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Higher Homotopy Commutativity of H-spaces and the Cyclohedra

Dedicated to Professor Yutaka Hemmi on his sixtieth birthday

by

Yusuke KAWAMOTO

Abstract

We define higher homotopy commutativity of H-spaces using the cyclohedra $\{W_n\}_{n\geq 1}$ constructed by Bott and Taubes. An H-space whose multiplication is homotopy commutative of the n-th order is called a B_n-space. We also give combinatorial decompositions of the permuto-associahedra $\{KP_n\}_{n\geq 1}$ introduced by Kapranov into unions of product spaces of cyclohedra. From the decomposition, we have a relation between the B_n-structures and another notion of higher homotopy commutativity represented by the permuto-associahedra.

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§1. Introduction

The concept of higher homotopy commutativity was introduced by Sugawara [26] and Williams [28] in the case of topological monoids. In the definition, Williams used permutohedra, which were introduced by Milgram [20] to construct approximations to iterated loop spaces. The homotopy commutativity of the third order in the sense of Williams is illustrated by the left hexagon in Figure 1.

Later Hemmi–Kawamoto [11] considered another type of higher homotopy commutativity of topological monoids using the resultohedra $\{N_{m,n}\}_{m,n\geq 1}$ constructed by Gel'fand–Kapranov–Zelevinsky [7]. In particular, we have higher homotopy commutativity represented by the simplices $\{\Delta^m\}_{m\geq 1}$ since $N_{m,1} \cong \Delta^m$

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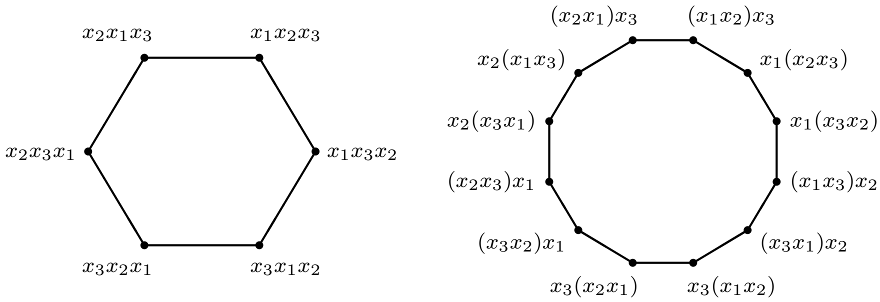

Figure 1. Homotopy commutativity of the third order.

for $m \geq 1$. A $C(n)$-space is a topological monoid with homotopy commutativity of the n-th order (see Section 4). From the definition, a topological monoid is a $C(2)$-space if and only if the multiplication is homotopy commutative. The $C(3)$-structure is illustrated in Figure 11. By Proposition 4.3, X is a $C(\infty)$-space if and only if the classifying space BX is a T-space in the sense of Aguadé [1].

In this paper, we show that the $C(n)$-structures can be defined only assuming that multiplication is homotopy associative of the n-th order.

According to Sugawara [25], there is a criterion for an H-space to have the homotopy type of a topological monoid. His criterion is higher homotopy associativity for multiplication. Later Stasheff [22] expanded the theory of Sugawara, and introduced the concept of A_n-spaces. An A_n-space is an H-space whose multiplication is homotopy associative of the n-th order. When defining A_n-spaces, he constructed special polytopes $\{K_n\}_{n\geq 1}$ called associahedra.

Bott–Taubes [4] introduced another family $\{W_n\}_{n\geq 1}$ of special polytopes called cyclohedra to study topological descriptions of self-linking invariants of knots. Since the cyclohedra are constructed by combining simplices and associahedra, we can use these polytopes to generalize the $C(n)$-structures to the case of A_n-spaces.

An A_n-space with homotopy commutativity of the n-th order is called a B_n-space (see Section 4). From the definition, a B_2-space is the same as a homotopy commutative H-space. Let X be an A_3-space with a B_2-structure. Using the associating homotopy $\mu_3 \colon K_3 \times X^3 \to X$ and the commuting homotopy $\varphi_2 \colon W_2 \times X^2 \to X$, we can define $\widetilde{\varphi}_3 \colon \partial W_3 \times X^3 \to X$ illustrated by the left hexagon in Figure 2. Then X is a B_3-space if and only if $\widetilde{\varphi}_3$ extends to $\varphi_3 \colon W_3 \times X^3 \to X$. We note that the above hexagon is similar to the one of Mac Lane [16, p. 38, (4.5)]. In this manner, X is called a B_n-space if there is a family $\{\varphi_i \colon W_i \times X^i \to X\}_{1\leq i\leq n}$ of maps with the relations stated in Definition 4.4. When X is a topological monoid, X is a B_n-space if and only if X is a $C(n)$-space.

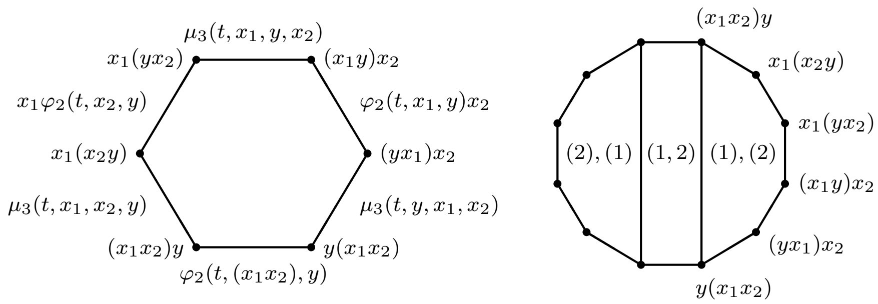

Figure 2. The B_3-structure on X and the decomposition of KP_3.

In [9], we also generalized higher homotopy commutativity in the sense of Williams to the case of A_n-spaces (see the right dodecagon in Figure 1). In the definition, we used the permuto-associahedra $\{KP_n\}_{n\geq 1}$ originally constructed by Kapranov [13] (see Section 3). An A_n-space with higher homotopy commutativity of this type is called an AC_n-space.

May [18] introduced the concept of E_n-space to give a criterion for a space to have the homotopy type of an n-fold loop space. The types of higher homotopy commutativity we are considering in this paper are just truncations of E_2-structures, just as A_n-spaces are truncated versions of E_1-spaces.

According to Hemmi [8, p. 108, (5.1)] and Kapranov–Voevodsky [14, Theorem 6.5], permutohedra can be combinatorially decomposed into unions of product spaces of simplices (see [14, p. 245, Figures 14 and 15]). To describe a relation between B_n-structures and AC_n-structures, we generalize their result to the case of permuto-associahedra.

We now recall some notation and terminology. Put $\mathbf{n} = (1, \dots, n) \in \mathbb{N}^n$ and

$$\mathbb{T}^m[n] = \big\{(t_1, \dots, t_m) \in \mathbb{N}^m \;\big|\; t_1 + \cdots + t_m = n\big\} \quad \text{for } m, n \geq 1.$$

A subsequence of $\mathbf{n}$ of length t is written as $\alpha = (\alpha(1), \dots, \alpha(t))$ with $\alpha(1) < \cdots < \alpha(t)$. A *partition* of $\mathbf{n}$ of type $(t_1, \dots, t_m) \in \mathbb{T}^m[n]$ is an ordered sequence $(\alpha_1, \dots, \alpha_m)$ consisting of disjoint subsequences α_i of $\mathbf{n}$ of length t_i for $1 \leq i \leq m$ with $\alpha_1 \cup \cdots \cup \alpha_m = \mathbf{n}$ as sets. Let $\mathbb{A}_n^{(t_1,\dots,t_m)}$ denote the set of all partitions of $\mathbf{n}$ of type $(t_1, \dots, t_m) \in \mathbb{T}^m[n]$. For example, $\mathbb{A}_2^{(2)} = \{((1, 2))\}$, $\mathbb{A}_3^{(3)} = \{((1, 2, 3))\}$, $\mathbb{A}_3^{(1,2)} = \{((1), (2, 3)), ((2), (1, 3)), ((3), (1, 2))\}$ and $\mathbb{A}_3^{(2,1)} = \{((1, 2), (3)), ((1, 3), (2)), ((2, 3), (1))\}$. Moreover, we see that

$$\mathbb{A}_n^{(1,\dots,1)} = \{((\sigma(1)), \dots, (\sigma(n))) \mid \sigma \in \mathscr{S}_n\} \quad \text{for } n \geq 1,$$

where $\mathscr{S}_n$ denotes the symmetric group on n letters. Put

$$\mathbb{A}_n = \{(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_n^{(t_1,\dots,t_m)} \mid (t_1, \dots, t_m) \in \mathbb{T}^m[n] \text{ with } m \geq 1\}.$$

Our result is as follows:

Theorem A. *Let $n \geq 2$. There is a family*

$$\{\mathscr{D}(\alpha_1, \dots, \alpha_m)\}_{(\alpha_1,\dots,\alpha_m)\in\mathbb{A}_{n-1}}$$

of subspaces of KP_n with the following properties:

(1) *If $(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_m)}$, then we have an isomorphism*

$$\iota^{(\alpha_1,\dots,\alpha_m)} \colon W_{m+1} \times KP_{t_1} \times \cdots \times KP_{t_m} \to \mathscr{D}(\alpha_1, \dots, \alpha_m).$$

(2) *KP_n decomposes as*

$$KP_n = \bigcup_{(\alpha_1,\dots,\alpha_m)\in\mathbb{A}_{n-1}} \mathscr{D}(\alpha_1, \dots, \alpha_m).$$

In the above theorem, isomorphism of polytopes means affine homeomorphism. The decomposition of KP_3 is illustrated by the right dodecagon in Figure 2 (see Figure 10 for the decomposition of KP_4). Then $\mathscr{D}((1,2)) \cong W_2 \times KP_2$ via $\iota^{((1,2))}$ and $\mathscr{D}((\sigma(1)),(\sigma(2))) \cong W_3 \times KP_1 \times KP_1$ by means of $\iota^{((\sigma(1)),(\sigma(2)))}$ for $\sigma \in \mathscr{S}_2$. It is remarkable that the decomposition of KP_3 also appears in Mac Lane [16, p. 40] and Bar-Natan [2, p. 171, Figure 6].

