

Transportation Cost Inequalities for Neutral Functional Stochastic Equations

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Abstract. By using Girsanov transformation and martingale representation, transportation cost inequalities with respect to both the uniform and the L^2 -distances on the global free path space are established for the segment process associated to a class of neutral functional stochastic differential equations. Neutral functional stochastic partial differential equations are also investigated.

Keywords. Neutral functional stochastic differential equation, transportation cost inequality, Girsanov transformation.

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1. Introduction

Let $(E, \mathcal{B}(E))$ be a measurable space with ρ a symmetric non-negative measurable function on $E \times E$. For any $p \geq 1$ and probability measures μ and ν on $(E, \mathcal{B}(E))$, the L^p -transportation cost (or, the L^p -Wasserstein distance if ρ is a distance) induced by ρ between these two measures is defined by

$$W_{p,\rho}(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left\{ \int_{E \times E} \rho^p(x, y) \pi(dx, dy) \right\}^{\frac{1}{p}},$$

where $\mathcal{C}(\mu, \nu)$ denotes the space of all couplings of μ and ν . In many practical situations, one wants to find reasonable and simple upper bounds for $W_{p,\rho}(\mu, \nu)$, where a fully satisfactory one is given by the transportation cost inequality first found by Talagrand [16] for the standard Gaussian measure μ on \mathbb{R}^d :

$$W_{2,\rho}(\mu, f\mu)^2 \leq 2\mu(f \log f), \quad f \geq 0, \quad \mu(f) = 1$$

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with $\rho(x, y) := |x - y|$. Since then, this type transportation cost inequality has been intensively investigated and applied for various different distributions. The importance of the study lies on intrinsic links of the transportation cost inequality to several crucial subjects, such as functional inequalities, concentration phenomena, optimal transport problem, and large deviations, see, e.g., [1, 2, 6, 8, 10, 13, 17, 19, 22] and references within.

In the past decade, a plenty of results have been published concerning Talagrand-type transportation cost inequalities on the path spaces of stochastic processes, see, e.g., [5, 25, 26] for diffusion processes on \mathbb{R}^d , [14] for multidimensional semi-martingales, [18] for diffusion processes with history-dependent drift, [21, 22] for diffusion processes on Riemannian manifolds, [24] for SDEs driven by pure jump processes, and [11] for SDEs driven by both Gaussian and jump noises. Recently, transportation cost inequalities for the reflecting diffusion processes on manifolds with boundary have been used in [23] to characterize the curvature of the generator and the convexity of the boundary.

Moreover, many different arguments have been developed to establish the transportation cost inequality. Among others, the Girsanov transformation argument introduced in [5] has been efficiently applied, see, e.g., [26] for infinite-dimensional dynamical systems, [14] for time-inhomogeneous diffusions, [18] for multi-valued SDEs and singular SDEs, and [15] for SDEs driven by a fractional Brownian motion. Following this line, in this paper we aim to establish transportation cost inequalities for the segment processes associated to a class of neutral functional SDEs, which is unknown so far. The point of our study is not the construction of the coupling as it is now more or less standard in the literature, but lies on the technical details to derive from the coupling reasonable estimates for which difficulties caused by the neutral part and functional coefficients have to be carefully managed.

Recall that a differential equation is called neutral if, besides the derivatives of the present state of the system, those of the past history are also involved (see [12]). Let $\mathcal{C} := C([- \tau, 0]; \mathbb{R}^d)$ for some constant $\tau > 0$, which is a Banach space with the uniform norm $\|\cdot\|_\infty$. Let \mathcal{C} be equipped with the Borel σ -field induced by $\|\cdot\|_\infty$. For any $h \in C([- \tau, \infty); \mathbb{R}^d)$ and $t \geq 0$, let $h_t \in \mathcal{C}$ such that $h_t(\theta) = h(t + \theta)$, $\theta \in [- \tau, 0]$. We consider the following neutral functional SDE on \mathbb{R}^d :

$$\begin{cases} d\{X(t) - G(X_t)\} = b(X_t)dt + \sigma(X_t)dW(t), & t \in [0, T], \\ X_0 = \xi \in \mathcal{C}, \end{cases} \quad (1)$$

where $G, b : \mathcal{C} \rightarrow \mathbb{R}^d$ and $\sigma : \mathcal{C} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ are Lipschitz continuous on bounded sets, and $W(\cdot)$ is an \mathbb{R}^m -valued Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Throughout this paper, we assume that for any initial data X_0 , a \mathcal{C} -valued random variable independent of $W(\cdot)$, this equation has a unique global solution. This can be

ensured by the strict contraction of G , i.e. $|G(\xi) - G(\eta)| \leq \kappa \|\xi - \eta\|_\infty$ holds for some constant $\kappa \in [0, 1)$ and all $\xi, \eta \in \mathcal{C}$, together with the usual monotonicity and coercivity conditions of b and σ , see, e.g., [20, Theorem 2.3]. We note that the segment process $(X_t)_{t \geq 0}$ of the solution is a Markov process.

As in [23], we allow the initial data of the equation to be random, i.e. we consider the transportation cost inequality for the law of the solution starting from a probability measure μ on \mathcal{C} . In Section 2 we study the transportation cost inequality with respect to the uniform distance on path space, while in Section 3 we consider the L^2 -distance. Finally, in Section 4, we extend our results to a class of neutral functional SPDEs.

2. The uniform distance

Let $T > 0$ be fixed. For any $\xi \in \mathcal{C}$, let Π_ξ^T be the distribution of $X_{[0,T]} := (X_t)_{t \in [0,T]}$ for the solution to (1) with $X_0 = \xi$. Then, for any $\mu \in \mathcal{P}(\mathcal{C})$, the set of all probability measures on \mathcal{C} , the distribution of $X_{[0,T]}$ with initial distribution μ is given by $\Pi_\mu^T = \int_{\mathcal{C}} \Pi_\xi^T \mu(d\xi)$.

For any probability density function F of Π_μ^T , i.e. F is a non-negative measurable function on the free path space $C([0, T]; \mathcal{C})$ such that $\Pi_\mu^T(F) := \int_{\mathcal{C}} F d\Pi_\mu^T = 1$, let μ_F^T be the marginal distribution of $F\Pi_\mu^T$ at time 0. We have

$$\mu_F^T(d\xi) = \Pi_\xi^T(F)\mu(d\xi) \in \mathcal{P}(\mathcal{C}).$$

Let $\|\cdot\|$ and $\|\cdot\|_{HS}$ denote the operator norm and the Hilbert-Schmidt norm respectively.

To establish the transportation cost inequality for Π_μ^T with respect to the uniform distance

$$\rho_\infty^T(\bar{\xi}, \bar{\eta}) := \sup_{t \in [0, T]} \|\bar{\xi}_t - \bar{\eta}_t\|_\infty, \quad \bar{\xi}, \bar{\eta} \in C([0, T]; \mathcal{C}), \tag{2}$$

we shall need the following conditions.

(A1) There exists a constant $\kappa \in [0, 1)$ such that

$$|G(\xi) - G(\eta)| \leq \kappa \|\xi - \eta\|_\infty, \quad \xi, \eta \in \mathcal{C}.$$

(A2) There exist constants $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \geq 0$ such that

$$\left. \begin{aligned} 2\langle \xi(0) - \eta(0) - G(\xi) + G(\eta), b(\xi) - b(\eta) \rangle \\ + \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \leq \lambda_1 \|\xi - \eta\|_\infty^2 \\ \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \leq \lambda_2 \|\xi - \eta\|_\infty^2, \end{aligned} \right\} \xi, \eta \in \mathcal{C}.$$

(A3) There exists a constant $\lambda_3 > 0$ such that $\|\sigma(\xi)\| \leq \lambda_3$ for all $\xi \in \mathcal{C}$.

