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Prime numbers along Rudin–Shapiro sequences

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Abstract. For a large class of digital functions f, we estimate the sums $\sum_{n\le x} \Lambda(n) f(n)$ (and $\sum_{n\le x} \mu(n) f(n)$), where Λ denotes the von Mangoldt function (and μ the Möbius function). We deduce from these estimates a Prime Number Theorem (and a Möbius randomness principle) for sequences of integers with digit properties including the Rudin–Shapiro sequence and some of its generalizations.

Keywords. Rudin–Shapiro sequence, prime numbers, Möbius function, exponential sums

1. Introduction

We denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{U}$ the set of complex numbers of modulus 1, by $\mathcal{P}$ the set of prime numbers and for any $a \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $m \ge 1$, by $\mathcal{P}(a, m)$ the set of prime numbers $p \equiv a \bmod m$. For $n \in \mathbb{N}$, $n \ge 1$, we denote by $\tau(n)$ the number of divisors of n, by $\omega(n)$ the number of distinct prime factors of n, by $\Lambda(n)$ the von Mangoldt function (defined by $\Lambda(n) = \log p$ if $n = p^k$ with $k \in \mathbb{N}$, $k \ge 1$ and $\Lambda(n) = 0$ otherwise) and by $\mu(n)$ the Möbius function (defined by $\mu(n) = (-1)^{\omega(n)}$ if n is squarefree and $\mu(n) = 0$ otherwise). For $x \in \mathbb{R}$ we denote by $\|x\|$ the distance of x to the nearest integer, by $\pi(x)$ the number of prime numbers less than or equal to x and we set $\mathrm{e}(x) = \exp(2i\pi x)$. If f and g are two functions taking strictly positive values such that f/g is bounded, we write $f = O(g)$ or $f \ll g$. Throughout this work we denote by q an integer greater than or equal to 2. Any $n \in \mathbb{N}$ can be written in base q as $n = \sum_{j\ge 0} \varepsilon_j(n) q^j$ with $\varepsilon_j(n) \in \{0, \dots, q-1\}$ for any $j \in \mathbb{N}$.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers of modulus at most 1 generated by a simple algorithm. Many recent works are devoted to the proof that special sequences $(u_n)_{n\in\mathbb{N}}$ satisfy the Möbius randomness principle (i.e. $\sum_{n\le x} \mu(n) u_n = o(x)$, see [15, p. 338]) or a Prime Number Theorem (i.e. an asymptotic formula for the sum $\sum_{n\le x} \Lambda(n) u_n$, more difficult to handle) (see [6], [7], [13], [24]). These works are related to the Sarnak conjecture (see [29]) which asserts that if $(u_n)_{n\in\mathbb{N}}$ is produced by a zero

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topological entropy dynamical system, then $\sum_{n\le x}\mu(n)u_n = o(x)$. In the case of sequences $(u_n)_{n\in\mathbb{N}}$ such that u_n is defined by a digital property of the integer n, it follows from [19, Theorem 1] and [14, Theorem 1] that

$$\sum_{n\le x}\mu(n)\,\mathrm{e}\Big(\alpha\sum_{j\ge 0}\varepsilon_j(n)\Big)=o(x). \tag{1}$$

Dartyge and Tenenbaum [7, Théorème 2.12] proved, using Daboussi's convolution method, a quantitative result that implies in particular the error term $O\,(x/\log\log x)$ in (1). In [24] we proved that, for any real number α such that $(q-1)\alpha\notin\mathbb{Z}$, there exists a real number $\eta(\alpha)<1$ such that

$$\sum_{n\le x}\Lambda(n)\,\mathrm{e}\Big(\alpha\sum_{j\ge 0}\varepsilon_j(n)\Big)=O(x^{\eta(\alpha)}), \tag{2}$$

answering a question due to Gelfond [11] (see [10] for an explicit value of $\eta(\alpha)$ and [9, 21, 22] for extensions to more general digital functions). The proof of (2), based on Vaughan's identity [15, (13.39)] and the estimate of type I and type II bilinear sums, can be applied to the Möbius function μ using [15, (13.40)], and this shows that, for any real number α such that $(q-1)\alpha\notin\mathbb{Z}$, there exists a real number $\eta(\alpha)<1$ such that the error term in (1) is $O\left(x^{\eta(\alpha)}\right)$.

Kalai [16], [17] asked a series of questions concerning the computational complexity of μ that can be translated in proving a Möbius randomness principle for some specific binary sequences. In [3] Bourgain proved that

$$\max_{S\subseteq\{0,\dots,\nu-1\}}\Big|\sum_{n<2^\nu}\mu(n)(-1)^{\sum_{i\in S}\varepsilon_i(n)}\Big|=O(2^{\nu-\nu^{1/10}}), \tag{3}$$

showing both a Möbius randomness principle and a Prime Number Theorem for these sequences (see [12] for a related result showing that μ is orthogonal to any Boolean function computable by constant depth and polynomial size circuits). Studying more precisely the distribution of the Fourier–Walsh coefficients $\sum_{n<2^\nu}\mu(n)(-1)^{\sum_{i\in S}\varepsilon_i(n)}$, Bourgain [4] proved that μ is orthogonal to any monotone Boolean function (see [5] for a lower bound for the number of primes captured by these functions). The estimate (3) means that for any polynomial $P\in\mathbb{Z}[X_0,\dots,X_{\nu-1}]$ of degree at most 1 we have

$$\sum_{n<2^\nu}\mu(n)(-1)^{P(\varepsilon_0(n),\dots,\varepsilon_{\nu-1}(n))}=O(2^{\nu-\nu^{1/10}}),$$

but the question asked by Kalai [18] concerning the case of polynomials of degree greater than 1 is open. The simplest case of polynomial of degree 2 is given by the Rudin–Shapiro sequence

$$\left((-1)^{\sum_{i\ge 1}\varepsilon_{i-1}(n)\varepsilon_i(n)}\right)_{n\in\mathbb{N}} \tag{4}$$

introduced independently by Shapiro [30] and by Rudin [28] for which Tao [18] suggests a strategy to prove a Möbius randomness principle, i.e.

$$\sum_{n\le x}\mu(n)(-1)^{\sum_{i\ge 1}\varepsilon_{i-1}(n)\varepsilon_i(n)}=o(x).$$

In this paper we will obtain as a special case in Theorem 3 a quantitative Prime Number Theorem (and a Möbius randomness principle) for the sequence

$$\left((-1)^{\sum_{i\geq\delta+1}\varepsilon_{i-\delta-1}(n)\varepsilon_i(n)}\right)_{n\in\mathbb{N}}$$

for any integer $\delta \geq 0$ (including the Rudin–Shapiro sequence for $\delta = 0$), and in Theorem 4 a quantitative Prime Number Theorem (and a Möbius randomness principle) for the sequence

$$\left((-1)^{\sum_{i\geq d-1}\varepsilon_{i-d+1}(n)\cdots\varepsilon_{i-1}(n)\varepsilon_i(n)}\right)_{n\in\mathbb{N}}$$

for any integer $d \geq 2$, providing an answer to Kalai's question for a special case of polynomial of degree d.

2. Statement of the results

One of the main ingredients in our proof in [24] of a Prime Number Theorem for the sequence $(\exp(\alpha\sum_{i\geq 0}\varepsilon_i(n)))_{n\in\mathbb{N}}$ was to establish that the L^1 norm of the discrete Fourier transform of this sequence is very small. Unfortunately this property is generally not true for other digital sequences and in particular for the Rudin–Shapiro sequence (4). Such a difference in the behaviour of the Fourier transforms is not surprising if we remember that the sequences $((-1)^{\sum_{i\geq 0}\varepsilon_i(n)})_{n\in\mathbb{N}}$ and $((-1)^{\sum_{i\geq 1}\varepsilon_{i-1}(n)\varepsilon_i(n)})_{n\in\mathbb{N}}$ have quite different spectral properties: the correlation measure of the first one is a singular measure, namely the Riesz product $\prod_{n\geq 0}(1-\cos 2^n t)$ (see [27, Section 3.3.3] or [20]), while for the second one it is the Lebesgue measure (see [27, Corollary 8.5]).

For $f : \mathbb{N} \to \mathbb{U}$ and any $\lambda \in \mathbb{N}$, let us denote by f_λ the q^λ-periodic function defined by

$$\forall n \in \{0,\ldots,q^\lambda-1\},\ \forall k \in \mathbb{Z},\qquad f_\lambda(n+kq^\lambda) = f(n). \tag{5}$$

Definition 1. A function $f : \mathbb{N} \to \mathbb{U}$ has the *carry property* if, uniformly for $(\lambda,\kappa,\rho) \in \mathbb{N}^3$ with $\rho < \lambda$, the number of integers $0 \leq \ell < q^\lambda$ such that there exists (k_1,k_2) in $\{0,\ldots,q^\kappa-1\}^2$ with

$$f(\ell q^\kappa+k_1+k_2)\,\overline{f(\ell q^\kappa+k_1)} \neq f_{\kappa+\rho}(\ell q^\kappa+k_1+k_2)\,\overline{f_{\kappa+\rho}(\ell q^\kappa+k_1)} \tag{6}$$

is at most $O(q^{\lambda-\rho})$, where the implied constant may depend only on q and f.

