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A Nonlinear Neumann-Type Problem of a System of High Order Hyperbolic Integro-Differential Equations

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The paper concerns a nonlinear Neumann-type boundary value problem for a system of hyperbolic integro-differential equations of order $2p$ with two independent variables. The problem is reduced to a system of integro functional equations and hence the existence and uniqueness of a local solution is proved by using the Banach fixed point theorem.

Key words: *Neumann problem, integro-differential equations, hyperbolic equations and systems, nonlinear boundary value problems*

AMS subject classification: 35L35, 35L75, 35G30

0. Introduction

Neumann and mixed boundary value problems for second order hyperbolic equations and systems have been dealt with in many papers (cp. [6, 11, 16 - 19, 22] and the references therein). Papers devoted to higher order equations were not so numerous and, except paper [9] where the right-hand side of the equation may depend on the unknown function but not on its derivatives, concerned only linear problems (cp. [1, 2, 5, 7 - 10, 12 - 15, 21]). In most of these papers the domain considered is a half-space.

In this paper we examine a nonlinear Neumann-type problem for a system of hyperbolic integro-differential equations of order $2p$ (where p is any positive integer) with two independent variables. The method of treating the problem is different from those in the quoted papers and similar to that in our paper [4] – we reduce the problem to a system of nonlinear integro-functional equations, via an auxiliary boundary value problem analogous to that in [20], and hence prove the existence of a local solution by using the Banach fixed point theorem.

To the best of our knowledge, the problem in question has not been examined so far.

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1. The problem

Let $n, p \in \mathbf{N}$ (where $\mathbf{N}$ denotes the set of all positive integers), set $\mathcal{D} = [0, A] \times [0, B]$ with $0 < A, B < \infty$ and consider the class $\mathcal{K}$ of all functions (vectors) $u = (u^k): \mathcal{D} \to \mathbf{R}^n$ such that the derivatives $D_x^r D_y^s u$ (where $D_x^r = \partial^r / \partial x^r$ and $D_y^s = \partial^s / \partial y^s$) exist for $r, s = 0, 1, \ldots, p$ and are continuous. We introduce the notation (cp. [20])

$$V = (v_r), \quad W = (w_r), \quad Z = (z_{rs}), \quad \Phi = (V, W), \tag{1.1}$$

where

$$v_r = D_x^p D_y^r u, \quad w_r = D_y^p D_x^r u, \quad z_{rs} = D_x^s D_y^r u \tag{1.2}$$

$(r, s = 0, 1, \ldots, p-1)$. We deal with the system of integro-differential equations

$$L^p u(x, y) = F[x, y, Z(x, y), \Phi(x, y), \Omega(x, y)] \tag{1.3}$$

where $L = D_x^1 D_y^1$ and

$$\Omega(x, y) = \int_0^x \int_0^y g[x, y; t, \tau, Z(t, \tau), \Phi(t, \tau)]\, d\tau\, dt \tag{1.4}$$

with F and g being given functions. By a *solution* of this system in $\mathcal{D}$ we mean a function $u \in \mathcal{K}$ satisfying (1.3) at each point $(x, y) \in \mathcal{D}$.

Let us consider a system of $2p$ curves

$$\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1} \qquad \text{and} \qquad \tilde{\Gamma}_0, \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{p-1}$$

of equations $y = f_i(x)$ and $x = h_i(y)$, respectively, where

$$f_i: [0, A] \to [0, B] \quad \text{and} \quad h_i: [0, B] \to [0, A] \qquad (i = 0, 1, \ldots, p-1)$$

are given functions of class C^1. Denote by $\mathbf{n}_i$ and $\tilde{\mathbf{n}}_i$ the unit vectors normal to Γ_i and $\tilde{\Gamma}_i$, respectively. We examine the following boundary value problem.

Problem ($\mathcal{P}$): Find a solution u of system (1.3) in $\mathcal{D}$ satisfying the boundary conditions

$$\begin{aligned} \left(\frac{d^{p-i}}{d\mathbf{n}^{p-i}}\right) L^i u[x, f_i(x)] &= M_i(x, Z[x, f_i(x)], \Phi[x, f_i(x)]) \\ \left(\frac{d^{p-i}}{d\tilde{\mathbf{n}}^{p-i}}\right) L^i u[h_i(y), y] &= N_i(y, Z[h_i(y), y], \Phi[h_i(y), y]) \end{aligned} \tag{1.5}$$

$((x, y) \in \mathcal{D};\ i = 0, 1, \ldots, p-1)$.

We make the following **assumptions I - IV**:

I. Let $c_i, \tilde{c}_i > 0$ and $s_i, \tilde{s}_i \geq 0$ $(i = 0, 1, \ldots, p-1)$ be constants such that the inequalities

$$\max(s_i, \tilde{s}_i) \leq 1, \quad x^{1-s_i} \leq b_i c_i A^{\omega_i}, \quad y^{1-\tilde{s}_i} \leq \tilde{b}_i \tilde{c}_i B^{\tilde{\omega}_i} \tag{1.6}$$

($(x,y) \in \mathcal{D}$) are satisfied for $i = 0,1,\dots,p-2$, where b_i, $\tilde{b}_i$ and ω_i, $\tilde{\omega}_i$ are positive constants. Moreover, let m_i, $\tilde{m}_i > 0$ and a_i, $\tilde{a}_i > 0$ ($i = 0,1,\dots,p-1$) be constants such that

$$\max_{[0,A]} |f_i'(x)| \le a_i, \quad \max_{[0,B]} |h_i'(y)| \le \tilde{a}_i \qquad \text{and} \qquad m_i < a_i^{-i/p}, \quad \tilde{m}_i < \tilde{a}_i^{-i/p}. \tag{1.7}$$

All the said constants except c_i and $\tilde{c}_i$ are required to be independent of A and B. We assume that the functions f_i and h_i are of class C^{p-i} and satisfy the conditions

$$f_i'(x) \ge \max(f_i(x)/m_i x, c_i x^{s_i}) \quad \text{and} \quad h_i'(y) \ge \max(h_i(y)/\tilde{m}_i y, \tilde{c}_i y^{\tilde{s}_i}) \tag{1.8}$$

($x \in (0,A], y \in (0,B], i = 0,1,\dots,p-1$).

