

Spectral Analysis of a Quantum System with a Double Line Singular Interaction

by

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Abstract

We consider a non-relativistic quantum particle interacting with a singular potential supported by two parallel straight lines in the plane. We locate the essential spectrum under the hypothesis that the interaction asymptotically approaches a constant value, and find conditions which guarantee either the existence of discrete eigenvalues or Hardy-type inequalities. For a class of our models admitting mirror symmetry, we also establish the existence of embedded eigenvalues and show that they turn into resonances after introducing a small perturbation.

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§1. Introduction

The problem we study in this paper belongs to the line of research often called *singular perturbations of Schrödinger operators*. Let us consider a non-relativistic quantum particle confined to a semiconductor structure $\Sigma \subset \mathbb{R}^3$. Suppose the particle has a possibility of *tunnelling*, so the whole \mathbb{R}^3 forms the configuration space. On the other hand, if the device Σ is *narrow* in a sense, we can make an idealization and assume that Σ is a set of lower dimension, for example, a surface, a curve or dots in \mathbb{R}^3 . Consequently, we come to the model of quantum systems with potential interaction supported by a *null set*. The interaction can vary on Σ ; let a function $V : \Sigma \rightarrow \mathbb{R}$ denote the potential strength. Then the Hamiltonian of

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such a quantum system can be symbolically written as

$$(1.1) \quad -\Delta + V\delta(\cdot - \Sigma),$$

where $-\Delta$ denotes the Laplace operator in $L^2(\mathbb{R}^3)$ and δ is the Dirac delta function. In view of singular interactions with translational symmetry, it also makes sense to consider one- and two-dimensional analogues of (1.1).

There are a lot of papers devoted to analysing the relation between the geometry of Σ and spectral properties of the Hamiltonian with delta interactions of *constant* strength [6, 11, 12, 13, 15, 16]; we also refer to the monographs [2, 3] with many references. Delta-type potentials supported on infinite curves and surfaces are used in particular for mathematical modelling of *leaky quantum wires and graphs* [9, 4].

The simplest known model belonging to the class described by (1.1) is given by N *quantum dots in one dimension* [2, Chap. I.3, II.2]. Then $\Sigma := \{x_i\}_{i=0}^{N-1}$ and $V(x_i) := -\alpha_i \in \mathbb{R}$. For one point interaction (i.e. $N = 1$, $\alpha_0 =: \alpha$) the system has one negative eigenvalue $-\alpha^2/4$ if, and only if, α is positive. In the case of two point interactions of *equal* strength (i.e. $N = 2$, $\alpha_0 = \alpha_1 =: \alpha$) at distance $2a$ from each other, there is one negative eigenvalue ξ_0 if, and only if, $0 < \alpha a \leq 1$ or two eigenvalues $\xi_0 < \xi_1$ whenever $\alpha a > 1$.

The problem we discuss in this paper can be considered as a *generalization of the two quantum dots* in two respects. First, our model is two-dimensional, with the set Σ being one-dimensional. Second, the generalized geometry enables us to consider potentials V of variable strength. More specifically, we consider the singular set Σ composed of two infinite lines

$$(1.2) \quad \Sigma := \Sigma_- \cup \Sigma_+ \quad \text{with} \quad \Sigma_{\pm} := \mathbb{R} \times \{\pm a\}$$

in \mathbb{R}^2 and

$$(1.3) \quad V(x) := \begin{cases} -\alpha + V_+(x) & \text{if } x \in \Sigma_+, \\ -\alpha + V_-(x) & \text{if } x \in \Sigma_-, \end{cases}$$

with $V_{\pm} : \Sigma_{\pm} \rightarrow \mathbb{R}$ and $\alpha > 0$. In the physical setting described above, the negative part $-\alpha$ of V models the confinement of the particle to Σ , while V_{\pm} can be thought of as a perturbation.

Our first aim is to find a *self-adjoint realization* in $L^2(\mathbb{R}^2)$ of (1.1) with (1.2)–(1.3). We will apply the form-sum method and the resulting operator will be called H_{α, V_+, V_-} . Note that the delta potential in our model does not vanish at infinity even if V_{\pm} do (just because α is assumed to be positive). This means that we may expect that the essential spectrum of H_{α, V_+, V_-} will differ from the spectrum of the

free Hamiltonian in \mathbb{R}^2 . In our setting, the role of the *unperturbed Hamiltonian* is played by $H_{\alpha,0,0}$.

For the unperturbed Hamiltonian, translational symmetry allows us to decompose the operator as follows:

$$(1.4) \quad H_{\alpha,0,0} \simeq (-\Delta^{\mathbb{R}}) \otimes 1 + 1 \otimes (-\Delta_{\alpha}^{\mathbb{R}}) \quad \text{on } L^2(\mathbb{R}) \otimes L^2(\mathbb{R}),$$

where $-\Delta^{\mathbb{R}}$ is the free one-dimensional Hamiltonian and $-\Delta_{\alpha}^{\mathbb{R}}$ governs the aforementioned one-dimensional system with two point interactions. The non-negative semi-axis is the spectrum of $-\Delta^{\mathbb{R}}$. On the other hand, as already mentioned, the spectrum of $-\Delta_{\alpha}^{\mathbb{R}}$ is

$$\sigma(-\Delta_{\alpha}^{\mathbb{R}}) = \sigma_{\text{disc}}(-\Delta_{\alpha}^{\mathbb{R}}) \cup [0, \infty),$$

with the negative eigenvalues (cf. Lemma 2.3 and Figure 1)

$$\sigma_{\text{disc}}(-\Delta_{\alpha}^{\mathbb{R}}) = \begin{cases} \{\xi_0\} & \text{if } 0 < \alpha a \leq 1, \\ \{\xi_0, \xi_1\} & \text{if } \alpha a > 1, \end{cases}$$

(the discrete spectrum is empty in the other situations, which are excluded here by the assumption $\alpha > 0$). Recalling the ordering $\xi_0 < \xi_1$ (if the latter eigenvalue exists), we conclude that (irrespective of the value of αa)

$$(1.5) \quad \sigma(H_{\alpha,0,0}) = \sigma_{\text{ess}}(H_{\alpha,0,0}) = [\xi_0, \infty).$$

The main results of the paper can be formulated as follows.

- *Definition of the Hamiltonian and its resolvent.* The definition of the Hamiltonian H_{α,V_+,V_-} by use of the form-sum method is given in Section 2. We also derive a Krein-like formula for the resolvent of H_{α,V_+,V_-} as a tool for further discussion.

- *Essential spectrum.* In Section 3 we find a weak condition guaranteeing the stability of the essential spectrum of H_{α,V_+,V_-} with respect to $H_{\alpha,0,0}$. We show that if V_{\pm} vanish at infinity then

$$(1.6) \quad \sigma_{\text{ess}}(H_{\alpha,V_+,V_-}) = \sigma_{\text{ess}}(H_{\alpha,0,0}) = [\xi_0, \infty).$$

The strategy of our proof is as follows. Using a Neumann bracketing argument together with the minimax principle, we get $\inf \sigma_{\text{ess}}(H_{\alpha,V_+,V_-}) \geq \xi_0$. The opposite inclusion $\sigma_{\text{ess}}(H_{\alpha,V_+,V_-}) \supseteq [\xi_0, \infty)$ is obtained by applying the Weyl criterion adapted to sesquilinear forms.

- *Discrete and embedded eigenvalues.* The point spectrum is investigated in Section 4. We show that the bottom of the spectrum of H_{α,V_+,V_-} starts below ξ_0

provided that the sum $V_+ + V_-$ is negative in an integral sense. Assuming additionally that V_{\pm} vanish at infinity and combining this with the previous result on the essential spectrum, we therefore obtain

$$(1.7) \quad \sigma_{\text{disc}}(H_{\alpha, V_+, V_-}) \neq \emptyset.$$

The proof is based on finding a suitable test function in the variational definition of the spectral threshold. We also find conditions which guarantee the existence of embedded eigenvalues in the system with mirror symmetry, i.e. when $V_+ = V_-$.

- *Hardy inequalities.* The case of repulsive singular potentials, i.e. $V_{\pm} \geq 0$, is studied in Section 5. In order to quantify the repulsive character of the singular potential, we derive Hardy-type inequalities

$$(1.8) \quad H_{\alpha, V_+, V_-} - \xi_0 \geq \varrho$$

in the form sense, where $\varrho : \mathbb{R} \rightarrow [0, \infty)$ is not identically zero. The functional inequality (1.8) is useful in the study of spectral stability of H_{α, V_+, V_-} ; indeed, it determines a class of potentials which can be added to our system without producing any spectrum below ξ_0 .

- *Resonances.* Finally, in Section 6 we show that breaking the mirror symmetry by introducing a “perturbant” function V_p on one of the lines, for example

$$V_+ = V_0 + \epsilon V_p, \quad V_- = V_0,$$

leads to resonances. These resonances are localized near the original embedded eigenvalues ν_k appearing when $\epsilon = 0$. More precisely, they are determined by poles of the resolvent which take the form

$$z_k = \nu_k + \mu_k(\epsilon) + i\nu_k(\epsilon),$$

where $\mu_k(\epsilon) = a_k\epsilon + \mathcal{O}(\epsilon^2)$ with a_k corresponding to the first order perturbation term and $\nu_k(\epsilon) = b_k\mathcal{O}(\epsilon^2)$ where $b_k < 0$ establishes the Fermi golden rule.

Let us conclude this introductory section by pointing out some special notations frequently used throughout the paper. We abbreviate $L^2 := L^2(\mathbb{R}^2)$ and $L^2_{\pm} := L^2(\mathbb{R} \times \{\pm a\})$. We also briefly write $W^{n,2} := W^{n,2}(\mathbb{R}^2)$ for the corresponding Sobolev spaces. The inner product and norm in L^2 are denoted by (\cdot, \cdot) and $\|\cdot\|$, respectively. Given a self-adjoint operator H , the symbols $\sigma_{\iota}(H)$ with $\iota \in \{\text{ess, ac, sc, p, disc}\}$ denote, respectively, the essential, absolutely continuous, singularly continuous, point and discrete spectrum of H . We use the symbol 1 to denote the identity operators acting in various Hilbert spaces.

§2. The Hamiltonian and its resolvent

Let V_+ and V_- be two real-valued functions from $L^\infty(\mathbb{R})$. Given a positive number a , we also denote by V_+ and V_- the functions $V_+ \otimes 1$ and $V_- \otimes 1$ on $\mathbb{R} \times \{+a\}$ and $\mathbb{R} \times \{-a\}$, respectively. Finally, let α be a *positive* constant.

§2.1. The self-adjoint realization of the Hamiltonian

Let us consider the quadratic form

$$\begin{aligned} \mathcal{E}_{\alpha, V_+, V_-}[\psi] &:= \int_{\mathbb{R}^2} |\nabla\psi|^2 + \int_{\mathbb{R} \times \{+a\}} (V_+ - \alpha)|I_+\psi|^2 + \int_{\mathbb{R} \times \{-a\}} (V_- - \alpha)|I_-\psi|^2, \\ \mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-}) &:= W^{1,2}. \end{aligned}$$

Here I_\pm are the trace operators associated with the Sobolev embedding $W^{1,2} \hookrightarrow L^2_\pm$. The corresponding sesquilinear form will be denoted by $\mathcal{E}_{\alpha, V_+, V_-}(\cdot, \cdot)$.

The form $\mathcal{E}_{\alpha, V_+, V_-}$ is clearly densely defined and symmetric. Moreover, the boundary integrals can be shown to be a relatively bounded perturbation of the form $\mathcal{E}_{0,0,0}$ with relative bound less than 1. This is a consequence of the boundedness of V_\pm and the following result.