We introduce a set of functions with uniformly small discrete Fourier transforms:

Definition 2. Given a non-decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{\lambda\to\infty}\gamma(\lambda) = \infty$ and $c > 0$ we denote by $\mathcal{F}_{\gamma,c}$ the set of functions $f : \mathbb{N} \to \mathbb{U}$ such that for any $(\kappa,\lambda) \in \mathbb{N}^2$ with $\kappa \leq c\lambda$ and $t \in \mathbb{R}$,

$$\left|q^{-\lambda}\sum_{0\leq u<q^\lambda} f(uq^\kappa)\,\mathrm{e}(-ut)\right| \leq q^{-\gamma(\lambda)}. \tag{7}$$

For example, for any α such that $(q-1)\alpha \in \mathbb{R}\setminus\mathbb{Z}$, it follows from [23, Lemmas 16 and 9] that the function $f(n) = \mathrm{e}(\alpha \sum_{i\geq 0} \varepsilon_i(n))$ has the carry property and is in $\mathcal{F}_{\gamma,c}$ for any $c > 0$ and γ such that for $\lambda \geq 2$,

$$\gamma(\lambda) = \frac{\pi^2}{12\log q}\left(1 - \frac{2}{q+1}\right)\|(q-1)\alpha\|^2\lambda - \frac{\pi^2}{48\log q}.$$

The goal of this paper is to present a new method which allows us to prove a Prime Number Theorem for a large class of sequences with digit properties including the Rudin–Shapiro sequence and some of its generalizations. Roughly speaking, we prove that if we control the carry properties of a function $f : \mathbb{N} \to \mathbb{U}$ (Definition 1) for which the discrete Fourier transform is uniformly small (Definition 2), then we have a Prime Number Theorem (Theorem 1) and a Möbius randomness principle (Theorem 2) for f. This general result can be applied in many situations. In Section 10 we will apply it to the case of Rudin–Shapiro sequences.

Theorem 1. *Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function satisfying $\lim_{\lambda\to\infty}\gamma(\lambda) = \infty$, and $f : \mathbb{N} \to \mathbb{U}$ be a function with the carry property which is in $\mathcal{F}_{\gamma,c}$ for some $c \geq 10$. Then for any $\vartheta \in \mathbb{R}$ we have*

$$\left|\sum_{n\leq x} \Lambda(n) f(n)\,\mathrm{e}(\vartheta n)\right| \ll c_1(q)(\log x)^{c_2(q)} x q^{-\gamma(2\lfloor(\log x)/80\log q\rfloor)/20} \tag{8}$$

with

$$c_1(q) = \max(\tau(q)\log q, \log^{10} q)^{1/4}(\log q)^{2-c_2(q)},$$
$$c_2(q) = 2 + \max(2, (1+\omega(q))/4).$$

Remark 1. Theorem 1 gives a non-trivial result if

$$\liminf_{\lambda\to\infty} \frac{\gamma(\lambda)}{\log\lambda} > \frac{20 c_2(q)}{\log q}. \tag{9}$$

Corollary 1. *Let $b : \mathbb{N} \to \mathbb{N}$ be such that, for any $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the function $f(n) = \mathrm{e}(\alpha b(n))$ has the carry property and is in $\mathcal{F}_{\gamma,c}$ for some $c \geq 10$ and γ satisfying (9). Then for any $a \in \mathbb{Z}$, $m \in \mathbb{N}$, $m \geq 1$ with $\gcd(a, m) = 1$, the sequence $(\alpha b(p))_{p\in\mathcal{P}(a,m)}$ is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.*

Corollary 2. *Let $b : \mathbb{N} \to \mathbb{N}$ and $(m, m') \in \mathbb{N}^2$, $m, m' \geq 1$, be such that, for any integer j', $1 \leq j' < m'$, the function $f(n) = \mathrm{e}\big(\frac{j'}{m'} b(n)\big)$ has the carry property and is in $\mathcal{F}_{\gamma,c}$ for some $c \geq 10$ and γ satisfying (9). Then for any $(a, a') \in \mathbb{Z}^2$ such that $\gcd(a, m) = 1$, we have, for $x \to \infty$,*

$$\mathrm{card}\{p \leq x : p \in \mathcal{P}(a,m),\ b(p) \equiv a' \bmod m'\} = (1 + o(1))\frac{\pi(x; a, m)}{m'}.$$

Corollary 3. *Let $b : \mathbb{N} \to \mathbb{N}$ and $(m, m') \in \mathbb{N}^2$, $m, m' \geq 1$ be such that, for any integer j', $1 \leq j' < m'$, the function $f(n) = \mathrm{e}\big(\frac{j'}{m'} b(n)\big)$ has the carry property and is in $\mathcal{F}_{\gamma,c}$ for some $c \geq 10$ and γ satisfying (9). Then for any $(a, a') \in \mathbb{Z}^2$ such that $\gcd(a, m) = 1$ the sequence $(\vartheta p)_{p\in\mathcal{B}(a,m,a',m')}$ is uniformly distributed modulo 1 if and only if $\vartheta \in \mathbb{R}\setminus\mathbb{Q}$, where $\mathcal{B}(a, m, a', m') = \{p \in \mathcal{P}(a, m) : b(p) \equiv a' \bmod m'\}$.*

In order to estimate sums of the form $\sum_n \Lambda(n) F(n)$ in Theorem 1 by using a combinatorial identity like Vaughan's identity (see [15, (13.39)]), it is sufficient to estimate bilinear sums of the form

$$\sum_m \sum_n a_m b_n F(mn)$$

(we have described this method in detail in [24]). These sums are said to be of *type I* if b_n is a smooth function of n. Otherwise they are said to be of *type II*. The key of this approach is that for type I sums the summation over the smooth variable n is of significant length, while for type II sums both summations have a significant length.

Using (13.40) instead of (13.39) of [15] we obtain a similar result for the Möbius function μ (a better exponent of the factor $\log x$ might be obtained with some extra work):

Theorem 2. *Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function satisfying $\lim_{\lambda\to\infty} \gamma(\lambda) = \infty$, $c \geq 10$ and $f : \mathbb{N} \to \mathbb{U}$ be a function with the carry property and lying in $\mathcal{F}_{\gamma,c}$. Then for any $\vartheta \in \mathbb{R}$ we have*

$$\Big|\sum_{n\leq x} \mu(n) f(n)\, \mathrm{e}(\vartheta n)\Big| \ll c_1(q)(\log x)^{c_2(q)} x q^{-\gamma(2\lfloor(\log x)/80\log q\rfloor)/20}, \tag{10}$$

with $c_1(q)$ and $c_2(q)$ defined in Theorem 1.

3. Notations and preliminary lemmas

For $a \in \mathbb{Z}$ and $\kappa \in \mathbb{N}$ we denote by $\mathrm{r}_\kappa(a)$ the unique integer $r \in \{0,\dots,q^\kappa - 1\}$ such that $a \equiv r \bmod q^\kappa$. More generally for integers $0 \leq \kappa_1 \leq \kappa_2$ we denote by $\mathrm{r}_{\kappa_1,\kappa_2}(a)$ the unique integer $u \in \{0,\dots,q^{\kappa_2-\kappa_1} - 1\}$ such that $a = kq^{\kappa_2} + uq^{\kappa_1} + v$ for some $v \in \{0,\dots,q^{\kappa_1} - 1\}$ and $k \in \mathbb{Z}$. We notice that $\mathrm{r}_{\kappa_1,\kappa_2}(a) = \lfloor \mathrm{r}_{\kappa_2}(a)/q^{\kappa_1}\rfloor$ and for any $u \in \{0,\dots,q^{\kappa_2-\kappa_1} - 1\}$,

$$\mathrm{r}_{\kappa_1,\kappa_2}(a) = u \;\Leftrightarrow\; \frac{a}{q^{\kappa_2}} \in \left[\frac{u}{q^{\kappa_2-\kappa_1}}, \frac{u+1}{q^{\kappa_2-\kappa_1}}\right) + \mathbb{Z}. \tag{11}$$

For $a \geq 0$, $\mathrm{r}_\kappa(a)$ is the integer obtained from the κ least significant digits of a, while $\mathrm{r}_{\kappa_1,\kappa_2}(a)$ is the integer obtained using the digits of a of indices $\kappa_1,\dots,\kappa_2 - 1$.

The following lemma is a classical method to detect real numbers in an interval modulo 1 by means of exponential sums. For $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$ we denote by χ_α the characteristic function of the interval $[0,\alpha)$ modulo 1:

$$\chi_\alpha(x) = \lfloor x \rfloor - \lfloor x - \alpha \rfloor. \tag{12}$$

Lemma 1. *For any $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$ and any integer $H \geq 1$ there exist real valued trigonometric polynomials $A_{\alpha,H}(x)$ and $B_{\alpha,H}(x)$ such that for any $x \in \mathbb{R}$,*

$$|\chi_\alpha(x) - A_{\alpha,H}(x)| \leq B_{\alpha,H}(x), \tag{13}$$

where

$$A_{\alpha,H}(x)=\sum_{|h|\le H}a_h(\alpha,H)\,\mathrm{e}(hx),\qquad B_{\alpha,H}(x)=\sum_{|h|\le H}b_h(\alpha,H)\,\mathrm{e}(hx),\tag{14}$$

with coefficients $a_h(\alpha,H)$ *and* $b_h(\alpha,H)$ *satisfying*

$$a_0(\alpha,H)=\alpha,\qquad |a_h(\alpha,H)|\le\min\left(\alpha,\frac{1}{\pi|h|}\right),\qquad |b_h(\alpha,H)|\le\frac{1}{H+1}.\tag{15}$$