II. The functions $M_i\colon [0,A] \times \mathbf{R}^{\tilde{n}} \to \mathbf{R}^n$ and $N_i\colon [0,B] \times \mathbf{R}^{\tilde{n}} \to \mathbf{R}^n$ (where $\tilde{n}$ denotes the total number of elements of Z, V and W) are continuous and satisfy the conditions

$$\begin{aligned} |M_i(x,(\mathbf{0})_1,(\mathbf{0})_2)| &\le K_1 x^{\alpha_1} \\ |N_i(y,(\mathbf{0})_1,(\mathbf{0})_2)| &\le K_1 y^{\alpha_1} \\ |M_i(x,\xi,\eta) - M_i(x,\bar{\xi},\bar{\eta})| &\le K_2(\Xi_1 + \Xi_2) \\ |N_i(y,\xi,\eta) - N_i(y,\bar{\xi},\bar{\eta})| &\le K_2(\Xi_1 + \Xi_2) \end{aligned} \tag{1.9}$$

($\xi = (\xi_{\nu,\mu})$; $\eta = (\vartheta,\omega)$ with $\vartheta = (\vartheta_\nu)$; $\omega = (\omega_\nu)$ ($\nu,\mu = 0,1,\dots,p-1$); $\bar{\xi}$, $\bar{\eta}$ and $\bar{\omega}$ are understood analogously, and $(\mathbf{0})_1$ and $(\mathbf{0})_2$ denote the appropriate systems of zeros), where

$$\begin{aligned} \Xi_1 &= \sum_{\nu,\mu} \left[\max(|\xi_{\nu,\mu}|, |\bar{\xi}_{\nu,\mu}|)\right]^{\alpha_2 - 1} |\xi_{\nu,\mu} - \bar{\xi}_{\nu,\mu}| \\ \Xi_2 &= \sum_{\nu} \left\{ \left[\max(|\vartheta_\nu|, |\bar{\vartheta}_\nu|)\right]^{\alpha_3 - 1} |\vartheta_\nu - \bar{\vartheta}_\nu| + \left[\max(|\omega_\nu|, |\bar{\omega}_\nu|)\right]^{\alpha_3 - 1} |\omega_\nu - \bar{\omega}_\nu| \right\}, \end{aligned} \tag{1.10}$$

the exponents α_1, α_2, α_3 fulfil the inequalities

$$\alpha_1 > p \max\left(2, 1 + \frac{\hat{s}_{p-1}}{p}\right) \quad \text{and} \quad \min(\alpha_2, \alpha_3) > \max\left(2, 1 + \frac{\hat{s}_{p-1}}{p}\right) \tag{1.11}$$

with $\hat{s}_{p-1} = \max(s_{p-1}, \tilde{s}_{p-1})$, and K_1 and K_2 are positive constants. Moreover, at the common points of the curves considered, the functions M_i and N_i satisfy suitable compatibility conditions.

III. The function $F\colon \mathcal{D} \times \mathbf{R}^{\tilde{n}+n} \to \mathbf{R}^n$ is continuous and satisfies the conditions

$$|F(x,y,(\mathbf{0})_1,(\mathbf{0})_2),(\mathbf{0})_3| \le K_3\left(x^{\beta_1} + y^{\beta_1}\right)$$

and

$$\begin{aligned} &|F(x,y,\xi,\eta,\zeta) - F(x,y,\bar{\xi},\bar{\eta},\bar{\zeta})| \\ &\quad \le K_4 \Bigg\{ \sum_{\nu,\mu} \left[\max(|\xi_{\nu,\mu}|, |\bar{\xi}_{\nu,\mu}|)\right]^{\beta_2 - 1} |\xi_{\nu,\mu} - \bar{\xi}_{\nu,\mu}| \\ &\quad + \sum_{\nu} \Big\langle \left[\max(|\vartheta_\nu|, |\bar{\vartheta}_\nu|)\right]^{\beta_3 - 1} |\vartheta_\nu - \bar{\vartheta}_\nu| \\ &\quad + \left[\max(|\omega_\nu|, |\bar{\omega}_\nu|)\right]^{\beta_3 - 1} |\omega_\nu - \bar{\omega}_\nu| \Big\rangle + |\zeta - \bar{\zeta}|^{\beta_4 - 1} \Bigg\} \end{aligned} \tag{1.12}$$

($(\mathbf{0})_3$ is the system of n zeros), where the exponents fulfil the inequalities

$$\beta_1, 2\beta_4 > \max(p, p-(1-\hat{s}_{p-1})) \quad \text{and} \quad \min_{r=2,3} \beta_r > \max(1, 1-\frac{1-\hat{s}_{p-1}}{p}) \tag{1.13}$$

and K_3 and K_4 are positive constants.

IV. The function $g: \mathcal{D}^2 \times \mathbf{R}^{\bar{n}} \to \mathbf{R}^n$ is continuous and satisfies the conditions

$$\begin{aligned} &|g(x,y;t,\tau,(\mathbf{0})_1,(\mathbf{0})_2| \le K_5 \\ &|g(x,y;t,\tau,\xi,\eta) - g(x,y;t,\tau,\bar{\xi},\bar{\eta})| \\ &\qquad \le K_6 \left[\sum_{\nu,\mu} |\xi_{\nu,\mu} - \bar{\xi}_{\nu,\mu}| + \sum_{\nu} (|\vartheta_\nu - \bar{\vartheta}_\nu| + |\omega_\nu - \bar{\omega}_\nu|) \right] \end{aligned} \tag{1.14}$$

where K_5 and K_6 are positive constants.

Corollary 1.1. *It follows from Assumptions II - IV that the inequalities*

$$\begin{aligned} |M_i(x,\xi,\eta)| &\le K_1 x^{\alpha_1} + K_2 \left[\sum_{\nu,\mu} |\xi_{\nu\mu}|^{\alpha_2} + \sum_{\nu} (|\vartheta_\nu|^{\alpha_3} + |\omega_\nu|^{\alpha_3}) \right] \\ |N_i(y,\xi,\eta)| &\le K_1 y^{\alpha_1} + K_2 \left[\sum_{\nu,\mu} |\xi_{\nu\mu}|^{\alpha_2} + \sum_{\nu} (|\vartheta_\nu|^{\alpha_3} + |\omega_\nu|^{\alpha_3}) \right] \\ |F(x,y,\xi,\eta,\zeta)| &\le K_3 (x^{\beta_1} + y^{\beta_1}) \\ &\quad + K_4 \left[\sum_{\nu,\mu} |\xi_{\nu,\mu}|^{\beta_2} + \sum_{\nu} (|\vartheta_\nu|^{\beta_3} + |\omega_\nu|^{\beta_3}) + |\zeta|^{\beta_4} \right] \\ |g(x,y;t,\tau,\xi,\eta)| &\le K_5 + K_6 \left[\sum_{\nu,\mu} |\xi_{\nu,\mu}| + \sum_{\nu} (|\vartheta_\nu| + |\omega_\nu|) \right] \end{aligned} \tag{1.15}$$

are satisfied.