Let $\lambda_1^+ = 0 \vee \lambda_1$ and $\lambda_1^- = 0 \vee (-\lambda_1)$. We will need the following two quantities:

$$\alpha(T) := \frac{2\lambda_3(1+\kappa)^2}{(1-\kappa)^2} \min \left\{ \frac{(4\sqrt{\lambda_2} + \sqrt{16\lambda_2 + \lambda_1^+})^2}{(\lambda_1^+)^2}, \frac{4T \exp \left[1 + \frac{2\lambda_1^- + 4\lambda_2}{(1-\kappa)^2} T \right]}{2T\lambda_1^+ + (1-\kappa)^2} \right\} \quad (3)$$

$$\beta(T) := 1 + \frac{(1+\kappa)^2}{(1-\kappa)^2} \min \left\{ \frac{(2\sqrt{\lambda_2} + \sqrt{4\lambda_2 + \lambda_1^+})^2}{\lambda_1^+}, 2 \exp \left[\frac{2\lambda_1^- + 16\lambda_2}{(1-\kappa)^2} T \right] \right\}. \quad (4)$$

The main result of this section is the following.

Theorem 2.1. *Assume (A1)–(A3) and let*

$$\rho(\xi, \eta) := \|\xi - \eta\|_\infty, \quad \xi, \eta \in \mathcal{C}.$$

Then for any $T > 0, \mu \in \mathcal{P}(\mathcal{C})$ and non-negative measurable function F on $\mathcal{C}([0, T]; \mathcal{C})$ such that $\Pi_\mu^T(F) = 1$,

$$W_{2, \rho_\infty^T}(F\Pi_\mu^T, \Pi_\mu^T) \leq \sqrt{\beta(T)}W_{2, \rho}(\mu, \mu_F^T) + \sqrt{\alpha(T)}\sqrt{\Pi_\mu^T(F \log F)}. \quad (5)$$

If moreover μ satisfies the transportation cost inequality

$$W_{2, \rho}(\mu, f\mu)^2 \leq c_\mu \mu(f \log f), \quad f \geq 0, \quad \mu(f) = 1 \quad (6)$$

for some constant $c_\mu > 0$, then

$$W_{2, \rho_\infty^T}(F\Pi_\mu^T, \Pi_\mu^T)^2 \leq \left(\sqrt{\alpha(T)} + \sqrt{c_\mu \beta(T)} \right)^2 \Pi_\mu^T(F \log F). \quad (7)$$

Proof. The proof is based on the following Lemma 2.2 and Lemma 2.3. By the triangle inequality it follows that

$$W_{2, \rho_\infty^T}(F\Pi_\mu^T, \Pi_\mu^T) \leq W_{2, \rho_\infty^T}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T) + W_{2, \rho_\infty^T}(\Pi_\mu^T, \Pi_{\mu_F^T}^T).$$

Then (5) follows from Lemma 2.2 and Lemma 2.3, and (7) is a direct consequence of (5) and (6). \square

Let $\mu = \delta_\xi$ for $\xi \in \mathcal{C}$. Then (6) holds for $c_\mu = 0$, so that (7) becomes

$$W_{2, \rho_\infty^T}(F\Pi_\xi^T, \Pi_\xi^T)^2 \leq \alpha(T)\Pi_\xi^T(F \log F).$$

This inequality also follows from the following lemma since in this case we have $\mu = \mu_F^T = \delta_\xi$.

Lemma 2.2. *Assume (A1)–(A3). For any $\mu \in \mathcal{P}(\mathcal{C})$ and $T > 0$,*

$$W_{2, \rho_\infty^T}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \leq \alpha(T)\Pi_\mu^T(F \log F), \quad F \geq 0, \quad \Pi_\mu^T(F) = 1.$$

Proof. The main idea of the proof is taken from [23, Proof of Theorem 1.1], which indeed goes back to [5]. According to (b) in the proof of [23, Theorem 1.1], we may and do assume that $\mu = \delta_\xi, \xi \in \mathcal{C}$. In this case $\Pi_\mu^T = \Pi_{\mu_F^T}^T = \Pi_\xi^T$. For a positive bounded measurable function F on $C([0, T]; \mathcal{C})$ such that $\Pi_\xi^T(F) = 1$ and $\inf F > 0$, define

$$m(t) := \mathbb{E}(F(X_{[0,T]})|\mathcal{F}_t) \quad \text{and} \quad L(t) := \int_0^t \frac{dm(s)}{m(s)}, \quad t \in [0, T],$$

where \mathbb{E} is the expectation taken for the probability measure \mathbb{P} . Then $m(t)$ and $L(t)$ are square-integrable \mathcal{F}_t -martingales under \mathbb{P} due to $\inf F > 0$ and the boundedness of F . Note by the Itô formula that

$$m(t) = e^{L(t) - \frac{1}{2}\langle L \rangle(t)}, \tag{8}$$

where $\langle L \rangle(t)$ denotes the quadratic variation process of $L(t)$, and, by the martingale representation theorem, e.g., [9, Theorem 6.6], there exists a unique \mathbb{R}^m -valued \mathcal{F}_t -predictable process $h(t)$ such that

$$L(t) = \int_0^t \langle h(s), dW(s) \rangle. \tag{9}$$

Since $F(X_{[0,T]})$ is \mathcal{F}_T -measurable and $\langle L \rangle(t) = \int_0^t |h(s)|^2 ds$, it then follows from (8) and (9) that

$$F(X_{[0,T]}) = m(T) = \exp \left[\int_0^T \langle h(s), dW(s) \rangle - \int_0^T |h(s)|^2 ds \right].$$

Let

$$d\mathbb{Q} = F(X_{[0,T]})d\mathbb{P}.$$

Then \mathbb{Q} is a probability measure on Ω due to $\Pi_\xi^T(F) = 1$. To prove the desired inequality, we need to characterize $\Pi_\xi^T(F \log F)$ and $W_{2, \rho_\infty^T}(F \Pi_\xi^T, \Pi_\xi^T)$ respectively.

(i) Recall that $F(X_{[0,T]}) = m(T)$, $m(t)$ is a square-integrable \mathcal{F}_t -martingale under \mathbb{P} , and observing that $h(s)$ is \mathcal{F}_s -measurable. We have

$$\mathbb{E}_\mathbb{Q}|h(s)|^2 = \mathbb{E}(m(T)|h(s)|^2) = \mathbb{E}(|h(s)|^2 \mathbb{E}(m(T)|\mathcal{F}_s)) = \mathbb{E}(|h(s)|^2 m(s)).$$

Moreover, by the Itô formula

$$\begin{aligned} d(m(s) \log m(s)) &= (1 + \log m(s))dm(s) + \frac{d\langle m \rangle(s)}{2m(s)} \\ &= (1 + \log m(s))dm(s) + \frac{m(s)}{2}|h(s)|^2 ds, \end{aligned} \tag{10}$$

where we have used the fact that $d\langle m \rangle(s) = m^2(s)d\langle L \rangle(s) = m^2(s)|h(s)|^2 ds$. Since $m(t)$ is a square-integrable \mathcal{F}_t -martingale under \mathbb{P} , integrating from 0 to T and taking expectations with respect to \mathbb{P} on both sides of (10), we get

$$\Pi_\xi^T(F \log F) = \mathbb{E}(m(T) \log m(T)) = \frac{1}{2} \int_0^T \mathbb{E}(m(t)|h(t)|^2) dt = \frac{1}{2} \int_0^T \mathbb{E}_\mathbb{Q}|h(t)|^2 dt. \tag{11}$$