Lemma 2.1. *For every $\psi \in W^{1,2}$ and $\epsilon \in (0, 1)$, we have*

$$(2.9) \quad \|I_\pm\psi\|_{L^2_\pm}^2 \leq \frac{1}{\epsilon} \|\psi\|_{L^2}^2 + \epsilon \|\partial_2\psi\|_{L^2}^2.$$

Proof. For every $\epsilon \in (0, 1)$ and $\psi \in C_0^\infty(\mathbb{R}^2)$, we have the bound

$$\begin{aligned} |\psi(x_1, \pm a)|^2 &= \int_{-\infty}^{\pm a} \partial_2|\psi|^2(x_1, x_2) dx_2 = 2\Re \int_{-\infty}^{\pm a} \overline{\psi(x_1, x_2)} \partial_2\psi(x_1, x_2) dx_2 \\ &\leq \frac{1}{\epsilon} \int_{\mathbb{R}} |\psi(x_1, x_2)|^2 dx_2 + \epsilon \int_{\mathbb{R}} |\partial_2\psi(x_1, x_2)|^2 dx_2. \end{aligned}$$

Integrating over x_1 , we therefore get (2.9) for $\psi \in C_0^\infty(\mathbb{R}^2)$. By density, the inequality extends to $W^{1,2}$. □

Remark 2.2 (Relation to the generalized Kato class). Inequality (2.9) represents a quantification of the embedding $W^{1,2} \hookrightarrow L^2_\pm$. If the support of the singular potential has a more complicated geometry we can derive a generalization of (2.9). Consider a Radon measure μ on \mathbb{R}^2 with support on a C^1 curve (finite or infinite) without self-intersections and “near-self-intersections” (see [5, Sec. 4] for precise assumptions). Such a measure belongs to the generalized Kato class (cf. [5, Thm. 4.1]), and consequently for any $a > 0$ there exists $b > 0$ such that

$$\int_{\mathbb{R}^2} |\psi|^2 d\mu \leq b\|\psi\|_{L^2}^2 + a\|\nabla\psi\|_{L^2}^2 \quad \text{for every } \psi \in W^{1,2}.$$

Since $\mathcal{E}_{0,0,0}$ is clearly closed and non-negative (it is in fact associated with the free Hamiltonian in \mathbb{R}^2), it follows from the KLMN theorem [25, Thm. X.17] together with Lemma 2.1 that $\mathcal{E}_{\alpha,V_+,V_-}$ is closed and bounded from below. Consequently, there exists a unique bounded-from-below self-adjoint operator H_{α,V_+,V_-} in L^2 which is associated with $\mathcal{E}_{\alpha,V_+,V_-}$. (Notice that the sign of α plays no role in the definition of H_{α,V_+,V_-} .)

Finally, let us note that H_{α,V_+,V_-} is indeed a natural realization of the formal expression (1.1) with (1.2)–(1.3). As a matter of fact, our form $\mathcal{E}_{\alpha,V_+,V_-}$ represents a closed extension of the form associated with the expression (1.1) initially considered as acting on smooth functions rapidly decaying at the infinity of \mathbb{R}^2 .

§2.2. The transverse Hamiltonian

Recall the decomposition (1.4) for the unperturbed Hamiltonian $H_{\alpha,0,0}$. Here the “transverse” operator $-\Delta_{\alpha}^{\mathbb{R}}$ is associated with the form

$$\varepsilon_{\alpha}[\phi] := \int_{\mathbb{R}} |\phi'|^2 - \alpha|\phi(+a)|^2 - \alpha|\phi(-a)|^2, \quad \mathfrak{D}(\varepsilon_{\alpha}) := W^{1,2}(\mathbb{R}).$$

Note that the boundary values have a meaning in view of the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$. It is easy to verify that $\mathfrak{D}(-\Delta_{\alpha}^{\mathbb{R}})$ consists of all functions $\psi \in W^{1,2}(\mathbb{R}) \cap W^{2,2}(\mathbb{R} \setminus \{-a, +a\})$ satisfying the interface conditions

$$(2.10) \quad \psi'(\pm a + 0) - \psi'(\pm a - 0) = -\alpha\psi(\pm a),$$

where $\psi'(a \pm 0) := \lim_{\epsilon \rightarrow 0^+} \psi'(a \pm \epsilon)$, and that $-\Delta_{\alpha}^{\mathbb{R}}\psi = -\psi''$ on $\mathbb{R} \setminus \{-a, +a\}$. An alternative way of introducing $-\Delta_{\alpha}^{\mathbb{R}}$ is via extension theory [2, Chap. II.2].

Due to [2, Thm. 2.1.3], we have $\sigma_{\text{ess}}(-\Delta_{\alpha}^{\mathbb{R}}) = \sigma_{\text{ac}}(-\Delta_{\alpha}^{\mathbb{R}}) = [0, \infty)$. The structure of the discrete spectrum depends on the value of αa .

Lemma 2.3. *The operator $-\Delta_{\alpha}^{\mathbb{R}}$ has exactly two negative eigenvalues if $\alpha a > 1$ and exactly one negative eigenvalue if $0 < \alpha a \leq 1$.*

Proof. Let $\kappa > 0$. Solving the eigenvalue problem $-\psi'' = -\kappa^2\psi$ on $\mathbb{R} \setminus \{-a, +a\}$ in terms of exponential functions decaying at infinity and using the continuity of ψ and (2.10), it is straightforward to see that the algebraic equation

$$(2.11) \quad \frac{\alpha^2}{4} e^{-4\kappa a} = \left(\kappa - \frac{\alpha}{2} \right)^2$$

represents a sufficient and necessary condition for $-\kappa^2 \in \sigma_{\text{p}}(-\Delta_{\alpha}^{\mathbb{R}})$. (Alternatively, one could directly use [2, eq. (2.1.33)].) Clearly, equation (2.11) is equivalent to

$g_1(\kappa) - g_2(\kappa) = 0$ with

$$(2.12) \quad g_1(\kappa) := \frac{\alpha}{2} e^{-2\kappa a} \quad \text{and} \quad g_2(\kappa) := \left| \kappa - \frac{\alpha}{2} \right|.$$

Since the exponential function g_1 is positive and decreasing and g_2 is increasing in $(\alpha/2, \infty)$ with $g_2(\alpha/2) = 0$, there exists exactly one solution $\kappa_0 > \alpha/2$ of (2.11) for any $\alpha > 0$. Since g_1 is strictly convex and $g_1(0) = g_2(0)$, and since the graph of g_2 is a straight line for $\kappa \in (0, \alpha/2)$ with $g_2'(0) = -1$, the existence of another (at most one) solution $\kappa_1 \in (0, \alpha/2)$ is determined by the derivative of g_1 at 0. Obviously, $g_1'(0) \geq -1$ if, and only if, $\alpha a \leq 1$. We set $\xi_0 := -\kappa_0^2$ and $\xi_1 := -\kappa_1^2$ (if the latter value exists). \square

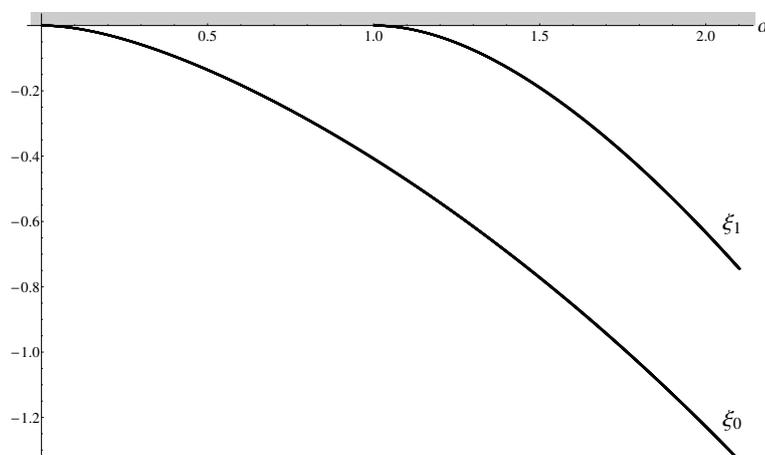


Figure 1. Emergence of the discrete eigenvalues ξ_0, ξ_1 of $-\Delta_\alpha^{\mathbb{R}}$ from the essential spectrum as α increases for $a = 1$.

We refer to Figure 1 for the dependence of the discrete spectrum on αa . The eigenfunctions corresponding to ξ_0 and ξ_1 will be denoted by ϕ_0 and ϕ_1 , respectively. For notational purposes, it will become convenient to introduce the index set

$$(2.13) \quad \mathcal{N} := \begin{cases} \{0\} & \text{if } \alpha a \leq 1, \\ \{0, 1\} & \text{if } \alpha a > 1. \end{cases}$$

Due to the symmetry of the system, ϕ_0 is even and ϕ_1 is odd. Moreover, ϕ_0 can be chosen positive [2, Thm. 2.1.3].

§2.3. A lower-bound Hamiltonian

In this subsection we derive an auxiliary result we shall use several times. It is based on an idea used in a similar context in [17].

Taking into account the structure of $\mathcal{E}_{\alpha, V_+, V_-}$, let us define

$$(2.14) \quad \tilde{\lambda}(v_+, v_-) := \inf_{\phi \in W^{1,2}(\mathbb{R}) \setminus \{0\}} \frac{\varepsilon_{\alpha}[\phi] + v_+ |\phi(+a)|^2 + v_- |\phi(-a)|^2}{\|\phi\|_{L^2(\mathbb{R})}^2} - \xi_0$$

for any real constants v_+ and v_- . The number $\tilde{\lambda}(v_+, v_-)$ is the lowest eigenvalue of the operator (1.1) on $L^2(\mathbb{R})$ shifted by ξ_0 , subject to two point interactions of strength $v_+ - \alpha$ and $v_- - \alpha$ and at distance $2a$ from each other.

It is clear that $\tilde{\lambda}(v_+, v_-)$ is a continuous and non-decreasing function of both v_+ and v_- . The symmetry relation $\tilde{\lambda}(v_+, v_-) = \tilde{\lambda}(v_-, v_+)$ holds true. Assume that both v_{\pm} are non-negative. Then by the variational definition of ξ_0 , we have $\tilde{\lambda}(v_+, v_-) \geq 0$ and $\tilde{\lambda}(0, 0) = 0$. For our purposes, it is important to point out that $\tilde{\lambda}(v_+, v_-)$ is positive whenever at least one of the arguments is. Indeed, if $\tilde{\lambda}(v_+, v_-) = 0$ and $v_+ > 0$ or $v_- > 0$, then we deduce from (2.14) that the minimum is achieved by the eigenfunction ϕ_0 of $-\Delta_{\alpha}^{\mathbb{R}}$ corresponding to ξ_0 and that $\phi_0(a) = 0$ or $\phi_0(-a) = 0$. Since ϕ_0 is positive, the latter leads to a contradiction.

If V_+ and V_- are real-valued functions, then $\tilde{\lambda}$ gives rise to a function $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ via setting

$$(2.15) \quad \lambda(x) := \tilde{\lambda}(V_+(x_1), V_-(x_1)).$$

It follows from the properties of $\tilde{\lambda}$ that if V_{\pm} are non-negative and V_- or V_+ is non-trivial (i.e., non-zero on a measurable set of positive Lebesgue measure), then λ is a non-trivial non-negative function.

In any case, using Fubini's theorem, we get a fundamental lower bound for our form $\mathcal{E}_{\alpha, V_+, V_-}$.