Remark 1.1. Let us assume that

$$\Gamma_i \underset{i}{\equiv} \Gamma_0, \tilde{\Gamma}_i \underset{i}{\equiv} \tilde{\Gamma}_0 \quad (i = 0,1,\dots,p-1) \qquad \text{and} \qquad f_0(A) = B, h_0(B) = A$$

and that the curves Γ_0 and $\tilde{\Gamma}_0$ do not intersect one another apart from the points $(0,0)$ and (A,B). Setting $\Gamma = \Gamma_0 \cup \tilde{\Gamma}_0$ and denoting by $\tilde{\mathcal{D}}$ the domain bounded by Γ, we can assert that problem $(\mathcal{P})$ is in the considered case a Neumann-type problem for the domain $\tilde{\mathcal{D}}$, with the boundary conditions (1.5) given on its boundary Γ. The compatibility conditions for M_i and N_i are in this case

$$M_i(0,\xi,\eta) = N_i(0,\xi,\eta) \qquad \text{and} \qquad M_i(A,\xi,\eta) = N_i(B,\xi,\eta)$$

$((\xi,\eta) \in \mathbf{R}^{\bar{n}}; i = 0,1,\dots,p-1)$.

Example 1.1. We give an example of curves satisfying Assumption I. Let

$$\Gamma_i \underset{i}{\equiv} \Gamma_0,\ \tilde{\Gamma}_i \underset{i}{\equiv} \tilde{\Gamma}_0\ \ (i = 0, 1, \ldots, p-1), \quad A = B \quad \text{and}\ \gamma \in (1, 2].$$

Set $f_0(x) = A^{1-\gamma} x^\gamma$ and

a) $h_0(y) = A^{1-\gamma} y^\gamma$

b) $h_0(y) = (A/\sin A^\gamma)\sin y^\gamma$.

Assumption I is satisfied with

$$b_0 = 1/\gamma, c_0 = \gamma/A^{\gamma-1}, \omega_0 = 1, s_0 = \gamma - 1, m_0 = 1/\gamma, a_0 = \gamma$$

and

a) the same parameters as for $f_0(x)$

b) $\tilde{b}_0 = b_0, \tilde{c}_0 = c_0, \tilde{\omega}_0 = 1, \tilde{s}_0 = s_0, \tilde{m}_0 = 1/(\gamma(1-\varepsilon)), \tilde{a}_0 = \gamma/(1-\varepsilon)$, where $0 < \varepsilon < 1 - (1/\gamma)^{1/2p}$ (we assume that A is sufficiently small, so that $0 < A < \left[\arccos(1-\varepsilon^2)\right]^{1/\gamma}$).

2. Auxiliary considerations

We begin this section with the following lemma whose inductive proof will be omitted.

Lemma 2.1. *If $m \in \mathbf{N}$, $v \in C^m(\mathcal{D})$ and $\mathbf{s}$ is a non-zero vector, then*

$$\begin{aligned} \frac{d^m}{ds^m} v &= \sum_{k=0}^{m} \binom{m}{k} D_x^k D_y^{m-k} v \cos^k \alpha \cos^{m-k} \beta \\ &+ \sum_{\nu=0}^{m-2} \frac{d^\nu}{ds^\nu} \left[\sum_{\mu=0}^{m-1-\nu} \binom{m-1-\nu}{\mu} D_x^\mu D_y^{m-1-\nu-\mu} v \, \frac{d}{ds}\left(\cos^\mu \alpha \cos^{m-1-\nu-\mu} \beta\right) \right] \end{aligned} \tag{2.1}$$

(as usual, we set $\sum_{\nu=0}^{m-2} u_\nu = 0$ for $m < 2$), where $(x, y) \in \mathcal{D}$, and α and β are the angles of $\mathbf{s}$ with the positive directions of the axes $0x$ and $0y$, respectively.

As an immediate consequence of Lemma 2.1, we get the following corollary (cp. relations (1.2)).

Corollary 2.1. *If $u \in \mathcal{K}$, then*

$$\begin{aligned} \frac{d^{p-i}}{d\hat{\mathbf{n}}_i^{p-i}} L^i u(x,y) &= v_i(x,y) \cos^{p-i} \alpha_i + w_i(x,y) \cos^{p-i} \beta_i \\ &+ \sum_{k=1}^{p-i-1} \binom{p-i}{k} z_{p-k,k+i}(x,y) \cos^k \alpha_i \cos^{p-i-k} \beta_i \\ &+ \sum_{\nu=0}^{p-i-2} \frac{d^\nu}{d\hat{\mathbf{n}}_i^\nu} \left[\sum_{\mu=0}^{p-i-1-\nu} \binom{p-i-1-\nu}{\mu} z_{p-i-1-\nu-\mu,\mu+i}(x,y) \right. \\ &\left. \times \frac{d}{d\hat{\mathbf{n}}_i} \left(\cos^\mu \alpha_i \cos^{p-i-1-\nu-\mu} \beta_i\right) \right] \end{aligned} \tag{2.2}$$

$((x,y)\in\mathcal{D})$, *where* $\hat{\mathbf{n}}_i=\mathbf{n}_i,\ \tilde{\mathbf{n}}_i;\ \alpha_i=(0x,\hat{\mathbf{n}}_i)$ *and* $\beta_i=(0y,\hat{\mathbf{n}}_i)$ $(i=0,1,\ldots,p-1)$.

Now, let $\mathcal{K}_1$ be the class of all functions $u\in\mathcal{K}$ such that

$$v_i(0,0)=w_i(0,0)=z_{ij}(0,0)=0 \tag{2.3}$$

$(i,j=0,1,\ldots,p-1)$, and assume that the normal vectors $\mathbf{n}_i$ and $\tilde{\mathbf{n}}_i$ are directed so that

$$\cos(x,\mathbf{n}_i)=\frac{-f_i'(x)}{e_i(x)},\ \cos(x,\tilde{\mathbf{n}}_i)=\frac{-1}{\tilde{e}_i(y)},\ \cos(y,\mathbf{n}_i)=\frac{1}{e_i(x)},\ \cos(y,\tilde{\mathbf{n}}_i)=\frac{h'(y)}{\tilde{e}_i(y)} \tag{2.4}$$

where

$$e_i(x)=\sqrt{1+(f_i'(x))^2}\qquad\text{and}\qquad\tilde{e}_i(y)=\sqrt{1+(h_i'(y))^2}. \tag{2.5}$$

Basing on Corollary 2.1 and using formulas (2.4), we can assert that, in the class $\mathcal{K}_1$, problem $(\mathcal{P})$ is equivalent to the following problem (Σ) (cp. with those in [3, 20]).

