}}{\sin \pi z}\right\}dz$$

gives

$$\frac{1-x}{1-xe^{\pm i\omega}} = \frac{1-e^{\mp i\omega}}{2}\int_{\operatorname{Re} z=1/4} x^{-z}\left\{\frac{-ie^{\mp i(\omega-\pi)z}}{\sin \pi z}\right\}dz + e^{\mp i\omega}. \tag{16}$$

Rewriting $K_{2,1}$ as

$$K_{2,1} = -\frac{1}{2\pi i}\left(\frac{1-\frac{k+1-\epsilon}{j+\delta}}{1-\frac{k+1-\epsilon}{j+\delta}e^{i\omega}}\frac{1}{(j+\delta)-(k+1-\epsilon)} - \frac{1-\frac{k+1-\epsilon}{j+\delta}}{1-\frac{k+1-\epsilon}{j+\delta}e^{-i\omega}}\frac{1}{(j+\delta)-(k+1-\epsilon)}\right)_{k,j=0}^{\infty} \tag{17}$$

and setting $x = \frac{k+1-\epsilon}{j+\delta}$ in (16) we have $K_{2,1} = K_{2,1}^{+} + K_{2,1}^{-}$ with

$$\begin{aligned} K_{2,1}^{+} &= -\frac{1}{2\pi i}\left(\frac{1-e^{-i\omega}}{2}\int_{\mathrm{Re}\,z=1/4}\frac{-ie^{-i(\omega-\pi)z}}{\sin\pi z}\frac{\left(\frac{k+1-\epsilon}{j+\delta}\right)^{-z}}{(j-k)+(\epsilon+\delta-1)}dz\right)_{k,j=0}^{\infty} \\ &\quad -\frac{1}{2\pi i}e^{-i\omega}\left(\frac{1}{(j-k)+(\epsilon+\delta-1)}\right)_{k,j=0}^{\infty} \end{aligned} \tag{18}$$

$$\begin{aligned} K_{2,1}^{-} &= -\frac{1}{2\pi i}\left(\frac{1-e^{i\omega}}{2}\int_{\mathrm{Re}\,z=1/4}\frac{-ie^{+i(\omega-\pi)z}}{\sin\pi z}\frac{\left(\frac{k+1-\epsilon}{j+\delta}\right)^{-z}}{(j-k)+(\epsilon+\delta-1)}dz\right)_{k,j=0}^{\infty} \\ &\quad -\frac{1}{2\pi i}e^{i\omega}\left(\frac{1}{(j-k)+(\epsilon+\delta-1)}\right)_{k,j=0}^{\infty}. \end{aligned} \tag{19}$$

Thus following Prössdorf and Rathsfeld [9, p.204] we obtain $K_{2,1}^{\pm} \in \mathfrak{A}$ by Lemma 2 and

$$\mathscr{A}_{K_{2,1}^{(\pm)}} = -\frac{1}{2}\frac{1-e^{\mp i\omega}}{2}\int_{\mathrm{Re}\,z=1/4}\frac{-ie^{\mp i(\omega-\pi)z}}{\sin\pi z}\mathscr{A}^{z}dz - \frac{1}{2}e^{\mp i\omega}\mathscr{A}^{0} \tag{20}$$

where $\mathscr{A}^{z} = \mathscr{A}_{\Lambda^{-z}T(f^{(1-\epsilon-\delta)})\Lambda^{z}}$. Analytically extending $z \to \mathscr{A}^{z}$ to a 1-periodic function, we have

$$\begin{aligned} \mathscr{A}_{K_{2,1}^{(\pm)}} &= -\frac{1}{2}e^{\mp i\omega}\mathscr{A}^{1} \\ &\quad -\frac{1}{4}\left[\int_{\mathrm{Re}\,z=1/4}\frac{-ie^{\mp i(\omega-\pi)z}}{\sin\pi z}\mathscr{A}^{z}dz - \int_{\mathrm{Re}\,z=5/4}\frac{-ie^{\mp i(\omega-\pi)z}}{\sin\pi z}\mathscr{A}^{z}dz\right]. \end{aligned} \tag{21}$$