Lemma 2.4. *For every $\psi \in W^{1,2}$, we have*

$$(2.16) \quad \mathcal{E}_{\alpha, V_+, V_-}[\psi] - \xi_0 \|\psi\|_{L^2}^2 \geq \int_{\mathbb{R}^2} |\partial_1 \psi|^2 + \int_{\mathbb{R}^2} \lambda |\psi|^2.$$

The above result shows that $H_{\alpha, V_+, V_-} - \xi_0$ is bounded from below by the one-dimensional Schrödinger operator

$$(-\Delta^{\mathbb{R}} + \lambda) \otimes 1 \quad \text{on } L^2(\mathbb{R}) \otimes L^2(\mathbb{R}).$$

§2.4. The Krein-like resolvent formula

The first auxiliary step is to reconstruct the resolvent of the unperturbed Hamiltonian, i.e. $R_{\alpha}(z) := (H_{\alpha, 0, 0} - z)^{-1}$ for $z \in \mathbb{C} \setminus [\xi_0, \infty)$. Once we have $R_{\alpha}(z)$ we

introduce potentials V_{\pm} and build a Krein-like resolvent of H_{α, V_+, V_-} as a perturbation of $R_{\alpha}(\cdot)$. To do this, we will follow the treatment by Posilicano [22].

2.4.1. The resolvent of the unperturbed Hamiltonian. As above, ξ_j , with $j \in \mathcal{N}$, stand for the discrete eigenvalues of $-\Delta_{\alpha}^{\mathbb{R}}$, and ϕ_j denote the corresponding eigenfunctions. Recall that the essential spectrum of $-\Delta_{\alpha}^{\mathbb{R}}$ is purely absolutely continuous. Let $E_{\alpha}^{\mathbb{R}}(\cdot)$ stand for the spectral resolution of $-\Delta_{\alpha}^{\mathbb{R}}$ corresponding to the continuous spectrum and let $P_{\alpha, j}$, with $j \in \mathcal{N}$, denote the eigen-projector $P_{\alpha, j} = \phi_j(\phi_j, \cdot)$. Analogously, $E^{\mathbb{R}}(\cdot)$ denotes the spectral resolution of $-\Delta^{\mathbb{R}}$. Using (1.4), one gets

$$R_{\alpha}(z) = \sum_{j \in \mathcal{N}} \int_{\mathbb{R}_+} (\beta + \xi_j - z)^{-1} dE^{\mathbb{R}}(\beta) \otimes P_{\alpha, j} + \int_{\mathbb{R}_+^2} (\beta + \beta' - z)^{-1} dE^{\mathbb{R}}(\beta) \otimes dE_{\alpha}^{\mathbb{R}}(\beta'),$$

where $z \in \mathbb{C} \setminus [\xi_0, \infty)$. This implies the decomposition

$$R_{\alpha}(z) = R_{\alpha}^d(z) + R_{\alpha}^c(z),$$

where the $R_{\alpha}^{\iota}(z)$, with $\iota \in \{d, c\}$, act on a separated variable function $f(x) = f_1(x_1)f_2(x_2)$ as

$$R_{\alpha}^d(z)f(x) = \sum_{j \in \mathcal{N}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dp_1 \frac{\widehat{f}_1(p_1)e^{ip_1x_1}}{p_1^2 + \xi_j - z} \phi_j(x_2)(\phi_j, f_2)_{L^2(\mathbb{R})},$$

$$R_{\alpha}^c(z)f = \int_{\mathbb{R}_+^2} \frac{1}{\beta + \beta' - z} dE^{\mathbb{R}}(\beta)f_1 \otimes dE_{\alpha}^{\mathbb{R}}(\beta')f_2,$$

with \widehat{f}_1 denoting the Fourier transform of f_1 . Set $\tau_j^2(z) := z - \xi_j$. In the following we assume that z is taken from the first sheet of the domain of $z \mapsto \tau_j(z)$, i.e. $\Im\tau_j(z) > 0$.

Using the standard representation of the Green function of the one-dimensional Laplace operator,

$$(-\Delta^{\mathbb{R}} - k^2)^{-1}(x_1, y_1) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ip(x_1 - y_1)}}{p^2 - k^2} dp = \frac{i}{2} \frac{e^{ik|x_1 - y_1|}}{k},$$

we conclude that $R_{\alpha}^d(z) := \sum_{j \in \mathcal{N}} R_{\alpha}^{j,d}(z)$ where the $R_{\alpha}^{j,d}(z)$ are integral operators with kernels

$$(2.17) \quad G_{\alpha}^{j,d}(z; x, y) = \frac{i}{2} \frac{e^{i\tau_j(z)|x_1 - y_1|}}{\tau_j(z)} \phi_j(x_2) \overline{\phi_j(y_2)}.$$

Moreover

$$(2.18) \quad R_{\alpha}^c(z)f = \frac{i}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{e^{i\tau(p,z)|x_1 - y_1|}}{\tau(p, z)} \psi(p, x_2) \overline{\psi(p, y_2)} f_1(y_1) f_2(y_2) dy_1 dy_2 dp,$$

where $\tau^2(p, z) = z - p^2$, $\Im\tau(p, z) > 0$ and $\psi(p, x_2)$ stands for the generalized eigenfunction of $-\Delta_\alpha^\mathbb{R}$ discussed in [2, Chap. II.2.4]. To be fully specific, $\psi(p, x_2)$ can be obtained from eq. (2.4.1) of [2, Chap. II] multiplying it by $\frac{1}{2\pi}$; see also [2, Appendix E, eq. (E.5)].

The resolvent of $H_{\alpha,0,0}$ can also be written in a Krein-like form. We start with the resolvent of the free system $R_0(z) = (H_{0,0,0} - z)^{-1}$, $z \in \mathbb{C} \setminus [0, \infty)$. By means of the embeddings $I_\pm : W^{1,2} \hookrightarrow L_\pm^2$ and their adjoints $I_\pm^* : L_\pm^2 \hookrightarrow W^{-1,2}$, we define

$$\hat{R}_{0,\pm}(z) := I_\pm R_0(z) : L^2 \rightarrow L_\pm^2, \quad \check{R}_{0,\pm}(z) := R_0(z) I_\pm^* : L_\pm^2 \rightarrow L^2.$$

Finally, the ‘‘bilateral’’ embeddings take the forms

$$R_{0,ij}(z) := I_i R_0(z) I_j^* : L^2(\mathbb{R} \times \{ja\}) \rightarrow L^2(\mathbb{R} \times \{ia\}), \quad i, j \in \{+, -\}.$$

Let $\Gamma_{0,\alpha}(z)$ denote the operator-valued matrix acting in $L_+^2 \oplus L_-^2$ as

$$\Gamma_{0,\alpha}(z) := \begin{pmatrix} -\alpha^{-1} & 0 \\ 0 & -\alpha^{-1} \end{pmatrix} + \begin{pmatrix} R_{0,++}(z) & R_{0,+-(z)} \\ R_{0,-+(z)} & R_{0,--}(z) \end{pmatrix}.$$

Note that the continuity of $I_i : W^{1,2} \hookrightarrow L_i^2$, $i \in \{+, -\}$, implies the continuity of the adjoint embedding $I_i^* : L_i^2 \hookrightarrow W^{-1,2}$. Therefore

$$(2.19) \quad \text{Ran } \check{R}_{0,i}(z) \subset W^{1,2}.$$

Moreover, since $\text{Ran } I_i^* \cap L^2 = \{0\}$, we obtain

$$(2.20) \quad \text{Ran } \check{R}_{0,i}(z) \cap W^{2,2} = \{0\}.$$

Theorem 2.5. *Let $z \in \mathbb{C} \setminus [\xi_0, \infty)$ be such that the operator $\Gamma_{0,\alpha}(z)$ is invertible with bounded inverse. Then the resolvent $R_\alpha(z)$ is given by*

$$(2.21) \quad R_\alpha(z) = R_0(z) - \sum_{i,j \in \{+,-\}} \check{R}_{0,i}(z) \Gamma_{0,\alpha}(z)_{ij}^{-1} \hat{R}_{0,j}(z),$$

where the $\Gamma_{0,\alpha}(z)_{ij}^{-1}$ are the matrix elements of the inverse of $\Gamma_{0,\alpha}(z)$.

Proof. In view of (2.19), the operator $R_\alpha(z)$ defined by (2.21) satisfies $\text{Ran } R_\alpha(z) \subset W^{1,2}$. Assume $\psi \in W^{1,2}$ and $\phi \in L^2$. Using $\mathcal{E}_{0,0,0}(\psi, R_0(z)\phi) - z(\psi, R_0(z)\phi) = (\psi, \phi)$, we get

$$\begin{aligned} \mathcal{E}_{\alpha,0,0}(\psi, R_\alpha(z)\phi) - z(\psi, R_\alpha(z)\phi) &= (\psi, \phi) \\ &- \sum_{i \in \{+,-\}} \alpha(\psi, \hat{R}_{0,i}(z)\phi)_{L_i^2} - \sum_{i,j \in \{+,-\}} (\psi, \Gamma_{0,\alpha}(z)_{ij}^{-1} \hat{R}_{0,j}(z)\phi)_{L_i^2} \\ &+ \sum_{i,j,k \in \{+,-\}} \alpha(\psi, R_{0,ij}(z) \Gamma_{0,\alpha}(z)_{jk}^{-1} \hat{R}_{0,k}(z)\phi)_{L_i^2}, \end{aligned}$$

where we use the embedding $W^{1,2} \hookrightarrow L^2_i$ for ψ in all expressions with scalar product in L^2_i . Applying the identity

$$(\psi, \hat{R}_{0,i}(z)\phi)_{L^2_i} = \sum_{j,k \in \{+,-\}} (\psi, \Gamma_{0,\alpha}(z)_{ik} \Gamma_{0,\alpha}(z)^{-1}_{kj} \hat{R}_{0,j}(z)\phi)_{L^2_i}$$

and using the explicit form of $\Gamma_{0,\alpha}(z)_{ij}$, one obtains

$$(2.22) \quad \mathcal{E}_{\alpha,0,0}(\psi, R_\alpha(z)\phi) - z(\psi, R_\alpha(z)\phi) = (\psi, \phi).$$

Moreover, note that $R_\alpha(z)$ is invertible. Indeed, employing (2.21) and (2.20) we conclude that each $f \in \text{Ran } R_\alpha(z)$ is a direct sum $f = f_1 \dot{+} f_2$ where $f_1 \in \text{Ran } R_0(z)$ and $f_2 \in W^{1,2} \setminus W^{2,2}$. This implies $\text{Ker } R_\alpha = \{0\}$ and together with (2.22) completes the proof. \square

2.4.2. The Krein-like formula for the resolvent of H_{α,V_+,V_-} . Relying on (2.21) and (2.19), we can determine analogous embedding operators as in the previous section but now instead of $R_0(z)$ we consider $R_\alpha(z)$. Namely, define $\check{R}_{\alpha,\pm}(z) := I_\pm R_\alpha(z) : L^2 \rightarrow L^2_\pm$, $\check{R}_{\alpha,\pm}(z) := R_\alpha(z) I_\pm^* : L^2_\pm \rightarrow L^2$ and $R_{\alpha,ij}(z) := I_i R_\alpha(z) I_j^* : L^2_j \rightarrow L^2_i$ where $i, j \in \{+, -\}$.

For any bounded function V , set $V^{1/2} := \text{sgn}(V)|V|^{1/2}$. Define the operator-valued matrix

$$B(z) := \begin{pmatrix} |V_+|^{1/2} R_{\alpha,++}(z) V_+^{1/2} & |V_+|^{1/2} R_{\alpha,+ -}(z) V_-^{1/2} \\ |V_-|^{1/2} R_{\alpha,- +}(z) V_+^{1/2} & |V_-|^{1/2} R_{\alpha,--}(z) V_-^{1/2} \end{pmatrix}$$

acting on $L^2_+ \oplus L^2_-$.

Theorem 2.6. *Let $z \in \mathbb{C} \setminus \sigma(H_{\alpha,V_+,V_-})$ be such that the operator $1 + B(z)$ is invertible with bounded inverse. Then*

$$(2.23) \quad R_{\alpha,V_+,V_-}(z) = R_\alpha(z) - \sum_{i,j \in \{+,-\}} \check{R}_{\alpha,i}(z) V_i^{1/2} [1 + B(z)]_{ij}^{-1} |V_j|^{1/2} \hat{R}_{\alpha,j}(z)$$

determines the resolvent of H_{α,V_+,V_-} ; here $[1 + B(z)]_{ij}^{-1}$ stand for the operator valued matrix elements of $[1 + B(z)]^{-1}$.