(ii) Recalling that $m(t)$ is a square-integrable \mathcal{F}_t -martingale under \mathbb{P} , we deduce from the Girsanov theorem that

$$\tilde{W}(t) := W(t) - \int_0^t h(s) ds \tag{12}$$

is an m -dimensional \mathcal{F}_t -Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Reformulate (1) as

$$\begin{cases} d\{X(t) - G(X_t)\} = \{b(X_t) + \sigma(X_t)h(t)\}dt + \sigma(X_t)d\tilde{W}(t), & t \in [0, T], \\ X_0 = \xi. \end{cases}$$

Noting that the law of $X_{[0,T]}$ under \mathbb{P} is Π_ξ^T and $d\mathbb{Q} = F(X_{[0,T]})d\mathbb{P}$, for any bounded measurable function G on $C([0, T]; \mathcal{C})$, we have

$$\mathbb{E}_\mathbb{Q}(G(X_{[0,T]})) = \mathbb{E}(FG)(X_{[0,T]}) = \Pi_\xi^T(FG).$$

Hence the law of $X_{[0,T]}$ under \mathbb{Q} is $F\Pi_\xi^T$. Next, consider the following equation

$$\begin{cases} d\{Y(t) - G(Y_t)\} = b(Y_t)dt + \sigma(Y_t)d\tilde{W}(t), & t \in [0, T], \\ Y_0 = \xi. \end{cases}$$

Since $\tilde{W}(t)$ is the Brownian motion under \mathbb{Q} , we conclude that the law of $Y_{[0,T]}$ under \mathbb{Q} is Π_ξ^T . This, together with $X_0 = Y_0$ and the law of $X_{[0,T]}$ under \mathbb{Q} is $F\Pi_\xi^T$, leads to

$$W_{2, \rho_\infty^T}(F\Pi_\xi^T, \Pi_\xi^T)^2 \leq \mathbb{E}_\mathbb{Q} \rho_\infty^T(X_{[0,T]}, Y_{[0,T]})^2 = \mathbb{E}_\mathbb{Q} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2 \right). \tag{13}$$

Now, combining (13) with (11), we need only to prove the inequality

$$\mathbb{E}_\mathbb{Q} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2 \right) \leq \frac{\alpha(T)}{2} \int_0^T \mathbb{E}_\mathbb{Q}|h(t)|^2 dt. \tag{14}$$

Let $M(t) = (X(t) - Y(t)) + (G(Y_t) - G(X_t))$. By (A1) and the inequality

$$(a + b)^2 \leq (1 + \epsilon) \left(a^2 + \frac{b^2}{\epsilon} \right), \quad \epsilon > 0, \tag{15}$$

we obtain that

$$|M(s)|^2 \leq (1+\kappa) \left(|X(s)-Y(s)|^2 + \frac{|G(Y_s)-G(X_s)|^2}{\kappa} \right) \leq (1+\kappa)^2 \|X_s - Y_s\|_\infty^2, \quad (16)$$

and

$$|X(s)-Y(s)|^2 = |M(s) + (G(X_s) - G(Y_s))|^2 \leq \kappa \|X_s - Y_s\|_\infty^2 + \frac{1}{1-\kappa} |M(s)|^2. \quad (17)$$

It thus follows from $X_0 = Y_0$ that

$$(1-\kappa)^2 \sup_{0 \leq s \leq t} |X(s)-Y(s)|^2 \leq \sup_{0 \leq s \leq t} |M(s)|^2 \leq (1+\kappa)^2 \sup_{0 \leq s \leq t} |X(s)-Y(s)|^2. \quad (18)$$

By (A2), (A3) and Itô's formula, one has

$$\begin{aligned} d|M(t)|^2 &\leq 2\langle M(t), (\sigma(X_t) - \sigma(Y_t))d\tilde{W}(t) \rangle \\ &\quad + \left(2\sqrt{\lambda_3} |M(t)| \cdot |h(t)| - \lambda_1 \|X_t - Y_t\|_\infty^2 \right) dt, \end{aligned}$$

which, together with the inequality $2ab \leq \delta a^2 + \frac{b^2}{\delta}$, $\delta > 0$, and (16), gives that

$$\begin{aligned} d|M(t)|^2 &\leq 2\langle M(t), (\sigma(X_t) - \sigma(Y_t))d\tilde{W}(t) \rangle \\ &\quad + \left(\frac{\lambda_3}{\delta} (1+\kappa)^2 |h(t)|^2 + (\delta - \lambda_1) \|X_t - Y_t\|_\infty^2 \right) dt, \quad \delta > 0. \end{aligned} \quad (19)$$

Due to the Burkhold-Davis-Gundy inequality and (A2), this implies that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq s \leq t} |M(s)|^2 \right) \\ &\leq 4\sqrt{\lambda_2} \mathbb{E}_{\mathbb{Q}} \left(\int_0^t |M(s)|^2 \|X_s - Y_s\|_\infty^2 ds \right)^{\frac{1}{2}} \\ &\quad + (\delta - \lambda_1)^+ \mathbb{E}_{\mathbb{Q}} \int_0^t \|X_s - Y_s\|_\infty^2 ds + \frac{\lambda_3}{\delta} (1+\kappa)^2 \int_0^t \mathbb{E}_{\mathbb{Q}} |h(s)|^2 ds \\ &\leq \left((\delta - \lambda_1)^+ + \frac{4\lambda_2}{\varepsilon} \right) \mathbb{E}_{\mathbb{Q}} \int_0^t \|X_s - Y_s\|_\infty^2 ds + \varepsilon \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq s \leq t} |M(s)|^2 \right) \\ &\quad + \frac{\lambda_3(1+\kappa)^2}{\delta} \int_0^t \mathbb{E}_{\mathbb{Q}} |h(s)|^2 ds, \quad \delta > 0, \varepsilon \in (0, 1). \end{aligned}$$

By an approximation argument using stopping times, we may assume that $\mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq s \leq t} |M(s)|^2 \right) < \infty$, so that this is equivalent to

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq s \leq t} |M(s)|^2 \right) &\leq \left(\frac{(\delta - \lambda_1)^+}{1 - \varepsilon} + \frac{4\lambda_2}{\varepsilon(1 - \varepsilon)} \right) \mathbb{E}_{\mathbb{Q}} \int_0^t \|X_s - Y_s\|_\infty^2 ds \\ &\quad + \frac{\lambda_3(1+\kappa)^2}{\delta(1 - \varepsilon)} \int_0^t \mathbb{E}_{\mathbb{Q}} |h(s)|^2 ds, \quad \delta > 0, \varepsilon \in (0, 1). \end{aligned}$$

Thus, (18) yields that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq s \leq t} |X(s) - Y(s)|^2 \right) \\ & \leq \frac{\varepsilon(\delta - \lambda_1)^+ + 4\lambda_2}{\varepsilon(1 - \varepsilon)(1 - \kappa)^2} \mathbb{E}_{\mathbb{Q}} \int_0^t \|X_s - Y_s\|_{\infty}^2 ds + \frac{\lambda_3(1 + \kappa)^2}{\delta(1 - \kappa)^2(1 - \varepsilon)} \int_0^t \mathbb{E}_{\mathbb{Q}} |h(s)|^2 ds, \end{aligned} \tag{20}$$