Problem (Σ): Find a solution $u\in\mathcal{K}_1$ of system (1.3) in $\mathcal{D}$ satisfying the boundary conditions

$$v_i[x,f_i(x)]=G_\Phi^i(x)\qquad\text{and}\qquad w_i[h_i(y),y]=H_\Phi^i(y) \tag{2.6}$$

$((x,y)\in\mathcal{D})$, where

$$\begin{aligned}&G_\Phi^i(0)=H_\Phi^i(0)=0\\&G_\Phi^i(x)=\check{G}_\Phi^i(x)+\hat{G}_\Phi^i(x)\quad\text{for } x\in(0,A]\\&H_\Phi^i(y)=\check{H}_\Phi^i(y)+\hat{H}_\Phi^i(y)\quad\text{for } y\in(0,B]\end{aligned} \tag{2.7}$$

with

$$\check{G}_\Phi^i(x)=-(-f_i'(x))^{i-p}w_i[x,f_i(x)],\quad \check{H}_\Phi^i(y)=-(-h_i'(y))^{i-p}v_i[h_i(y),y] \tag{2.8}$$

$$\begin{aligned}\hat{G}_\Phi^i(x)=\left(\frac{-e_i(x)}{f_i'(x)}\right)^{p-i}\Bigg\{&M_i(x,Z[x,f_i(x)],\Phi[x,f_i(x)])\\&-\sum_{k=1}^{p-i-1}\binom{p-i}{k}z_{p-k,k+i}[x,f_i(x)]\frac{(-f_i'(x))^k}{(e_i(x))^{p-i}}\\&-\sum_{\nu=0}^{p-i-2}\Bigg(\frac{d^\nu}{d\mathbf{n}_i^\nu}\Bigg[\frac{-f_i'(x)}{e_i(x)}\sum_{\mu=0}^{p-i-1-\nu}\binom{p-i-1-\nu}{\mu}\\&\times z_{p-i-1-\nu-\mu,\mu+i}(x,y)\left(\frac{(-f_i'(x))^\mu}{(e_i(x))^{p-i-1-\nu}}\right)'\Bigg]\Bigg)_{y=f_i(x)}\Bigg\}\end{aligned} \tag{2.9$'$}$$

and

$$\hat{H}^i_\Phi(x) = \left(\frac{\tilde{e}_i(y)}{h_i'(y)}\right)^{p-i} \Bigg\{ N_i(y, Z[h_i(y),y], \Phi[h_i(y),y]) - \sum_{k=1}^{p-i-1} \binom{p-i}{k} z_{p-k,k+i}[h_i(y),y](-1)^k \frac{(h_i'(y))^{p-i-k}}{(\tilde{e}_i(y))^{p-i}} - \sum_{\nu=0}^{p-i-2} \left(\frac{d^\nu}{d\tilde{\mathbf{n}}_i^\nu}\left[\frac{-1}{\tilde{e}_i(y)} \sum_{\mu=0}^{p-i-1-\nu} \binom{p-i-1-\nu}{\mu} \times z_{p-i-1-\nu-\mu,\mu+i}(x,y) \left((-1)^\mu \frac{(h_i'(y))^{p-i-1-\nu-\mu}}{(\tilde{e}_i(y))^{p-i-1-\nu}}\right)'\right)_{x=h_i(y)}\right]\Bigg\} \tag{2.9''}$$

$(i = 0, 1, \ldots, p-1)$.

We shall use the following lemma whose validity follows from Taylor's formula with the integral remainder.

Lemma 2.2. *If $u \in \mathcal{K}_1$, then*

$$\begin{aligned} z_{rs}(x,y) &= \int_0^x \frac{(x-\xi)^{p-s-1}}{(p-s-1)!} v_r(\xi,y)\,d\xi + \sum_{k=0}^{p-s-1} \int_0^y \frac{(y-\eta)^{p-r-1}}{(p-r-1)!} w_{s+k}(0,\eta)\,d\eta\, \frac{x^k}{k!} \\ &= \int_0^y \frac{(y-\eta)^{p-r-1}}{(p-r-1)!} w_s(x,\eta)\,d\eta + \sum_{k=0}^{p-r-1} \int_0^x \frac{(x-\xi)^{p-s-1}}{(p-s-1)!} v_{r+k}(\xi,0)\,d\xi\, \frac{y^k}{k!} \end{aligned} \tag{2.10}$$

$((x,y) \in \mathcal{D};\ r,s = 0,1,\ldots,p-1)$. *If, moreover, u is a solution of system (1.3) in $\mathcal{D}$, then*

$$\begin{aligned} v_r(x,y) &= \sum_{k=0}^{p-r-1} v_{r+k}(x,0)\frac{y^k}{k!} + \int_0^y \frac{(y-\eta)^{p-r-1}}{(p-r-1)!} F[x,\eta,Z(x,\eta),\Phi(x,\eta),\Omega(x,\eta)]d\eta \\ w_r(x,y) &= \sum_{k=0}^{p-r-1} w_{r+k}(0,y)\frac{x^k}{k!} + \int_0^x \frac{(x-\xi)^{p-r-1}}{(p-r-1)!} F[\xi,y,Z(\xi,y),\Phi(\xi,y),\Omega(\xi,y)]d\xi \end{aligned} \tag{2.11}$$

$((x,y) \in \mathcal{D};\ r = 0,1,\ldots,p-1)$.

If z_{rs} $(r, s = 0, 1, \dots, p-1)$ are expressed in terms of Φ by formulae (2.10), then we shall write $Z = \Lambda^1_\Phi = ({}^1\lambda^{rs}_\Phi)$. The expression Ω (cp. (1.4)) with $Z = \Lambda^1_\Phi$ will be denoted by Λ^*_Φ. Finally, $\Lambda^2_\Phi = ({}^2\lambda^{rs}_\Phi)$ and $\Lambda^3_\Phi = ({}^3\lambda^{rs}_\Phi)$ will stand for V and W, respectively, with v_r and w_r given by (2.11) with $Z = \Lambda^1_\Phi$ and $\Omega = \Lambda^*_\Phi$.