Proof. Combining (2.21) and (2.19) and using the definition of $R_{\alpha,V_+,V_-}(z)$ given by (2.23) we have $\text{Ran } \check{R}_{\alpha,i}(z) \subset W^{1,2}$ and consequently $\text{Ran } R_{\alpha,V_+,V_-}(z) \subset W^{1,2}$.

Denote $\Upsilon(z)_{ij} := [1 + B(z)]_{ij}^{-1}$ and assume $\psi \in W^{1,2}$ and $\phi \in L^2$. Employing the identity $\mathcal{E}_{\alpha,0,0}(\psi, R_\alpha(z)\phi) - z(\psi, R_\alpha(z)\phi) = (\psi, \phi)$ we get

$$\begin{aligned} \mathcal{E}_{\alpha, V_+, V_-}(\psi, R_{\alpha, V_+, V_-}(z)\phi) - z(\psi, R_{\alpha, V_+, V_-}(z)\phi) &= (\psi, \phi) \\ &+ \sum_{i \in \{+, -\}} (\psi, V_i \hat{R}_{\alpha, i}(z)\phi)_{L^2_i} - \sum_{i, j \in \{+, -\}} (\psi, V_i^{1/2} \Upsilon(z)_{ij} |V_j|^{1/2} \hat{R}_{\alpha, j}(z)\phi)_{L^2_i} \\ &- \sum_{i, j, k \in \{+, -\}} (\psi, V_i R_{\alpha, ij}(z) V_j^{1/2} \Upsilon(z)_{jk} |V_k|^{1/2} \hat{R}_{\alpha, k}(z)\phi)_{L^2_i}. \end{aligned}$$

Furthermore, writing

$$(\psi, V_i \hat{R}_{\alpha, i}(z)\phi)_{L^2_i} = \sum_{j, k \in \{+, -\}} (\psi, V_i^{1/2} \Upsilon(z)_{ik}^{-1} \Upsilon(z)_{kj} |V_j|^{1/2} \hat{R}_{\alpha, j}(z)\phi)_{L^2_i}$$

and using the explicit form of $\Upsilon(z)_{ij}$, one obtains

$$\mathcal{E}_{\alpha, V_+, V_-}(\psi, R_{\alpha, V_+, V_-}(z)\phi) - z(\psi, R_{\alpha, V_+, V_-}(z)\phi) = (\psi, \phi).$$

Repeating the same argument as in the proof Theorem 2.5, we conclude that $R_{\alpha, V_+, V_-}(z)$ is invertible. \square

The resolvent formula derived in the above theorem can be written in a short, more familiar form. To this end it is convenient to write $V^{1/2} = V_+^{1/2} \oplus V_-^{1/2}$ (and analogously for $|V|^{1/2}$) and introduce $\hat{R}_\alpha(z) : L^2 \rightarrow L^2_+ \oplus L^2_-$ defined by $\hat{R}_\alpha(z)\psi = \hat{R}_{\alpha,+}(z)\psi \oplus \hat{R}_{\alpha,-}(z)\psi$ and $\check{R}_\alpha(z) : L^2_+ \oplus L^2_- \rightarrow L^2$ defined by $\check{R}_\alpha(z)(f_+ \oplus f_-) = \check{R}_{\alpha,+}(z)f_+ + \check{R}_{\alpha,-}(z)f_-$. With these notations, identity (2.23) reads

$$(2.24) \quad R_{\alpha, V_+, V_-}(z) = R_\alpha(z) - \check{R}_\alpha(z) V^{1/2} [1 + B(z)]^{-1} |V|^{1/2} \hat{R}_\alpha(z).$$

Note that (2.24) gives the Krein-like resolvent studied by Posilicano [22, 23]. (To be fully specific we identify the embedding η introduced in [22] with $W^{1,2} \hookrightarrow L^2(\mathbb{R} \times \{+a\}, |V_+|^{1/2} dx) \oplus L^2(\mathbb{R} \times \{-a\}, |V_-|^{1/2} dx)$.) Furthermore, redefining all the embeddings introduced above by means of η , we get the resolvent formula derived in [22, Thm. 2.1]. Applying the results of [23], we get

$$z \in \rho(H_\alpha) \cap \rho(H_{\alpha, V_+, V_-}) \Leftrightarrow z \in \rho(1 + B(z))$$

and

$$(2.25) \quad z \in \sigma_p(H_{\alpha, V_+, V_-}) \Leftrightarrow \text{Ker}(1 + B(z)) \neq \{0\}.$$

2.4.3. The system with mirror symmetry. Let us consider a system with mirror symmetry, i.e.

$$(2.26) \quad V_+ = V_0 = V_-,$$

where $V_0 : \mathbb{R} \rightarrow \mathbb{R}$ is a given function. For this case we use a special notation, $H_{\alpha, V_0} := H_{\alpha, V_0, V_0}$. Using (2.23), we can reconstruct the resolvent $R_{\alpha, V_0}(z) := (H_{\alpha, V_0} - z)^{-1}$.

Now, we introduce a “slight” perturbation of the symmetry taking

$$(2.27) \quad \begin{aligned} V_+ &= V + V_\epsilon, \quad \text{where } V_\epsilon := \epsilon V_p, \quad \epsilon > 0, \\ V_- &= V, \end{aligned}$$

where the “perturbant” V_p is a function from $L^\infty(\mathbb{R})$. In the following we abbreviate $H_{\alpha;\epsilon} := H_{\alpha, V+V_\epsilon, V}$.

Considering the system governed by H_{α, V_0} as a starting point, we construct the resolvent $R_{\alpha;\epsilon}$ of $H_{\alpha;\epsilon}$,

$$(2.28) \quad R_{\alpha;\epsilon}(z) = R_{\alpha, V_0}(z) - \check{R}_{\alpha,+}(z) V_\epsilon^{1/2} \Gamma(z)^{-1} |V_\epsilon|^{1/2} \hat{R}_{\alpha,+}(z)$$

where

$$(2.29) \quad \Gamma(z) := 1 + |V_\epsilon|^{1/2} R_{\alpha, V_0}(z) V_\epsilon^{1/2}$$

and R_{α, V_0} acts as R_{α, V_0, V_0} but maps L_+^2 to L_+^2 . The proof of (2.28) can be obtained by repeating the arguments of the proof of Theorem 2.6.

§3. The essential spectrum

The spectrum of the unperturbed Hamiltonian $H_{\alpha, 0, 0}$ is given by (1.4). In the following we show that the essential spectrum (1.5) is stable under perturbations V_\pm which *vanish at infinity* in the following sense:

$$(3.30) \quad \lim_{L \rightarrow \infty} \operatorname{ess\,sup}_{\mathbb{R} \setminus [-L, L]} |V_\pm| = 0.$$

Theorem 3.1 (Essential spectrum). *Assume (3.30) holds. Then*

$$\sigma_{\operatorname{ess}}(H_{\alpha, V_+, V_-}) = [\xi_0, \infty).$$

We prove this theorem in two steps.

§3.1. A lower bound for the essential spectrum threshold

First, we show that the threshold of the essential spectrum does not descend below the energy ξ_0 .

Lemma 3.2. *Assume (3.30) holds. Then*

$$\inf \sigma_{\operatorname{ess}}(H_{\alpha, V_+, V_-}) \geq \xi_0.$$

Proof. Given a number $L > a$, let H^N denote the operator H_{α, V_+, V_-} with a supplementary Neumann condition imposed on the lines $\{\pm L\} \times \mathbb{R}$ and segments $(-L, L) \times \{\pm L\}$. It is conventionally introduced as the operator associated with the quadratic form \mathcal{E}^N which acts in the same way as $\mathcal{E}_{\alpha, V_+, V_-}$ but has a larger

domain, $\mathfrak{D}(\mathcal{E}^N) := \bigoplus_{k=0}^4 W^{1,2}(\Omega_k)$, where Ω_k are the connected components of \mathbb{R}^2 divided by the curves where the Neumann condition is imposed:

$$\begin{aligned} \Omega_0 &:= (-L, L)^2, & \Omega_1 &:= (-L, L) \times (-\infty, -L), & \Omega_3 &:= (-\infty, -L) \times \mathbb{R}, \\ \Omega_2 &:= (-L, L) \times (L, \infty), & \Omega_4 &:= (L, \infty) \times \mathbb{R}. \end{aligned}$$

We have the decomposition $H^N = \bigoplus_{k=0}^4 H_k^N$, where H_k^N are the operators associated on $L^2(\Omega_k)$ with the quadratic forms

$$\begin{aligned} \mathcal{E}_k^N[\psi] &:= \int_{\Omega_k} |\nabla \psi|^2 + \int_{(\mathbb{R} \times \{+a\}) \cap \Omega_k} (V_+ - \alpha) |I_+ \psi|^2 \\ &\quad + \int_{(\mathbb{R} \times \{-a\}) \cap \Omega_k} (V_- - \alpha) |I_- \psi|^2, \\ \mathfrak{D}(\mathcal{E}_k^N) &:= W^{1,2}(\Omega_k). \end{aligned}$$

Since $H_{\alpha, V_+, V_-} \geq H^N$ and the spectrum of H_0^N is purely discrete, the minimax principle gives the estimate

$$\inf \sigma_{\text{ess}}(H_{\alpha, V_+, V_-}) \geq \min_{k \in \{1, \dots, 4\}} \{ \inf \sigma_{\text{ess}}(H_k^N) \} \geq \min_{k \in \{1, \dots, 4\}} \{ \inf \sigma(H_k^N) \}.$$

It is easy to see that the spectra of H_1^N and H_2^N coincide with the non-negative semi-axis $[0, \infty)$. Hence, it remains to analyze the bottoms of the spectra of H_3^N and H_4^N . Using analogous arguments to those leading to Lemma 2.4, we have the lower bound ($k \in \{3, 4\}$)

$$\mathcal{E}_k^N[\psi] - \xi_0 \|\psi\|_{L^2(\Omega_k)}^2 \geq \int_{\Omega_k} |\partial_1 \psi|^2 + \int_{\Omega_k} \lambda |\psi|^2 \geq \text{ess inf}_{\Omega_k} \lambda \|\psi\|_{L^2(\Omega_k)}^2.$$

Consequently,

$$\inf \sigma_{\text{ess}}(H_{\alpha, V_+, V_-}) \geq \min \left\{ 0, \xi_0 + \min \left\{ \text{ess inf}_{\Omega_3} \lambda, \text{ess inf}_{\Omega_4} \lambda \right\} \right\}.$$

Now, since V_{\pm} vanish at infinity, the same is true for the function λ . Hence, for every ϵ , there exists L such that for a.e. $|x_1| > L$ ($x_2 \in \mathbb{R}$), we have $|\lambda(x)| < \epsilon$. Consequently, if $\epsilon < |\xi_0|$,

$$\inf \sigma_{\text{ess}}(H_{\alpha, V_+, V_-}) \geq \xi_0 - \epsilon.$$

The claim then follows by the fact that ϵ can be chosen arbitrarily small. □

§3.2. The opposite inclusion

Lemma 3.3. *Assume (3.30) holds. Then*

$$\sigma_{\text{ess}}(H_{\alpha, V_+, V_-}) \supseteq [\xi_0, \infty).$$

Proof. Our proof is based on the Weyl criterion adapted to quadratic forms in [7] and applied to quantum waveguides in [21]. By this general characterization of

essential spectrum and since the set $[\xi_0, \infty)$ has no isolated points, it is enough to find for every $\xi \in [\xi_0, \infty)$ a sequence $\{\psi_n\}_{n=1}^\infty \subseteq \mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})$ such that

- (i) $\forall n \in \mathbb{N} \setminus \{0\}, \|\psi_n\|_{L^2} = 1,$
- (ii) $\|(H_{\alpha, V_+, V_-} - \xi)\psi_n\|_{-1} \xrightarrow{n \rightarrow \infty} 0.$

The symbol $\|\cdot\|_{-1}$ denotes the norm in $\mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})^*$, the dual of $\mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})$ equipped with the norm

$$\|\psi\|_{+1} := \sqrt{\mathcal{E}_{\alpha, V_+, V_-}[\psi] + (1 + C_0)\|\psi\|_{L^2}^2},$$

where C_0 denotes any positive constant such that $H_{\alpha, V_+, V_-} + C_0$ is a non-negative operator. We choose the constant C_0 sufficiently large, so that $\|\cdot\|_{+1}$ is equivalent to the usual norm in $W^{1,2}$. This is possible in view of the boundedness of V_\pm and Lemma 2.1.