From Theorem A and an inductive argument, we see that KP_n can be decomposed into a union of product spaces of $\{W_i\}_{1\leq i\leq n}$ in a combinatorial way. Then W_n can be regarded as a subspace of KP_n via $\iota^{((1),\dots,(n-1))} \colon W_n \times KP_1 \times \cdots \times KP_1 \to \mathscr{D}((1), \dots, (n-1)) \subset KP_n$.

From Theorem A, we have the following result:

Theorem B. *If X is a B_n-space, then X is an AC_n-space for $n \geq 1$.*

The above result generalizes [11, Proposition 4.5] to the case of A_n-spaces. By Example 4.12, the converse of Theorem B is not true.

This paper is organized as follows: In Section 2, we recall combinatorial properties of the associahedra $\{K_n\}_{n\geq1}$ and the cyclohedra $\{W_n\}_{n\geq1}$. In order to prove Theorem A in Section 3, we define a poset $(\mathscr{F}_n, \preceq_f)$ describing the faces of W_n. Then we study the face operators and degeneracy operators of W_n. In Section 3, we recall the permuto-associahedra $\{KP_n\}_{n\geq1}$, and give a proof of Theorem A. It is also shown that the degeneracy operators of KP_n can be reconstructed from those of W_n using Theorem A. Section 4 is devoted to studying higher homotopy commutativity of A_n-spaces. In the case of topological monoids, we first recall the definition of $C(n)$-spaces (see Definition 4.1 and Remark 4.2). Using cyclohedra instead of simplices, we define B_n-spaces (see Definition 4.4 and Remark 4.5). It is shown that the property of being a B_n-space is preserved by covering spaces. We also give some examples of B_n-spaces (see Examples 4.7, 4.8 and 4.12). Then we recall the definition of AC_n-spaces, and prove Theorem B using Theorem A.

§2. Cyclohedra

We first recall the associahedra $\{K_n\}_{n\geq 1}$ and the cyclohedra $\{W_n\}_{n\geq 1}$ constructed by Stasheff [22] and Bott–Taubes [4], respectively.

Stasheff [22, I, Section 6] constructed the associahedra $\{K_n\}_{n\geq 1}$ in order to define A_n-spaces (see Section 3). From the construction, the *associahedron* K_n is a polytope of dimension $n-2$ whose faces correspond to meaningful bracketings of the word $x_1\cdots x_n$ for $n\geq 2$. More precisely, a codimension t face of K_n is represented by inserting t pairs of brackets in a meaningful way into the word $x_1\cdots x_n$ so that any pair of brackets includes at least two elements each of which is x_i or a bracketed sequence for $t\geq 1$. In particular, each vertex of K_n is represented by one of the meaningful complete ways of bracketing the word $x_1\cdots x_n$. For convenience, we also put $K_1=\{*\}$.

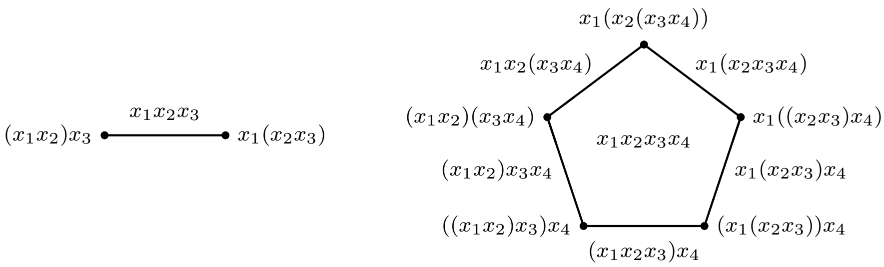

Figure 3. The associahedra K_3 and K_4.

Denote the set of all meaningful bracketings of the word $x_1\cdots x_n$ by $\mathscr{K}_n$. Then $(\mathscr{K}_n,\preceq_k)$ is a poset (partially ordered set) ordered by defining $\xi\preceq_k\xi'$ if ξ' is obtained from ξ by removing some pairs of brackets or $\xi'=\xi$. Let $K_k(r,s)$ be the facet (codimension-one face) of K_n represented by

$$x_1\cdots x_{k-1}(x_k\cdots x_{k+s-1})x_{k+s}\cdots x_n\in\mathscr{K}_n \qquad \text{for } (r,s,k)\in\mathbb{K}_n,$$

where

$$\mathbb{K}_n=\{(r,s,k)\in\mathbb{N}^3 \mid r,s\geq 2 \text{ with } r+s=n+1 \text{ and } k\leq r\}.$$

Then the boundary ∂K_n is given by

$$\partial K_n=\bigcup_{(r,s,k)\in\mathbb{K}_n} K_k(r,s).$$

According to Stasheff [22, I, Section 2], $K_k(r,s) \cong K_r \times K_s$ via a face operator $\partial_k(r,s)\colon K_r \times K_s \to K_k(r,s)$ for $(r,s,k) \in \mathbb{K}_n$ and there is a family of degeneracy operators $\{\theta_j\colon K_n \to K_{n-1}\}_{1\le j\le n}$.

Later Bott–Taubes [4, Section 1] introduced another family $\{W_n\}_{n\ge 1}$ of special complexes closely related to the associahedra. According to Stasheff [24, p. 58], W_n is called a *cyclohedron* for $n \ge 1$.

Stasheff [24, Section 10] and Markl [17, Section 1] reconstructed W_n as the convex hull of a finite set of points in $\mathbb{R}^n$, and gave a poset representing all the faces of W_n. By their results, W_n is a polytope of dimension $n-1$ whose faces correspond to meaningful bracketings of the string $x_1 \cdots x_n$ arranged on a circle for $n \ge 1$ (see also Devadoss [6, Section 1]). Such bracketings are called *cyclic bracketings*. In particular, W_n is represented by the string $x_1 \cdots x_n$ without brackets, and a codimension t face of W_n is represented by a cyclic bracketing of the string $x_1 \cdots x_n$ including just t pairs of brackets for $t \ge 1$. In this manner, the vertices of W_n correspond to all complete ways of cyclic bracketings of the string $x_1 \cdots x_n$.

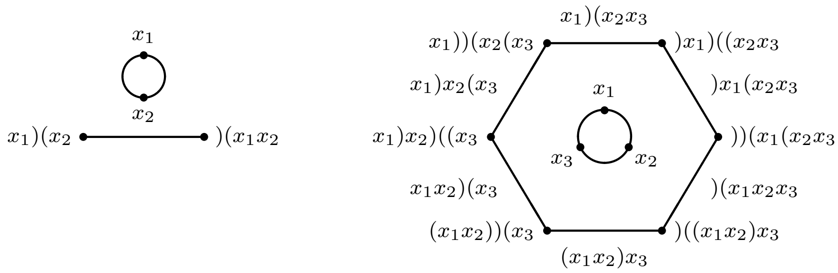

Figure 4. The cyclic bracketings of the strings x_1x_2 and $x_1x_2x_3$.

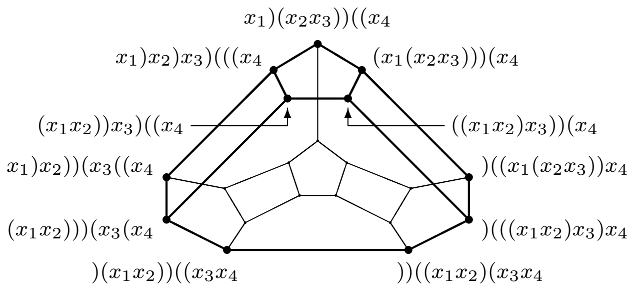

Figure 5. The cyclic bracketings of the string $x_1x_2x_3x_4$.

We denote the set of all cyclic bracketings of the string $x_1 \cdots x_n$ by $\mathscr{W}_n$. Then $(\mathscr{W}_n, \preceq_w)$ is a poset, where the poset structure $\preceq_w$ is defined in a similar way to the one of $(\mathscr{K}_n, \preceq_k)$. Put

$$\begin{aligned}\mathbb{W}_n &= \{(r,s,k) \in \mathbb{N}^3 \mid r,s \geq 2 \text{ with } r+s=n+1 \text{ and } k \leq r-1\},\\ \mathbb{W}'_n &= \{(r,s,k) \in \mathbb{N}^3 \mid r \geq 2 \text{ with } r+s=n+1 \text{ and } k \leq r\}.\end{aligned}$$

Let $W_k(r,s)$ and $W'_k(r,s)$ denote the facets of W_n represented by

$$x_1 \cdots x_{k-1}(x_k \cdots x_{k+s-1})x_{k+s} \cdots x_n \in \mathscr{W}_n \quad \text{for } (r,s,k) \in \mathbb{W}_n, \tag{2.1}$$

$$x_1 \cdots x_{k-1})x_k \cdots x_{k+s-2}(x_{k+s-1} \cdots x_n \in \mathscr{W}_n \quad \text{for } (r,s,k) \in \mathbb{W}'_n, \tag{2.2}$$

respectively. Then the boundary ∂W_n is given by

$$\partial W_n = \bigcup_{(r,s,k)\in\mathbb{W}_n} W_k(r,s) \cup \bigcup_{(r,s,k)\in\mathbb{W}'_n} W'_k(r,s).$$

Remark 2.1. We have a simple proof of the well-known result that $|\mathrm{vert}(K_n)| = \frac{1}{n}\binom{2n-2}{n-1}$ using W_n, where $\mathrm{vert}(Q)$ denotes the set of all vertices of a polytope Q and $|S|$ is the number of elements of a set S. Let v be a vertex of W_n. Then v is represented by one of the complete ways of cyclic bracketing of the string $x_1 \cdots x_n$. Replacing x_i with $\bullet$ for $1 \leq i \leq n-1$ and removing x_n and all the closing brackets ")" from v, we have a bijection between $\mathrm{vert}(W_n)$ and the set of all permutations of $\{\overbrace{(,\dots,(}^{n-1},\overbrace{\bullet,\dots,\bullet}^{n-1}\}$. For example, the vertices of W_4 represented by $(x_1x_2)))(x_3(x_4$ and $x_1)x_2))(x_3((x_4$ correspond to $(\bullet\bullet(\bullet($ and $\bullet\bullet(\bullet(($, respectively. Then $|\mathrm{vert}(W_n)| = \binom{2n-2}{n-1}$, which implies the required result since $|\mathrm{vert}(W_n)| = n|\mathrm{vert}(K_n)|$.