Proof. In order to apply [31, Theorem 19] we need to normalize χ_α: let us define, for any $x\in\mathbb{R}$,

$$\widetilde{\chi_\alpha}(x)=\lim_{t\to0+}\tfrac12(\chi_\alpha(x-t)+\chi_\alpha(x+t)).$$

Applying [31, (7.24)] we get

$$|\widetilde{\chi_\alpha}(x)-A_{\alpha,H}(x)|\le B_{\alpha,H}(x),$$

with the coefficients $a_h(\alpha,H)$ and $b_h(\alpha,H)$ defined by $a_0(\alpha,H)=\alpha$,

$$\begin{aligned}a_h(\alpha,H)&=a_h^*(\alpha,H)\,\mathrm{e}\left(\frac{-h\alpha}{2}\right),\\ a_h^*(\alpha,H)&=\frac{\sin\pi h\alpha}{\pi h}\left(\pi\frac{|h|}{H+1}\left(1-\frac{|h|}{H+1}\right)\cot\left(\pi\frac{|h|}{H+1}\right)+\frac{|h|}{H+1}\right),\end{aligned}\tag{16}$$

$$\begin{aligned}b_h(\alpha,H)&=b_h^*(\alpha,H)\,\mathrm{e}\left(\frac{-h\alpha}{2}\right),\\ b_h^*(\alpha,H)&=\frac{1}{H+1}\left(1-\frac{|h|}{H+1}\right)\cos(\pi h\alpha).\end{aligned}\tag{17}$$

In order to see that $A_{\alpha,H}(x)$ is real valued we notice that $a_{-h}^*(\alpha,H)=a_h^*(\alpha,H)$ and

$$\begin{aligned}A_{\alpha,H}(x)&=a_0(\alpha,H)+\sum_{h=1}^{H}a_h^*(\alpha,H)\big(\mathrm{e}(h(x-\alpha/2))+\mathrm{e}(-h(x-\alpha/2))\big)\\&=a_0(\alpha,H)+2\sum_{h=1}^{H}a_h^*(\alpha,H)\cos(2\pi h(x-\alpha/2)).\end{aligned}$$

Since $B_{\alpha,H}(x)\ge0$, obviously $B_{\alpha,H}(x)$ is real valued. By the argument above we have

$$B_{\alpha,H}(x)=b_0(\alpha,H)+2\sum_{h=1}^{H}b_h^*(\alpha,H)\cos(2\pi h(x-\alpha/2)).$$

Observing that $\chi_\alpha(x)=\lim_{t\to0+}\widetilde{\chi_\alpha}(x+t)$ for any $x\in\mathbb{R}$, we obtain (13).

The upper bound of $|a_h(\alpha,H)|$ given by (15) follows from [31, Theorem 6], and the upper bound of $|b_h(\alpha,H)|$ given by (15) follows from (17). □

In dimension 2, we can detect points in a rectangle (modulo 1) using the following:

Lemma 2. *For any* $(\alpha_1, \alpha_2) \in [0,1)^2$, *any integers* $H_1, H_2 \geq 1$, *and any* $(x, y) \in \mathbb{R}^2$ *we have*

$$\begin{aligned} |\chi_{\alpha_1}(x)\chi_{\alpha_2}(y) - A_{\alpha_1,H_1}(x)A_{\alpha_2,H_2}(y)| \\ \leq \chi_{\alpha_1}(x)B_{\alpha_2,H_2}(y) + B_{\alpha_1,H_1}(x)\chi_{\alpha_2}(y) + B_{\alpha_1,H_1}(x)B_{\alpha_2,H_2}(y), \end{aligned} \tag{18}$$

where $A_{\alpha,H}(\cdot)$ *and* $B_{\alpha,H}(\cdot)$ *are the real valued trigonometric polynomials defined by* (14).

Proof. For $(x, y) \in \mathbb{R}^2$ we have

$$\begin{aligned} &\chi_{\alpha_1}(x)\chi_{\alpha_2}(y) - A_{\alpha_1,H_1}(x)A_{\alpha_2,H_2}(y) \\ &\quad = \chi_{\alpha_1}(x)(\chi_{\alpha_2}(y) - A_{\alpha_2,H_2}(y)) + (\chi_{\alpha_1}(x) - A_{\alpha_1,H_1}(x))\chi_{\alpha_2}(y) \\ &\quad\quad - (\chi_{\alpha_1}(x) - A_{\alpha_1,H_1}(x))(\chi_{\alpha_2}(y) - A_{\alpha_2,H_2}(y)). \end{aligned}$$

Since $\chi_{\alpha_1}(x), \chi_{\alpha_2}(y) \geq 0$, by (13) we get (18). □

The following lemma is a generalization of van der Corput's inequality.

Lemma 3. *For any* $(z_1, \ldots, z_N) \in \mathbb{C}^N$ *and any integers* $k, R \geq 1$ *we have*

$$\Big|\sum_{1\leq n\leq N} z_n\Big|^2 \leq \frac{N+kR-k}{R}\Bigg(\sum_{1\leq n\leq N}|z_n|^2 + 2\sum_{1\leq r<R}\Big(1-\frac{r}{R}\Big)\sum_{1\leq n\leq N-kr}\Re(z_{n+kr}\overline{z_n})\Bigg), \tag{19}$$

where $\Re(z)$ *denotes the real part of* z.

Proof. See for example [23, Lemma 17]. □

We will often make use of the following upper bound of geometric series of ratio $\mathrm{e}(\xi)$ for $(L_1, L_2) \in \mathbb{Z}^2$ with $L_1 \leq L_2$ and $\xi \in \mathbb{R}$:

$$\Big|\sum_{L_1<\ell\leq L_2} \mathrm{e}(\ell\xi)\Big| \leq \min(L_2 - L_1, |\sin\pi\xi|^{-1}). \tag{20}$$

Lemmas 4 and 5 allow one to estimate on average the minimums arising from (20).

Lemma 4. *For any* $(a, m) \in \mathbb{Z}^2$ *with* $m \geq 1$, $b \in \mathbb{R}$ *and* $U \in \mathbb{R}$ *with* $U > 0$ *we have*

$$\sum_{0\leq n\leq m-1} \min\Bigg(U, \Big|\sin\pi\frac{an+b}{m}\Big|^{-1}\Bigg) \ll \gcd(a,m)U + m\log m. \tag{21}$$

Proof. This follows from [24, Lemma 6]. □

Lemma 5. *For any* $(A, m) \in \mathbb{N}^2$ *with* $m \geq 1$ *and* $A \geq 1$, $b \in \mathbb{R}$ *and* $U \in \mathbb{R}$ *with* $U > 0$ *we have*

$$\frac{1}{A}\sum_{1\leq a\leq A}\sum_{0\leq n<m} \min\Bigg(U, \Big|\sin\pi\frac{an+b}{m}\Big|^{-1}\Bigg) \ll \tau(m)U + m\log m. \tag{22}$$

Proof. By (21) it is enough to observe that

$$\sum_{1\le a\le A}\gcd(a,m)=\sum_{\substack{d\,|\,m\\ d\le A}} d\sum_{\substack{1\le a\le A\\ (a,m)=d}} 1\le \sum_{\substack{d\,|\,m\\ d\le A}} d\sum_{\substack{1\le a\le A\\ d\,|\,a}} 1=\sum_{\substack{d\,|\,m\\ d\le A}} d\left\lfloor\frac{A}{d}\right\rfloor\le A\,\tau(m),$$

which implies (22). □

The following lemma is a classical application of the large sieve inequality:

Lemma 6. *For any* $(z_1,\dots,z_N)\in\mathbb{C}^N$ *and any integer* $Q\ge 1$ *we have*

$$\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{n=1}^{N} z_n\,\mathrm{e}\Big(\frac{an}{q}\Big)\right|^2\le (N-1+Q^2)\sum_{n=1}^{N}|z_n|^2. \tag{23}$$

Proof. See [25, Theorem 3 and Section 8]. □

Let $f:\mathbb{N}\to\mathbb{U}$, $\lambda\in\mathbb{N}$ and f_λ defined by (5). The *discrete Fourier transform* of f_λ is defined for $t\in\mathbb{R}$ by

$$\widehat{f_\lambda}(t)=\frac{1}{q^\lambda}\sum_{0\le u<q^\lambda} f_\lambda(u)\,\mathrm{e}\Big(-\frac{ut}{q^\lambda}\Big)=\frac{1}{q^\lambda}\sum_{0\le u<q^\lambda} f(u)\,\mathrm{e}\Big(-\frac{ut}{q^\lambda}\Big). \tag{24}$$

For $\lambda\in\mathbb{N}$ and $t\in\mathbb{R}$ we have

$$\sum_{0\le h<q^\lambda}|\widehat{f_\lambda}(h+t)|^2=1, \tag{25}$$

so that, if f satisfies (7), then

$$1=\sum_{0\le h<q^\lambda}\left|q^{-\lambda}\sum_{0\le u<q^\lambda} f(uq^\kappa)\,\mathrm{e}\Big(-\frac{u(h+t)}{q^\lambda}\Big)\right|^2\le\sum_{0\le h<q^\lambda} q^{-2\gamma(\lambda)}=q^{\lambda-2\gamma(\lambda)}$$

and

$$\gamma(\lambda)\le\lambda/2. \tag{26}$$

4. Carry propagation lemmas

Lemma 7. *Let* $\mu,\nu,\mu'\ge 1$ *be integers with* $\mu'\le\mu+\nu$. *For any set* $\mathcal{B}$ *contained in* $\{0,\dots,q^{\mu+\nu-\mu'}-1\}$, *the number* $\mathcal{N}$ *of* $(m,n)\in\{q^{\mu-1},\dots,q^\mu-1\}\times\{q^{\nu-1},\dots,q^\nu-1\}$ *such that* $mn=a+q^{\mu'}b$ *with* $0\le a<q^{\mu'}$ *and* $b\in\mathcal{B}$ *satisfies*

$$\mathcal{N}\le(q^{\mu'}\log q+q^\mu-q^{\mu-1}+q^{\mu'-\mu+1})\,\mathrm{card}\,\mathcal{B}.$$