Then, by the Gronwall inequality,

$$\mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2 \right) \leq \frac{\lambda_3(1 + \kappa)^2 \exp \left[\frac{\varepsilon(\delta - \lambda_1)^+ + 4\lambda_2}{\varepsilon(1 - \varepsilon)(1 - \kappa)^2} T \right]}{\delta(1 - \kappa)^2(1 - \varepsilon)} \int_0^T \mathbb{E}_{\mathbb{Q}} |h(t)|^2 dt$$

holds for all $\delta > 0$ and $\varepsilon \in (0, 1)$. Taking $\varepsilon = \frac{1}{2}$ and $\delta = \lambda_1^+ + \frac{(1 - \kappa)^2}{2T}$, we obtain

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2 \right) \\ & \leq \frac{4\lambda_3(1 + \kappa)^2 T \exp \left[1 + \frac{2\lambda_1^+ + 16\lambda_2}{(1 - \kappa)^2} T \right]}{(1 - \kappa)^2 \{2T\lambda_1^+ + (1 - \kappa)^2\}} \int_0^T \mathbb{E}_{\mathbb{Q}} |h(t)|^2 dt. \end{aligned} \tag{21}$$

On the other hand, if $\lambda_1 > 0$, taking $\delta = \frac{\lambda_1}{2}$ in (19) we obtain

$$\mathbb{E}_{\mathbb{Q}} \int_0^t \|X_s - Y_s\|_{\infty}^2 ds \leq \frac{4\lambda_3(1 + \kappa)^2}{\lambda_1^2} \int_0^t \mathbb{E}_{\mathbb{Q}} |h(s)|^2 ds.$$

Combining this with (20) where $\delta = \lambda_1$ we derive

$$\mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2 \right) \leq \frac{\lambda_3(1 + \kappa)^2}{\lambda_1(1 - \kappa)^2} \left(\frac{16\lambda_2}{\varepsilon(1 - \varepsilon)\lambda_1} + \frac{1}{1 - \varepsilon} \right) \int_0^T \mathbb{E}_{\mathbb{Q}} |h(s)|^2 ds.$$

Taking the optimal choice

$$\varepsilon = \frac{4\sqrt{\lambda_2}}{4\sqrt{\lambda_2} + \sqrt{16\lambda_2 + \lambda_1}},$$

we conclude that

$$\mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^2 \right) \leq \frac{\lambda_3(1 + \kappa)^2 (4\sqrt{\lambda_2} + \sqrt{16\lambda_2 + \lambda_1})^2}{\lambda_1^2(1 - \kappa)^2} \int_0^T \mathbb{E}_{\mathbb{Q}} |h(s)|^2 ds.$$

Combining this with (21) we prove (14), and hence, finish the proof. □

Lemma 2.3. *Let (A1) and (A2) hold. Then*

$$W_{2, \rho_{\infty}^T}(\Pi_{\nu}^T, \Pi_{\mu}^T)^2 \leq \beta(T) W_{2, \rho}(\nu, \mu)^2, \quad \nu, \mu \in \mathcal{P}(\mathcal{C}).$$

Proof. Let $\{X(t)\}_{t \geq 0}, \{Y(t)\}_{t \geq 0}$ be the solutions to (1) with $X_0 = \xi$ and $Y_0 = \eta$, where ξ and η are \mathcal{C} -valued random variables with distributions μ and ν respectively and are independent of $W(\cdot)$ such that

$$\mathbb{E}(\|\xi - \eta\|_\infty^2) = W_{2,\rho}(\nu, \mu)^2.$$

Then it suffices to show that

$$\mathbb{E}\left(\sup_{t \in [0, T]} \|X_t - Y_t\|_\infty^2\right) \leq \beta(T)\mathbb{E}(\|\xi - \eta\|_\infty^2). \tag{22}$$

Let $h = 0$. We have $\tilde{W} = W$ so that (19) still holds for W in place of \tilde{W} . Combining it with (16), we obtain that when $\lambda_1 > 0$,

$$\mathbb{E} \int_0^t \|X_s - Y_s\|_\infty^2 ds \leq \frac{1}{\lambda_1} \mathbb{E}|M(0)|^2 \leq \frac{(1 + \kappa)^2}{\lambda_1} \mathbb{E}\|\xi - \eta\|_\infty^2. \tag{23}$$

Similarly, since in the present case $h = 0$ and according to (16), $|M(0)|^2 \leq (1 + \kappa)^2 \|\xi - \eta\|_\infty^2$, we may take $\delta = 0$ in the argument leading to (20) to derive that

$$\begin{aligned} & \mathbb{E}\left(\sup_{0 \leq s \leq t} |X(s) - Y(s)|^2\right) \\ & \leq \frac{\varepsilon \lambda_1^- + 4\lambda_2}{\varepsilon(1 - \varepsilon)(1 - \kappa)^2} \mathbb{E} \int_0^t \|X_s - Y_s\|_\infty^2 ds + \frac{(1 + \kappa)^2}{(1 - \varepsilon)(1 - \kappa)^2} \mathbb{E}\|\xi - \eta\|_\infty^2 \end{aligned} \tag{24}$$

for $\varepsilon \in (0, 1)$. When $\lambda_1 > 0$, combining this with (23) we arrive at

$$\begin{aligned} \mathbb{E}\left(\sup_{t \in [0, T]} \|X_t - Y_t\|_\infty^2\right) & \leq \mathbb{E}\left(\sup_{s \in [0, T]} |X(s) - Y(s)|^2\right) + \mathbb{E}\|\xi - \eta\|_\infty^2 \\ & \leq \left\{1 + \frac{(1 + \kappa)^2}{(1 - \kappa)^2} \left(\frac{1}{1 - \varepsilon} + \frac{4\lambda_2}{\varepsilon(1 - \varepsilon)\lambda_1}\right)\right\} \mathbb{E}\|\xi - \eta\|_\infty^2. \end{aligned}$$

Taking

$$\varepsilon = \frac{2\sqrt{\lambda_2}}{2\sqrt{\lambda_2} + \sqrt{4\lambda_2 + \lambda_1}}$$

we deduce that

$$\mathbb{E}\left(\sup_{t \in [0, T]} \|X_t - Y_t\|_\infty^2\right) \leq \left(1 + \frac{(1 + \kappa)^2(2\sqrt{\lambda_2} + \sqrt{4\lambda_2 + \lambda_1})^2}{\lambda_1(1 - \kappa)^2}\right) \mathbb{E}\|\xi - \eta\|_\infty^2 \tag{25}$$

for $\lambda_1 > 0$. In general, by the Gronwall inequality, (24) yields that

$$\begin{aligned} \mathbb{E}\left(\sup_{t \in [0, T]} \|X_t - Y_t\|_\infty^2\right) & \leq \mathbb{E}\left(\sup_{s \in [0, T]} |X(s) - Y(s)|^2\right) + \mathbb{E}\|\xi - \eta\|_\infty^2 \\ & \leq \left(1 + \frac{(1 + \kappa)^2}{(1 - \varepsilon)(1 - \kappa)^2} \exp\left[\frac{\varepsilon \lambda_1^- + 4\lambda_2}{\varepsilon(1 - \varepsilon)(1 - \kappa)^2} T\right]\right) \mathbb{E}\|\xi - \eta\|_\infty^2. \end{aligned}$$

Taking $\varepsilon = \frac{1}{2}$ we obtain

$$\mathbb{E}\left(\sup_{t \in [0, T]} \|X_t - Y_t\|_\infty^2\right) \leq \left(1 + \frac{2(1 + \kappa)^2}{(1 - \kappa)^2} \exp\left[\frac{2\lambda_1^- + 16\lambda_2}{(1 - \kappa)^2} T\right]\right) \mathbb{E}\|\xi - \eta\|_\infty^2.$$