We next give an alternative description of the poset $(\mathscr{W}_n, \preceq_w)$ to be used in the proof of Theorem A in Section 3.

Consider the rectangle $\mathbb{E}_n = [0, n-1] \times I$ for $n \geq 2$. A *lattice path* in $\mathbb{E}_n$ is a map $\ell\colon [0,n] \to \mathbb{E}_n$ such that $\ell(0) = (0,0)$, $\ell(n) = (n-1, 1)$ and if we write $\ell(s) = (\ell_1(s), \ell_2(s))$ for $s \in [0,n]$, then $\ell(i+t)$ is either $(\ell_1(i)+t, \ell_2(i))$ or $(\ell_1(i), \ell_2(i)+t)$ for $0 \leq i < n$ and $t \in I$. We denote the set of all lattice paths in $\mathbb{E}_n$ by $\mathscr{L}_n$.

In $\mathbb{E}_n$, we label the interval $[i-1, i] \times \{j\}$ by x_i for $1 \leq i \leq n-1$ and $j = 0, 1$, and the interval $\{i\} \times I$ by y for $0 \leq i \leq n-1$ as in Figure 6. Then each lattice

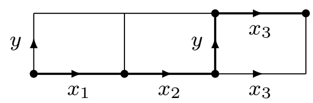

Figure 6. The lattice path $\ell = x_1x_2yx_3$.

path $\ell \in \mathscr{L}_n$ is labeled by a word $x_1 \cdots x_{i-1} y x_i \cdots x_{n-1}$ for some i with $1 \le i \le n$. In this label of ℓ, the symbol x_i means the horizontal unit move from the line $x = i-1$ to the line $x = i$ for $1 \le i \le n-1$, and y is the vertical move between the lines $y = 0$ and $y = 1$. For example, the lattice path $\ell \in \mathscr{L}_4$ in Figure 6 is labeled by $x_1 x_2 y x_3$.

Put

$$\mathbb{H}^m[n] = \{(h_1, \dots, h_m) \in (\mathbb{Z}^+)^m \mid h_1 + \cdots + h_m = n\} \quad \text{for } m, n \ge 1,$$

where $\mathbb{Z}^+ = \{h \in \mathbb{Z} \mid h \ge 0\}$.

Let $\xi \in \mathscr{W}_n$ be such that x_n is covered by just t pairs of brackets for $t \ge 0$. Then we can write

$$\xi = \xi_1)\xi_2)\cdots)\xi_t)\xi_{t+1}(\xi_{t+2}(\cdots(\xi_{2t}(\xi_{2t+1}x_n, \tag{2.3}$$

where ξ_j is a meaningful bracketing of the word $x_{h_1+\cdots+h_{j-1}+1} \cdots x_{h_1+\cdots+h_j}$ for $1 \le j \le 2t+1$ and $(h_1, \dots, h_{2t+1}) \in \mathbb{H}^{2t+1}[n-1]$ with $h_j + h_{2t+2-j} > 0$ for $1 \le j \le t$.

We now define $\mathscr{F}_n = \{f(\xi) \mid \xi \in \mathscr{W}_n\}$, where

$$f(\xi) = (\xi_1(\xi_2(\cdots(\xi_t[\xi_{t+1}|y]\xi_{t+2})\cdots)\xi_{2t})\xi_{2t+1})$$

if $\xi \in \mathscr{W}_n$ is written as in (2.3). Then $(\mathscr{F}_n, \preceq_f)$ is a poset ordered by defining $f(\xi) \preceq_f f(\xi')$ if $\xi \preceq_w \xi'$ for $\xi, \xi' \in \mathscr{W}_n$.

Remark 2.2. Let $\xi \in \mathscr{W}_n$ be written as in (2.3). From the definition, we have the following relations:

(1) $f(\xi) \prec_f (\xi_1(\cdots(\xi_i\xi_{i+1}(\cdots(\xi_t[\xi_{t+1}|y]\xi_{t+2})\cdots)\xi_{2t+1-i}\xi_{2t+2-i})\cdots)\xi_{2t+1})$ for $1 \le i \le t-1$.

(2) $f(\xi) \prec_f (\xi_1(\cdots(\xi_{t-1}[\xi_t\xi_{t+1}\xi_{t+2}|y]\xi_{t+3})\cdots)\xi_{2t+1})$.

(3) If ξ_i' is obtained from ξ_i by removing some pair of brackets or $\xi_i' = \xi_i$ for $1 \le i \le 2t+1$, then $f(\xi) \preceq_f (\xi_1'(\cdots(\xi_t'[\xi_{t+1}'|y]\xi_{t+2}')\cdots)\xi_{2t+1}')$.

Since $f\colon (\mathscr{W}_n, \preceq_w) \to (\mathscr{F}_n, \preceq_f)$ is an isomorphism of posets, we can assume that the faces of W_n are labeled by $(\mathscr{F}_n, \preceq_f)$. Recall that W_n is represented by $x_1 \cdots x_n \in \mathscr{W}_n$. Then it is labeled by $f(x_1 \cdots x_n) = [x_1 \cdots x_{n-1}|y] \in \mathscr{F}_n$. By (2.1) and (2.2), the facets $W_k(r,s)$ and $W_k'(r,s)$ are labeled by

$$[x_1 \cdots x_{k-1}(x_k \cdots x_{k+s-1})x_{k+s} \cdots x_{n-1}|y] \in \mathscr{F}_n \quad \text{for } (r,s,k) \in \mathbb{W}_n$$

and

$$(x_1 \cdots x_{k-1}[x_k \cdots x_{k+s-2}|y]x_{k+s-1} \cdots x_{n-1}) \in \mathscr{F}_n \quad \text{for } (r,s,k) \in \mathbb{W}_n',$$

respectively. In this manner, a vertex of W_n is labeled by a meaningful complete way of bracketing of some lattice path $\ell \in \mathscr{L}_n$.

The cyclohedra W_n whose faces are labeled by $(\mathscr{F}_n, \preceq_f)$ for $n = 2, 3$ and 4 are illustrated in Figures 7 and 8. For simplicity, we denote $[\emptyset|y]$ by y, and omit

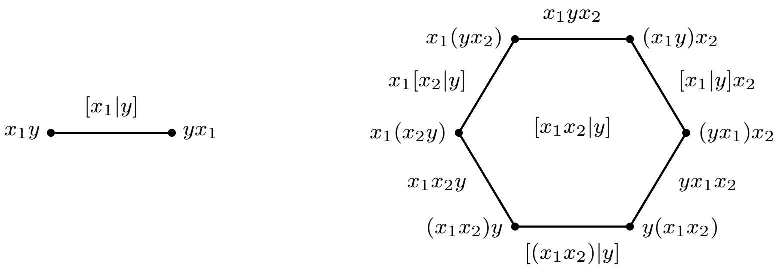

Figure 7. The cyclohedra W_2 and W_3.

the outermost pair of brackets. Then W_2 labeled by $[x_1|y]$ is the left interval in Figure 7, which represents a commuting homotopy between x_1y and yx_1.

When $n = 3$, the cyclohedron W_3 labeled by $[x_1x_2|y]$ is illustrated by the right hexagon in Figure 7. The bottom edge labeled by $[(x_1x_2)|y]$ represents a commuting homotopy between $(x_1x_2)y$ and $y(x_1x_2)$, and the next left edge labeled by x_1x_2y is an associating homotopy between $(x_1x_2)y$ and $x_1(x_2y)$. The next edge labeled by $x_1[x_2|y]$ is regarded as a commuting homotopy between $x_1(x_2y)$ and $x_1(yx_2)$.

Remark 2.3. The cyclohedra $\{W_n\}_{n\geq 1}$ realizing the posets $\{(\mathscr{F}_n, \preceq_f)\}_{n\geq 1}$ are closely related to the commuto-associahedra $\{CA_n\}_{n\geq 1}$ introduced by Bar-Natan [2, Sections 5 and 6] (see also [3, Section 4] and [6, p. 73, 4.2]). In particular, the 2-skeleton of W_n is a subspace of CA_n for $n \geq 1$.

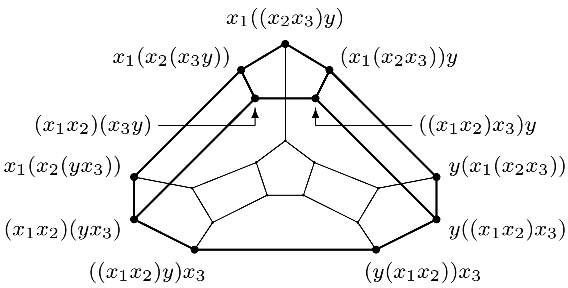

Figure 8. The cyclohedron W_4.