Proof. For each $m \in \{q^{\mu-1}, \dots, q^{\mu}-1\}$, the number $\mathcal{N}_m$ of integers n such that $mn = a + q^{\mu'} b$ with $0 \le a < q^{\mu'}$ and $b \in \mathcal{B}$ satisfies

$$\mathcal{N}_m \le \sum_{b \in \mathcal{B}} \operatorname{card}\{a \in \mathbb{N} : 0 \le a < q^{\mu'},\ a + q^{\mu'} b \equiv 0 \bmod m\}.$$

This gives

$$\mathcal{N}_m \le \sum_{b \in \mathcal{B}} \left(1 + \frac{q^{\mu'}}{m}\right) = \left(1 + \frac{q^{\mu'}}{m}\right) \operatorname{card} \mathcal{B}.$$

It follows that

$$\mathcal{N} = \sum_{q^{\mu-1} \le m < q^{\mu}} \mathcal{N}_m \le \sum_{q^{\mu-1} \le m < q^{\mu}} \left(1 + \frac{q^{\mu'}}{m}\right) \operatorname{card} \mathcal{B},$$

so that

$$\mathcal{N} \le (q^{\mu} - q^{\mu-1}) \operatorname{card} \mathcal{B} + q^{\mu'} \left(\frac{1}{q^{\mu-1}} + \int_{q^{\mu-1}}^{q^{\mu}} \frac{dt}{t}\right) \operatorname{card} \mathcal{B},$$

and the result follows. □

Lemma 8. *If $f : \mathbb{N} \to \mathbb{U}$ has the carry property (Definition 1), then for $(\mu, \nu, \rho) \in \mathbb{N}^3$ with $2\rho < \nu$ the set $\mathcal{E}$ of $(m, n) \in \{q^{\mu-1}, \dots, q^{\mu}-1\} \times \{q^{\nu-1}, \dots, q^{\nu}-1\}$ such that there exists $k < q^{\mu+\rho}$ with $f(mn+k)\overline{f(mn)} \ne f_{\mu+2\rho}(mn+k)\overline{f_{\mu+2\rho}(mn)}$ satisfies*

$$\operatorname{card} \mathcal{E} \ll (\log q) q^{\mu+\nu-\rho}. \tag{27}$$

Proof. Applying Definition 1 with $\lambda = \nu - \rho$ and $\kappa = \mu + \rho$, let $\mathcal{B}$ be the set of $\ell < q^{\nu-\rho}$ such that there exists $(k_1, k_2) \in \{0, \dots, q^{\kappa}-1\}^2$ for which (6) is true. By Definition 1 we have $\operatorname{card} \mathcal{B} \ll q^{\nu-2\rho}$. We need to count the $(m, n) \in \{q^{\mu-1}, \dots, q^{\mu}-1\} \times \{q^{\nu-1}, \dots, q^{\nu}-1\}$ such that mn is of the form $mn = k_1 + q^{\mu'} \ell$ with $\ell \in \mathcal{B}$. Applying Lemma 7 with $\mu' = \mu + \rho$ we get

$$\operatorname{card} \mathcal{E} \ll (q^{\mu+\rho} \log q + q^{\mu} - q^{\mu-1} + q^{\rho+1}) \operatorname{card} \mathcal{B} \ll (\log q) q^{\mu+\nu-\rho},$$

which gives (27). □

Lemma 9. *Let $f : \mathbb{N} \to \mathbb{U}$ have the carry property and $(\mu, \nu, \mu_0, \mu_1, \mu_2) \in \mathbb{N}^5$ with $\mu_0 \le \mu_1 \le \mu \le \mu_2$, $\mu \le \nu$ and $2(\mu_2 - \mu) \le \mu_0$. For any $(a, b, c) \in \mathbb{N}^3$ the set $\mathcal{E}(a, b, c)$ of $(m, n) \in \{q^{\mu-1}, \dots, q^{\mu}-1\} \times \{q^{\nu-1}, \dots, q^{\nu}-1\}$ such that*

$$\begin{aligned} &f_{\mu_2}(mn + am + bn + c)\, \overline{f_{\mu_2}(q^{\mu_0} \mathrm{r}_{\mu_0,\mu_2}(mn + am + bn + c))} \\ &\qquad \ne f_{\mu_1}(mn + am + bn + c)\, \overline{f_{\mu_1}(q^{\mu_0} \mathrm{r}_{\mu_0,\mu_2}(mn + am + bn + c))} \end{aligned}$$

satisfies

$$\operatorname{card} \mathcal{E}(a, b, c) \ll \max(\tau(q), \log q)\, \mu_2^{\omega(q)} q^{\mu+\nu+\mu_0-\mu_1}. \tag{28}$$

Proof. Let $\mathcal{B}$ be the set of $\ell \in \{0, \dots, q^{\mu_2-\mu_0} - 1\}$ for which there exists (k_1, k_2) in $\{0, \dots, q^{\mu_0} - 1\}^2$ with

$$f_{\mu_2}(q^{\mu_0}\ell + k_1 + k_2)\ \overline{f_{\mu_2}(q^{\mu_0}\ell + k_1)} \neq f_{\mu_1}(q^{\mu_0}\ell + k_1 + k_2)\ \overline{f_{\mu_1}(q^{\mu_0}\ell + k_1)}.$$

For $0 \le \ell \le q^{\mu_2-\mu_0} - 2$ we have $0 \le q^{\mu_0}\ell + k_1 + k_2 \le q^{\mu_2} - 2$. Therefore we have $f_{\mu_2}(q^{\mu_0}\ell+k_1+k_2) = f(q^{\mu_0}\ell+k_1+k_2)$ and $f_{\mu_2}(q^{\mu_0}\ell+k_1) = f(q^{\mu_0}\ell+k_1)$ except possibly if $\ell = q^{\mu_2-\mu_0} - 1$. Since f has the carry property, $\operatorname{card}\mathcal{B} = O(q^{\mu_2-\mu_0-(\mu_1-\mu_0)}) = O(q^{\mu_2-\mu_1})$. Observing for $k = mn+am+bn+c$ that $k = \mathrm{r}_{0,\mu_0}(k)+q^{\mu_0}\,\mathrm{r}_{\mu_0,\mu_2}(k)+q^{\mu_2}k'$, we notice that $\mathcal{E}(a, b, c) \subseteq \mathcal{E}'(a, b, c)$ where $\mathcal{E}'(a, b, c)$ is the set of (m, n) such that $\mathrm{r}_{\mu_0,\mu_2}(mn + am + bn + c) \in \mathcal{B}$. Then

$$\operatorname{card}\mathcal{E}'(a, b, c) = \sum_{\ell\in\mathcal{B}} \operatorname{card}\{(m, n) : \mathrm{r}_{\mu_0,\mu_2}(mn + am + bn + c) = \ell\},$$

which by (11) and (12) can be written

$$\operatorname{card}\mathcal{E}'(a, b, c) = \sum_{\ell\in\mathcal{B}} \sum_{m} \sum_{n} \chi_{q^{\mu_0-\mu_2}}\left(\frac{mn + am + bn + c}{q^{\mu_2}} - \frac{\ell}{q^{\mu_2-\mu_0}}\right).$$

Using Lemma 1 it follows that for any integer $H \ge 1$ there exist $a_h(q^{\mu_0-\mu_2}, H)$ and $b_h(q^{\mu_0-\mu_2}, H)$ satisfying (15) such that

$$\begin{aligned}
&\operatorname{card}\mathcal{E}'(a, b, c)\\
&\quad\le \sum_{\ell\in\mathcal{B}} \sum_{m} \sum_{n} \sum_{|h|\le H} a_h(q^{\mu_0-\mu_2}, H)\,\mathrm{e}\left(\frac{h(mn + am + bn + c)}{q^{\mu_2}} - \frac{h\ell}{q^{\mu_2-\mu_0}}\right)\\
&\qquad+ \sum_{\ell\in\mathcal{B}} \sum_{m} \sum_{n} \sum_{|h|\le H} b_h(q^{\mu_0-\mu_2}, H)\,\mathrm{e}\left(\frac{h(mn + am + bn + c)}{q^{\mu_2}} - \frac{h\ell}{q^{\mu_2-\mu_0}}\right).
\end{aligned}$$

Taking $H = q^{\mu_2-\mu_0}$, the contribution of the terms $h = 0$ in both sums is bounded by

$$q^{\mu+\nu+\mu_0-\mu_2} \operatorname{card}\mathcal{B} \ll q^{\mu+\nu+\mu_0-\mu_1}.$$