Combining this with (25) we prove (22), and hence, finish the proof. \square

Remark 2.4. Obviously, when $\lambda_1 > 0$ both $\alpha(T)$ and $\beta(T)$ are bounded in T , so that Theorem 2.1 works also for $T = \infty$, i.e. on the global free path space $C([0, \infty); \mathcal{C})$. Precisely, let Π_μ and Π_ξ denote the distribution of $X_{[0, \infty)}$ with initial distributions μ and δ_ξ respectively, let $\mu_F(d\xi) = \Pi_\xi(F)\mu(d\xi)$, and let

$$\rho_\infty(\bar{\xi}, \bar{\eta}) = \sup_{t \geq 0} \rho_\infty(\bar{\xi}_t, \bar{\eta}_t), \quad \bar{\xi}, \bar{\eta} \in C([0, \infty); \mathcal{C}).$$

If $\lambda_1 > 0$, then Theorem 2.1 implies

$$\begin{aligned} W_{2, \rho_\infty}(F\Pi_\mu, \Pi_\mu) &\leq \frac{\sqrt{2\lambda_3}(1 + \kappa)(4\sqrt{\lambda_2} + \sqrt{16\lambda_2 + \lambda_1})}{(1 - \kappa)\lambda_1} \sqrt{\Pi_\mu(F \log F)} \\ &\quad + \left(1 + \frac{(1 + k)(2\sqrt{\lambda_2} + \sqrt{4\lambda_2 + \lambda_1})}{(1 - \kappa)\sqrt{\lambda_1}}\right) W_{2, \rho}(\mu, \mu_F). \end{aligned}$$

In general, for any $\lambda_1 \in \mathbb{R}$, we can find $\lambda > 0$ and constants $C_1(\lambda), C_2(\lambda) > 0$ such that

$$W_{2, \rho_{\infty, \lambda}}(F\Pi_\mu, \Pi_\mu) \leq C_1(\lambda) \sqrt{\Pi_\mu(F \log F)} + C_2(\lambda) W_{2, \rho}(\mu, \mu_F), \quad (26)$$

where

$$\rho_{\infty, \lambda}(\bar{\xi}, \bar{\eta}) := \sup_{t \geq 0} \{e^{-\lambda t} \rho_\infty(\bar{\xi}_t, \bar{\eta}_t)\}, \quad \bar{\xi}, \bar{\eta} \in C([0, \infty); \mathcal{C}).$$

Indeed, for any $\lambda > \frac{\lambda_1^- + 8\lambda_2}{(1 - k)^2}$,

$$\sum_{n=1}^{\infty} e^{-2\lambda n} \{\alpha(n) + \beta(n)\} < \infty. \quad (27)$$

Noting that $\rho_{\infty, \lambda}(\bar{\xi}, \bar{\eta})^2 \leq \sum_{n=1}^{\infty} e^{-2\lambda(n-1)} \rho_\infty^n(\bar{\xi}_{[0, n]}, \bar{\eta}_{[0, n]})^2$, we have

$$W_{2, \rho_{\infty, \lambda}}^2 \leq \sum_{n=1}^{\infty} e^{-2\lambda(n-1)} W_{2, \rho_\infty^n}^2.$$

Combining this with Theorem 2.1 and (27), we may find finite constants $C_1(\lambda) > 0$ and $C_2(\lambda) > 0$ such that (26) holds.

3. The weighted L^2 -distance on $C([0, \infty); \mathcal{C})$

Since for a fixed $T > 0$ the L^2 -distance on $C([0, T]; \mathcal{C})$ is dominated by the uniform norm, the corresponding transportation cost inequality is weaker than that derived in Section 2. So, in this section we only consider the global path space $C([0, \infty); \mathcal{C})$. Let

$$\rho_2(\xi, \eta)^2 = \frac{1}{\tau} \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|^2 d\theta, \quad \xi, \eta \in \mathcal{C}, \tag{28}$$

and for $\lambda \geq 0$ let

$$\rho_{2,\lambda}(\bar{\xi}, \bar{\eta})^2 = \int_0^\infty e^{-\lambda t} \rho_2(\bar{\xi}_t, \bar{\eta}_t)^2 dt, \quad \bar{\xi}, \bar{\eta} \in C([0, \infty); \mathcal{C}). \tag{29}$$

As mentioned in Remark 2.4, let Π_μ and Π_ξ denote the distribution of $X_{[0,\infty)}$ with initial distributions μ and δ_ξ respectively. Let $\mu_F(d\xi) = \Pi_\xi(F)\mu(d\xi)$.

To derive the transportation cost inequality w.r.t. $\rho_{2,\lambda}$, we need the following assumptions to replace (A1) and (A2) in the last section.

(B1) There exists $k \in [0, 1)$ such that

$$|G(\xi) - G(\eta)| \leq k\rho_2(\xi, \eta), \quad \xi, \eta \in \mathcal{C}.$$

(B2) There exist constants $k_1 \in \mathbb{R}, k_2 \geq 0$ and a probability measure Λ on $[-\tau, 0]$ such that

$$\begin{aligned} & 2\langle (\xi(0) - \eta(0)) - G(\xi) + G(\eta), b(\xi) - b(\eta) \rangle + \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \\ & \leq -k_1|\xi(0) - \eta(0)|^2 + k_2 \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|^2 \Lambda(d\theta). \end{aligned}$$

A simple example such that (B1) and (B2) hold is that

$$\begin{aligned} G(\xi) &= \frac{k}{\tau} \int_{-\tau}^0 \xi(\theta) d\theta, \\ b(\xi) &= c_1\xi(0) + \int_{-\tau}^0 \xi(\theta) \Lambda_1(d\theta), \\ \sigma(\xi) &= c_3\xi(0) + \int_{-\tau}^0 \xi(\theta) \Lambda_2(d\theta) \end{aligned}$$

for some constants $k \in (0, 1), c_1 \in \mathbb{R}$ and some finite measures Λ_1, Λ_2 on $[-\tau, 0]$.

Theorem 3.1. *Assume (B1), (B2) and (A3). Let $\tilde{\rho}_2(\xi, \eta)^2 = |\xi(0) - \eta(0)|^2 + \rho_2(\xi, \eta)^2, \xi, \eta \in \mathcal{C}$. Let $\mu \in \mathcal{P}(\mathcal{C})$ and F be non-negative measurable function F on $C([0, \infty); \mathcal{C})$ such that $\Pi_\mu(F) = 1$.*

(1) *If $k_1 > k_2$ then*

$$W_{2, \rho_{2,0}}(\Pi_\mu, F\Pi_\mu) \leq \frac{\sqrt{2\lambda_3}\{1 + (1 + k)^2\}}{k_1 - k_2} \sqrt{\Pi_\mu(F \log F)} + \sqrt{\tau + \frac{k_2\tau + 1 + k}{k_1 - k_2}} W_{2, \tilde{\rho}_2}(\mu, \mu_F).$$

(2) *If $k_1 \leq k_2$ then for any $\lambda > \frac{k_2 - k_1}{(1 - k)^2}$,*

$$W_{2, \rho_{2,\lambda}}(\Pi_\mu, F\Pi_\mu) \leq \frac{\sqrt{2\lambda_3}\{1 + (1 + k)^2\}}{k_1 - k_2 + \lambda(1 - k)^2} \sqrt{\Pi_\mu(F \log F)} + \sqrt{\tau + \frac{\lambda k(1 - k)\tau + k_2\tau + 1 + k}{\lambda(1 - k)^2 + k_1 - k_2}} W_{2, \tilde{\rho}_2}(\mu, \mu_F).$$

As explained in the proof of Theorem 2.1 that the result follows immediately from Lemmas 3.3 and 3.4 below. To prove these lemmas, we first collect some simple facts.