Now, let us consider the following system of integro-functional equations

$$v_i(x,y) = \mathbf{T}^i_\Phi(x,y), \quad w_i(x,y) = \hat{\mathbf{T}}^i_\Phi(x,y) \tag{2.12}$$

$((x,y) \in \mathcal{D};\ i = 0, 1 \dots, p-1)$ with the unknown vector Φ (cp. (1.1), (1.2)), where

$$\begin{aligned} \mathbf{T}^i_\Phi(x,y) &= \mathbf{G}^i_\Phi(x) + \int_{f_i(x)}^{y} \vartheta^i_\Phi(x,\eta)\,d\eta \\ \hat{\mathbf{T}}^i_\Phi(x,y) &= \mathbf{H}^i_\Phi(y) + \int_{h_i(y)}^{x} \hat{\vartheta}^i_\Phi(\xi,y)\,d\xi \end{aligned} \tag{2.13}$$

$(i = 0, 1, \dots, p-1)$. Here

$$\mathbf{G}^i_\Phi(0) = \mathbf{H}^i_\Phi(0) = 0$$

and

$$\begin{aligned} \mathbf{G}^i_\Phi(x) &= \check{\mathbf{G}}^i_\Phi(x) + \hat{\mathbf{G}}^i_\Phi(x) \qquad \text{for all } x \in (0, A] \\ \mathbf{H}^i_\Phi(y) &= \check{\mathbf{H}}^i_\Phi(y) + \hat{\mathbf{H}}^i_\Phi(y) \qquad \text{for all } y \in (0, B] \end{aligned}$$

where $\check{\mathbf{G}}^i_\Phi$ and $\check{\mathbf{H}}^i_\Phi$ denote the expressions (2.8), respectively, with $V = \Lambda^2_\Phi$, $W = \Lambda^3_\Phi$ (we set in (2.11) $Z = \Lambda^1_\Phi$; $\Omega = \Lambda^*_\Phi$), and $\hat{\mathbf{G}}^i_\Phi$ and $\hat{\mathbf{H}}^i_\Phi$ the expressions (2.9), respectively, with $Z = \Lambda^1_\Phi$. Moreover, ϑ^i_Φ and $\hat{\vartheta}^i_\Phi$ are given by

$$\vartheta^i_\Phi(x,\eta) = \begin{cases} v_{i+1}(x,\eta) & \text{for } i = 0, 1, \dots, p-2 \\ F[x,\eta,\Lambda^1_\Phi(x,\eta),\Phi(x,\eta),\Lambda^*_\Phi(x,\eta)] & \text{for } i = p-1 \end{cases} \tag{2.14}$$

$$\hat{\vartheta}^i_\Phi(\xi,y) = \begin{cases} w_{i+1}(\xi,y) & \text{for } i = 0, 1, \dots, p-2 \\ F[\xi,y,\Lambda^1_\Phi(\xi,y),\Phi(\xi,y),\Lambda^*_\Phi(\xi,y)] & \text{for } i = p-1 \end{cases} \tag{2.15}$$

The following lemma holds good, the validity of which follows from that of Lemma 8 in [3].

Lemma 2.3. *If u is a solution of problem (Σ), then Φ is a continuous solution of system (2.12). Conversely, if Φ is a continuous solution of system (2.12), then the function $z_{00} = {}^1\lambda^{00}_\Phi$ is a solution of problem (Σ).*

3. Solution of the problem

In this section we shall prove the existence and uniqueness of a solution of problem (Σ) (and hence of problem $(\mathcal{P})$) by using the Banach fixed point theorem.

Let $\mathcal{S}$ be the set of all systems Φ (cp. (1.1)), where the components

$$v_i: \mathcal{D}_* \to \mathbf{R}^n \quad \text{and} \quad w_i: \mathcal{D}_* \to \mathbf{R}^n \qquad (\mathcal{D}_* = \mathcal{D} \setminus \{(0,0)\}; i = 0, 1, \dots, p-1)$$

are continuous functions such that

$$B_\Phi := \max_{0 \le i \le p-1} \max \left(\sup_{\mathcal{D}_*} \left[(x^p + y^p)^{-1} |v_i(x,y)| \right], \sup_{\mathcal{D}_*} \left[(x^p + y^p)^{-1} |w_i(x,y)| \right] \right) < \infty.$$

We define the distance by the formula

$$\begin{aligned} d(\Phi, \bar{\Phi}) = B_{\Phi - \bar{\Phi}} = \max_{0 \le i \le p-1} \max \Big(& \sup_{\mathcal{D}_*} \left[(x^p + y^p)^{-1} |v_i(x,y) - \bar{v}_i(x,y)| \right], \\ & \sup_{\mathcal{D}_*} \left[(x^p + y^p)^{-1} |w_i(x,y) - \bar{w}_i(x,y)| \right] \Big) \end{aligned} \tag{3.1}$$

($\Phi = (V, W)$ and $\bar{\Phi} = (\bar{V}, \bar{W})$). It is easily observed that $\mathcal{S}$ is a complete metric space. Let us consider the set $\mathcal{Z}$ of all points $\Phi \in \mathcal{S}$ such that

$$B_\Phi \le \kappa, \tag{3.2}$$

where $\kappa \in (0,1)$. This is a closed subset of $\mathcal{S}$ and hence it is itself a complete metric space with the metric given by (3.1).

In view of system (2.12), we map $\mathcal{Z}$ by the transformation $\mathbf{T}$ defined by formulas (cp. (2.13) - (2.15))

$$\tilde{v}_i(x,y) = \mathbf{T}^i_\Phi(x,y) \quad \text{and} \quad \tilde{w}_i(x,y) = \hat{\mathbf{T}}^i_\Phi(x,y) \tag{3.3}$$

($(x,y) \in \mathcal{D}_*$; $i = 0, 1 \dots, p-1$). In the sequel, $\tilde{\Phi}$ will denote the system $(\tilde{V}, \tilde{W})$ where $\tilde{V} = (\tilde{v}_i)$ and $\tilde{W} = (\tilde{w}_i)$. We shall find sufficient conditions for the inclusion $\mathbf{T}(\mathcal{Z}) \subset \mathcal{Z}$.

In order to estimate the functions $\tilde{v}_i$ and $\tilde{w}_i$, let us first observe that the following inequality is valid (cp. (1.8), (2.10) and (3.2)):

$$|{}^1\lambda^{rs}_\Phi[x, f_i(x)]| \le \kappa \left[x^{2p-s}(1 + (m_i a_i)^p) + x^{2p-r}(m_i a_i)^{2p-r} \sum_{k=0}^{p-s-1} \frac{A^k}{k!} \right] \tag{3.4}$$