We handle both sums over h similarly, exchanging the order of summations and using the bounds $|a_h(q^{\mu_0-\mu_2}, H)| \le q^{\mu_0-\mu_2}$ and $|b_h(q^{\mu_0-\mu_2}, H)| \le H^{-1} = q^{\mu_0-\mu_2}$. We obtain the upper bound

$$\begin{aligned}
&\operatorname{card}\mathcal{E}'(a, b, c)\\
&\quad\ll q^{\mu+\nu+\mu_0-\mu_1} + \frac{\operatorname{card}\mathcal{B}}{q^{\mu_2-\mu_0}} \sum_{1\le|h|\le q^{\mu_2-\mu_0}} \sum_{n} \left|\sum_{m} \mathrm{e}\left(\frac{h(mn + am + bn + c)}{q^{\mu_2}}\right)\right|,
\end{aligned}$$

which gives

$$\begin{aligned}
&\operatorname{card}\mathcal{E}'(a, b, c)\\
&\quad\ll q^{\mu+\nu+\mu_0-\mu_1} + \frac{q^{\mu_2-\mu_1}}{q^{\mu_2-\mu_0}} \sum_{1\le|h|\le q^{\mu_2-\mu_0}} \sum_{n} \min\left(q^{\mu}, \left|\sin\pi\frac{h(n + a)}{q^{\mu_2}}\right|^{-1}\right).
\end{aligned}$$

The summation on n runs over at most $\lceil q^{\nu-\mu_2}\rceil$ periods modulo q^{μ_2}. By (22),

$$\operatorname{card}\mathcal{E}'(a,b,c) \ll q^{\mu+\nu+\mu_0-\mu_1} + q^{\mu_2-\mu_1}(q^{\nu-\mu_2}+1)(\tau(q^{\mu_2})q^{\mu} + q^{\mu_2}\log q^{\mu_2}).$$

By multiplicativity of the function τ we have $\tau(q^{\mu_2}) \le \tau(q)\mu_2^{\omega(q)}$ and we obtain

$$\operatorname{card}\mathcal{E}'(a,b,c) \ll q^{\mu+\nu+\mu_0-\mu_1}\big(1 + q^{-\mu_0}(1+q^{\mu_2-\nu})(\tau(q)\mu_2^{\omega(q)} + q^{\mu_2-\mu}\log q^{\mu_2})\big),$$

and using $\mu_1 \le \mu \le \nu$ we may replace μ and ν by μ_1 in the parentheses; this leads to

$$\operatorname{card}\mathcal{E}'(a,b,c) \ll q^{\mu+\nu+\mu_0-\mu_1}(1 + \mu_2^{\omega(q)}\max(\tau(q),\log q)q^{2\mu_2-2\mu-\mu_0}),$$

which, using the hypothesis $2(\mu_2-\mu) \le \mu_0$, gives (28). □

5. Sums of type I

We take a non-decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{\lambda\to\infty}\gamma(\lambda) = \infty$, $c \ge 2$ and $f : \mathbb{N} \to \mathbb{U}$ with the carry property and lying in $\mathcal{F}_{\gamma,c}$ (see Definitions 1 and 2). Let

$$1 \le M \le N, \qquad M \le (MN)^{1/3}, \tag{29}$$

and let μ and ν be the unique integers such that

$$q^{\mu-1} \le M < q^{\mu} \quad \text{and} \quad q^{\nu-1} \le N < q^{\nu}.$$

For any $\vartheta \in \mathbb{R}$ and any interval $I(M,N) \subseteq [0, MN]$ we set

$$S_I(\vartheta) = \sum_{M/q<m\le M}\Big|\sum_{\substack{n\\ mn\in I(M,N)}} f(mn)\,\mathrm{e}(\vartheta mn)\Big|.$$

Proposition 1. *Assuming (29) and $c \ge 2$ we have, uniformly for $\vartheta \in \mathbb{R}$,*

$$S_I(\vartheta) \ll (\log q)^{5/2}(\mu+\nu)^2\, q^{\mu+\nu-\frac{1}{2}\gamma(\frac{\mu+\nu}{3})}. \tag{30}$$

Proof. Let $0 \le \ell < q^{\mu+\nu}$. For $M/q < m \le M$, we have $\ell = mn$ with $mn \in I(M,N)$ if and only if $\ell \in I(M,N)$ and $\ell \equiv 0 \bmod m$. Therefore the inner sum (over n) in $S_I(\vartheta)$ is

$$\sum_{0\le\ell<q^{\mu+\nu}} f(\ell)\,\mathrm{e}(\vartheta\ell) \sum_{u\in I(M,N)} \frac{1}{q^{\mu+\nu}} \sum_{0\le h<q^{\mu+\nu}} \mathrm{e}\Big(\frac{h(u-\ell)}{q^{\mu+\nu}}\Big)\frac{1}{m}\sum_{0\le k<m}\mathrm{e}\Big(\frac{k\ell}{m}\Big),$$

i.e.

$$\sum_{0\le h<q^{\mu+\nu}}\Big(\sum_{u\in I(M,N)}\mathrm{e}\Big(\frac{hu}{q^{\mu+\nu}}\Big)\Big)\frac{1}{m}\sum_{0\le k<m}\frac{1}{q^{\mu+\nu}}\sum_{0\le\ell<q^{\mu+\nu}} f(\ell)\,\mathrm{e}\Big(\vartheta\ell - \frac{h\ell}{q^{\mu+\nu}} + \frac{k\ell}{m}\Big).$$

This gives

$$S_I(\vartheta) \le \sum_{0\le h<q^{\mu+\nu}} \min\Big(q^{\mu+\nu}, \Big|\sin\frac{\pi h}{q^{\mu+\nu}}\Big|^{-1}\Big) S_I'(h - \vartheta q^{\mu+\nu}),$$

where

$$S'_I(\vartheta') = \sum_{M/q<m\le M} \frac{1}{m} \sum_{0\le k<m} \left| \widehat{f_{\mu+\nu}}\left(\vartheta' - \frac{k}{m} q^{\mu+\nu}\right)\right|. \tag{31}$$

Then we have, uniformly for $\vartheta \in \mathbb{R}$,

$$S_I(\vartheta) \le \left(\max_{\vartheta'\in\mathbb{R}} S'_I(\vartheta')\right) \sum_{0\le h<q^{\mu+\nu}} \min\left(q^{\mu+\nu}, \left|\sin \frac{\pi h}{q^{\mu+\nu}}\right|^{-1}\right),$$

which gives

$$S_I(\vartheta) \ll \left(\max_{\vartheta'\in\mathbb{R}} S'_I(\vartheta')\right) q^{\mu+\nu} \log q^{\mu+\nu}. \tag{32}$$

It remains to estimate $S'_I(\vartheta')$ uniformly for $\vartheta' \in \mathbb{R}$. For any κ such that

$$1 \le \kappa \le \tfrac{2}{3}(\mu+\nu), \tag{33}$$

by (24) we can write

$$\widehat{f_{\mu+\nu}}(t) = \frac{1}{q^{\mu+\nu}} \sum_{0\le u<q^\kappa} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(u+vq^\kappa)\, \mathrm{e}\left(-\frac{(u+vq^\kappa)t}{q^{\mu+\nu}}\right).$$

This gives

$$\widehat{f_{\mu+\nu}}(t) = \frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^\kappa)\, \mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}}\right) \times \frac{1}{q^\kappa} \sum_{0\le u<q^\kappa} f(u+vq^\kappa)\overline{f(vq^\kappa)}\, \mathrm{e}\left(-\frac{ut}{q^{\mu+\nu}}\right).$$

Given ρ_1 such that

$$1 \le \rho_1 \le \mu+\nu-\kappa, \tag{34}$$

by Definition 1 the number of $v \in \{0,\dots,q^{\mu+\nu-\kappa}-1\}$ such that there exists u in $\{0,\dots,q^\kappa-1\}$ for which

$$f(u+vq^\kappa)\,\overline{f(vq^\kappa)} \ne f_{\kappa+\rho_1}(u+vq^\kappa)\,\overline{f_{\kappa+\rho_1}(vq^\kappa)}$$

is at most $O(q^{\mu+\nu-\kappa-\rho_1})$. Hence the set $\widetilde{\mathcal{W}}_\kappa$ of (u,v) with this property satisfies

$$\operatorname{card} \widetilde{\mathcal{W}}_\kappa \ll q^{\mu+\nu-\rho_1}. \tag{35}$$

Therefore for any $t \in \mathbb{R}$, any κ satisfying (33) and any ρ_1 satisfying (34) we have

$$\widehat{f_{\mu+\nu}}(t) = G_{\kappa,1}(t) + G_{\kappa,2}(t) \tag{36}$$

with

$$G_{\kappa,1}(t) = \frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^{\kappa})\, \mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}}\right) \times \frac{1}{q^{\kappa}} \sum_{0\le u<q^{\kappa}} f_{\kappa+\rho_1}(u+vq^{\kappa})\overline{f_{\kappa+\rho_1}(vq^{\kappa})}\, \mathrm{e}\left(-\frac{ut}{q^{\mu+\nu}}\right),$$

$$G_{\kappa,2}(t) = \frac{1}{q^{\mu+\nu}} \sum_{(u,v)\in\widetilde{\mathcal{W}}_{\kappa}} f(vq^{\kappa})\, \mathrm{e}\left(-\frac{(u+vq^{\kappa})t}{q^{\mu+\nu}}\right) \times \left(f(u+vq^{\kappa})\overline{f(vq^{\kappa})} - f_{\kappa+\rho_1}(u+vq^{\kappa})\overline{f_{\kappa+\rho_1}(vq^{\kappa})}\right).$$