Let $n \in \mathbb{N} \setminus \{0\}$. Given $k \in \mathbb{R}$, we set $\xi := \xi_0 + k^2$. Recall that the function ϕ_0 denotes the ground state of $-\Delta_\alpha^\mathbb{R}$ corresponding to ξ_0 . Since the interactions V_\pm vanish at infinity, good candidates for the sequence ψ_n seem to be plane waves in the x_1 -direction “localized at infinity” and modulated by ϕ_0 in the x_2 -direction. More precisely,

$$\psi_n(x) := \varphi_n(x_1)\phi_0(x_2)e^{ikx_1},$$

where $\varphi_n(x_1) := n^{-1/2}\varphi(x_1/n - n)$ with φ being a non-zero C^∞ function with compact support in $(-1, 1)$. Note that $\text{supp } \varphi_n \subset (n^2 - n, n^2 + n)$. Clearly, $\psi_n \in W^{1,2} = \mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})$. To satisfy (i), we assume that both ϕ_0 and φ have $L^2(\mathbb{R})$ norm 1. It remains to verify condition (ii).

By the definition of the dual norm, we have

$$\|(H_{\alpha, V_+, V_-} - \xi)\psi_n\|_{-1} = \sup_{\eta \in W^{1,2} \setminus \{0\}} \frac{|\mathcal{E}_{\alpha, V_+, V_-}(\eta, \psi_n) - \xi(\eta, \psi_n)_{L^2}|}{\|\eta\|_{+1}}.$$

An explicit computation using integration by parts yields

$$\begin{aligned} &\mathcal{E}_{\alpha, V_+, V_-}(\eta, \psi_n) - \xi(\eta, \psi_n)_{L^2} \\ &= (\eta, [-\ddot{\varphi}_n - 2ik\dot{\varphi}_n]\phi_0e^{ikx_1})_{L^2} + \int_{\mathbb{R} \times \{a\}} V_+ I_+(\bar{\eta}\psi_n) + \int_{\mathbb{R} \times \{-a\}} V_- I_-(\bar{\eta}\psi_n). \end{aligned}$$

Using the Schwarz inequality and the normalization of ϕ_0 , we get the estimates

$$\begin{aligned} |(\eta, [-\ddot{\varphi}_n - 2ik\dot{\varphi}_n]\phi_0e^{ikx_1})_{L^2}| &\leq \|\eta\|_{L^2}(\|\ddot{\varphi}_n\|_{L^2(\mathbb{R})} + 2|k|\|\dot{\varphi}_n\|_{L^2(\mathbb{R})}), \\ \left| \int_{\mathbb{R} \times \{\pm a\}} V_\pm I_\pm(\bar{\eta}\psi_n) \right| &\leq \|I_\pm \eta\|_{L^2_\pm} \|V_\pm \varphi_n\|_{L^2(\mathbb{R})} |\phi_0(\pm a)|. \end{aligned}$$

By the choice of C_0 and Lemma 2.1, both $\|\eta\|_{L^2}$ and $\|I_{\pm}\eta\|_{L^2_{\pm}}$ can be bounded by a constant times $\|\eta\|_{+1}$. Hence, there is a constant C , depending on a , α , k , $\|V_+\|_{\infty}$ and $\|V_-\|_{\infty}$, such that

$$\begin{aligned} & \| (H_{\alpha, V_+, V_-} - \xi)\psi_n \|_{-1} \\ & \leq C (\|\ddot{\varphi}_n\|_{L^2(\mathbb{R})} + \|\dot{\varphi}_n\|_{L^2(\mathbb{R})} + \|V_+\varphi_n\|_{L^2(\mathbb{R})} + \|V_-\varphi_n\|_{L^2(\mathbb{R})}). \end{aligned}$$

The first two norms on the right hand side tend to zero as $n \rightarrow \infty$ because

$$\|\dot{\varphi}_n\|_{L^2(\mathbb{R})} = n^{-1}\|\dot{\varphi}\|_{L^2(\mathbb{R})}, \quad \|\ddot{\varphi}_n\|_{L^2(\mathbb{R})} = n^{-2}\|\ddot{\varphi}\|_{L^2(\mathbb{R})}.$$

The remaining terms tend to zero because of the estimate

$$\|V_{\pm}\varphi_n\|_{L^2(\mathbb{R})} \leq \operatorname{ess\,sup}_{\operatorname{supp}\varphi_n} |V_{\pm}|,$$

in which we have employed the normalization of φ_n , and the fact that $\inf \operatorname{supp}\varphi_n$ tends to infinity as $n \rightarrow \infty$. \square

§4. The point spectrum

In this section we study the existence of eigenvalues corresponding to bound states. We will be interested in the discrete spectrum as well as in embedded eigenvalues.

§4.1. The discrete spectrum

In the following we establish a condition which guarantees the existence of discrete eigenvalues outside the essential spectrum.

Theorem 4.1 (Discrete spectrum). *Assume that $V_- + V_+ \in L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} (V_- + V_+)(x) \, dx < 0.$$

Then

$$\inf \sigma(H_{\alpha, V_+, V_-}) < \xi_0.$$

Consequently, if in addition V_{\pm} vanish at infinity, then H_{α, V_+, V_-} has an isolated eigenvalue of finite multiplicity below ξ_0 , i.e. (1.7) holds.

Proof. The proof is based on a variational argument. Our aim is to find a test function $\psi \in W^{1,2}$ such that

$$Q[\psi] := \mathcal{E}_{\alpha, V_+, V_-}[\psi] - \xi_0 \|\psi\|_{L^2(\mathbb{R}^2)}^2 < 0.$$

For any $n \in \mathbb{N} \setminus \{0\}$, we set

$$\psi_n(x) := \varphi_n(x_1)\phi_0(x_2),$$

where ϕ_0 is, as before, a positive eigenfunction of $-\Delta_{\alpha}^{\mathbb{R}}$, normalized to 1 in $L^2(\mathbb{R})$, and

$$(4.31) \quad \varphi_n(x_1) := \begin{cases} 1 & \text{if } |x_1| \leq n, \\ \frac{2n - |x_1|}{n} & \text{if } n \leq |x_1| \leq 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\psi_n \in W^{1,2}$. Using $\varepsilon_{\alpha}[\phi_0] = \xi_0 \|\phi_0\|_{L^2(\mathbb{R})}^2$ and the fact that ϕ_0 is even, it is easy to check the identity

$$Q[\psi_n] = \int_{\mathbb{R}} |\varphi_n'|^2 + |\phi_0(a)|^2 \int_{\mathbb{R}} (V_- + V_+) |\varphi_n|^2.$$

As $n \rightarrow \infty$, the first term on the right hand side tends to zero, while the second converges (by the dominated convergence theorem) to a multiple of $\int_{\mathbb{R}} (V_- + V_+) < 0$. Hence, $Q[\psi_n]$ is negative for n sufficiently large. \square

Remark 4.2. The integrability of $V_+ + V_-$ is just a technical assumption in Theorem 4.1. It is clear from the proof that it is only important to ensure that the quantity

$$\int_{\mathbb{R}} (V_- + V_+) |\varphi_n|^2$$

becomes negative as $n \rightarrow \infty$, the value $-\infty$ for the limit being admissible in principle. For instance, one can alternatively assume that $V_+ + V_-$ is non-trivial and non-positive on Ω_0 for the present proof to work.

§4.2. Embedded eigenvalues

The aim of this section is to show that the system with mirror symmetry (cf. Section 2.4.3) has embedded eigenvalues under certain assumptions. For $V_0 \in L^{\infty}(\mathbb{R})$ vanishing at infinity, we consider the Hamiltonian $H_{\alpha, V_0} \equiv H_{\alpha, V_0, V_0}$. Recall that the essential spectrum of H_{α, V_0} is given by (1.6).

We simultaneously consider an auxiliary operator H_{α, V_0}^D on $L^2(\mathbb{R} \times \mathbb{R}^+)$, which acts as H_{α, V_0} on $\mathbb{R} \times \mathbb{R}^+$, subject to Dirichlet boundary conditions on $\mathbb{R} \times \{0\}$. It is introduced as the operator on $L^2(\mathbb{R} \times \mathbb{R}^+)$ associated with the form

$$\begin{aligned} \mathcal{E}_{\alpha, V_0}^D[\psi] &:= \int_{\mathbb{R} \times \mathbb{R}^+} |\nabla \psi|^2 + \int_{\mathbb{R} \times \{a\}} (V_0 - \alpha) |I_+ \psi|^2, \\ \mathfrak{D}(\mathcal{E}_{\alpha, V_0}^D) &:= W_0^{1,2}(\mathbb{R} \times \mathbb{R}^+), \end{aligned}$$

where we keep the same notation I_+ for the embedding $I_+ : W_0^{1,2}(\mathbb{R} \times \mathbb{R}^+) \hookrightarrow L_+^2$. Since V_0 vanishes at infinity, it can be shown in the same way as in Section 3 that

$$\sigma_{\text{ess}}(H_{\alpha, V}^D) = [\mu_0, \infty),$$

where $\mu_0 \in (\xi_0, 0]$ is the spectral threshold of the one-dimensional operator h_α^D on $L^2(\mathbb{R}_+)$ associated with the form

$$\varepsilon_\alpha^D[\phi] := \int_{\mathbb{R}_+} |\phi'|^2 - \alpha|\phi(a)|^2, \quad \mathfrak{D}(\varepsilon_\alpha^D) := W_0^{1,2}(\mathbb{R}_+).$$

Proposition 4.3. *One has $\sigma_{\text{ess}}(h_\alpha^D) = [0, \infty)$ and*

$$\sigma_{\text{disc}}(h_\alpha^D) = \begin{cases} \{\xi_1\} & \text{if } \alpha a > 1, \\ \emptyset & \text{if } \alpha a \leq 1. \end{cases}$$

Consequently,

$$(4.32) \quad \mu_0 = \begin{cases} \xi_1 & \text{if } \alpha a > 1, \\ 0 & \text{if } \alpha a \leq 1. \end{cases}$$

Proof. By the methods of Section 3, it is easy to see that the essential spectrum of h_α^D is the non-negative semi-axis. Note that the existence of a negative eigenvalue of h_α^D is equivalent to the existence of an eigenfunction of $-\Delta_\alpha^{\mathbb{R}}$ corresponding to a negative eigenvalue and vanishing on $\mathbb{R} \times \{0\}$. The latter holds if, and only if, the operator $-\Delta_\alpha^{\mathbb{R}}$ has an *odd* eigenfunction ϕ_1 , i.e. $\alpha a > 1$. This means that h_α^D has a negative eigenvalue which is ξ_1 if, and only if, $\alpha a > 1$; otherwise $\sigma_{\text{disc}}(h_\alpha^D) = \emptyset$. \square

Theorem 4.4 (Embedded eigenvalues). *Let V_0 vanish at infinity. Assume that H_{α, V_0}^D has a (discrete) eigenvalue ν below μ_0 . Then ν is an eigenvalue of H_{α, V_0} . More specifically, if $\nu \geq \xi_0$ (respectively, $\nu < \xi_0$), then ν is an embedded (respectively, discrete) eigenvalue of H_{α, V_0} .*

Proof. Let ψ be an eigenfunction of H_{α, V_0}^D corresponding to ν . By mirror symmetry, the odd extension of ψ to \mathbb{R}^2 is an eigenfunction of H_{α, V_0} corresponding to the same value ν . The rest follows from the fact that the essential spectrum of H_{α, V_0}^D is strictly contained in the essential spectrum of H_{α, V_0} . \square

The following result makes Theorem 4.4 non-void.