Since the set of all faces of $W_k(r,s)$ is described by the poset $(\mathscr{F}_r \times \mathscr{K}_s, \sqsubseteq)$, it follows that $W_k(r,s) \cong W_r \times K_s$ for $(r,s,k) \in \mathbb{W}_n$, where the poset structure of $\mathscr{F}_r \times \mathscr{K}_s$ is given by defining $(\lambda, \xi) \sqsubseteq (\lambda', \xi')$ if $\lambda \preceq_f \lambda'$ and $\xi \preceq_k \xi'$. In a similar way, we see that $W'_k(r,s) \cong K_r \times W_s$ for $(r,s,k) \in \mathbb{W}'_n$. Define face operators $\varepsilon_k(r,s)\colon W_r \times K_s \to W_k(r,s)$ and $\varepsilon'_k(r,s)\colon K_r \times W_s \to W'_k(r,s)$ of W_n by using these isomorphisms. From the construction, we have the following proposition:

Proposition 2.4. *The face operators* $\{\varepsilon_k(r,s)\}_{(r,s,k)\in\mathbb{W}_n}$, $\{\varepsilon'_k(r,s)\}_{(r,s,k)\in\mathbb{W}'_n}$ *and* $\{\partial_k(r,s)\}_{(r,s,k)\in\mathbb{K}_n}$ *satisfy the following relations:*

$$\begin{aligned}(2.4)\qquad &\varepsilon_k(r,s)(\varepsilon_l(p,q)(a,b),c)\\ &= \begin{cases}\varepsilon_{l+s-1}(p+s-1,q)(\varepsilon_k(p,s)(a,c),b) & \text{if } k \le l-1,\\ \varepsilon_l(p,q+s-1)(a,\partial_{k-l+1}(q,s)(b,c)) & \text{if } l \le k \le l+q-1,\\ \varepsilon_l(p+s-1,q)(\varepsilon_{k-q+1}(p,s)(a,c),b) & \text{if } k \ge l+q,\end{cases}\\ &\qquad\qquad \textit{for } (r,s,k)\in\mathbb{W}_n \textit{ and } (p,q,l)\in\mathbb{W}_r;\end{aligned}$$

$$\begin{aligned}(2.5)\qquad &\varepsilon_k(r,s)(\varepsilon'_l(p,q)(a,b),c)\\ &= \begin{cases}\varepsilon'_{l+s-1}(p+s-1,q)(\partial_k(p,s)(a,c),b) & \text{if } k \le l-1,\\ \varepsilon'_l(p,q+s-1)(a,\varepsilon_{k-l+1}(q,s)(b,c)) & \text{if } l \le k \le l+q-2,\\ \varepsilon'_l(p+s-1,q)(\partial_{k-q+2}(p,s)(a,c),b) & \text{if } k \ge l+q-1,\end{cases}\\ &\qquad\qquad \textit{for } (r,s,k)\in\mathbb{W}_n \textit{ and } (p,q,l)\in\mathbb{W}'_r;\end{aligned}$$

$$\begin{aligned}(2.6)\qquad &\varepsilon'_k(r,s)(a,\varepsilon'_l(p,q)(b,c)) = \varepsilon'_{k+l-1}(r+p-1,q)(\partial_k(r,p)(a,b),c)\\ &\qquad\qquad \textit{for } (r,s,k)\in\mathbb{W}'_n \textit{ and } (p,q,l)\in\mathbb{W}'_s.\end{aligned}$$

We now explain the proposition in the case of $n=3$ and 4.

In the right hexagon of Figure 7, the bottom edge labeled by $[(x_1x_2)|y]$ is isomorphic to $W_2 \times K_2$ by means of the face operator $\varepsilon_1(2,2)$, and the edge labeled by x_1x_2y is isomorphic to $K_3 \times W_1$ via $\varepsilon'_3(3,1)$. The intersection of these two edges is a vertex which is the image of $(\varepsilon'_2(2,1)(*,*),*)$ under $\varepsilon_1(2,2)$ and of $(\partial_1(2,2)(*,*),*)$ under $\varepsilon'_3(3,1)$.

The next left vertex is the intersection of the two edges labeled by x_1x_2y and $x_1[x_2|y]$, the image of $(\partial_2(2,2)(*,*),*)$ in $K_3 \times W_1$ under $\varepsilon'_3(3,1)$ and of $(*,\varepsilon'_2(2,1)(*,*))$ in $K_2 \times W_2$ under $\varepsilon'_2(2,2)$. The next vertex, the intersection of the two edges $x_1[x_2|y]$ and x_1yx_2, is the image of $(*,\varepsilon'_1(2,1)(*,*))$ in $K_2 \times W_2$ under $\varepsilon'_2(2,2)$ and of $(\partial_2(2,2)(*,*),*)$ in $K_3 \times W_1$ under $\varepsilon'_2(3,1)$.

In the case of W_4, the front hexagon, the right rectangle and the top pentagon of Figure 8 are labeled by $[(x_1x_2)x_3|y]$, $[(x_1x_2x_3)|y]$ and $x_1x_2x_3y$, respectively. Then the facet $[(x_1x_2)x_3|y]$ is isomorphic to $W_3 \times K_2$ via $\varepsilon_1(3,2)$, while the facet labeled by $[(x_1x_2x_3)|y]$ is isomorphic to $W_2 \times K_3$ via $\varepsilon_1(2,3)$. The intersection of these two facets is an edge which is the image $(\varepsilon_1(2,2)(a,*),*)$ under $\varepsilon_1(3,2)$ and of $(a,\partial_1(2,2)(*,*))$ under $\varepsilon_1(2,3)$ for $a \in W_2$.

The facet $x_1x_2x_3y$ is isomorphic to $K_4 \times W_1$ via $\varepsilon'_4(4,1)$, and the intersection of $[(x_1x_2)x_3|y]$ and $x_1x_2x_3y$ is an edge which is the image $(\varepsilon'_3(3,1)(b,*),*)$ under $\varepsilon_1(3,2)$ and of $(\partial_1(3,2)(b,*),*)$ under $\varepsilon'_4(4,1)$ for $b \in K_3$.

In a similar way to the proof of [20, Lemma 4.5], we have the following proposition:

Proposition 2.5. *There are degeneracy operators* $\{\delta_j \colon W_n \to W_{n-1}\}_{1\le j\le n-1}$ *and* $\delta_n \colon W_n \to K_{n-1}$ *with the following relations:*

$$\delta_n\varepsilon_1(2,n-1)(a,b) = b; \tag{2.7}$$

$$\delta_k\varepsilon_k(n-1,2)(a,*) = \delta_{k+1}\varepsilon_k(n-1,2)(a,*) = a \quad \text{for } 1 \le k \le n-2; \tag{2.8}$$

$$\delta_j\varepsilon_k(r,s)(a,b) = \begin{cases} \varepsilon_{k-1}(r-1,s)(\delta_j(a),b) & \text{if } 1 \le j \le k-1, \\ \varepsilon_k(r,s-1)(a,\theta_{j-k+1}(b)) & \text{if } k \le j \le k+s-1, \\ \varepsilon_k(r-1,s)(\delta_{j-s+1}(a),b) & \text{if } k+s \le j \le n-1, \\ \partial_k(r-1,s)(\delta_r(a),b) & \text{if } j = n, \end{cases} \tag{2.9}$$

for $(r,s,k) \in \mathbb{W}_n$ *excluding* (2.7) *and* (2.8);

$$\delta_{n-1}\varepsilon'_1(2,n-1)(*,b) = \delta_1\varepsilon'_2(2,n-1)(*,b) = b; \tag{2.10}$$

$$\delta_n\varepsilon'_k(n,1)(a,*) = \theta_k(a) \quad \text{for } 1 \le k \le n; \tag{2.11}$$

$$\delta_n\varepsilon'_k(n-1,2)(a,b) = a \quad \text{for } 1 \le k \le n-1; \tag{2.12}$$

$$\delta_j\varepsilon'_k(r,s)(a,b) = \begin{cases} \varepsilon'_{k-1}(r-1,s)(\theta_j(a),b) & \text{if } 1 \le j \le k-1, \\ \varepsilon'_k(r,s-1)(a,\delta_{j-k+1}(b)) & \text{if } k \le j \le k+s-2, \\ \varepsilon'_k(r-1,s)(\theta_{j-s+2}(a),b) & \text{if } k+s-1 \le j \le n-1, \\ \partial_k(r,s-1)(a,\delta_s(b)) & \text{if } j = n, \end{cases} \tag{2.13}$$

for $(r,s,k) \in \mathbb{W}'_n$ *excluding* (2.10)–(2.12).

Proof. We prove the case of $\{\delta_j\}_{1\le j\le n-1}$ by induction on n. When $n=2$, we put $\delta_1(a) = *$. Let $n > 2$, and assume inductively that $\{\delta_j \colon W_{n'} \to W_{n'-1}\}_{1\le j\le n'-1}$ are constructed for any $n' < n$.

We now define $\{\widetilde{\delta}_j \colon \partial W_n \to W_{n-1}\}_{1\le j\le n-1}$ by (2.8)–(2.10) and (2.13). Since W_n is regarded as the cone of ∂W_n, if $a \in W_n$, then we can write $a = (b,t)$ with $b \in \partial W_n$ and $t \in I$. Set $\widetilde{\delta}_j(b) = (c,u)$ with $c \in \partial W_{n-1}$ and $u \in I$. Let $\delta_j \colon W_n \to W_{n-1}$ be defined by $\delta_j(a) = (c,tu)$. Then $\{\delta_j\}_{1\le j\le n-1}$ satisfies the required conditions. In the case of $\delta_n \colon W_n \to K_{n-1}$, the proof is similar. □

§3. Permuto-associahedra

We recall the permuto-associahedra $\{KP_n\}_{n\geq 1}$ constructed by Kapranov [13] and Reiner–Ziegler [21].

Kapranov [13, Section 2] constructed a family $\{KP_n\}_{n\geq 1}$ of special complexes such that KP_n is homeomorphic to the ball of dimension $n-1$ for $n \geq 1$. Later Reiner–Ziegler [21, Theorem 2] reconstructed KP_n as the convex hull of a finite set of points in $\mathbb{R}^n$ (see also Ziegler [29, Definition 9.13 and Example 9.14]). The polytopes $\{KP_n\}_{n\geq 1}$ are called *permuto-associahedra*.