($r, s = 0, 1, \dots, p-1$), whence, and from Assumption I, Corollary 1.1 and relations (2.8), (2.11) and (3.2), we obtain the sequence of inequalities (in which, as well as in the sequel, const denotes a positive constant independent of κ)

$$\begin{aligned} |\check{\mathbf{G}}^i_\Phi(x)| \le & (m_i a_i)^i m_i^{p-i} \left(1 + \sum_{k=1}^{p-i-1} \frac{A^k}{k!} \right) \kappa x^p \\ & + \text{const} \left(\frac{x}{f_i'(x)} \right)^{p-i} \left[x^{\beta_1} + (f_i(x))^{\beta_1} + \kappa \left(x^{(p+1)\beta_2} + x^{p\beta_3} + x^{(p+1)\beta_4} \right) \right] \\ \le & \left[(m_i a_i)^i m_i^{p-i} \kappa + \text{const}(1+\kappa) A^{\omega_i} \right] x^p, \end{aligned} \tag{3.5}$$

where $\check{\omega}_i = \min(\omega_i, 1)$ $(i = 0, 1, \ldots, p-2)$. For $i = p-1$, we have

$$\begin{aligned} |\check{\mathbf{G}}_\Phi^{p-1}(x)| &\le (m_{p-1}a_{p-1})^{p-1} m_{p-1}\kappa x^p + \text{const}\left(\frac{x}{f'_{p-1}(x)}\right)\Big[x^{\beta_1} + (f_{p-1}(x))^{\beta_1} \\ &\quad + \kappa\big(x^{(p+1)\beta_2} + x^{p\beta_3} + x^{(p+1)\beta_4}\big)\Big] \\ &\le (m_{p-1}a_{p-1})^{p-1} m_{p-1}\kappa x^p + \text{const}(1+\kappa)x^{(1-s_{p-1}+\bar{\beta})}, \end{aligned} \tag{3.5'}$$

where $\bar{\beta} = \min\big(\beta_1, (p+1)\min(\beta_2, \beta_4), \beta_3\big)$. Inequalities (1.13) and (3.5') yield

$$|\check{\mathbf{G}}_\Phi^{p-1}(x)| \le \big[(m_{p-1}a_{p-1})^{p-1} m_{p-1}\kappa + \text{const}(1+\kappa)A^{\check{\omega}_{p-1}}\big]x^p, \tag{3.6}$$

where $\check{\omega}_{p-1}$ is a positive constant, and using (3.5) and (3.6) we obtain

$$|\check{\mathbf{G}}_\Phi^i(x)| \le \big[(m_i a_i)^i m_i^{p-i}\kappa + \text{const}(1+\kappa)A^{\theta'_i}\big]x^p \tag{3.7}$$

$(x \in (0, A]$; $i = 0, 1, \ldots, p-1)$, where θ'_i are positive constants.

We proceed to the examination of $\hat{\mathbf{G}}_\Phi^i(x)$ (cp. (2.9) and (2.13)). Let us observe that, by Assumption I, Corollary 1.1 and relations (2.5) and (3.4), we obtain

$$\left|\left(\frac{-e_i(x)}{f'_i(x)}\right)^{p-i} M_i\big(x, \Lambda_\Phi^1[x, f_i(x)], \Phi[x, f_i(x)]\big)\right| \le \text{const}(1+\kappa)A^{\theta''_i}x^p, \tag{3.8}$$

where θ''_i are positive constants and $i = 0, 1, \ldots, p-1$. Moreover, basing on Assumption I and using formulae (2.10) and (3.2), we get the sequence of inequalities

$$\begin{aligned} &\left|\left(\frac{-e_i(x)}{f'_i(x)}\right)^{p-i} \sum_{k=1}^{p-i-1}\binom{p-i}{k}\, {}^1\lambda_\Phi^{\beta-k,k+i}[x, f_i(x)]\frac{(-f'_i(x))^k}{(e_i(x))^{p-i}}\right| \\ &\le \text{const}\,\kappa \sum_{k=1}^{p-i-1}\binom{p-i}{k}\left[\left(\frac{x}{f'_i(x)}\right)^{p-i-k}(x^p + (f_i(x))^p) + \frac{(f_i(x))^{p+k}}{(f'_i(x))^{p-i-k}}\right] \\ &\le \text{const}\,\kappa \sum_{k=1}^{p-i-1}\big[(b_i A^{\omega_i})^{p-i-k} + A^k\big]x^p \\ &\le \text{const}\,\kappa\, A^{\omega_i}x^p. \end{aligned} \tag{3.9}$$

Thus, it remains to estimate the expressions

$$\begin{aligned} \Delta_\Phi^i(x) = \left(\frac{-e_i(x)}{f'_i(x)}\right)^{p-i} \sum_{\nu=0}^{p-i-2}\Bigg\{\frac{d^\nu}{d\mathbf{n}_i^\nu}\Bigg[\left(\frac{-f'_i(x)}{e_i(x)}\right)^{p-i-1-\nu}\sum_{\mu=0}\binom{p-i-1-\nu}{\mu} \\ \times\, {}^1\lambda_\Phi^{p-i-1-\nu-\mu,\mu+i}(x, y)\left(\frac{(-f'_i(x))^\mu}{(e_i(x))^{p-i-1-\nu}}\right)'\Bigg]\Bigg\}_{y=f_i(x)} \end{aligned} \tag{3.10}$$

$(i = 0, 1, \ldots, p-2)$. Let us examine the part of $\Delta^i_\Phi(x)$ given by

$$\tilde{\Delta}^i_\Phi(x) = \left(\frac{-e_i(x)}{f_i'(x)}\right)^{p-i} \sum_{\nu=0}^{p-i-2} \Bigg\{ D_y^\nu \Bigg[\left(\frac{-f_i'(x)}{e_i(x)}\right)^{p-i-1-\nu} \sum_{\mu=0}^{p-i-1-\nu} \binom{p-i-1-\nu}{\mu} \times {}^1\lambda_\Phi^{p-i-1-\nu-\mu,\mu+i}(x,y) \Bigg] \left(\frac{(-f_i'(x))^\mu}{(e_i(x))^{p-i-1-\nu}}\right)' (e_i(x))^{-\nu} \Bigg\}_{y=f_i(x)} . \tag{3.11}$$

Evidently,

$$|\tilde{\Delta}^i_\Phi(x)| \le \tilde{\delta}^i_\Phi(x) + \hat{\delta}^i_\Phi(x), \tag{3.12}$$

where

$$\tilde{\delta}^i_\Phi(x) = (f_i'(x))^{p+i+1} \sum_{\nu=0}^{p-i-2} (p-i-1-\nu) \left| {}^1\lambda_\Phi^{p-i-2,i+1}[x, f_i(x)] f_i''(x) \right| \tag{3.13}$$

and

$$\begin{aligned} \hat{\delta}^i_\Phi(x) = {} & (f_i'(x))^{p+i+1} \\ & \times \sum_{\nu=0}^{p-i-2} \Bigg\{ \sum_{\mu=0}^{p-i-1-\nu} \binom{p-i-1-\nu}{\mu} \\ & \times \left| {}^1\widetilde{\lambda}_\Phi^{p-i-1-\mu,\mu+i}[x, f_i(x)] \right| \frac{p-i-1-\nu}{(e_i(x))^2} (f_i'(x))^{\mu+1} \\ & + \sum_{\mu=2}^{p-i-1-\nu} \binom{p-i-1-\nu}{\mu} \\ & \times \left| {}^1\lambda_\Phi^{p-i-1-\mu,\mu+i}[x, f_i(x)] \right| \mu (f_i'(x))^{\mu-1} \Bigg\} |f_i''(x)|. \end{aligned} \tag{3.14}$$