Lemma 3.2. *Assume (B1). Let $t > 0, \lambda \geq 0, \bar{\xi}, \bar{\eta} \in C([0, t]; \mathcal{C})$, and Λ be a probability measure on $[-\tau, 0]$. Let*

$$\bar{M}(s) = \bar{\xi}(s) - \bar{\eta}(s) - G(\bar{\xi}_s) + G(\bar{\eta}_s).$$

Then

- (a) $\int_0^t e^{-\lambda s} ds \int_{-\tau}^0 |\bar{\xi}(s+\theta) - \bar{\eta}(s+\theta)|^2 \Lambda(d\theta) \leq \tau \rho_2(\bar{\xi}_0, \bar{\eta}_0)^2 + \int_0^t e^{-\lambda s} |\bar{\xi}(s) - \bar{\eta}(s)|^2 ds.$
- (b) $\int_0^t e^{-\lambda s} |\bar{M}(s)|^2 ds \leq (1 + k)^2 \int_0^t e^{-\lambda s} |\bar{\xi}(s) - \bar{\eta}(s)|^2 ds + (1 + k)k\tau \rho_2(\bar{\xi}_0, \bar{\eta}_0)^2.$
- (c) $\int_0^t e^{-\lambda s} |\bar{\xi}(s) - \bar{\eta}(s)|^2 ds \leq \frac{1}{(1 - k)^2} \int_0^t e^{-\lambda s} |\bar{M}(s)|^2 ds + \frac{k\tau}{1 - k} \rho_2(\bar{\xi}_0, \bar{\eta}_0)^2.$

Proof. (a) By the Fubini theorem, we have

$$\int_0^t e^{-\lambda s} ds \int_{-\tau}^0 |\bar{\xi}(s+\theta) - \bar{\eta}(s+\theta)|^2 \Lambda(d\theta) = \int_{-\tau}^0 \Lambda(d\theta) \int_\theta^{t+\theta} e^{-\lambda(s-\theta)} |\bar{\xi}(s) - \bar{\eta}(s)|^2 ds \leq \int_0^t e^{-\lambda s} |\bar{\xi}(s) - \bar{\eta}(s)|^2 ds + \tau \rho_2(\bar{\xi}_0, \bar{\eta}_0)^2.$$

(b) By (B1) and applying (15) to $\varepsilon = k$, we obtain

$$|\bar{M}(s)|^2 \leq (1 + k) \{ |\bar{\xi}(s) - \bar{\eta}(s)|^2 + k \rho_2(\bar{\xi}_s, \bar{\eta}_s)^2 \}. \tag{30}$$

Then

$$\int_0^t |\bar{M}(s)|^2 e^{-\lambda s} ds \leq (1+k) \int_0^t \{|\bar{\xi}(s) - \bar{\eta}(s)|^2 + k\rho_2(\bar{\xi}_s, \bar{\eta}_s)^2\} e^{-\lambda s} ds.$$

On the other hand, taking $\Lambda(d\theta) = \frac{1}{\tau} d\theta$ on $[-\tau, 0]$, we have

$$\int_0^t \rho_2(\bar{\xi}_s, \bar{\eta}_s)^2 e^{-\lambda s} ds \leq \int_0^t |\bar{\xi}(s) - \bar{\eta}(s)|^2 e^{-\lambda s} ds + \tau\rho_2(\bar{\xi}_0, \bar{\eta}_0)^2. \tag{31}$$

Therefore, the second assertion follows.

(c) By (B1) and (15) with $\varepsilon = \frac{k}{1-k}$, we have

$$|\bar{\xi}(s) - \bar{\eta}(s)|^2 \leq k\rho_2(\bar{\xi}_s, \bar{\eta}_s)^2 + \frac{1}{1-k} |\bar{M}(s)|^2.$$

Combining this with (31) we arrive at

$$\begin{aligned} & \int_0^t |\bar{\xi}(s) - \bar{\eta}(s)|^2 e^{-\lambda s} ds \\ & \leq k \int_0^t |\bar{\xi}(s) - \bar{\eta}(s)|^2 e^{-\lambda s} ds + k\tau\rho_2(\bar{\xi}_0, \bar{\eta}_0)^2 + \frac{1}{1-k} \int_0^t |\bar{M}(s)|^2 e^{-\lambda s} ds. \end{aligned}$$

This implies the third assertion. □

Lemma 3.3. Assume (B1), (B2) and (A3).

(I) If $k_1 > k_2$ then for any measurable $F \geq 0$ with $\Pi_\mu(F) = 1$,

$$W_{2,\rho_{2,0}}(F\Pi_\mu, \Pi_{\mu_F})^2 \leq \frac{2\lambda_3\{1 + (1+k)^2\}^2}{(k_1 - k_2)^2} \Pi_\mu(F \log F).$$

(II) If $k_1 \leq k_2$ then for any $\lambda > \frac{k_2 - k_1}{(1-k)^2}$ and measurable $F \geq 0$ with $\Pi_\mu(F) = 1$,

$$W_{2,\rho_{2,\lambda}}(F\Pi_\mu, \Pi_{\mu_F}^T)^2 \leq \frac{2\lambda_3\{1 + (1+k)^2\}^2}{\{k_1 - k_2 + \lambda(1-k)^2\}^2} \Pi_\mu(F \log F).$$

Proof. By an approximation argument, it suffices to prove the result for Π_μ^T and $\rho_{2,\lambda}^T$ in place of Π_μ and $\rho_{2,\lambda}$ respectively with arbitrary $T > 0$, where

$$\rho_{2,\lambda}^T(\bar{\xi}, \bar{\eta})^2 := \int_0^T e^{-\lambda t} \rho_2(\bar{\xi}_t, \bar{\eta}_t)^2 dt, \quad \bar{\xi}, \bar{\eta} \in C([0, T]; \mathcal{C}).$$

As indicated in the proof of Lemma 2.2 that we may and do assume $\mu = \delta_\xi$. Let $h, \tilde{W}(t), \mathbb{Q}, X(t), Y(t)$ and $M(t)$ be constructed in the proof of Lemma 2.2. It suffices to prove that

$$\mathbb{E}_{\mathbb{Q}} \int_0^T e^{-\lambda t} \rho_2(X_t, Y_t)^2 dt \leq C(\lambda) \mathbb{E}_{\mathbb{Q}} \int_0^T |h(t)|^2 dt \tag{32}$$

for

$$C(\lambda) = \begin{cases} \frac{\lambda_3\{1+(1+k)^2\}^2}{(k_1-k_2)^2}, & \text{if } k_1 > k_2, \lambda = 0, \\ \frac{\lambda_3\{1+(1+k)^2\}^2}{\{k_1-k_2+\lambda(1-k)^2\}^2}, & \text{if } k_1 \leq k_2, \lambda > \frac{k_2-k_1}{(1-k)^2}. \end{cases}$$