From the construction, there is a natural way of describing all the faces of KP_n. Let $\mathbb{KP}_n = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{A}_n \mid m \geq 2\}$. By the above results, a facet of KP_n is represented by $(\alpha_1, \ldots, \alpha_m) \in \mathbb{KP}_n$, and a codimension-two face is represented by inserting a pair of brackets in $(\alpha_1, \ldots, \alpha_m) \in \mathbb{KP}_n$ as

$$(\alpha_1, \ldots, \alpha_{k-1}, (\alpha_k, \ldots, \alpha_{k+s-1}), \alpha_{k+s}, \ldots, \alpha_m) \quad \text{for } (m-s+1, s, k) \in \mathbb{K}_m.$$

In general, a codimension t face of KP_n is represented by inserting $t-1$ pairs of brackets in a meaningful way into some $(\alpha_1, \ldots, \alpha_m) \in \mathbb{KP}_n$ for $t \geq 1$. In this manner, each vertex of KP_n corresponds to a meaningful complete way of bracketing of some $(\alpha_1, \ldots, \alpha_n) \in \mathbb{A}_n^{(1,\ldots,1)}$.

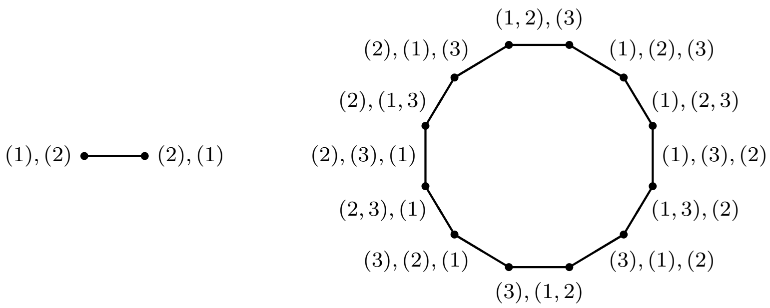

Figure 9. The permuto-associahedra KP_2 and KP_3.

Let $KP(\alpha_1, \ldots, \alpha_m)$ denote the facet represented by $(\alpha_1, \ldots, \alpha_m) \in \mathbb{KP}_n$. Then the boundary ∂KP_n is given by

$$\partial KP_n = \bigcup_{(\alpha_1,\ldots,\alpha_m)\in\mathbb{KP}_n} KP(\alpha_1, \ldots, \alpha_m). \tag{3.1}$$

By [13, p. 139] and [9, Proposition 2.1], $KP(\alpha_1, \ldots, \alpha_m) \cong K_m \times KP_{t_1} \times \cdots \times KP_{t_m}$ via a face operator $\varepsilon^{(\alpha_1,\ldots,\alpha_m)}\colon K_m \times KP_{t_1} \times \cdots \times KP_{t_m} \to KP(\alpha_1, \ldots, \alpha_m)$ for $(\alpha_1, \ldots, \alpha_m) \in \mathbb{A}_n^{(t_1,\ldots,t_m)}$.

To prove Theorem A, we show the following lemma (cf. [22, I, Proposition 25]):

Lemma 3.1. *There is a family* $\{\eta_m\colon W_3\times K_m\to W_{m+1}\}_{m\geq 2}$ *of homeomorphisms with the following relations:*

$$\eta_m(\varepsilon_1(2,2)(a,*),b)=\varepsilon_1(2,m)(a,b), \tag{3.2}$$

$$\eta_m(a,\partial_k(r,s)(b,c))=\varepsilon_k(r+1,s)(\eta_r(a,b),c)\quad \textit{for } (r,s,k)\in\mathbb{K}_m. \tag{3.3}$$

Proof. We work by induction on m. When $m=2$, we define $\eta_2(a,*)=a$ for $a\in W_3$. Let $m>2$, and assume inductively that $\{\eta_j\}_{2\leq j<m}$ are constructed.

We now define $\widetilde{\eta}_m\colon \mathscr{V}_m\to W_{m+1}$ by (3.2) and (3.3), where $\mathscr{V}_m=W_1(2,2)\times K_m\cup W_3\times\partial K_m\subset W_3\times K_m$. Then $\mathscr{V}_m$ is homeomorphic to the ball of dimension $m-1$, and the image of $\widetilde{\eta}_m$ is given by

$$\widetilde{\eta}_m(\mathscr{V}_m)=\bigcup_{(r,s,k)\in\mathbb{W}_{m+1}}W_k(r,s).$$

Let $\eta_m\colon W_3\times K_m\to W_{m+1}$ be defined by $\eta_m(b,t)=(\widetilde{\eta}_m(b),t)$ with $b\in\mathscr{V}_m$ and $t\in I$ since $W_3\times K_m$ and W_{m+1} are homeomorphic to $\mathscr{V}_m\times I$ and $\widetilde{\eta}_m(\mathscr{V}_m)\times I$, respectively. Then $\{\eta_j\}_{2\leq j\leq m}$ satisfy the required relations. □

Proof of Theorem A. We work by induction on n. When $n=2$, put $\mathscr{D}((1))=W_2$ and define $\iota^{((1))}\colon W_2\times KP_1\to\mathscr{D}((1))$ by $\iota^{((1))}(a,*)=a$. Since $KP_2=W_2=I$, the result is clear.

Let $n>2$, and assume inductively that the result is proved for any $n'<n$.

We first define a complex $\mathscr{U}_n$ with the properties of Theorem A. Put $\mathscr{U}_n=W_1(2,2)\times KP_{n-1}\cup W_3\times\partial KP_{n-1}$. Then $\mathscr{U}_n$ is homeomorphic to the ball of dimension $n-1$. Let $\iota^{((1,\dots,n-1))}\colon W_2\times KP_{n-1}\to\mathscr{D}((1,\dots,n-1))$ be defined by $\iota^{((1,\dots,n-1))}(a,b)=(\varepsilon_1(2,2)(a,*),b)$, where $\mathscr{D}((1,\dots,n-1))=W_1(2,2)\times KP_{n-1}\subset\mathscr{U}_n$. If $(\alpha_1,\dots,\alpha_m)\in\mathbb{A}_{n-1}^{(t_1,\dots,t_m)}$ with $m\geq 2$, then $\iota^{(\alpha_1,\dots,\alpha_m)}\colon W_{m+1}\times KP_{t_1}\times\cdots\times KP_{t_m}\to\mathscr{D}(\alpha_1,\dots,\alpha_m)$ is defined by

$$\iota^{(\alpha_1,\dots,\alpha_m)}(\eta_m(a,b),c_1,\dots,c_m)=(a,\varepsilon^{(\alpha_1,\dots,\alpha_m)}(b,c_1,\dots,c_m)), \tag{3.4}$$

where $\mathscr{D}(\alpha_1,\dots,\alpha_m)=W_3\times KP(\alpha_1,\dots,\alpha_m)\subset\mathscr{U}_n$ and $\eta_m\colon W_3\times K_m\to W_{m+1}$ denotes the homeomorphism of Lemma 3.1. By (3.1), we have

$$\mathscr{U}_n=\bigcup_{(\alpha_1,\dots,\alpha_m)\in\mathbb{A}_{n-1}}\mathscr{D}(\alpha_1,\dots,\alpha_m).$$

To see $\mathscr{U}_n=KP_n$, we show that there is a family

$$\{\mathscr{U}(\alpha_1,\dots,\alpha_m)\}_{(\alpha_1,\dots,\alpha_m)\in\mathbb{KP}_n}$$

of subspaces of $\partial\mathscr{U}_n$ with the following properties:

(1) If $(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_n^{(t_1,\dots,t_m)}$ with $m \geq 2$, then we have an isomorphism

$$\varepsilon^{(\alpha_1,\dots,\alpha_m)} \colon K_m \times KP_{t_1} \times \cdots \times KP_{t_m} \to \mathscr{U}(\alpha_1, \dots, \alpha_m).$$

(2) $\partial\mathscr{U}_n$ decomposes as

$$\partial\mathscr{U}_u = \bigcup_{(\alpha_1,\dots,\alpha_m)\in\mathbb{KP}_n} \mathscr{U}(\alpha_1, \dots, \alpha_m).$$

Put

$$\widetilde{\mathscr{D}}(\alpha_1, \dots, \alpha_m) = \iota^{(\alpha_1,\dots,\alpha_m)}\Big(\bigcup_{(r,s,k)\in\mathbb{W}'_{m+1}} W'_k(r,s) \times KP_{t_1} \times \cdots \times KP_{t_m}\Big) \subset \partial\mathscr{U}_n \quad \text{for } (\alpha_1, \dots, \alpha_m) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_m)}.$$

Since

$$\eta_m\Big(\bigcup_{(r,s,k)\in\mathbb{W}'_3} W'_k(r,s) \times K_m\Big) = \bigcup_{(r,s,k)\in\mathbb{W}'_{m+1}} W'_k(r,s)$$

by Lemma 3.1, we have

$$\partial\mathscr{U}_n = \bigcup_{(\alpha_1,\dots,\alpha_m)\in\mathbb{A}_{n-1}} \widetilde{\mathscr{D}}(\alpha_1, \dots, \alpha_m). \tag{3.5}$$

Given $(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_n^{(t_1,\dots,t_m)}$ with $m \geq 2$, we have $\alpha_k(t_k) = n$ for some k with $1 \leq k \leq m$. When $t_k = 1$, define $\varepsilon^{(\alpha_1,\dots,\alpha_m)} \colon K_m \times KP_{t_1} \times \cdots \times KP_{t_{k-1}} \times \{*\} \times KP_{t_{k+1}} \times \cdots \times KP_{t_m} \to \partial\mathscr{U}_n$ by

$$\begin{aligned}\varepsilon^{(\alpha_1,\dots,\alpha_m)}&(a, c_1, \dots, c_{k-1}, *, c_{k+1}, \dots, c_m)\\ &= \iota^{(\gamma_1,\dots,\gamma_{m-1})}(\varepsilon'_k(m,1)(a,*), c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_m),\end{aligned}$$

where $(\gamma_1, \dots, \gamma_{m-1}) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_{k-1},t_{k+1},\dots,t_m)}$ is given by

$$\gamma_i(s) = \begin{cases} \alpha_i(s) & \text{if } 1 \leq i \leq k-1, \\ \alpha_{i+1}(s) & \text{if } k \leq i \leq m-1. \end{cases}$$