In the strip $\{z : 1/4 < \mathrm{Re}\,z < 5/4\}$, the function $z \to \mathscr{A}^{z}(\tau,\mu)$ is constant if $\tau \neq 1$ and has a pole at $z_0 = \frac{1}{2} + \frac{i}{2\pi}\log(\frac{\mu}{1-\mu})$ at $\tau = 1$. With Lemma 2, this can be seen as follows: Using

$$\mathscr{A}_{\Lambda^{-z}T(a)\Lambda^{z}}(\tau,\mu) = \begin{cases} a(\tau) & \text{if } \tau \neq 1 \\ \frac{\mu a(\tau+0)+(1-\mu)a(\tau-0)e^{-i2\pi z}}{\mu+(1-\mu)e^{-i2\mu z}} & \text{if } \tau = 1 \end{cases} \tag{22}$$

with $a(\tau) = f^{1-\epsilon-\delta}(\tau)$, $\tau = e^{i2\pi x}$ and

$$a(1+0) = \frac{ie^{i\pi\nu}}{\sin\pi\nu} \text{ and } a(1-0) = \frac{ie^{-i\pi\nu}}{\sin\pi\nu},\ \nu = 1-\epsilon-\delta$$

we obtain

$$\frac{\mu a(\tau+0)+(1-\mu)a(\tau-0)e^{-i2\pi z}}{\mu+(1-\mu)e^{-i2\pi z}} = \frac{i\sin(\pi(\nu+1/2)+\pi z-\frac{i}{2}\log\frac{\mu}{1-\mu})}{\sin\pi\nu\,\cos(\pi z-\frac{i}{2}\log\frac{\mu}{1-\mu})}. \tag{23}$$

On the other hand $a(\tau) = f^{1-\epsilon-\delta}(\tau)$ is continuous for $\tau \neq 1$. Consequently, the residue theorem yields $(0 \leq \mu \leq 1)$

$$\begin{aligned} \mathscr{A}_{K_{2,1}^{(\pm)}}(1,\mu) &= -\frac{1}{2}e^{\mp i\omega}\mathscr{A}^{1}(1,\mu) \\ &\quad -2\pi i\left(\frac{ie^{\mp i(\omega-\pi)}}{4\pi}\right)\mathscr{A}^{1}(1,\mu) - 2\pi i\left(\frac{ie^{\mp i(\omega-\pi)z_0}}{4\sin\pi z_0}\frac{i\sin\pi(\nu+1)}{-\pi\sin\pi\nu}\right) \end{aligned} \tag{24}$$

$$\mathscr{A}^{(\overset{+}{-})}_{K_{2,1}}(\tau,\mu) = -\frac{1}{2}e^{\overset{-}{+}i\omega}\mathscr{A}^1(\tau,\mu) - 2\pi i\left(\frac{ie^{\overset{-}{+}i(\omega-\pi)}}{4\pi}\right)\mathscr{A}^1(\tau,\mu) = 0,\ \tau \neq 1 \tag{25}$$

$$\mathscr{A}_{K_{2,1}}(1,\mu) = \mathscr{A}^{(+)}_{K_{2,1}}(1,\mu) - \mathscr{A}^{(-)}_{K_{2,1}}(1,\mu) = \frac{\sin(\omega-\pi)z_0}{\sin \pi z_0} \text{ if } \tau = 1. \tag{26}$$

Similarly we can show $K_{1,2} \in \mathfrak{A}$ and compute its symbol $\mathscr{A}_{K_{1,2}}$. Finally we have altogether

$$\mathscr{A}_{A_1}(\tau,\mu) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \tau \neq 1,\ 0 \leq \mu \leq 1 \\ \begin{pmatrix} 1 & -\frac{\sin(\omega-\pi)z_0}{\sin \pi z_0} \\ -\frac{\sin(\omega-\pi)z_0}{\sin \pi z_0} & 1 \end{pmatrix} & \text{if } \tau = 1,\ 0 \leq \mu \leq 1. \end{cases} \tag{27}$$

Now we observe for all $0 < \omega < 2\pi$ with $z_0 = \frac{1}{2} + \frac{i}{2\pi}\log(\frac{\mu}{1-\mu})\,, 0 \leq \mu \leq 1\,,$

$$\det \mathscr{A}_{A_1}(1,\mu) = 1 - \left(\frac{\sin(\omega-\pi)z_0}{\sin \pi z_0}\right)^2 \neq 0, \tag{28}$$

$$\det \mathscr{A}_{A_1}(\tau,\mu) = 1,\ \tau \neq 1\ . \tag{29}$$

Hence $\mathscr{A}_{A_1}$ is a Fredholm operator of index zero for all ω with $0 < \omega < 2\pi$. But for $\omega = \pi$ we have $A_1 = I$, hence A_1 is invertible. Therefore A_1 is invertible on $\tilde{l}^2$ for all ω with $0 < \omega < 2\pi$ ∎

Now, let Γ be a polygon having a parameter representation γ which is twice continuously differentiable outside the vertices. Let us assume that the vertices are grid points and that grid points and collocation points are chosen such that a quadrature scheme corresponding to (6) is given on Γ. Before presenting the stability result for this scheme we introduce some notation. For $\tau \in \Gamma$, let us define $\omega_\tau \in (0, 2\pi)$ by

$$\omega_\tau = arg\left(-\frac{\gamma'(\tau-0)}{\gamma'(\tau+0)}\right)$$

and set

$$A^\tau = I + K_{\Gamma_{\omega_\tau}}\,.$$

The model problem for the quadrature problem on Γ is the method (6) applied to the operator $A^\tau \in \mathscr{L}(L^2(\Gamma_{\omega_\tau}))$. The matrix of the corresponding system of equations we denote by A_1^τ. In the proof of Theorem 1 we have shown that $A_1^\tau \in \mathfrak{A}_{2\times 2}$ is invertible on $\tilde{l}^2$.