Basing on Assumption I and relations (2.5), (2.10) and (3.2), we have

$$\tilde{\delta}^i_\Phi(x) \le \text{const}\ \kappa \left[\left(\frac{x}{f_i'(x)}\right)^{p-i-1} + x^{i+1} \right] x^p,$$

whence (cp. the derivation of (3.9)) we get

$$\tilde{\delta}^i_\Phi(x) \le \text{const}\, \kappa A^{\check{\omega}_i} x^p \tag{3.15}$$

with $\check{\omega}_i$ being understood as in (3.5). In the same way we obtain

$$\hat{\delta}^i_\Phi(x) \le \text{const}\, \kappa \left[\left(\frac{x}{f_i'(x)}\right)^{p-i-\mu} + x^{i+1} \right] x^p \le \text{const}\, \kappa A^{\check{\omega}_i} x^p. \tag{3.16}$$

Thus, by (3.12) – (3.16), the expression $\tilde{\Delta}^i_\Phi$ (cp. (3.11)) satisfies the inequality

$$|\tilde{\Delta}^i_\Phi(x)| \le \text{const}\, \kappa A^{\check{\omega}_i} x^p \tag{3.17}$$

$(i = 0, 1, \ldots, p-2)$.

Using a similar argument and basing on the inequality (cp. (2.4)) $|\cos(x, \mathbf{n}_i)| \leq f_i'(x)$, we can conclude that (cp. (3.10) and (3.11))

$$|\Delta_\Phi^i(x) - \tilde{\Delta}_\Phi^i(x)| \leq \text{const}\, \kappa A^{\omega_i} x^p. \tag{3.18}$$

On joining relations (3.8) - (3.11), (3.17) and (3.18), we get

$$|\hat{\mathbf{G}}_\Phi^i(x)| \leq \text{const}(1+\kappa) A^{\theta_i'''} x^p \tag{3.19}$$

$(x \in (0, A];\ i = 0, 1, \ldots, p-1)$, where θ_i''' are positive constants, and (3.7) and (3.19) yield the following estimate of the first term in the first of relations (2.13) (cp. (2.7)):

$$|\mathbf{G}_\Phi^i(x)| \leq \left[(m_i a_i)^i m_i^{p-i} \kappa + \text{const}(1+\kappa) A^{\theta_i}\right] x^p \tag{3.20}$$

$(x \in [0, A];\ i = 0, 1, \ldots, p-1)$, where θ_i are positive constants.

As for the second term in the first of relations (2.13), we easily conclude, basing on Assumptions I, III, IV and formulae (2.15),(3.2), that

$$\left| \int_{f_i(x)}^{y} \vartheta_\Phi^i(x, \eta)\, d\eta \right| \leq \text{const}(1+\kappa) \mathbf{A}(x^p + y^p) \tag{3.21}$$

$((x, y) \in \mathcal{D}_*)$, where $\mathbf{A} = \max(A, B)$. As a consequence of (3.3), (2.13), (3.20) and (3.21), we have

$$|\tilde{v}_i(x, y)| \leq \left[(m_i a_i)^i m_i^{p-i} \kappa + \text{const}(1+\kappa) \mathbf{A}^{\theta_i}\right] (x^p + y^p) \tag{3.22}$$

$((x, y) \in \mathcal{D}_*;\ i = 0, 1, \ldots, p-1)$. By a similar argument we show that (cp. (3.3))

$$|\tilde{w}_i(x, y)| \leq \left[(\tilde{m}_i \tilde{a}_i)^i \tilde{m}_i^{p-i} \kappa + \text{const}(1+\kappa) \mathbf{A}^{\theta_i}\right] (x^p + y^p) \tag{3.23}$$

$((x, y) \in \mathcal{D}_*;\ i = 0, 1, \ldots, p-1)$. It follows from (3.22) and (3.23) that the functions $\tilde{v}_i$ and $\tilde{w}_i$ satisfy relations (3.2) if the inequality

$$\max(m_i^p a_i^i, \tilde{m}_i^p \tilde{a}_i^i)\kappa + C(1+\kappa)\mathbf{A}^\theta \leq \kappa \tag{3.24}$$

$(i = 0, 1, \ldots, p-1)$ is fulfilled, where $\theta = \min_{0 \leq i \leq p-1} \theta_i$ and C is a positive constant independent of κ. It is evident (cp. (1.7)) that inequality (3.24) holds if $\mathbf{A}$ is sufficiently small, so that

$$\mathbf{A} < \left\{ \frac{\kappa[1 - \max(m_i^p a_i^i, \tilde{m}_i^p \tilde{a}_i^i)]}{C(1+\kappa)} \right\}^{1/\theta}. \tag{3.25}$$

Moreover, by the definition of $\mathcal{Z}$ and relations (2.7), (2.14) - (2.16), (3.3) and (3.24), we can assert that $\tilde{v}_i$ and $\tilde{w}_i$ $(i = 0, 1, \ldots, p-1)$ are continuous in $\mathcal{D}_*$. Thus, inequality (3.25) implies the inclusion $\mathbf{T}(\mathcal{Z}) \subset \mathcal{Z}$.

Now, assuming the validity of (3.25), we shall find sufficient conditions under which the transformation $\mathbf{T}$ (cf. (3.3)) is a contraction.

Let $\Phi = (V, W)$ and $\bar{\Phi} = (\bar{V}, \bar{W})$ be arbitrary points of $\mathcal{Z}$, and $\tilde{\Phi} = (\tilde{V}, \tilde{W})$ and $\tilde{\bar{\Phi}} = (\tilde{\bar{V}}, \widetilde{\bar{W}})$ their images, respectively, in the transformation $\mathbf{T}$. In order to estimate the expression $|\tilde{v}_i - \tilde{\bar{v}}_i|$, let us observe that by (1.8), (2.10) and (3.1) the following inequalities are valid (cf. (3.4)):

$$
\begin{aligned}
&|{}^1\lambda^{rs}_{\Phi}[x, f_i(x)] - {}^1\lambda^{rs}_{\bar{\Phi}}[x, f_i(x)]| \\
&\quad\le \left[\int_0^x \frac{(x-\xi)^{p-s-1}}{(p-s-1)!}(\xi^p + (f_i(x))^p)\,d\xi \right. \\
&\qquad \left. + \sum_{k=0}^{p-s-1} \int_0^{f_i(x)} \frac{(f_i(x)-\xi)^{p-r-1}}{(p-r-1)!}\eta^p\,d\eta\, \frac{x^k}{k!}\right] d(\Phi, \bar{\Phi}) \\
&\quad\le \left[x^{2p-s}(1 + (m_i a_i)^p) + x^{2p-r}(m_i a_i)^{2p-r}\sum_{k=0}^{p-s-1}\frac{\mathbf{A}^k}{k!}\right] d(\Phi, \bar{\Phi})
\end{aligned}
\tag{3.26}
$$