If $t_k \geq 2$, then

$$KP_{t_k} = \bigcup_{(\beta_1,\dots,\beta_r)\in\mathbb{A}_{t_k-1}} \mathscr{D}(\beta_1, \dots, \beta_r)$$

by inductive hypothesis, where

$$\mathscr{D}(\beta_1, \dots, \beta_r) = \iota^{(\beta_1,\dots,\beta_r)}(W_{r+1} \times KP_{u_1} \times \cdots \times KP_{u_r}) \subset KP_{t_k} \quad \text{for } (\beta_1, \dots, \beta_r) \in \mathbb{A}_{t_k-1}^{(u_1,\dots,u_r)}.$$

Let $\varepsilon^{(\alpha_1,\dots,\alpha_m)}\colon K_m \times KP_{t_1} \times \cdots \times KP_{t_m} \to \partial\mathscr{U}_n$ be defined by

$$\varepsilon^{(\alpha_1,\dots,\alpha_m)}(a, c_1, \dots, c_{k-1}, \iota^{(\beta_1,\dots,\beta_r)}(b, d_1, \dots, d_r), c_{k+1}, \dots, c_m)$$
$$= \iota^{(\gamma_1,\dots,\gamma_{m+r-1})}(\varepsilon'_k(m, r+1)(a, b), c_1, \dots, c_{k-1}, d_1, \dots, d_r, c_{k+1}, \dots, c_m),$$

where $(\gamma_1, \dots, \gamma_{m+r-1}) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_{k-1},u_1,\dots,u_r,t_{k+1},\dots,t_m)}$ is given by

$$\gamma_i(s) = \begin{cases} \alpha_i(s) & \text{if } 1 \le i \le k-1, \\ \alpha_k\beta_{i-k+1}(s) & \text{if } k \le i \le k+r-1, \\ \alpha_{i-r+1}(s) & \text{if } k+r \le i \le m+r-1. \end{cases} \tag{3.6}$$

Put

$$\mathscr{U}(\alpha_1, \dots, \alpha_m) = \varepsilon^{(\alpha_1,\dots,\alpha_m)}(K_m \times KP_{t_1} \times \cdots \times KP_{t_m}) \subset \partial\mathscr{U}_n \quad \text{for } (\alpha_1, \dots, \alpha_m) \in \mathbb{A}_n^{(t_1,\dots,t_m)}.$$

Then

$$\partial\mathscr{U}_n = \bigcup_{(\alpha_1,\dots,\alpha_m)\in\mathbb{KP}_n} \mathscr{U}(\alpha_1, \dots, \alpha_m)$$

by (3.5). This completes the proof of Theorem A. □

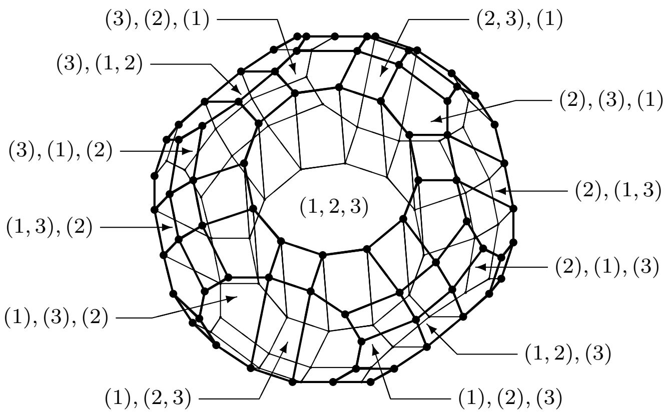

Figure 10. The decomposition of KP_4.

Remark 3.2. Assume that $(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_m)}$ with $m \ge 2$ and $(\beta_1, \dots, \beta_r) \in \mathbb{A}_{t_k}^{(u_1,\dots,u_r)}$ with $r \ge 2$. Then by (3.4) and [9, Proposition 2.1], we have the following relations:

$$\iota^{(\alpha_1,\dots,\alpha_m)}(a, c_1, \dots, c_{k-1}, \varepsilon^{(\beta_1,\dots,\beta_r)}(b, d_1, \dots, d_r), c_{k+1}, \dots, c_m) \\ = \iota^{(\gamma_1,\dots,\gamma_{m+r-1})}(\varepsilon_k(m+1, r)(a, b), c_1, \dots, c_{k-1}, d_1, \dots, d_r, c_{k+1}, \dots, c_m), \tag{3.7}$$

where $(\gamma_1, \dots, \gamma_{m+r-1}) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_{k-1},u_1,\dots,u_r,t_{k+1},\dots,t_m)}$ is defined by (3.6).

According to Hemmi–Kawamoto [9, Proposition 2.3], there is a family $\{\omega_j \colon KP_n \to KP_{n-1}\}_{1\le j\le n}$ of degeneracy operators of KP_n. From Theorem A and an inductive argument, we can reconstruct $\{\omega_j\}_{1\le j\le n}$ using the degeneracy operators $\{\delta_j\}_{1\le j\le n}$ of W_n.

When $n = 2$, we put $\omega_j(a) = *$ for $j = 1, 2$. Assume inductively that $\{\omega_j \colon KP_{n'} \to KP_{n'-1}\}_{1\le j\le n'}$ are constructed for any $n' < n$. Let $(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_m)}$.

We first consider the case of $1 \le j \le n-1$. Then $\alpha_k(t) = j$ for some k, t with $1 \le k \le m$ and $1 \le t \le t_k$. If $t_k \ge 2$, then $\omega_j \colon KP_n \to KP_{n-1}$ is defined by

$$\omega_j\iota^{(\alpha_1,\dots,\alpha_m)}(a, c_1, \dots, c_m) = \iota^{(\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_m)}(a, c_1, \dots, c_{k-1}, \omega_t(c_k), c_{k+1}, \dots, c_m),$$

where $(\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_m) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_{k-1},t_k-1,t_{k+1},\dots,t_m)}$ is given by

$$\widetilde{\alpha}_k(s) = \begin{cases} \alpha_k(s) & \text{if } \alpha_k(s) < j, \\ \alpha_k(s+1) - 1 & \text{if } \alpha_k(s) \ge j, \end{cases}$$

and

$$\widetilde{\alpha}_i(s) = \begin{cases} \alpha_i(s) & \text{if } \alpha_i(s) < j, \\ \alpha_i(s) - 1 & \text{if } \alpha_i(s) > j, \end{cases} \qquad \text{for } 1 \le i \le m \text{ with } i \ne k. \tag{3.8}$$

When $t_k = 1$, we put

$$\omega_j\iota^{(\alpha_1,\dots,\alpha_m)}(a, c_1, \dots, c_m) \\ = \iota^{(\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_{k-1},\widetilde{\alpha}_{k+1},\dots,\widetilde{\alpha}_m)}(\delta_k(a), c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_m),$$

where $(\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_{k-1}, \widetilde{\alpha}_{k+1}, \dots, \widetilde{\alpha}_m) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_{k-1},t_{k+1},\dots,t_m)}$ is given by (3.8).

In the case of $\omega_n \colon KP_n \to KP_{n-1}$, we define $\omega_n\iota^{((1,\dots,n-1))}(a, c) = c$ and

$$\omega_n\iota^{(\alpha_1,\dots,\alpha_m)}(a, c_1, \dots, c_m) = \varepsilon^{(\alpha_1,\dots,\alpha_m)}(\delta_{m+1}(a), c_1, \dots, c_m) \qquad \text{for } m \ge 2.$$

§4. Higher homotopy commutativity

Let Δ^m denote the m-simplex

$$\Delta^m = \left\{(t_0, \dots, t_m) \in (\mathbb{R}^+)^{m+1} \mid t_0 + \cdots + t_m = 1\right\} \qquad \text{for } m \ge 0,$$

where $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \ge 0\}$. Then we have face operators $\{\partial_k \colon \Delta^{m-1} \to \Delta^m\}_{0\le k\le m}$ and degeneracy operators $\{\sigma_j \colon \Delta^m \to \Delta^{m-1}\}_{1\le j\le m}$ (cf. [8, p. 109]).

Definition 4.1. Let $n \geq 1$. A topological monoid X is called a $C(n)$-*space* if there is a family $\{\psi_i\colon \Delta^{i-1} \times X^i \to X\}_{1\leq i\leq n}$ of maps with the following relations:

$$\psi_1(*, y) = y; \tag{4.1}$$

$$\psi_i(\partial_k(a), x_1, \dots, x_{i-1}, y) = \begin{cases} x_1\psi_{i-1}(a, x_2, \dots, x_{i-1}, y) & \text{if } k = 0, \\ \psi_{i-1}(a, x_1, \dots, (x_k x_{k+1}), \dots, x_{i-1}, y) & \text{if } 0 < k < i-1, \\ \psi_{i-1}(a, x_1, \dots, x_{i-2}, y)x_{i-1} & \text{if } k = i-1; \end{cases} \tag{4.2}$$

$$\psi_i(a, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_{i-1}, y) = \psi_{i-1}(\sigma_j(a), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{i-1}, y) \quad \text{for } 1 \leq j \leq i-1; \tag{4.3}$$

$$\psi_i(a, x_1, \dots, x_{i-1}, *) = x_1 \cdots x_{i-1}. \tag{4.4}$$

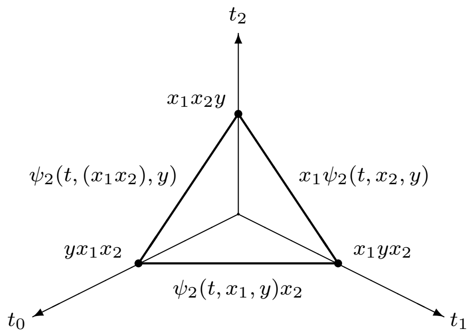

Figure 11. The $C(3)$-structure on X.

Remark 4.2. By Definition 4.1, a $C(n)$-space is the same as a $C_1(n)$-space in the sense of Hemmi–Kawamoto [11, Definition 4.3] for $n \geq 1$.