Proposition 4.5. *Assume $\alpha a > 1$. Suppose $V_0 \in L^1(\mathbb{R})$ vanishes at infinity and $\int_{\mathbb{R}} V_0(x) dx < 0$. Then $\sigma_{\text{disc}}(H_{\alpha, V_0}^D) \neq \emptyset$.*

Proof. It follows from Proposition 4.3 that, under the condition $\alpha a > 1$, h_α^D has a negative eigenvalue ξ_1 with eigenfunction ϕ_1 (restricted to $\mathbb{R} \times \mathbb{R}_+$). Then the claim follows by using the test function $\psi(x) := \varphi_n(x_1)\phi_1(x_2)$, where φ_n is introduced in (4.31), as in the proof of Theorem 4.1. \square

Finally, to ensure the existence of embedded eigenvalues by Theorem 4.4, it remains to verify that the discrete eigenvalue ν of H_{α, V_0}^D (which exists under the hypotheses of Proposition 4.5) can be made larger than or equal to ξ_0 . However, this happens, for instance, in the weak-coupling regime.

Corollary 4.6. *Assume $a\alpha > 1$. Suppose $V_0 \in L^1(\mathbb{R})$ vanishes at infinity and $\int_{\mathbb{R}} V_0(x) dx < 0$. Then there exists $\varepsilon > 0$ such that $H_{\alpha, \varepsilon V_0}$ has an embedded eigenvalue in $[\xi_0, \mu_0)$.*

§5. Hardy inequalities

In this section we study the case when V_+ and V_- are non-negative. It is easy to see that in this situation the spectrum does not start below ξ_0 . The purpose of this subsection is to show that a stronger result holds in the present setting. We derive a functional, Hardy-type inequality for H_{α, V_+, V_-} with non-trivial non-negative V_+ and V_- , quantifying the repulsive character of the line interactions in this case.

The Hardy-type inequality follows immediately from Lemma 2.4. Indeed, neglecting the kinetic term in (2.16), we arrive at

Theorem 5.1 (Local Hardy inequality). *Assume that V_{\pm} are non-negative and that V_- or V_+ is non-trivial. Then*

$$H_{\alpha, V_+, V_-} - \xi_0 \geq \lambda$$

in the sense of quadratic forms, where λ is a non-trivial and non-negative function (cf. (2.15)).

This Hardy inequality is called *local* since it reflects the local behaviour of the functions V_{\pm} . In particular, if V_{\pm} are compactly supported then λ is compactly supported as a function of the first variable.

In any case, a *global* Hardy inequality follows by applying the classical Hardy inequality.

Theorem 5.2 (Global Hardy inequality). *Assume that V_{\pm} are non-negative, and that there exists $x_1^0 \in \mathbb{R}$ and positive numbers V_0 and R such that*

$$\forall x_1 \in (x_1^0 - R, x_1^0 + R), \quad V_+(x_1) \geq V_0 \quad \text{or} \quad V_-(x_1) \geq V_0.$$

Then

$$H_{\alpha, V_+, V_-} - \xi_0 \geq c\rho \quad \text{with} \quad \rho(x) := \frac{1}{1 + (x_1 - x_1^0)^2}$$

in the sense of quadratic forms. Here the constant c depends on V_0 , R and a .

Proof. The proof follows the strategy developed in [8, Sec. 3.3] to establish a similar global Hardy inequality in curved waveguides. For clarity of exposition, we divide the proof into several steps. Denote $I := (x_1^0 - R, x_1^0 + R)$.

Step 1. The main ingredient in the proof is the classical one-dimensional Hardy inequality

$$(5.33) \quad \forall \phi \in W_0^{1,2}(\mathbb{R}_+), \quad \int_{\mathbb{R}_+} |\phi'(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+} \frac{|\phi(x)|^2}{x^2} dx.$$

We apply it in our case as follows. Let us define an auxiliary function $\eta : \mathbb{R} \rightarrow \mathbb{R}$ by $\eta(x_1) := |x_1 - x_1^0|/R$ if $|x_1 - x_1^0| < R$ and by setting it equal to 1 otherwise; we keep the same notation η for the function $x \mapsto \eta(x_1)$ on \mathbb{R}^2 . For any $\psi \in C_0^\infty(\mathbb{R}^2)$, let us write $\psi = \eta\psi + (1 - \eta)\psi$. Applying the classical Hardy inequality to the function $\eta\psi$ and using Fubini's theorem we get

$$(5.34) \quad \begin{aligned} \int_{\mathbb{R}^2} \rho|\psi|^2 &\leq 2 \int_{\mathbb{R}^2} \frac{|\eta\psi|^2}{(x_1 - x_1^0)^2} + 2 \int_{\mathbb{R}^2} |(1 - \eta)\psi|^2 \\ &\leq 16 \int_{\mathbb{R}^2} |\partial_1 \eta|^2 |\psi|^2 + 16 \int_{\mathbb{R}^2} |\eta|^2 |\partial_1 \psi|^2 + 2 \int_{I \times \mathbb{R}} |\psi|^2 \\ &\leq 16 \int_{\mathbb{R}^2} |\partial_1 \psi|^2 + (2 + 16/R^2) \int_{I \times \mathbb{R}} |\psi|^2. \end{aligned}$$

By density, the inequality extends to all $\psi \in W^{1,2} = \mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})$.

Step 2. By Theorem 5.1, we have

$$(5.35) \quad \mathcal{E}_{\alpha, V_+, V_-}[\psi] - \xi_0 \|\psi\|_{L^2}^2 \geq \operatorname{ess\,inf}_{I \times \mathbb{R}} \lambda \int_{I \times \mathbb{R}} |\psi|^2 \geq \lambda_0 \int_{I \times \mathbb{R}} |\psi|^2$$

for every $\psi \in \mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})$, where $\lambda_0 := \tilde{\lambda}(V_0, 0) = \tilde{\lambda}(0, V_0)$. Of course, λ_0 is a positive number under the stated hypotheses.

On the other hand, neglecting the non-negative potential term in (2.16), we have

$$(5.36) \quad \mathcal{E}_{\alpha, V_+, V_-}[\psi] - \xi_0 \|\psi\|_{L^2}^2 \geq \int_{\mathbb{R}^2} |\partial_1 \psi|^2$$

for every $\psi \in \mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})$.

Step 3. Interpolating between the bounds (5.35) and (5.36), and using (5.34) in the latter, we arrive at

$$\mathcal{E}_{\alpha, V_+, V_-}[\psi] - \xi_0 \|\psi\|_{L^2}^2 \geq \frac{\epsilon}{16} \int_{\mathbb{R}^2} \rho|\psi|^2 + \left[(1 - \epsilon)\lambda_0 - \epsilon \left(\frac{1}{8} + \frac{1}{R^2} \right) \right] \int_{I \times \mathbb{R}} |\psi|^2$$

for every $\psi \in \mathfrak{D}(\mathcal{E}_{\alpha, V_+, V_-})$ and $\epsilon \in (0, 1)$. It is clear that the right hand side of this inequality can be made non-negative by choosing ϵ sufficiently small. Choosing ϵ

such that the expression in the square brackets vanishes, the inequality yields the claim of the theorem with

$$c := \frac{\lambda_0/16}{\lambda_0 + 1/8 + 1/R^2}.$$

It remains to observe that λ_0 depends on V_0 and a through the definition (2.14). \square

As a direct consequence of Theorem 5.2, we get

Corollary 5.3. *Assume the hypotheses of Theorem 5.2. Let W be the operator in L^2 of multiplication by any bounded function w for which there exists a positive constant C such that $|w(x)| \leq C|x_1|^{-2}$ for a.e. $x \in \mathbb{R}^2$. Then there exists $\epsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$,*

$$\inf \sigma(H_{\alpha, V_+, V_-} + \epsilon W) \geq \xi_0.$$

Assume that V_{\pm} vanish at infinity. Since also the potential W of the corollary is bounded and vanishes at infinity, it is easy to see that the essential spectrum is not changed, i.e., $\sigma_{\text{ess}}(H_{\alpha, V_+, V_-} + \epsilon W) = [\xi_0, \infty)$, independently of the value of ϵ and irrespective of the signs of V_{\pm} . It follows from the corollary that a certain critical value of ϵ is needed in order to generate discrete spectrum of $H_{\alpha, V_+, V_-} + \epsilon W$ if the Hardy inequality for H_{α, V_+, V_-} exists. On the other hand, it is easy to see that $H_{\alpha, 0, 0} + \epsilon W$ has eigenvalues below ξ_0 for arbitrarily small ϵ provided that W is non-trivial and non-positive.

§6. Resonances induced by broken symmetry

As already stated (cf. Corollary 4.6), the Hamiltonian H_{α, V_0} of the system with mirror symmetry (2.26) admits embedded eigenvalues. In the following we show that breaking this symmetry by (2.27) will turn the eigenvalues into resonances.

The strategy we employ here is as follows. Our first aim is to show that the operator-valued function $z \mapsto R_{\alpha; \epsilon}(z)$ has a second sheet analytic continuation in the following sense: for any $f, g \in C_0(\mathbb{R}^2)$ the operator $fR_{\alpha; \epsilon}(z)g$ can be analytically continued to the lower half-plane as a bounded operator in L^2 . Such a continuation will be denoted by $R_{\alpha; \epsilon}^{II}(\cdot) \equiv R_{\alpha; \epsilon, f, g}^{II}(\cdot)$. Of course, the above formulation implies that the function $z \mapsto (f, R_{\alpha; \epsilon}(z)g)$ has a second sheet continuation. The analogous definition will be employed for the second sheet continuation of $R_{\alpha}(\cdot)$. To recover resonances in the system governed by $H_{\alpha; \epsilon}$, we look for poles of $R_{\alpha; \epsilon}^{II}(\cdot)$. These poles are defined by the condition

$$(6.37) \quad \text{Ker } \Gamma^{II}(z) \neq \{0\}, \quad \Im z < 0,$$

where $\Gamma^{II}(\cdot)$ is the second sheet continuation of the analytic operator valued-function $z \mapsto \Gamma(z)$. Our aim is to find z satisfying (6.37).

§6.1. The second sheet continuation of $\Gamma(\cdot)$

Henceforth we assume

$$(6.38) \quad V_{\pm} e^{C|x|} \in L^{\infty}(\mathbb{R}) \quad \text{for some } C > 0.$$

The first auxiliary statement is contained in the following lemma.