$(r, s = 0, 1, \dots, p-1)$, whence and from Assumptions I, III, IV, and relations (2.8), (2.11), (2.12) we obtain (cf. (3.7))

$$
|\check{\mathbf{G}}^i_{\Phi}(x) - \check{\mathbf{G}}^i_{\bar{\Phi}}(x)| \le \left[(m_i a_i)^i m_i^{p-i} + \text{const}\,\mathbf{A}^{\theta'_i}\right] x^p d(\Phi, \bar{\Phi}) \tag{3.27}
$$

$(x \in (0, A];\ i = 0, 1, \dots, p-1)$. Furthermore, basing on Assumptions I – IV and formulas (2.9), (2.13) and (3.26), we get the inequality

$$
|\hat{\mathbf{G}}^i_{\Phi}(x) - \hat{\mathbf{G}}^i_{\bar{\Phi}}(x)| \le \text{const}\,\mathbf{A}^{\theta'''_i} x^p d(\Phi, \bar{\Phi}) \tag{3.28}
$$

$(x \in (0, A];\ i = 0, 1, \dots, p-1)$, where θ'''_i are as in (3.19). On joining (3.27) and (3.28) we have (cp. (2.7))

$$
|\mathbf{G}^i_{\Phi}(x) - \mathbf{G}^i_{\bar{\Phi}}(x)| \le \left[(m_i a_i)^i m_i^{p-i} + \text{const}\,\mathbf{A}^{\theta_i}\right] x^p d(\Phi, \bar{\Phi}) \tag{3.29}
$$

$(x \in [0, A];\ i = 0, 1, \dots, p-1)$, where θ_i are as in (3.20).

As for the second term in (2.3), we easily conclude that (cf. (3.21))

$$
\left|\int_{f_i(x)}^y \sigma^i_{\Phi}(x, \eta)\,d\eta - \int_{f_i(x)}^y \sigma^i_{\bar{\Phi}}(x, \eta)\,d\eta\right| \le \text{const}\,\mathbf{A}(x^p + y^p) d(\Phi, \bar{\Phi}) \tag{3.30}
$$

$((x, y) \in \mathcal{D}_*)$, whence, and from (2.7), (2.13), (3.3) and (3.29), we obtain

$$
|\tilde{v}_i(x, y) - \tilde{\bar{v}}_i(x, y)| \le \left[(m_i a_i)^i m_i^{p-i} + \text{const}\,\mathbf{A}^{\theta_i}\right](x^p + y^p) d(\Phi, \bar{\Phi}) \tag{3.31}
$$

$((x, y) \in \mathcal{D}_*;\ i = 0, 1, \dots, p-1)$. By a similar argument we show that

$$
|\tilde{w}_i(x, y) - \tilde{\bar{w}}_i(x, y)| \le \left[(\tilde{m}_i \tilde{a}_i)^i \tilde{m}_i^{p-i} + \text{const}\,\mathbf{A}^{\theta_i}\right](x^p + y^p) d(\Phi, \bar{\Phi}) \tag{3.32}
$$

$((x,y) \in \mathcal{D}_*;\ i = 0,1,\dots,p-1)$, and using (3.1), (3.31) and (3.32) we can conclude that

$$d(\tilde{\Phi}, \tilde{\tilde{\Phi}}) \le \left[\max\left((m_i a_i)^i m_i^{p-i}, (\tilde{m}_i \tilde{a}_i)^i \tilde{m}_i^{p-i}\right) + \text{const}\, \mathbf{A}^{\theta_i}\right] d(\Phi, \bar{\Phi}). \tag{3.33}$$

It follows from (3.33) that the transformation $\mathbf{T}$ (see (3.3)) is a contraction if the inequality

$$\max\left((m_i a_i)^i m_i^{p-i}, (\tilde{m}_i \tilde{a}_i)^i \tilde{m}_i^{p-i}\right) + \tilde{C}\, \mathbf{A}^{\theta} < 1 \tag{3.34}$$

$(i = 0,1,\dots,p-1)$ is fulfilled, where θ is as in (3.24) and $\tilde{C}$ is a positive constant independent of $\mathbf{A}$. Evidently (cp. (1.7)), inequality (3.34) holds good if $\mathbf{A}$ is so small that

$$\mathbf{A} < \left[\frac{1 - \max(m_i^p a_i^i, \tilde{m}_i^p \tilde{a}_i^i)}{\tilde{C}}\right]^{1/\theta}. \tag{3.35}$$

So, if inequalities (3.25) and (3.35) are fulfilled, then by the Banach fixed point theorem applied to the space $\mathcal{Z}$ and transformation $\mathbf{T}$, there is a unique system $\Phi^0 = (V^0, W^0) \in \mathcal{Z}$ satisfying the system of integral-functional equations (2.12) in D_*. Setting

$$V^* = (v_r^*) \qquad \text{where} \quad v_r^*(x,y) = \begin{cases} 0 & \text{for } x = y = 0 \\ v_r^0(x,y) & \text{for } (x,y) \in \mathcal{D}_* \end{cases}$$

and

$$W^* = (w_r^*) \qquad \text{where} \quad w_r^*(x,y) = \begin{cases} 0 & \text{for } x = y = 0 \\ w_r^0(x,y) & \text{for } (x,y) \in \mathcal{D}_* \end{cases}$$

$(r = 0,1,\dots,p-1)$, we get a system $\Phi^* = (V^*, W^*)$ of continuous functions satisfying (2.12) in $\mathcal{D}$. As a result (cp. Lemma 2.3), problem (Σ) has a uniqe solution $z_{00}^* = {}^1\lambda_{\Phi^*}^{00} \in \mathcal{K}_1$ which, by the equivalence of problems $(\mathcal{P})$ and (Σ), is also a unique solution of problem $(\mathcal{P})$.

Thus, we can formulate the following final theorem.

Theorem. *If Assumptions I - IV are satisfied and $\mathbf{A} = \max(A,B)$ is sufficiently small, so that inequalities (3.25) and (3.35) hold good, then problem $(\mathcal{P})$ has a solution. This solution is unique in the class $\mathcal{K}_1$.*

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