Let us introduce in $G_{\kappa,1}(t)$ the residue w of v mod q^{ρ_1} in order to make the variables u and v independent:

$$G_{\kappa,1}(t) = \sum_{0\le w<q^{\rho_1}} \frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^{\kappa})\, \mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}}\right) \frac{1}{q^{\rho_1}} \sum_{0\le h<q^{\rho_1}} \mathrm{e}\left(h\frac{v-w}{q^{\rho_1}}\right) \times \frac{1}{q^{\kappa}} \sum_{0\le u<q^{\kappa}} f_{\kappa+\rho_1}(u+wq^{\kappa})\overline{f_{\kappa+\rho_1}(wq^{\kappa})}\, \mathrm{e}\left(-\frac{ut}{q^{\mu+\nu}}\right).$$

Writing

$$c_{\kappa,\rho_1}(u,h) = \frac{1}{q^{\rho_1}} \sum_{0\le w<q^{\rho_1}} f_{\kappa+\rho_1}(u+wq^{\kappa})\overline{f_{\kappa+\rho_1}(wq^{\kappa})}\, \mathrm{e}\left(-\frac{hw}{q^{\rho_1}}\right) \tag{37}$$

leads to

$$G_{\kappa,1}(t) = \sum_{0\le h<q^{\rho_1}} \left(\frac{1}{q^{\kappa}} \sum_{0\le u<q^{\kappa}} c_{\kappa,\rho_1}(u,h)\, \mathrm{e}\left(\frac{-ut}{q^{\mu+\nu}}\right)\right) \times \left(\frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^{\kappa})\, \mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}} + \frac{hv}{q^{\rho_1}}\right)\right).$$

By (33) we have $\kappa \le 2(\mu+\nu-\kappa)$ and we may use (7) (with $c \ge 2$) for the sum over v with $\kappa = \kappa$ and $\lambda = \mu+\nu-\kappa$. We get

$$|G_{\kappa,1}(t)| \ll q^{-\gamma(\mu+\nu-\kappa)} \sum_{0\le h<q^{\rho_1}} \left| \frac{1}{q^{\kappa}} \sum_{0\le u<q^{\kappa}} c_{\kappa,\rho_1}(u,h)\, \mathrm{e}\left(\frac{-ut}{q^{\mu+\nu}}\right)\right|.$$

From (31) we can write

$$S_I'(\vartheta') \le \sum_{1\le d\le M} \sum_{M/q<m\le M} \frac{1}{m} \sum_{\substack{0\le k<m\\ (k,m)=d}} \left|\widehat{f_{\mu+\nu}}\left(\vartheta' - \frac{k}{m}q^{\mu+\nu}\right)\right|.$$

In order to estimate $S_I'(\vartheta')$, for each $1 \le d \le M$, we will use (36) with κ_d defined to be the unique integer such that

$$q^{\kappa_d-1} < M^2/d^2 \le q^{\kappa_d}. \tag{38}$$

Hence

$$S'_I(\vartheta') \le S'_{I,1}(\vartheta') + S'_{I,2}(\vartheta')$$

with

$$S'_{I,1}(\vartheta') = \sum_{1\le d\le M} \sum_{M/q<m\le M} \frac{1}{m} \sum_{\substack{0\le k<m\\(k,m)=d}} \left| G_{\kappa_d,1}\left(\vartheta' - \frac{k}{m} q^{\mu+\nu}\right)\right|,$$

$$S'_{I,2}(\vartheta') = \sum_{1\le d\le M} \sum_{M/q<m\le M} \frac{1}{m} \sum_{\substack{0\le k<m\\(k,m)=d}} \left| G_{\kappa_d,2}\left(\vartheta' - \frac{k}{m} q^{\mu+\nu}\right)\right|.$$

Then

$$\left| G_{\kappa_d,1}\left(\vartheta' - \frac{k}{m} q^{\mu+\nu}\right)\right| \le q^{-\gamma(\mu+\nu-\kappa_d)} \sum_{0\le h<q^{\rho_1}} \frac{1}{q^{\kappa_d}} \left| \sum_{0\le u<q^{\kappa_d}} c_{\kappa_d,\rho_1}(u,h)\, \mathrm{e}\left(-\frac{u\vartheta'}{q^{\mu+\nu}} + \frac{uk}{m}\right)\right|.$$

Since κ_d is decreasing in d, by (38) and (29) we can check that (33) is satisfied:

$$1 \le \kappa_d \le \kappa_1 \le 2\mu \le \tfrac{2}{3}(\mu+\nu). \tag{39}$$

But γ is non-decreasing, which implies

$$S'_{I,1}(\vartheta') \le q^{-\gamma(\frac{\mu+\nu}{3})} \sum_{1\le d\le M} \frac{S''_{I,1}(M,d)}{d\, q^{\kappa_d}}, \tag{40}$$

where

$$S''_{I,1}(M,d) = \sum_{0\le h<q^{\rho_1}} \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \frac{1}{m'} \sum_{\substack{0\le k'<m'\\(k',m')=1}} \left| \sum_{0\le u<q^{\kappa_d}} c_{\kappa_d,\rho_1}(u,h)\, \mathrm{e}\left(-\frac{u\vartheta'}{q^{\mu+\nu}} + \frac{uk'}{m'}\right)\right|.$$

Since

$$\sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \sum_{\substack{0\le k'<m'\\(k',m')=1}} \frac{1}{m'^2} \le \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \frac{1}{m'} \ll \log q, \tag{41}$$

by the Cauchy–Schwarz inequality we get

$$|S''_{I,1}(M,d)|^2 \ll (\log q)\, q^{\rho_1} \sum_{0\le h<q^{\rho_1}} \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \sum_{\substack{0\le k'<m'\\(k',m')=1}} \left| \sum_{0\le u<q^{\kappa_d}} c_{\kappa_d,\rho_1}(u,h)\, \mathrm{e}\left(-\frac{u\vartheta'}{q^{\mu+\nu}} + \frac{uk'}{m'}\right)\right|^2.$$

By (23) it follows that

$$|S''_{I,1}(M,d)|^2 \ll (\log q) q^{\rho_1} \sum_{0\le h<q^{\rho_1}} (q^{\kappa_d} + M^2/d^2) \sum_{0\le u<q^{\kappa_d}} |c_{\kappa_d,\rho_1}(u,h)|^2.$$

But by (37) and (25) we have

$$\sum_{0\le h<q^{\rho_1}} |c_{\kappa_d,\rho_1}(u,h)|^2 = 1,$$

hence summing over u and using (38) we obtain

$$|S''_{I,1}(M,d)| \ll (\log q)^{1/2} q^{\kappa_d+\rho_1/2},$$

which by (40) leads to

$$S'_{I,1}(\vartheta') \ll (\log q)^{1/2} q^{-\gamma(\frac{\mu+\nu}{3})} \sum_{1\le d\le M} \frac{q^{\rho_1/2}}{d} \ll \mu\,(\log q)^{3/2} q^{\rho_1/2-\gamma(\frac{\mu+\nu}{3})}. \tag{42}$$

In order to estimate $S'_{I,2}(\vartheta')$ we denote by $\mathcal{W}_{\kappa_d}$ the set of integers $w = u + vq^{\kappa_d}$ such that $(u,v) \in \widetilde{\mathcal{W}}_{\kappa_d}$. Using the bijective correspondence between $\mathcal{W}_{\kappa_d}$ and $\widetilde{\mathcal{W}}_{\kappa_d}$ given by $w \mapsto (\mathrm{r}_{\kappa_d}(w), \mathrm{r}_{\kappa_d,\mu+\nu}(w))$ we can write

$$G_{\kappa_d,2}(t) = \frac{1}{q^{\mu+\nu}} \sum_{0\le w<q^{\mu+\nu}} c'_{\kappa_d,\rho_1}(w)\,\mathrm{e}\left(-\frac{wt}{q^{\mu+\nu}}\right),$$

where

$$\begin{aligned} &c'_{\kappa_d,\rho_1}(w)\\ &\quad = f(q^{\kappa_d}\,\mathrm{r}_{\kappa_d,\mu+\nu}(w))\left(f(w)\overline{f(q^{\kappa_d}\,\mathrm{r}_{\kappa_d,\mu+\nu}(w))} - f_{\kappa_d+\rho_1}(w)\overline{f_{\kappa_d+\rho_1}(q^{\kappa_d}\,\mathrm{r}_{\kappa_d,\mu+\nu}(w))}\right) \end{aligned}$$

satisfies $|c'_{\kappa_d,\rho_1}(w)| \le 2$ for $0 \le w < q^{\mu+\nu}$ and $c'_{\kappa_d,\rho_1}(w) = 0$ for $w \notin \mathcal{W}_{\kappa_d}$. Then we write

$$S'_{I,2}(\vartheta') \le \sum_{1\le d\le M} \frac{S''_{I,2}(M,d)}{d\,q^{\mu+\nu}}, \tag{43}$$

where

$$S''_{I,2}(M,d) = \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \frac{1}{m'} \sum_{\substack{0\le k'<m'\\ (k',m')=1}} \left| \sum_{0\le w<q^{\mu+\nu}} c'_{\kappa_d,\rho_1}(w)\,\mathrm{e}\left(-\frac{w\vartheta'}{q^{\mu+\nu}} + \frac{wk'}{m'}\right)\right|.$$

It follows from the Cauchy–Schwarz inequality and (41) that

$$|S''_{I,2}(M,d)|^2 \ll (\log q) \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \sum_{\substack{0\le k'<m'\\ (k',m')=1}} \left| \sum_{0\le w<q^{\mu+\nu}} c'_{\kappa_d,\rho_1}(w)\,\mathrm{e}\left(-\frac{w\vartheta'}{q^{\mu+\nu}} + \frac{wk'}{m'}\right)\right|^2.$$