Lemma 6.1. *Suppose (6.38) holds. Then for any $i, j \in \{+, -\}$ and $z \in \mathbb{C} \setminus [\xi_0, \infty)$ the operator $|V_i|^{1/2} R_{\alpha, ij}(z) V_j^{1/2}$ is Hilbert–Schmidt. Consequently, so is $B(z)$.*

Proof. Using (2.21) we have

$$\begin{aligned} & |V_i|^{1/2} R_{\alpha, ij}(z) V_j^{1/2} \\ &= \underbrace{|V_i|^{1/2} R_{0, ij}(z) V_j^{1/2}}_{A_1} - \sum_{k, l \in \{+, -\}} \underbrace{|V_i|^{1/2} R_{0, ik}(z)}_{A_2} \Gamma_{0, \alpha}(z)_{kl}^{-1} \underbrace{R_{0, lj}(z) V_j^{1/2}}_{A_3}. \end{aligned}$$

It is well known that $R_0(z)$ is an integral operator:

$$R_0(z)f(x) = \int_{\mathbb{R}^2} G_0(z; x-w)f(w) dw, \quad G_0(z; x) = \frac{1}{2\pi} K_0(\sqrt{z}|x|),$$

where $K_0(\cdot)$ stands for the Macdonald function (cf. [1, Sec. 9.6]), $\Im\sqrt{z} > 0$. Consequently, $R_{0, ij}(z)$ is an integral operator with kernel $G_{0, ij}(z; \cdot - \cdot)$ defined as the “bilateral” embedding of $G_0(z; \cdot - \cdot)$ acting from L_j^2 to L_i^2 . Using the properties of K_0 (see [1, eq. (9.6.8)]), we conclude that $G_0(z; \cdot)$ has a logarithmic singularity at the origin and is continuous elsewhere; moreover it exponentially decays at infinity. This implies that $G_{0, ij}(z; \cdot) \in L^2(\mathbb{R})$ and consequently $|G_{0, ij}(z; \cdot)|^2 \in L^1(\mathbb{R})$. Since $V_{\pm} \in L^{\infty}(\mathbb{R}) \cap L^1(\mathbb{R})$ we have

$$\begin{aligned} \||V_i|^{1/2} R_{0, ij}(z) V_j^{1/2}\|_{\text{HS}}^2 &\leq \int_{\mathbb{R}^2} |V_i(x)| |G_{0, ij}(z; x-y)|^2 |V_j(y)| dx dy, \\ &\leq \|V_j\|_{\infty} \||V_i|^{1/2} R_{0, ij}(z)\|_{\text{HS}}^2 \\ &\leq \|V_j\|_{\infty} \|V_i\|_{L^1(\mathbb{R})} \||G_{0, ij}(z)|^2\|_{L^1(\mathbb{R})}, \end{aligned}$$

where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert–Schmidt norm and the last step employs the Young inequality (cf. [25, Chap. IX.4, Ex. 1]). The above inequalities show that A_1, A_2, A_3 are Hilbert–Schmidt operators; note $\|R_{0, ij}(z) V_j^{1/2}\|_{\text{HS}} = \||V_j|^{1/2} R_{0, ji}(z)\|_{\text{HS}}$. Moreover, since the $\Gamma_{0, \alpha}(z)_{kl}^{-1}$ are bounded, the operators $|V_i|^{1/2} R_{\alpha, ij}(z) V_j^{1/2}$ are Hilbert–Schmidt as well. \square

Remark 6.2. Note that to prove the above lemma we use only $V_{\pm} \in L^{\infty}(\mathbb{R}) \cap L^1(\mathbb{R})$; the stronger assumption (6.38) will be used in the following.

Suppose \mathcal{B} is an open subset of $[\xi_0, \infty)$ and $E_{\mathcal{B}}$ denotes the spectral measure of H_{α, V_+, V_-} . Denote $\mathcal{H}_{ac} := \{\psi \in L^2 : \mathcal{B} \mapsto (\psi, E_{\mathcal{B}}\psi) \text{ is absolutely continuous}\}$.

Lemma 6.3. *Suppose (6.38) holds. There is a finite set $\xi \subset \mathbb{R}$ such that:*

- (i) *For any interval $[a, b] \subset [\xi_0, \mu_0]$ which is disjoint from \mathcal{E} we have $\text{Ran } E_{(a,b)} \subset \mathcal{H}_{ac}$.*
- (ii) *There exists a region $\Omega_- \subset \mathbb{C}_-$ with boundary containing (ξ_0, μ_0) and an operator-valued function $R_{\alpha, V_+, V_-}^{II}(\cdot)$ analytic in $\Omega \setminus \mathcal{E}$, where $\Omega = \mathbb{C}_+ \cup (\xi_0, \mu_0) \cup \Omega_-$, which is the analytic continuation of $R_{\alpha, V_+, V_-}(\cdot)$.*

Proof. The operator $R_{\alpha}(\cdot)$ is analytic in $\mathbb{C} \setminus [\xi_0, \infty)$. The Stone formula implies that $s\text{-}\lim_{\varepsilon \rightarrow 0} (R_{\alpha}(\lambda + i\varepsilon) - R_{\alpha}(\lambda - i\varepsilon)) \neq 0$ for $\lambda \in (\xi_0, \mu_0)$. In the following we write $R_{\alpha}(\lambda \pm i0)$ for $s\text{-}\lim_{\varepsilon \rightarrow 0} R_{\alpha}(\lambda \pm i\varepsilon)$ and analogously for resolvents of other operators.

Our first aim is to show that $R_{\alpha}(\cdot)$ can be analytically continued from \mathbb{C}_+ through (ξ_0, μ_0) to the lower half-plane in the sense described at the beginning of this section. Recall

$$(6.39) \quad R_{\alpha}(z) = R_{\alpha}^c(z) + R^d(z), \quad R^d(z) = \sum_{j \in \mathcal{N}} R_{\alpha}^{j,d}(z).$$

Note that $R_{\alpha}^c(z)$ is analytic for $z \in \mathbb{C} \setminus [0, \infty)$ (cf. (2.18)). Furthermore, if $\alpha a > 1$ then the component $R_{\alpha}^{1,d}(z)$ of $R_{\alpha}^d(z)$ is analytic for $z \in \mathbb{C} \setminus [\xi_1, \infty)$. On the other hand, the analytic continuation of $R_{\alpha}^{0,d}(\cdot)$ through (ξ_0, ∞) is defined by the second sheet values of τ_0 , i.e. $\Im \tau_0^{II} \leq 0$. More precisely, for $f, g \in C_0(\mathbb{R}^2)$ the operator $fR_{\alpha}^{0,d}(\cdot)g$ has a second sheet analytical continuation which is an integral operator with kernel $f(x)G_{\alpha}^{0,d,II}(z; x - y)g(y)$ and $G_{\alpha}^{0,d,II}$ is defined by (2.17), with τ_0 replaced by τ_0^{II} . The resulting operator is denoted by $R_{\alpha}^{0,d,II}(\cdot)$. Consequently, we define the second sheet continuation of $R_{\alpha}(z)$ as $R_{\alpha}^{II}(z) = R_{\alpha}^c(z) + R_{\alpha}^{d,II}(z)$ where $R_{\alpha}^{d,II}(z) = R_{\alpha}^{0,d,II}(z) + R_{\alpha}^{1,d}(z)$ and $z \in \mathbb{C}_+ \cup (\xi_0, \mu_0) \cup \mathbb{C}_-$. The operator $R_{\alpha}^{II}(z)$ is analytic bounded in L^2 in the sense described at the beginning of the section.

By means of $G_{\alpha}^{0,d,II}(z; x - y)$ we define the operator $|V_i|^{1/2}R_{\alpha, ij}^{0,d,II}(z)V_j^{1/2} : L_j^2 \rightarrow L_i^2, i, j \in \{+, -\}$, with

$$\begin{aligned} & \| |V_i|^{1/2}R_{\alpha, ij}^{0,d,II}(z)V_j^{1/2} \|_{\text{HS}}^2 \\ &= \frac{|\phi_0(ia)\phi_0(ja)|^2}{4|\tau_0^{II}(z)|^2} \int_{\mathbb{R}^2} e^{-C(|x_1|+|y_1|)} e^{-2\Im \tau_0^{II}(z)|x_1-y_1|} |h_i(x_1)h_j(y_1)| dx_1 dy_1, \end{aligned}$$

where $h_{\pm} := V_{\pm}e^{C|\cdot|} \in L^{\infty}(\mathbb{R})$ (cf. (6.38)). The above expression is finite if $\Im \tau_0^{II}(z) > -C/2$. For $|V_i|^{1/2}R_{\alpha, ij}^{1,d}(z)V_j^{1/2}$ the analogous expression is always finite because $R_{\alpha}^{1,d}(z)$ is analytic on $\mathbb{C}_+ \cup (\xi_0, \mu_0) \cup \mathbb{C}_-$, and $\Im \tau_1(z) > 0$.

Furthermore, since $|V_i|^{1/2}R_{\alpha,ij}(z)V_j^{1/2}$ for $z \in \mathbb{C} \setminus [\xi_0, \infty)$ is a Hilbert–Schmidt operator (cf. Lemma 6.1), as also is $|V_i|^{1/2}R_{\alpha,ij}^d(z)V_j^{1/2}$, we conclude that $|V_i|^{1/2}R_{\alpha,ij}^c(z)V_j^{1/2}$ is a Hilbert–Schmidt operator with boundary values $|V_i|^{1/2}R_{\alpha,ij}^c(\lambda \pm i0)V_j^{1/2}$, $\lambda \in (\xi_0, \mu_0)$, being compact. Consequently, $B_{ij}(\lambda + i0)$, $\lambda \in (\xi_0, \mu_0)$, is compact and $B_{ij}(\cdot)$ has second sheet continuation $B_{ij}^{II}(\cdot) = |V_i|^{1/2}R_{\alpha,ij}^{II}(\cdot)V_j^{1/2}$ through (ξ_0, μ_0) . Finally, we conclude that $B^{II}(z)$ is compact for $z \in \Omega = \mathbb{C}_+ \cup (\xi_0, \mu_0) \cup \Omega_-$, where Ω_- is a region in \mathbb{C}_- with boundary containing (ξ_0, μ_0) and determined by the condition $\Im\tau_0(z) > -C/2$.

Note that the operator $f\check{R}_{\alpha,i}(z)V_i^{1/2}$, where $f \in C_0(\mathbb{R}^2)$, has an analytic second sheet continuation. Indeed, as the above discussion shows, the only non-trivial component is given by $f\check{R}_{\alpha,i}^{0,d}(z)V_i^{1/2}$. Since its Hilbert–Schmidt norm is

$$\begin{aligned} & \|f\check{R}_{\alpha,i}^{0,d,II}(z)V_i^{1/2}\|_{\text{HS}}^2 \\ &= \frac{|\phi_0(i\alpha)|^2}{4|\tau_0^{II}(z)|^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{-C|y_1|} e^{-2\Im\tau_0^{II}(z)|x_1-y_1|} |f(x)|^2 |h_i(y_1)| \, dx \, dy_1, \end{aligned}$$

where $x = (x_1, x_2)$ and $h_i := V_i e^{C|\cdot|} \in L^\infty(\mathbb{R})$ we conclude that $f\check{R}_{\alpha,i}^{0,d,II}(z)V_i^{1/2}$ is analytic for $z \in \Omega$ and bounded as an operator acting from L_i^2 to L^2 . The analogous statement can be obtained for $|V_i|^{1/2}\hat{R}_{\alpha,i}(z)f$.