Applying a local principle for spline approximation methods given in Prõssdorf and Rathsfeld [9] we obtain Corollary 3 as a consequence of Theorem 1. Note that the proof follows from the analysis given in the proof of Theorem 1.2 in [9].

Corollary 3 : *We have the following assertions.*
(a) The method (6) is stable if and only if the operators $A_\Gamma = (I + K) \in \mathscr{L}(L^2(\Gamma))$ *and* $A_1^\tau \in \mathscr{L}(\tilde{l}^2)$ *for all* $\tau \in \Gamma$, *and are invertible.*

(b) If the quadrature method is stable and if $f \in \mathbf{R}(\Gamma)$, *then (6) is uniquely solvable for* n *large enough and the approximate solutions* u_n *converge to* $u = (I+K)^{-1}f$ *as* $n \to \infty$.

Since $I + K : L^2(\Gamma) \to L^2(\Gamma)$ is continuous and bijective, and the invertibility of $A_1^\tau \in \mathscr{L}(\tilde{l}^2)$ for all $\tau \in \Gamma$ follows from Theorem 1, we have

Theorem 4 : *The quadrature method (6) is stable and has a unique solution* u_n *for* n *sufficiently large, and* $u_n \to u$ *as* $n \to \infty$.

2. Collocation methods on an angle

For simplicity we consider only the collocation method for (1) with piecewise constant trial functions. We establish the stability of the model problem on $\Gamma = \Gamma_\omega$. With this result one can prove convergence of the collocation method for (1) on general curves with corners following the arguments in Prössdorf and Rathsfeld [9]. We omit the corresponding details.

For the ϵ-collocation method $(0 < \epsilon < 1)$, we look for an approximate solution $u_n = \sum_{k\in\mathbf{Z}} \xi_k^{(n)} \chi_k^{(n)} \in imL_n \subset L^2(\Gamma_\omega)$ satisfying the equation

$$(Au_n)(\tau_k^{(n)}) = f(\tau_k^{(n)}), \; k \in \mathbf{Z}. \tag{30}$$

The latter system we rewrite as $A_n u_n = T_n f$ where $A_n := T_n A | imL_n \in \mathcal{L}(imL_n)$. Again $A_n \in \mathcal{L}(\tilde{l}^2)$ and A_n does not depend on n. Hence the sequence $\{A_n\}$ with $A_n \in \mathcal{L}(imL_n)$ is stable if and only if $A_1 \in \mathcal{L}(\tilde{l}^2)$ is invertible.

Theorem 5 : *The operator $A_1 \in \mathcal{L}(\tilde{l}^2)$ is invertible in $\tilde{l}^2$ for any $0 < \omega < 2\pi$.*

Proof: Firstly, we show $A_1 \in \mathfrak{A}_{2\times 2}$ and $\det \mathscr{A}_{A_1}$ is independent of ω. For the sake of brevity we consider only $\epsilon = 1/2$. Then one of the typical terms in the collocation schemes is given by (for $n = 1, j \geq 0$)

$$\mathrm{Im} \int_0^\infty \frac{\chi_j^{(1)}(\tau)}{\tau - \tau_k^{(1)}} d\tau = \mathrm{Im} \int_0^{1/2} \left\{ \frac{1}{t_j^\delta - \tau_k^{(1)}} + \frac{1}{t_j^{1-\delta} - \tau_k^{(1)}} \right\} d\delta \tag{31}$$

where for $0 < \delta < 1$

$$t_j^\delta = \begin{cases} j + \delta & \text{if } j \geq 0 \\ -(j+\delta)e^{i\omega} & \text{if } j < 0. \end{cases} \tag{32}$$

Let

$$M_{2,1}^\delta = \left(\frac{1}{t_j^\delta - \tau_k^{(1)}} \right)_{j\in\mathbf{Z}^+, k\in\mathbf{Z}^-} \tag{33}$$

and consider the operator-valued function

$$\delta \to K_{2,1}(\delta) = \frac{-1}{2\pi i} \left\{ (M_{2,1}^\delta + M_{2,1}^{1-\delta}) - \overline{(M_{2,1}^\delta + M_{2,1}^{1-\delta})} \right\} \tag{34}$$

defined on [0,1/2]. The proof of Theorem 1 shows $A_1(\delta) \in \mathfrak{A}_{2\times 2} \subset \mathcal{L}(\tilde{l}^2)$ where

$$A_1(\delta) = \begin{pmatrix} I & K_{2,1}(\delta) \\ K_{1,2}(\delta) & I \end{pmatrix}. \tag{35}$$