By (23) we get

$$\begin{aligned} |S''_{I,2}(M,d)|^2 &\ll (\log q)(q^{\mu+\nu} + M^2/d^2) \sum_{0\le w<q^{\mu+\nu}} |c'_{\kappa_d,\rho_1}(w)|^2\\ &\ll (\log q)(q^{\mu+\nu} + M^2/d^2) \sum_{w\in\mathcal{W}_{\kappa_d}} 2^2. \end{aligned}$$

By (35) it follows that

$$|S''_{I,2}(M,d)| \ll (\log q)^{1/2} q^{\mu+\nu-\rho_1/2},$$

which by (40) leads to

$$S'_{I,2}(\vartheta') \ll (\log q)^{1/2} \sum_{1\le d\le M} \frac{q^{-\rho_1/2}}{d} \ll \mu(\log q)^{3/2} q^{-\rho_1/2}. \tag{44}$$

Taking

$$\rho_1 = \gamma\left(\frac{\mu+\nu}{3}\right), \tag{45}$$

by (26) we have $\rho_1 \le (\mu+\nu)/6$, so that by (39), ρ_1 satisfies (34). By (32), (42) and (44) it follows that uniformly for $\vartheta \in \mathbb{R}$ we get (30). □

6. Sums of type II

We take a non-decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{\lambda\to\infty} \gamma(\lambda) = \infty$, $c \ge 10$ (this condition appears in (89)) and $f : \mathbb{N} \to \mathbb{U}$ with the carry property and lying in $\mathcal{F}_{\gamma,c}$ (see Definitions 1 and 2). Let $1 \le M \le N$ and let μ and ν be the unique integers such that

$$q^{\mu-1} \le M < q^{\mu} \quad \text{and} \quad q^{\nu-1} \le N < q^{\nu}.$$

Let us assume that

$$\tfrac{1}{4}(\mu+\nu) \le \mu \le \nu \le \tfrac{3}{4}(\mu+\nu) \tag{46}$$

(replacing $(1/4, 3/4)$ by $(\xi, 1-\xi)$ with $1/4 < \xi < 1/3$ would provide a better exponent for q in (47)). We also assume that the multiplicative dependence of the variables in type II sums has been removed by the classical method described (for example) in [24, Section 5]. Let $\vartheta \in \mathbb{R}$, $a_m, b_n \in \mathbb{C}$ with $|a_m|, |b_n| \le 1$ and

$$S_{II}(\vartheta) = \sum_m \sum_n a_m b_n f(mn)\, \mathrm{e}(\vartheta mn)$$

where we sum over $m \in (M/q, M]$ and $n \in (N/q, N]$. We will prove

Proposition 2. *Assuming (46) and $c \ge 10$ we have, uniformly for $|a_m|, |b_n| \le 1$ and $\vartheta \in \mathbb{R}$,*

$$|S_{II}(\vartheta)| \ll \max(\tau(q)\log q, \log^3 q)^{1/4} (\mu+\nu)^{\frac{1}{4}(1+\max(\omega(q),2))} q^{\mu+\nu-\gamma(2\lfloor\mu/15\rfloor)/20}. \tag{47}$$

As is often the case in this approach, getting an upper bound for sums of type II is the most difficult part. The proof is quite long and complicated, and will be developed over several sections and completed at formula (90). By the Cauchy–Schwarz inequality we have

$$|S_{II}(\vartheta)|^2 \le M \sum_m \Big| \sum_n b_n f(mn)\, \mathrm{e}(\vartheta mn) \Big|^2.$$

Let ρ be an integer such that

$$1 \le 7\rho \le \mu \tag{48}$$

and let

$$R = q^{\rho} \tag{49}$$

so that

$$1 \le R \ll N. \tag{50}$$

Applying Lemma 3 to the summation over n with $k = 1$ and then summing over m we get

$$|S_{II}(\vartheta)|^2 \ll \frac{M^2N^2}{R} + \frac{MN}{R} \sum_{1\le r<R} \left(1 - \frac{r}{R}\right) \Re(S_1(r))$$

with

$$S_1(r) = \sum_{m} \sum_{n \in I_1(N,r)} b_{n+r}\overline{b_n} f(mn + mr)\overline{f(mn)}\, \mathrm{e}(\vartheta mr),$$

where $I_1(N, r) = (N/q, N - r]$. Let

$$\mu_2 = \mu + 2\rho. \tag{51}$$

If f has the carry property (Definition 1), then by Lemma 8 the number of (m, n) for which $f(mn + mr)\overline{f(mn)} \ne f_{\mu_2}(mn + mr)\overline{f_{\mu_2}(mn)}$ is $O(q^{\mu+\nu-\rho})$. Hence

$$S_1(r) = S_1'(r) + O(q^{\mu+\nu-\rho}),$$

where

$$S_1'(r) = \sum_{m} \sum_{n \in I_1(N,r)} b_{n+r}\overline{b_n} f_{\mu_2}(mn + mr)\overline{f_{\mu_2}(mn)}\, \mathrm{e}(\vartheta mr).$$

Using again the Cauchy–Schwarz inequality for the summation over r leads to

$$|S_{II}(\vartheta)|^4 \ll \frac{M^4N^4}{R^2} + \frac{M^2N^2}{R^2} R \sum_{1\le r<R} |S_1'(r)|^2. \tag{52}$$

It remains to give an upper bound for $|S_1'(r)|^2$. We reverse the order of summation in $S_1'(r)$ and obtain

$$|S_1'(r)| \le \sum_{n \in I_1(N,r)} \left| \sum_{m} f_{\mu_2}(mn + mr)\overline{f_{\mu_2}(mn)}\, \mathrm{e}(\vartheta mr) \right|.$$

We may extend the summation over n to $(N/q, N]$ and apply the Cauchy–Schwarz inequality:

$$|S_1'(r)|^2 \ll N \sum_{N/Q<n\le N} \left| \sum_{m} f_{\mu_2}(mn + mr)\overline{f_{\mu_2}(mn)}\, \mathrm{e}(\vartheta mr) \right|^2.$$

Applying to the summation over m Lemma 3 with positive integers $k = q^{\mu_1}$ and S such that

$$1 \leq q^{\mu_1} S \ll M, \tag{53}$$

and then summing over n and r, we get

$$\sum_{1 \leq r < R} |S_1'(r)|^2 \ll \frac{M^2 N^2 R}{S} + \frac{MN}{S} \Re(S_2) \tag{54}$$

with

$$S_2 = \sum_{1 \leq r < R} \sum_{1 \leq s < S} \left(1 - \frac{s}{S}\right) \mathrm{e}(\vartheta q^{\mu_1} rs)\, S_2'(r,s)$$

and

$$S_2'(r,s) = \sum_{m \in I_2(M,s)} \sum_n f_{\mu_2}((m+sq^{\mu_1})(n+r)) \overline{f_{\mu_2}(m(n+r)) f_{\mu_2}((m+sq^{\mu_1})n)} f_{\mu_2}(mn),$$

where $I_2(M,s) = (M/q, M - sq^{\mu_1}]$.

Using (52) and (54) we obtain, uniformly for $\vartheta \in \mathbb{R}$,

$$|S_{II}(\vartheta)|^4 \ll \frac{M^4 N^4}{R^2} + \frac{M^4 N^4}{S} + \frac{M^3 N^3}{RS} \sum_{1 \leq r < R} \sum_{1 \leq s < S} |S_2'(r,s)|. \tag{55}$$

Writing $f_{\mu_2} = f_{\mu_1}(f_{\mu_2} \overline{f_{\mu_1}})$ and observing that $f_{\mu_1}((m+sq^{\mu_1})(n+r)) = f_{\mu_1}(m(n+r))$ and $f_{\mu_1}((m+sq^{\mu_1})n) = f_{\mu_1}(mn))$ we get

$$S_2'(r,s) = \sum_{m \in I_2(M,s)} \sum_n f_{\mu_1,\mu_2}(mn + mr + q^{\mu_1} sn + q^{\mu_1} rs) \overline{f_{\mu_1,\mu_2}(mn + mr)} \times \overline{f_{\mu_1,\mu_2}(mn + q^{\mu_1} sn)} f_{\mu_1,\mu_2}(mn)$$

with

$$f_{\mu_1,\mu_2} = f_{\mu_2} \overline{f_{\mu_1}}. \tag{56}$$

We take

$$\mu_1 = \mu - 2\rho, \tag{57}$$

$$S = R^2 = q^{2\rho}, \tag{58}$$

so that condition (53) is fullfilled.

For $0 \leq r < R$ and $0 \leq s < S$ and $\mu_0 \leq \mu_1$ denote by $\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s)$ the set of (m,n) with $M/q < m \leq M$ and $N/q < n \leq N$ such that

$$f_{\mu_1,\mu_2}(mn + q^{\mu_1} sn + q^{\mu_1} rs) \neq f_{\mu_1,\mu_2}(q^{\mu_0} \mathrm{r}_{\mu_0,\mu_2}(mn + q^{\mu_1} sn + q^{\mu_1} rs)).$$

The set $\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s)$ is a set of exceptions: if μ_0 is taken sufficiently small, the function f_{μ_1,μ_2} will depend on digits of indices in $\mu_0, \ldots, \mu_2 - 1$, except for (m,n) in $\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s)$. Of course if $\mu_0 = 0$ we have $\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s) = \emptyset$ but we want to choose μ_0 more carefully so that this set is still small enough. More precisely, let $\rho' \in \mathbb{N}$ (to be chosen later) be such that

$$0 \leq \rho' \leq \rho. \tag{59}$$

Since $f:\mathbb{N}\to\mathbb{U}$ has the carry property, by taking

$$\mu_0=\mu_1-2\rho' \tag{60}$$

in Lemma 9 we have

$$\operatorname{card}\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s)\ll\max(\tau(q),\log q)\,(\mu+\nu)^{\omega(q)}\,q^{\mu+\nu-2\rho'}. \tag{61}$$

Remark 2. A direct argument depending on a better knowledge of f might permit one to choose a greater value of μ_0, leading to a sharper final estimate for such a more specific function f.