(i) Since the condition $\text{Ker}[1+B(z)] \neq \{0\}$ determines the poles of $R_{\alpha,V_+,V_-}(\cdot)$ (cf. (2.25)), which is the resolvent of a self-adjoint operator, the former has no solution for $z \in \mathbb{C}_+$. Combining compactness of $B(z)$ and the analytic Fredholm theorem (see, e.g., [24, Thm. VI.14]) with the fact that $z \mapsto B(z)$ is analytic for $z \in \mathbb{C}_+ \cup (\xi_0, \mu_0)$, we conclude that the operator $[1+B(z)]^{-1}$ exists and it is bounded analytic in $z \in \mathbb{C}_+$ with boundary values $z = \lambda + i0$, provided $\lambda \in (\xi_0, \mu_0)$ avoids a finite set \mathcal{E} of real numbers; for an analogous discussion see [26, Thm. XIII.21]. Moreover, the operator $|f|R_\alpha(\lambda + i\varepsilon)|f|$, $f \in C_0(\mathbb{R}^2)$, is uniformly bounded for $0 < \varepsilon < 1$ and $\lambda \in (\xi_0, \mu_0)$. The operators $|f|\check{R}_{\alpha,i}(z)V_i^{1/2}$ and $|V_i|^{1/2}\hat{R}_{\alpha,i}(z)|f|$ are uniformly bounded as well. Combining the above statements with the resolvent formula (2.23), we come to the conclusion that, for any $f \in C_0(\mathbb{R}^2)$, the function

$$\begin{aligned} & |f|R_{\alpha,V_+,V_-}(z)|f| = |f|R_\alpha(z)|f| \\ & \quad - \sum_{i,j \in \{+,-\}} (|f|\check{R}_{\alpha,i}(z)V_i^{1/2})[1+B(z)]_{ij}^{-1}(|V_j|^{1/2}\hat{R}_{\alpha,j}(z)|f|) \end{aligned}$$

is uniformly bounded with respect to $z = \lambda + i\varepsilon$, $0 < \varepsilon < 1$, $\lambda \in [a, b]$. Furthermore, since

$$|(g, R_{\alpha,V_+,V_-}(z)g)| \leq \| |g|^{1/2} \|^2 \| |g|^{1/2} R_{\alpha,V_+,V_-}(z) |g|^{1/2} \|$$

for any $g \in C_0(\mathbb{R}^2)$, the assumption of [26, Thm. XIII.19] is fulfilled, which yields the claim. (Note that in the last expression the same notation $\|\cdot\|$ was used for the operator norm and for the norm of a function.)

(ii) Using again the Fredholm theorem and the compactness of $B^H(\cdot)$, we find that the operator $[1 + B^H(z)]^{-1}$ exists and it is bounded analytic in $\Omega \setminus \mathcal{E}$. For $z \in \Omega \setminus \mathcal{E}$ we construct $R_{\alpha, V_+, V_-}^{II}(z)$ as

$$R_{\alpha, V_+, V_-}^{II}(z) = R_{\alpha}^{II}(z) - \check{R}_{\alpha}^{II}(z)V^{1/2}[1 + B^H(z)]^{-1}|V|^{1/2}\hat{R}_{\alpha}^{II}(z)$$

(cf. (2.24)), where $\check{R}_{\alpha}^{II}(z)V^{1/2}$ and $\hat{R}_{\alpha}^{II}(z)|V|^{1/2}$ are the second sheet continuations already discussed. □

Corollary 6.4. $\sigma_{\text{sc}}(H_{\alpha, V_+, V_-}) \cap [\xi_0, \mu_0] = \emptyset$.

Proof. It follows from the previous theorem that $\sigma_{\text{sc}}(H_{\alpha, V_+, V_-}) \cap [\xi_0, \mu_0] \subset \mathcal{E}$ is a finite set; this implies the claim. □

With no danger of confusion, we employ the notation $\lambda \mapsto E(\lambda)$ for the spectral resolution of H_{α, V_0} for $\lambda \in [\xi_0, \infty) \setminus \mathcal{E}$. Then the operator $R_{\alpha, V_0}(z)$ admits the decomposition

$$(6.40) \quad R_{\alpha, V_0}(z) = \sum_{i=1}^N \frac{1}{\nu_i - z} P_i + \int_{[\xi_0, \infty)} \frac{dE(\lambda)}{\lambda - z},$$

where $\{\nu_i\}_{i=1}^N = \sigma_{\text{p}}(H_{\alpha, V_0})$, $P_i = \omega_i(\omega_i, \cdot)$, and ω_i are the corresponding eigenvectors. From the definition of $E(\cdot)$ and Corollary 6.4, we conclude that $(\xi_0, \mu_0) \ni \lambda \mapsto E(\lambda)$ projects onto \mathcal{H}_{ac} . Given $f, g \in L^2$, let us denote by $F(\cdot)_{f, g}$ the Radon–Nikodym derivative of $(\xi_0, \mu_0) \ni \lambda \mapsto (f, E(\lambda)g)$. The limit of $z = \gamma + i\varepsilon$ as $\varepsilon \rightarrow 0$ with $\gamma \in (\xi_0, \mu_0) \setminus \mathcal{E}$ and $f, g \in C_0(\mathbb{R}^2)$ takes the form

$$(6.41) \quad (f, R_{\alpha, V_0}(\gamma + i0)g) = \sum_{i=1}^N \frac{1}{\nu_i - \gamma} (f, \omega_i)(\omega_i, g) + \mathcal{P} \int_{[\xi_0, \mu_0)} \frac{F_{f, g}(\lambda) d\lambda}{\lambda - \gamma} + \int_{\mu_0}^{\infty} \frac{d(f, E(\lambda)g)}{\lambda - \gamma} + \pi i F_{f, g}(\gamma),$$

where \mathcal{P} denotes principal value. By the edge-of-the-wedge theorem (cf. [27]), we get

$$(6.42) \quad (f, R_{\alpha, V_0}(\gamma + i0)g) = (f, R_{\alpha, V_0}^{II}(\gamma - i0)g), \quad f, g \in C_0(\mathbb{R}^2),$$

where $R_{\alpha, V_0}^{II}(\cdot)$ denotes the second sheet continuation of $R_{\alpha, V_0}(\cdot)$ given in Lemma 6.3. The operator $R_{\alpha, V_0}(\cdot)$ is the resolvent of the mirror symmetry system.

Now we introduce the potential $V_\epsilon = \epsilon V_p$ living on $\mathbb{R} \times \{+a\}$. By means of $R_{\alpha, V_0}^H(z)$ we determine

$$(6.43) \quad \Gamma^H(z) = 1 + |V_\epsilon|^{1/2} R_{\alpha, V_0}^H(z) V_\epsilon^{1/2}, \quad z \in \Omega \setminus \mathcal{E},$$

where the last term is the analytic second sheet continuation of the operator $|V_\epsilon|^{1/2} R_{\alpha, V_0}(z) V_\epsilon^{1/2} : L_+^2 \rightarrow L_+^2$.

§6.2. Zeros of $\Gamma^H(\cdot)$; the Fermi golden rule

Henceforth we assume that the set of embedded eigenvalues of H_{α, V_0} is not empty (this is true, for instance, under the hypotheses of Corollary 4.6). Then there exists an integer $k_0 \leq N$ such that for all $i \geq k_0$ we have $\nu_i \in (\xi_0, \mu_0)$. Given $k \geq k_0$, define

$$(6.44) \quad A_k(z) := R_{\alpha, V_0}(z) - \frac{1}{\nu_k - z} P_k.$$

Analogously we define $A_k^H(z)$ replacing $R_{\alpha, V_0}(z)$ in (6.44) by $R_{\alpha, V_0}^H(z)$.

The main result of this section is the following theorem.

Theorem 6.5. *Suppose (6.38) holds. Assume that $\nu_k \in (\xi_0, \mu_0)$ is an embedded eigenvalue of H_{α, V_0} . Then the resolvent of $H_{\alpha; \epsilon}$ has a pole z_k satisfying*

$$z_k = \nu_k + \epsilon(\omega_k, V_p \omega_k)_{L_+^2} + \epsilon^2 (\Gamma_r + i\Gamma_i) + \mathcal{O}(\epsilon^3) \quad \text{as } \epsilon \rightarrow 0,$$

where

$$\Gamma_r := - \sum_{i \in \{1, \dots, N\} \setminus k} \frac{1}{\nu_i - \nu_k} |(\omega_k, V_p \omega_i)_{L_+^2}|^2 - \mathcal{P} \int_{[\xi_0, \infty)} \frac{d(\omega_k, V_p E(\lambda) V_p \omega_k)_{L_+^2}}{\lambda - \nu_k},$$

$$\Gamma_i := -\pi F_+(\nu_k) \quad \text{with} \quad F_+(\nu_k) := \frac{d}{d\nu} (\omega_k, V_p E(\nu) V_p \omega_k)_{L_+^2} \Big|_{\nu=\nu_k}.$$

(Here the functions ω_k are understood as embeddings into L_+^2 . Similarly, $E(\cdot)$ is a family of operators acting from L_+^2 to L_+^2 .)

Proof. Note that the function $z \mapsto (f, A_k^H(z)g)$, $f, g \in C_0(\mathbb{R}^2)$, is analytic in a neighbourhood M of ν_k . Furthermore, for ϵ small enough, say $\epsilon \leq \epsilon_0$, where $\epsilon_0 > 0$, the operator $G_{\epsilon, k}(z) := 1 + |V_\epsilon|^{1/2} A_k^H(z) V_\epsilon^{1/2}$ is invertible. We define the function $\eta_k : [0, \epsilon_0) \times M \rightarrow \mathbb{C}$ by

$$\eta_k(\epsilon, z) := z - \nu_k - (\omega_k, V_\epsilon^{1/2} G_{\epsilon, k}(z)^{-1} |V_\epsilon|^{1/2} \omega_k)_{L_+^2}.$$

Suppose $z \in M \setminus \nu_k$. A straightforward calculation using (6.43) and (6.44) shows that $\phi \in \text{Ker } \Gamma^H(z)$ if, and only if,

$$\phi + \frac{1}{\nu_k - z}(\omega_k, V_\epsilon^{1/2}\phi)G_{\epsilon,k}(z)^{-1}|V_\epsilon|^{1/2}\omega_k = 0.$$

This means that $\text{Ker } \Gamma^H(z) \neq \{0\}$ if, and only if, z is a solution of

$$(6.45) \quad \eta_k(\epsilon, z) = 0.$$

After expanding $G_{\epsilon,k}(z)^{-1}$ with respect to ϵ , the function $\eta(\epsilon, z)$ reads

$$(6.46) \quad \eta_k(\epsilon, z) = z - \nu_k - \epsilon(\omega_k, V_p\omega_k)_{L^2_+} + \epsilon^2(\omega_k, V_pA_k^H(z)V_p\omega_k)_{L^2_+} + \mathcal{O}(\epsilon^3).$$

The function η_k is analytic in $z \in M$ and it is C^1 in both variables. It is clear that $\eta_k(0, \nu_k) = 0$ and $\partial_z \eta_k(0, \nu_k) = 1$. Applying the implicit function theorem to (6.45) and using (6.46), we find that there exists an open neighbourhood $\mathcal{U}_0 \subset \mathbb{R}_+$ of zero and a unique function $z_k : \mathcal{U}_0 \rightarrow \mathbb{C}$ of ϵ given by

$$z_k = \nu_k + \epsilon(\omega_k, V_p\omega_k)_{L^2_+} - \epsilon^2(\omega_k, V_pA_k^H(\nu_k - i0)V_p\omega_k)_{L^2_+} + \mathcal{O}(\epsilon^3),$$

and being a zero of $\Gamma^H(\cdot)$. Employing (6.44), (6.41) and (6.42), we get the statement of the theorem. \square

The above results can be summarized as follows. Suppose the mirror symmetry system (2.26) has an embedded eigenvalue ν . Once we break the symmetry by introducing the ‘‘perturbant’’ $V_\epsilon = \epsilon V_p$, the pole of the resolvent shifts from the spectrum and makes the second sheet continuation a pole of the resolvent. The imaginary component of the pole is related to the resonance width $\Gamma_w := -2\Im z_k = \mathcal{O}(\epsilon^2)$. This means that for ϵ small the resonance is physically essential. Employing the lowest order perturbation in ϵ , we can write

$$\Gamma_w \approx 2\pi \frac{d(\omega_k, V_pE(\lambda)V_p\omega_k)_{L^2_+}}{d\lambda} \Big|_{\lambda=\nu_k} \epsilon^2.$$

This gives the Fermi golden rule. Moreover, Γ_w^{-1} determines the life time of the resonance state.

Remark 6.6. Resonance phenomena induced by broken symmetry were studied in [13, 20]. However, in the models analyzed there, the singular potentials are constants and the broken symmetry has a rather geometrical character. The methods derived in this paper essentially differ from the technics applied in [13, 20]. Finally, let us mention that the resonance phenomena and the decay law were recently reviewed in [10]. Systems with singular potentials were considered as examples of solvable models.

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