By (B2), (A3) and Itô's formula, we obtain

$$\begin{aligned} & d|M(t)|^2 - 2\langle M(t), \{\sigma(X_t) - \sigma(Y_t)\}d\tilde{W}(t)\rangle \\ & \leq \left\{ k_2 \int_{-\tau}^0 |X(t+\theta) - Y(t+\theta)|^2 \Lambda(d\theta) + 2\sqrt{\lambda_3}|M(t)| \cdot |h(t)| - k_1 |X(t) - Y(t)|^2 \right\} dt \\ & \leq \left\{ k_2 \int_{-\tau}^0 |X(t+\theta) - Y(t+\theta)|^2 \Lambda(d\theta) + \frac{\lambda_3}{\delta} |h(t)|^2 + \delta |M(t)|^2 - k_1 |X(t) - Y(t)|^2 \right\} dt \end{aligned}$$

for $\delta > 0$. Thus, for any $\lambda \geq 0$,

$$\begin{aligned} & d\{e^{-\lambda t}|M(t)|^2\} - 2e^{-\lambda t}\langle M(t), \{\sigma(X_t) - \sigma(Y_t)\}d\tilde{W}(t)\rangle \\ & \leq e^{-\lambda t} \left\{ k_2 \int_{-\tau}^0 |X(t+\theta) - Y(t+\theta)|^2 \Lambda(d\theta) \right. \\ & \quad \left. + \frac{\lambda_3}{\delta} |h(t)|^2 + (\delta - \lambda)|M(t)|^2 - k_1 |X(t) - Y(t)|^2 \right\} dt, \quad \delta > 0. \end{aligned} \tag{33}$$

(a) Let $k_1 > k_2$ and $\lambda = 0$. Combining (33) with Lemma 3.2 and noting that $X_0 = Y_0$, we obtain

$$\begin{aligned} 0 & \leq \mathbb{E}_{\mathbb{Q}} \int_0^T e^{-\lambda t} \left\{ k_2 \int_{-\tau}^0 |X(t+\theta) - Y(t+\theta)|^2 \Lambda(d\theta) \right. \\ & \quad \left. + \frac{\lambda_3 |h(t)|^2}{\delta} + \delta |M(t)|^2 - k_1 |X(t) - Y(t)|^2 \right\} dt \\ & \leq \{k_2 - k_1 + \delta(1+k)^2\} \int_0^T e^{-\lambda t} \mathbb{E}_{\mathbb{Q}} |X(t) - Y(t)|^2 dt + \frac{\lambda_3}{\delta} \int_0^T \mathbb{E}_{\mathbb{Q}} |h(t)|^2 dt. \end{aligned}$$

Taking

$$\delta = \frac{k_1 - k_2}{1 + (1+k)^2},$$

we arrive at

$$\int_0^T e^{-\lambda t} \mathbb{E}_{\mathbb{Q}} |X(t) - Y(t)|^2 dt \leq \frac{\lambda_3\{1+(1+k)^2\}^2}{(k_1-k_2)^2} \int_0^T \mathbb{E}_{\mathbb{Q}} |h(t)|^2 dt.$$

Since by Lemma 3.2 and $X_0 = Y_0$ we have $\int_0^T \rho_2(X_s, Y_s)^2 ds \leq \int_0^T |X(t) - Y(t)|^2 dt$. This implies (32) for $\lambda = 0$ and the desired constant $C(0)$.

(b) Let $k_1 \leq k_2$ and $\lambda > \frac{k_2 - k_1}{(1-k)^2}$. Similarly to (a), by taking

$$\delta = \frac{k_1 - k_2 + \lambda(1-k)^2}{1 + (1-k)^2}$$

in (33), we obtain

$$\mathbb{E}_{\mathbb{Q}} \int_0^T e^{-\lambda t} \rho_2(X_t, Y_t)^2 dt \leq \mathbb{E}_{\mathbb{Q}} \int_0^T e^{-\lambda t} |X(t) - Y(t)|^2 dt \leq C(\lambda) \int_0^T \mathbb{E}_{\mathbb{Q}} |h(t)|^2 dt.$$

Therefore, (32) holds. \square

Lemma 3.4. *Assume (B1) and (B2). Let*

$$\tilde{\rho}_2(\xi, \eta)^2 = |\xi(0) - \eta(0)|^2 + \rho_2(\xi, \eta)^2.$$

Then for any $\lambda \in [0, \infty) \cap (\frac{k_2 - k_1}{(1-k)^2}, \infty)$,

$$W_{2, \rho_{2, \lambda}}(\Pi_{\mu}, \Pi_{\nu})^2 \leq \left(\tau + \frac{\lambda k(1-k)\tau + k_2\tau + 1 + k}{\lambda(1-k)^2 + k_1 - k_2} \right) W_{2, \tilde{\rho}_2}(\mu, \nu)^2, \quad \mu, \nu \in \mathcal{P}(\mathcal{C}).$$

Proof. Let ξ, η be \mathcal{C} -valued random variables with distributions μ and ν respectively, which are independent of $W([0, \infty))$ such that

$$\mathbb{E} \tilde{\rho}_2(\xi, \eta)^2 = W_{2, \tilde{\rho}_2}(\mu, \nu)^2. \quad (34)$$

By (B2) and Itô's formula,

$$\begin{aligned} & d\{e^{-\lambda t} |M(t)|^2\} - 2e^{-\lambda t} \langle M(t), \{\sigma(X_t) - \sigma(Y_t)\} dW(t) \rangle \\ & \leq e^{-\lambda t} \left\{ k_2 \int_{-\tau}^0 |X(t+\theta) - Y(t+\theta)|^2 \Lambda(d\theta) - k_1 |X(t) - Y(t)|^2 - \lambda |M(t)|^2 \right\} dt. \end{aligned}$$

Then, it follows from Lemma 3.2 that

$$\begin{aligned} & \mathbb{E} \left\{ |M(0)|^2 + \int_0^T e^{-\lambda t} \{k_2 - k_1 - \lambda(1-k)^2\} |X(t) - Y(t)|^2 dt \right. \\ & \quad \left. + \{\lambda k(1-k)\tau + k_2\tau\} \rho_2(\xi, \eta)^2 \right\} \\ & \geq 0. \end{aligned}$$

Since due to (30)

$$|M(0)|^2 \leq (1+k)|\xi(0) - \eta(0)|^2 + k(1+k)\rho_2(\xi, \eta)^2 \leq (1+k)\tilde{\rho}_2(\xi, \eta)^2,$$

this implies that

$$\mathbb{E} \int_0^T e^{-\lambda t} |X(t) - Y(t)|^2 dt \leq \frac{\lambda k(1 - k)\tau + k_2\tau + 1 + k}{\lambda(1 - k)^2 + k_1 - k_2} \mathbb{E} \tilde{\rho}_2(\xi, \eta)^2, \quad T > 0.$$

Combining this with Lemma 3.2(a) for $\Lambda(d\theta) = \frac{1}{\tau}d\theta$ on $[-\tau, 0]$, we conclude that

$$\begin{aligned} W_{2, \rho_2, \lambda}(\Pi_\mu, \Pi_\nu)^2 &\leq \mathbb{E} \int_0^\infty e^{-\lambda t} \rho_2(X_t, Y_t)^2 dt \\ &\leq \left(\tau + \frac{\lambda k(1 - k)\tau + k_2\tau + 1 + k}{\lambda(1 - k)^2 + k_1 - k_2} \right) \mathbb{E} \tilde{\rho}_2(\xi, \eta)^2. \end{aligned}$$

Therefore, the proof is finished according to (34). □

4. An extension of Theorem 3.1 to neutral functional SPDEs

In this section we shall discuss the transportation cost inequalities for the laws of segment processes of a class of neutral functional SPDEs in infinite-dimensional setting. Let $(H, \langle \cdot, \cdot \rangle, |\cdot|)$ be a real separable Hilbert space, let $\mathcal{C} = C([-\tau, 0]; H)$ be equipped with the uniform norm $\rho(\xi, \eta) := \|\xi - \eta\|_\infty$, and let ρ_∞^T, ρ_2 and $\rho_{2, \lambda}$ be defined by (2), (28) and (29) respectively. Let $\mathcal{L}(H)$ (resp. $\mathcal{L}_{HS}(H)$) be the set of all bounded (resp. Hilbert-Schmidt) operators on H equipped with the operator norm $\|\cdot\|$ (resp. Hilbert-Schmidt norm $\|\cdot\|_{HS}$).