A $C(1)$-space is just a topological monoid. Since $\psi_2(\partial_0(1), x, y) = xy$ and $\psi_2(\partial_1(1), x, y) = yx$ for $x, y \in X$, a topological monoid X is a $C(2)$-space if and only if the multiplication of X is homotopy commutative. Any abelian topological monoid is a $C(\infty)$-space whose $C(\infty)$-structure $\{\psi_i\}_{i\geq 1}$ is given by

$$\psi_i(a, x_1, \dots, x_{i-1}, y) = x_1 \cdots x_{i-1}y \quad \text{for } i \geq 1.$$

In particular, Eilenberg–Mac Lane spaces are $C(\infty)$-spaces (cf. [23, Corollary 13.10]).

According to Aguadé [1, p. 939], a space Y is called a T-*space* if

$$\Omega Y \to \mathrm{Map}(S^1, Y) \xrightarrow{e} Y$$

is fiber homotopy equivalent to the trivial fibration, where ΩY is the based loop space of Y and $e\colon \mathrm{Map}(S^1, Y) \to Y$ denotes evaluation at the base point. While an H-space is always a T-space, the converse is not true.

Let $\widetilde{\Omega}Y$ denote the based loop space of Y in the sense of Moore defined by

$$\widetilde{\Omega}Y = \{\alpha\colon [0,r] \to Y \mid r \in \mathbb{R}^+ \text{ and } \alpha(0) = \alpha(r) = *\}$$

(cf. [23, Definition 4.1]).

By Remark 4.2, we have the following proposition:

Proposition 4.3 ([11, Corollary 1.1]). *A connected topological monoid X is a $C(\infty)$-space if and only if the classifying space BX is a T-space. In particular, if Y is an H-space, then $\widetilde{\Omega}Y$ is a $C(\infty)$-space.*

Stasheff [22, I, Section 2] defined A_n-spaces using the associahedra $\{K_i\}_{1\le i\le n}$. An A_n-*form* on a space X is a family of maps $\{\mu_i\colon K_i \times X^i \to X\}_{1\le i\le n}$ with the following relations:

(4.5) $\mu_1(*, x) = x$;

(4.6) $\mu_i(\partial_k(r,s)(a,b), x_1, \dots, x_i)$
$= \mu_r(a, x_1, \dots, x_{k-1}, \mu_s(b, x_k, \dots, x_{k+s-1}), x_{k+s}, \dots, x_i)$ for $(r,s,k) \in \mathbb{K}_i$;

(4.7) $\mu_i(a, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i)$
$= \mu_{i-1}(\theta_j(a), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i)$ for $1 \le j \le i$.

A space with an A_n-form is called an A_n-*space* for $n \ge 1$. From the definition, an A_1-space is just a space. Since $\mu_2(*, x, *) = \mu_2(*, *, x) = x$ for $x \in X$, $\mu_3(\partial_1(2,2)(*,*), x_1, x_2, x_3) = (x_1x_2)x_3$ and $\mu_3(\partial_2(2,2)(*,*), x_1, x_2, x_3) = x_1(x_2x_3)$ for $x_1, x_2, x_3 \in X$, we see that an A_2-space and an A_3-space are the same as an H-space and a homotopy associative H-space, respectively.

If there is a family $\{\mu_i\}_{i\ge 1}$ of maps such that $\{\mu_i\}_{1\le i\le n}$ is an A_n-form on X for any $n \ge 1$, then X is called an A_∞-*space*. By [23, Theorem 11.4], X is an A_∞-space if and only if $X \simeq \widetilde{\Omega}(BX)$.

Using the cyclohedra $\{W_i\}_{1\le i\le n}$, we generalize Definition 4.1 to the case of A_n-spaces.

Definition 4.4. Let $n \ge 1$. Assume that X is an A_n-space with an A_n-form $\{\mu_i\}_{1\le i\le n}$. Then X is called a B_n-*space* if there is a family of maps $\{\varphi_i\colon W_i \times X^i \to X\}_{1\le i\le n}$ with the following relations:

(4.8) $\varphi_1(*, y) = y$;

(4.9) $\varphi_i(\varepsilon_k(r,s)(a,b), x_1, \dots, x_{i-1}, y)$
$= \varphi_r(a, x_1, \dots, x_{k-1}, \mu_s(b, x_k, \dots, x_{k+s-1}), x_{k+s}, \dots, x_{i-1}, y)$
for $(r,s,k) \in \mathbb{W}_i$;

(4.10) $\varphi_i(\varepsilon'_k(r,s)(a,b),x_1,\dots,x_{i-1},y)$
$= \mu_r(a,x_1,\dots,x_{k-1},\varphi_s(b,x_k,\dots,x_{k+s-2},y),x_{k+s-1},\dots,x_{i-1})$ for $(r,s,k)\in \mathbb{W}'_i$;

(4.11) $\varphi_i(a,x_1,\dots,x_{j-1},*,x_{j+1},\dots,x_{i-1},y)$
$= \varphi_{i-1}(\delta_j(a),x_1,\dots,x_{j-1},x_{j+1},\dots,x_{i-1},y)$ for $1\le j\le i-1$;

(4.12) $\varphi_i(a,x_1,\dots,x_{i-1},*) = \mu_{i-1}(\delta_n(a),x_1,\dots,x_{i-1})$.

Remark 4.5. (1) A B_1-space is just a space. Since $\varphi_2(\varepsilon'_2(2,1)(*,*),x,y) = xy$ and $\varphi_2(\varepsilon'_1(2,1)(*,*),x,y) = yx$ for $x,y\in X$, a B_2-space is the same as a homotopy commutative H-space.

(2) When X is a topological monoid, X is a B_n-space if and only if X is a $C(n)$-space.

Let X and Y be A_n-spaces. According to Stasheff [22, II, Definition 4.1], a map $f\colon X\to Y$ is called an A_n-*homomorphism* if $f\mu_i^X = \mu_i^Y(1_{K_i}\times f^i)$ for $1\le i\le n$, where $\{\mu_i^X\}_{1\le i\le n}$ and $\{\mu_i^Y\}_{1\le i\le n}$ are A_n-forms on X and Y, respectively.

Definition 4.6. Let $n\ge 1$. Assume that X and Y are B_n-spaces with B_n-structures $\{\varphi_i^X\}_{1\le i\le n}$ and $\{\varphi_i^Y\}_{1\le i\le n}$, respectively. An A_n-homomorphism $f\colon X\to Y$ is called a B_n-*homomorphism* if $f\varphi_i^X = \varphi_i^Y(1_{W_i}\times f^i)$ for $1\le i\le n$.

Example 4.7. Let $(\widetilde{X},\rho,X)$ be a covering space. If X is a B_n-space, then $\widetilde{X}$ is also a B_n-space so that the projection $\rho\colon\widetilde{X}\to X$ is a B_n-homomorphism for $n\ge 1$.

Proof. We give an outline of the proof. Since the result is clear for $n=1$, we assume $n>1$. Let $\{\mu_i\}_{1\le i\le n}$ and $\{\varphi_i\}_{1\le i\le n}$ be an A_n-form and a B_n-structure on X, respectively.

Put $g_i = \mu_i(1_{K_i}\times\rho^i)$ for $1\le i\le n$. Let $\alpha\in\pi_1(K_i\times\widetilde{X}^i)$. Since $\pi_1(K_i\times\widetilde{X}^i)\cong\pi_1(\widetilde{X})^i$, we can write $\alpha=(a_1,\dots,a_i)$ with $a_j\in\pi_1(\widetilde{X})$ for $1\le j\le i$. Let $\mu'_i\colon K_i\times X^i\to X$ be defined by $\mu'_i(b,x_1,\dots,x_i) = (\cdots((x_1x_2)x_3)\cdots)x_i$. Since X is an H-space and $\mu_i\simeq\mu'_i$, we have $g_{i\#}(\alpha) = \rho_\#(a_1)+\cdots+\rho_\#(a_i) = \rho_\#(a_1*\cdots*a_i)\in\rho_\#(\pi_1(\widetilde{X}))$, where $+$ and $*$ denote the multiplications of $\pi_1(X)$ and $\pi_1(\widetilde{X})$, respectively. Then $g_{i\#}(\pi_1(K_i\times\widetilde{X}^i))\subset\rho_\#(\pi_1(\widetilde{X}))$, and so we have a lifting $\widetilde{\mu}_i\colon K_i\times\widetilde{X}^i\to\widetilde{X}$ with $\rho\widetilde{\mu}_i = g_i$ for $1\le i\le n$ (cf. [12, Chapter III, Section 16, Theorem 16.2]).

In a similar way, we have a map $\widetilde{\varphi}_i\colon W_i\times\widetilde{X}^i\to\widetilde{X}$ with $\rho\widetilde{\varphi}_i = \varphi_i(1_{W_i}\times\rho^i)$ for $1\le i\le n$. From the uniqueness of lifting, $\{\widetilde{\mu}_i\}_{1\le i\le n}$ and $\{\widetilde{\varphi}_i\}_{1\le i\le n}$ are an A_n-form and a B_n-structure on $\widetilde{X}$, respectively. ∎

Consider the double suspension $\Sigma^2\colon (S^{2m-1})^\wedge_p \to \widetilde{\Omega}^2(S^{2m+1})^\wedge_p$ which is the double adjoint of the identity $1_{(S^{2m+1})^\wedge_p}$ on $(S^{2m+1})^\wedge_p \simeq \Sigma^2(S^{2m-1})^\wedge_p$ for $m \geq 1$, where p is a prime and $Y^\wedge_p$ denotes the p-completion of the space Y in the sense of Bousfield–Kan [5, Chapter VI, Section 6]. By Proposition 4.3 and Remark 4.5, we deduce that $\widetilde{\Omega}^2(S^{2m+1})^\wedge_p$ is a B_∞-space.

According to Stasheff [22, I, Theorem 17], $(S^{2m-1})^\wedge_p$ is an A_{p-1}-space such that $\Sigma^2\colon (S^{2m-1})^\wedge_p \to \widetilde{\Omega}^2(S^{2m+1})^\wedge_p$ is an A_{p-1}-homomorphism.