Moreover as in [9] one verifies the continuity of the function $\delta \to A(\delta)$. Furthermore (31), (34) show for A_1 given by (30)

$$A_1 = \int_0^{1/2} A_1(\delta)\, d\delta \in \mathfrak{A}_{2\times 2}, \quad \mathscr{A}_{A_1} = \int_0^{1/2} \mathscr{A}_{A_1(\delta)}\, d\delta$$

where $\mathscr{A}_{A_1(\delta)}$ is given by (27). Hence the invertibility of A_1 follows in the same way as at the end of the proof of Theorem 1 ∎

	Quadrature Method				Collocation Method			
h	$\delta = .50$ $\epsilon = .25$	α_N	$\delta = .25$ $\epsilon = 0.50$	α_N	$\epsilon = .25$	α_N	$\epsilon = 0.50$	α_N
3/2	.4598E-0		.4629E-0		.2087E-0		.1040E-0	
		0.58		0.33		1.01		1.00
3/4	.3078E-0		.3682E-0		.1035E-0		.5203E-1	
		0.55		0.37		1.01		1.00
1/2	.2465E-0		.3174E-0		.6869E-1		.3471E-1	
		0.53		0.38		1.01		1.00
3/8	.2114E-0		.2843E-0		.5135E-1		.2604E-1	

Table 1: Relative L^2 error and experimental convergence rate for Example 1

3. Numerical Results

Below we present two examples which illustrate the quadrature and collocation methods discussed above. In both examples we seek the solution of the Laplace's equation in the exterior domain $\mathbb{R}^2 \setminus \overline{\Omega}$. Explicitly, consider the following *Neumann problem: For* $g \in H^{-1/2}(\Gamma)$ *find* $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Omega)$ *satisfying*

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma = \partial\Omega, \tag{36}$$

and

$$u(x) \sim A \log|x| + O(1/|x|) \text{ as } |x| \to \infty$$

where $\frac{\partial u}{\partial n}$ denotes the normal derivative of u on Γ. The function u satisfying (36) can be represented as the solution of the second kind integral equation

$$(I - K)u = f \text{ on } \Gamma \tag{37}$$

where

$$f(x) = -Vg(x) \quad \text{and } Vg(x) = -\frac{1}{\pi}\int_\Gamma g(y)\ln|x-y|ds_y.$$

Example 1 : Γ is the triangle with vertices (0,0), (3,0), (0,4). Here we take for the true solution $u(x) = \mathrm{Re}(\log(x - (0.5, 0.5)))$.

Example 2 : Γ is taken to be the square with vertices (-1,-1), (1,-1), (1,1), (-1,1). For the true solution we use $u(x) = \mathrm{Re}(\sqrt{x^2-1} - x)$.

Given in Table 1 (Table 2) is the relative L^2 error for Example 1 (Example 2) for the quadrature methods with parameters $\epsilon = 0.25, \delta = 0.5$, and $\epsilon = 0.5, \delta = 0.25$, as well as for the collocation method with parameters $\epsilon = 0.25$, and $\epsilon = 0.5$. Also given are the experimental convergence rates α_N.

In Example 1 the solution u is analytic on Γ, whereas in Example 2 it has singularities at the points (-1,0) and (1,0).

The numerical results given in Tables 1, 2 indicate convergence of the quadrature and collocation method as was proven in Sections 1 and 2. Chandler and Graham give in [2] that the optimal order of convergence in the uniform norm for the 'Nyström' interpolant of the collocation method for (37) is 0.5, when piecewise constant trial functions on a uniform grid are used.Theoretical estimates for the asymptotic order of convergence of the L^2 error are not proven.

	Quadrature Method				Collocation Method			
	$\delta = .50$		$\delta = .25$					
h	$\epsilon = .25$	α_N	$\epsilon = 0.50$	α_N	$\epsilon = .25$	α_N	$\epsilon = 0.50$	α_N
1/2	.2908E-0		.3037E-0		.2878E-0		.2378E-0	
		0.77		0.69		0.79		0.82
1/4	.1708E-0		.1884E-0		.1667E-0		.1346E-1	
		0.81		0.66		0.84		0.86
1/8	.9760E-1		.1189E-0		.9323E-1		.7425E-1	
		0.82		0.63		0.86		0.88
1/12	.7007E-1		.9200E-1		.6567E-1		.5202E-1	

Table 2: Relative L^2 error and experimental convergence rate for Example 2

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