Let $(A, \mathcal{D}(A))$ be a self-adjoint operator on H with spectrum $\sigma(A) \subset (-\infty, -\lambda_0]$ for some constant $\lambda_0 > 0$, and let $G, b : \mathcal{C} \rightarrow H$ and $\sigma : \mathcal{C} \rightarrow \mathcal{L}(H)$ be Lipschitz continuous. Consider the neutral functional SPDE

$$\begin{cases} d\{Z(t) - G(Z_t)\} = \{AZ(t) + b(Z_t)\}dt + \sigma(Z_t)dW(t), & t \in [0, T], \\ Z_0 = \xi \in \mathcal{C}, \end{cases} \quad (35)$$

where $(W(t))_{t \geq 0}$ is the cylindrical Wiener process on H with respect to a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Throughout the section, we assume that equation (35) has a unique mild solution, which, by definition, is a continuous adapted H -valued process $\{Z(t)\}_{t \geq -\tau}$ such that $Z_0 = \xi$ and

$$\begin{aligned} Z(t) &= e^{tA}\{\xi(0) - G(\xi)\} + G(Z_t) + \int_0^t Ae^{(t-s)A}G(Z_s)ds \\ &\quad + \int_0^t e^{(t-s)A}b(Z_s)ds + \int_0^t e^{(t-s)A}\sigma(Z_s)dW(s), \quad t \geq 0 \end{aligned}$$

holds. For concrete conditions implying the existence and uniqueness of mild solution, we refer to, e.g., [4, Theorem 3.2] and [3, Theorem 6].

Let Π_μ be the distribution of $\{Z_t\}_{t \geq 0}$ with initial distribution μ . To establish the transportation cost inequality, we further need the following conditions.

(C1) There exist constants $\bar{\lambda}_1 \in \mathbb{R}$ and $\bar{\lambda}_2 \geq 0$ such that

$$\begin{aligned} 2\langle \xi(0) - \eta(0) + G(\eta) - G(\xi), A\xi(0) - A\eta(0) + b(\xi) - b(\eta) \rangle \\ + \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \leq \bar{\lambda}_1 \|\xi - \eta\|_\infty, \\ \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \leq \bar{\lambda}_2 \|\xi - \eta\|_\infty, \end{aligned}$$

for $\xi, \eta \in \mathcal{C}$ with $\xi(0), \eta(0) \in \mathcal{D}(A)$.

(C2) There exist constants $\bar{\kappa}_1 \in \mathbb{R}, \bar{\kappa}_2 \geq 0$ and a probability measure $\bar{\Lambda}$ on $[-\tau, 0]$ such that

$$\begin{aligned} 2\langle \xi(0) - \eta(0) + G(\eta) - G(\xi), A\xi(0) - A\eta(0) + b(\xi) - b(\eta) \rangle + \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \\ \leq -\bar{\kappa}_1 |\xi(0) - \eta(0)|^2 + \bar{\kappa}_2 \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|^2 \bar{\Lambda}(d\theta) \end{aligned}$$

for $\xi, \eta \in \mathcal{C}$ with $\xi(0), \eta(0) \in \mathcal{D}(A)$.

Obviously, (C1) (resp. (C2)) holds provided b, σ and AG (i.e. G takes value in $\mathcal{D}(A)$) are Lipschitz continuous w.r.t. ρ (resp. ρ_2).

Let $\xi \in \mathcal{C}$ and $T > 0$ be fixed, and as before let Π_ξ^T denote the law of $Z_{[0,T]} := (Z_t)_{t \in [0,T]}$. For any $F \geq 0$ such that $\Pi_\xi^T(F) = 1$, let $\mathbb{Q}, m(t)$ be defined in the proof of Lemma 2.2 with $X_{[0,T]}$ replaced by $Z_{[0,T]}$. For the H -valued \mathcal{F}_t -Brownian motion \tilde{W} defined by (12) and on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, (35) can be rewritten as

$$\begin{cases} d\{Z(t) + G(Z_t)\} = \{AZ(t) + b(Z_t) + \sigma(Z_t)h(t)\}dt + \sigma(Z_t)d\tilde{W}(t), \\ Z_0 = \xi. \end{cases} \quad (36)$$

Consider the following equation

$$\begin{cases} d\{Y(t) + G(Y_t)\} = \{AY(t) + b(Y_t)\}dt + \sigma(Y_t)d\tilde{W}(t), \\ Y_0 = \xi. \end{cases} \quad (37)$$

Then $\tilde{M}(t) := Z(t) - Y(t) + G(Y_t) - G(Z_t)$ solves the following equation

$$\begin{cases} d\tilde{M}(t) = \{A(Z(t) - Y(t)) + b(Z_t) - b(Y_t) + \sigma(Z_t)h(t)\}dt + (\sigma(Z_t) - \sigma(Y_t))d\tilde{W}(t), \\ Y_0 = Z_0. \end{cases}$$

Then repeating the proofs of Theorem 2.1 and Theorem 3.1 respectively, we obtain the following results.

Theorem 4.1. *Assume (A1), (A3) and (C1). Let $\mu \in \mathcal{P}(\mathcal{C})$ and F be non-negative measurable function F on $C([0, \infty); \mathcal{C})$ such that $\Pi_\mu(F) = 1$. Then*

$$W_{2, \rho_\infty^T}(F \Pi_\mu^T, \Pi_\mu^T) \leq \sqrt{\beta(T)} W_{2, \rho}(\mu, \mu_F^T) + \sqrt{\alpha(T)} \sqrt{\Pi_\mu^T(F \log F)},$$

where $\alpha(T)$ and $\beta(T)$ are defined by (3) and (4) with λ_1 and λ_2 replaced by $\bar{\lambda}_1$ and $\bar{\lambda}_2$ respectively.

Theorem 4.2. Assume (B1), (C2) and (A3). Let $\mu \in \mathcal{P}(\mathcal{C})$ and F be non-negative measurable function F on $C([0, \infty); \mathcal{C})$ such that $\Pi_\mu(F) = 1$.

(1) If $\bar{\kappa}_1 > \bar{\kappa}_2$, then

$$W_{2,\rho_{2,0}}(\Pi_\mu, F\Pi_\mu) \leq \frac{\sqrt{2\lambda_3}\{1 + (1 + \kappa)^2\}}{\bar{\kappa}_1 - \bar{\kappa}_2} \sqrt{\Pi_\mu(F \log F)} \\ + \sqrt{\tau + \frac{\bar{\kappa}_2\tau + 1 + \kappa}{\bar{\kappa}_1 - \bar{\kappa}_2}} W_{2,\rho_2}(\mu, \mu_F).$$

(2) If $\bar{\kappa}_1 \leq \bar{\kappa}_2$, then for any $\lambda > \frac{\bar{\kappa}_2 - \bar{\kappa}_1}{(1 - \kappa)^2}$,

$$W_{2,\rho_{2,\lambda}}(\Pi_\mu, F\Pi_\mu) \leq \frac{\sqrt{2\lambda_3}\{1 + (1 + \kappa)^2\}}{\bar{\kappa}_1 - \bar{\kappa}_2 + \lambda(1 - \kappa)^2} \sqrt{\Pi_\mu(F \log F)} \\ + \sqrt{\tau + \frac{\lambda\kappa(1 - \kappa)\tau + \bar{\kappa}_2\tau + 1 + \kappa}{\lambda(1 - \kappa)^2 + \bar{\kappa}_1 - \bar{\kappa}_2}} W_{2,\rho_2}(\mu, \mu_F).$$

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