Example 4.8. Let p be a prime. Then $(S^{2m-1})^\wedge_p$ is a B_{p-1}-space such that the double suspension $\Sigma^2\colon (S^{2m-1})^\wedge_p \to \widetilde{\Omega}^2(S^{2m+1})^\wedge_p$ is a B_{p-1}-homomorphism for $m \geq 1$.

Proof. Since the result is clear for $p = 2$, we consider the case of $p > 2$. As in the proof of [22, I, Theorem 17], we assume that $(S^{2m-1})^\wedge_p$ is a subspace of $\widetilde{\Omega}^2(S^{2m+1})^\wedge_p$ and $\Sigma^2\colon (S^{2m-1})^\wedge_p \to \widetilde{\Omega}^2(S^{2m+1})^\wedge_p$ is the inclusion.

For simplicity, we write $X = (S^{2m-1})^\wedge_p$ and $Y = \widetilde{\Omega}^2(S^{2m+1})^\wedge_p$. Let $\{\kappa_i\}_{i\geq 1}$ be a B_∞-structure on Y. By induction on i, we construct a B_{p-1}-structure $\{\varphi_i\}_{1\leq i\leq p-1}$ on X with $\Sigma^2\varphi_i = \kappa_i(1_{W_i} \times (\Sigma^2)^i)$ for $1 \leq i \leq p-1$.

Put $\varphi_1(*, x) = x$ for $x \in X$. Assume inductively that $\{\varphi_j\}_{1\leq j<i}$ is constructed. Let $F_i = \partial W_i \times X^i \cup W_i \times X^{[i]}$, where $Z^{[i]}$ denotes the i-fold fat wedge of a space Z given by

$$Z^{[i]} = \{(z_1, \ldots, z_i) \in Z^i \mid z_j = * \text{ for some } j \text{ with } 1 \leq j \leq i\} \qquad \text{for } i \geq 1.$$

Then we have $(W_i \times X^i)/F_i \simeq (S^{2mi-1})^\wedge_p$.

Define $\widetilde{\varphi}_i\colon W_i \times X^i \to Y$ by $\widetilde{\varphi}_i = \kappa_i(1_{W_i} \times (\Sigma^2)^i)$. By inductive hypothesis, we have $\widetilde{\varphi}_i(F_i) \subset X$. Then the obstructions to obtain $\varphi_i\colon W_i \times X^i \to X$ with $\Sigma^2\varphi_i \simeq \widetilde{\varphi}_i$ rel F_i appear in the following cohomology groups:

$$H^k(W_i \times X^i, F_i; \pi_k(Y, X)) \cong \widetilde{H}^k((S^{2mi-1})^\wedge_p; \pi_k(Y, X)) \qquad \text{for } k \geq 1 \tag{4.13}$$

(cf. [12, p. 197, E.6]). Now, (4.13) is non-trivial only if $k = 2mi-1 \leq 2mp-2m-1 \leq 2mp-3$. On the other hand, $\pi_k(Y, X) = 0$ for $k \leq 2mp-3$ by Toda [27, Proposition 13.1]. This implies that (4.13) is trivial for any k, and we have a map φ_i. From the homotopy extension property, we have a map $\widetilde{\kappa}_i\colon W_i \times Y^i \to Y$ with $\widetilde{\kappa}_i \simeq \kappa_i$ rel $\partial W_i \times Y^i \cup W_i \times Y^{[i]}$ and $\Sigma^2\varphi_i = \widetilde{\kappa}_i(1_{W_i} \times (\Sigma^2)^i)$. This completes the induction, and we have a B_{p-1}-structure $\{\varphi_i\}_{1\leq i\leq p-1}$ on X. □

Hemmi–Kawamoto [9, Definition 3.1] introduced another type of higher homotopy commutativity of A_n-spaces using the permuto-associahedra $\{KP_i\}_{1\leq i\leq n}$. Let X be an A_n-space with an A_n-form $\{\mu_i\}_{1\leq i\leq n}$ for $n \geq 1$. Then X is called an AC_n-*space* if there is a family $\{\nu_i\colon KP_i \times X^i \to X\}_{1\le i\le n}$ of maps with the following relations:

(4.14) $\nu_1(*, x) = x$;

(4.15)
$$\nu_i(\varepsilon^{(\alpha_1,\dots,\alpha_m)}(a, b_1, \dots, b_m), x_1, \dots, x_i) = \mu_m(a, \nu_{t_1}(b_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(t_1)}), \dots, \nu_{t_m}(b_m, x_{\alpha_m(1)}, \dots, x_{\alpha_m(t_m)}))$$
for $(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_i^{(t_1,\dots,t_m)}$ with $m \ge 2$;

(4.16)
$$\nu_i(a, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = \nu_{i-1}(\omega_j(a), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i) \quad \text{for } 1 \le j \le i.$$

Remark 4.9. (1) An AC_1-space is just a space. Since $\nu_2(\varepsilon^{((1),(2))}(*, *, *), x_1, x_2) = x_1x_2$ and $\nu_2(\varepsilon^{((2),(1))}(*, *, *), x_1, x_2) = x_2x_1$ for $x_1, x_2 \in X$, an AC_2-space is the same as a homotopy commutative H-space.

(2) When X is a topological monoid, X is an AC_n-space if and only if it is a C_n-space in the sense of Williams [28, Definition 5].

Proof of Theorem B. We work by induction on n. The result is clear for $n = 1$. Assume inductively that the result is proved for any $n' < n$.

Let X be a B_n-space with a B_n-structure $\{\varphi_i\}_{1\le i\le n}$. By inductive hypothesis, X is an AC_{n-1}-space with an AC_{n-1}-structure $\{\nu_i\}_{1\le i\le n-1}$. From Theorem A and Remark 3.2, we can define $\nu_n\colon KP_n \times X^n \to X$ by
$$\nu_n(\iota^{(\alpha_1,\dots,\alpha_m)}(a, b_1, \dots, b_m), x_1, \dots, x_n) = \varphi_m(a, \nu_{t_1}(b_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(t_1)}), \dots, \nu_{t_m}(b_m, x_{\alpha_m(1)}, \dots, x_{\alpha_m(t_m)}), x_n)$$
for $(\alpha_1, \dots, \alpha_m) \in \mathbb{A}_{n-1}^{(t_1,\dots,t_m)}$ with $m \ge 1$

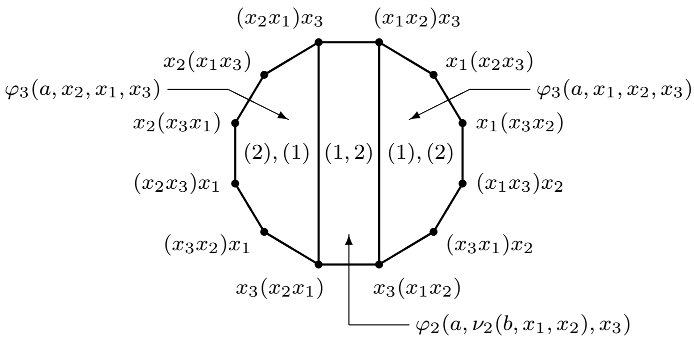

Figure 12. The B_3-structure on X.

(see Figure 12). From the proof of Theorem A, we see that $\{\nu_i\}_{1\le i\le n}$ is an AC_n-structure on X. □

Let p be a prime. An H-space X is called *p-Postnikov* if there is an integer $l_X \geq 1$ such that $\pi_j(X)$ is finitely generated over the p-adic integers $\mathbb{Z}_p^\wedge$ for $1 \leq j \leq l_X$ and $\pi_j(X) = 0$ for $j > l_X$. For example, Eilenberg–Mac Lane spaces $K(\mathbb{Z}_p^\wedge, m)$ and $K(\mathbb{Z}/p^i, m)$ are p-Postnikov H-spaces for $i, m \geq 1$.

Remark 4.10. By the result of McGibbon–Neisendorfer [19, Theorem 1], if X is a connected p-Postnikov H-space whose cohomology $H^*(X; \mathbb{F}_p)$ is finite-dimensional, then X is homotopy equivalent to a p-completed torus.

Let $(\mathbb{C}P^\infty)_p^\wedge$ denote the p-completion of the infinite-dimensional complex projective space. Its cohomology is given by $H^*((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p) \cong \mathbb{F}_p[u]$ with $\deg u = 2$. Denote the homotopy fiber of the map $f_t \colon (\mathbb{C}P^\infty)_p^\wedge \to K(\mathbb{Z}/p, 2t)$ corresponding to the class $u^t \in H^{2t}((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p)$ by Y_t for $t \geq 1$. Put $X_t = \widetilde{\Omega} Y_t$.

Remark 4.11. (1) X_t is a p-Postnikov H-space.

(2) Y_t is an H-space if and only if $t = p^i$ for some $i \geq 1$.

By Remarks 4.5 and 4.9, we have the following example:

Example 4.12 ([11, Propositions 5.3 and 5.5]). (1) If $t = 1$ or $t \equiv 0 \bmod p$, then X_t is a B_∞-space.

(2) If $1 < t < p$, then X_t is a B_{t-1}-space, but not an AC_t-space.

(3) If $t > p$ with $t \not\equiv 0 \bmod p$, then X_t is an AC_∞-space, which is also a B_{t-1}-space, but not a B_t-space.

From Theorem B, all results stated for AC_n-spaces also hold for B_n-spaces (cf. [9], [10] and [15]).

For example, if X is a connected B_p-space whose cohomology $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra $\mathscr{A}_p^*$, then $X_p^\wedge$ is a p-Postnikov H-space by [15, Theorem B]. Moreover, if $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over $\mathbb{F}_p$, then $X_p^\wedge$ is homotopy equivalent to a finite product of $(S^1)_p^\wedge$s, $(\mathbb{C}P^\infty)_p^\wedge$s and $B\mathbb{Z}/p^i$s with $i \geq 1$ using [9, Theorem B]. On the other hand, $(S^{2m-1})_p^\wedge$ is a B_{p-1}-space which is not p-Postnikov for any $m > 1$ by Example 4.8 and Remark 4.10.

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