For $k\in\mathbb{Z}$ we define a $q^{\mu_2-\mu_0}$-periodic function g by

$$g(k)=f_{\mu_1,\mu_2}(q^{\mu_0}k). \tag{62}$$

Write $\mathrm{r}_{\mu_0,\mu_2}(mn)=u_0$ so that

$$\mathrm{r}_{\mu_0,\mu_2}(mn+q^{\mu_1}sn)=\mathrm{r}_{\mu_0,\mu_2}(q^{\mu_0}u_0+q^{\mu_1}sn)=\mathrm{r}_{\mu_2-\mu_0}(u_0+q^{\mu_1-\mu_0}sn)$$

and

$$f_{\mu_1,\mu_2}(q^{\mu_0}\,\mathrm{r}_{\mu_0,\mu_2}(mn+q^{\mu_1}sn))=g(u_0+q^{\mu_1-\mu_0}sn).$$

Similarly if we set $\mathrm{r}_{\mu_0,\mu_2}(mn+mr)=u_1$ then we obtain

$$\begin{aligned}\mathrm{r}_{\mu_0,\mu_2}(mn+mr+q^{\mu_1}sn+q^{\mu_1}sr)&=\mathrm{r}_{\mu_0,\mu_2}(q^{\mu_0}u_1+q^{\mu_1}sn+q^{\mu_1}sr)\\&=\mathrm{r}_{\mu_2-\mu_0}(u_1+q^{\mu_1-\mu_0}sn+q^{\mu_1-\mu_0}sr)\end{aligned}$$

and

$$f_{\mu_1,\mu_2}(q^{\mu_0}\,\mathrm{r}_{\mu_0,\mu_2}(mn+mr+q^{\mu_1}sn+q^{\mu_1}sr))=g(u_1+q^{\mu_1-\mu_0}sn+q^{\mu_1-\mu_0}sr).$$

Using (61) and (11), we can write

$$S_2'(r,s)=S_3(r,s)+O(\max(\tau(q),\log q)\,(\mu+\nu)^{\omega(q)}\,q^{\mu+\nu-2\rho'}), \tag{63}$$

where

$$\begin{aligned}S_3(r,s)=&\sum_{m\in I_2(M,s)}\sum_{n}\sum_{0\le u_0<q^{\mu_2-\mu_0}}\chi_{q^{\mu_0-\mu_2}}\left(\frac{mn}{q^{\mu_2}}-\frac{u_0}{q^{\mu_2-\mu_0}}\right)\\&\times\sum_{0\le u_1<q^{\mu_2-\mu_0}}\chi_{q^{\mu_0-\mu_2}}\left(\frac{mn+mr}{q^{\mu_2}}-\frac{u_1}{q^{\mu_2-\mu_0}}\right)\\&\times g(u_1+q^{\mu_1-\mu_0}sn+q^{\mu_1-\mu_0}sr)\,\overline{g}(u_1)\,\overline{g}(u_0+q^{\mu_1-\mu_0}sn)\,g(u_0),\end{aligned}$$

with $\chi_{q^{\mu_0-\mu_2}}$ defined by (12) and $\alpha=q^{\mu_0-\mu_2}$. Let H be an integer with $q^{\mu_2-\mu_0}\le H\le q^{\mu}$, to be chosen later. Using (18) with $\alpha_1=\alpha_2=q^{\mu_0-\mu_2}$ we have

$$S_3(r,s)=S_4(r,s)+O(E_4(r,0))+O(E_4(0,r))+O(E_4'(r)), \tag{64}$$

where

$$S_4(r,s) = \sum_{m\in I_2(M,s)} \sum_n \sum_{0\le u_0<q^{\mu_2-\mu_0}} A_{q^{\mu_0-\mu_2},H}\left(\frac{mn}{q^{\mu_2}} - \frac{u_0}{q^{\mu_2-\mu_0}}\right)$$
$$\times \sum_{0\le u_1<q^{\mu_2-\mu_0}} A_{q^{\mu_0-\mu_2},H}\left(\frac{mn+mr}{q^{\mu_2}} - \frac{u_1}{q^{\mu_2-\mu_0}}\right)$$
$$\times\, g(u_1 + q^{\mu_1-\mu_0}sn + q^{\mu_1-\mu_0}sr)\, \overline{g}(u_1)\overline{g}(u_0 + q^{\mu_1-\mu_0}sn)\, g(u_0).$$

For the error terms, since $\chi_{q^{\nu_0-\nu_2}} \ge 0$ and $B_{q^{\nu_0-\nu_2},H} \ge 0$, it is possible to extend the summation over m to the full interval, removing the dependence on s:

$$E_4(r,r') = \sum_{M/q<m\le M} \sum_n \sum_{0\le u_0<q^{\mu_2-\mu_0}} B_{q^{\mu_0-\mu_2},H}\left(\frac{mn+mr}{q^{\mu_2}} - \frac{u_0}{q^{\mu_2-\mu_0}}\right)$$
$$\times \sum_{0\le u_1<q^{\mu_2-\mu_0}} \chi_{q^{\mu_0-\mu_2}}\left(\frac{mn+mr'}{q^{\mu_2}} - \frac{u_1}{q^{\mu_2-\mu_0}}\right)$$

and

$$E_4'(r) = \sum_{M/q<m\le M} \sum_n \sum_{0\le u_0<q^{\mu_2-\mu_0}} B_{q^{\mu_0-\mu_2},H}\left(\frac{mn}{q^{\mu_2}} - \frac{u_0}{q^{\mu_2-\mu_0}}\right)$$
$$\times \sum_{0\le u_1<q^{\mu_2-\mu_0}} B_{q^{\mu_0-\mu_2},H}\left(\frac{mn+mr}{q^{\mu_2}} - \frac{u_1}{q^{\mu_2-\mu_0}}\right).$$

6.1. Estimate of $E_4(r,r')$

Since

$$\sum_{0\le u_1<q^{\mu_2-\mu_0}} \chi_{q^{\mu_0-\mu_2}}\left(\frac{mn+mr'}{q^{\mu_2}} - \frac{u_1}{q^{\mu_2-\mu_0}}\right) = 1,$$

we have

$$E_4(r,r') = \sum_m \sum_n \sum_{0\le u_0<q^{\mu_2-\mu_0}} B_{q^{\mu_0-\mu_2},H}\left(\frac{mn+mr}{q^{\mu_2}} - \frac{u_0}{q^{\mu_2-\mu_0}}\right),$$

which by (14) gives

$$E_4(r,r') = \sum_{|h_0|\le H} b_{h_0}(q^{\mu_0-\mu_2},H) \sum_m \sum_n \sum_{0\le u_0<q^{\mu_2-\mu_0}} \mathrm{e}\left(h_0\frac{mn+mr}{q^{\mu_2}} - h_0\frac{u_0}{q^{\mu_2-\mu_0}}\right).$$

By (15) we have $|b_{h_0}(q^{\mu_0-\mu_2},H)| \le 1/H$. Using

$$\sum_{0\le u_0<q^{\mu_2-\mu_0}} \mathrm{e}\left(\frac{-h_0u_0}{q^{\mu_2-\mu_0}}\right) = \begin{cases} q^{\mu_2-\mu_0} & \text{if } h_0 \equiv 0 \bmod q^{\mu_2-\mu_0},\\ 0 & \text{if } h_0 \not\equiv 0 \bmod q^{\mu_2-\mu_0},\end{cases}$$

and writing $h_0 = h'_0 q^{\mu_2-\mu_0}$ we get

$$|E_4(r, r')| \ll E_5 \tag{65}$$

with

$$E_5 = \frac{q^{\mu_2-\mu_0}}{H} \sum_{|h'_0|\le H/q^{\mu_2-\mu_0}} \sum_m \left|\sum_n \mathrm{e}\left(\frac{h'_0 mn}{q^{\mu_0}}\right)\right|. \tag{66}$$

After summation over n, we have

$$E_5 \ll \frac{q^{\mu_2-\mu_0}}{H} \sum_{|h'|\le H/q^{\mu_2-\mu_0}} \sum_m \min\left(N, \left|\sin\pi \frac{h'm}{q^{\mu_0}}\right|^{-1}\right).$$

The summation over m runs over at most $q^{\mu-\mu_0}$ periods modulo q^{μ_0}, hence

$$E_5 \ll q^{\mu-\mu_0} \frac{q^{\mu_2-\mu_0}}{H} \sum_{|h'|\le H/q^{\mu_2-\mu_0}} \sum_{0\le m'<q^{\mu_0}} \min\left(N, \left|\sin\pi \frac{h'm'}{q^{\mu_0}}\right|^{-1}\right).$$

Using the trivial estimate for $h' = 0$ and (22) when $h' \ne 0$, we obtain

$$E_5 \ll q^{\mu+\nu} \frac{q^{\mu_2-\mu_0}}{H} + q^{\mu-\mu_0}(\tau(q^{\mu_0})N + q^{\mu_0}\log q^{\mu_0}).$$

Choosing

$$H = q^{\mu_2-\mu_0+2\rho}, \tag{67}$$

this gives

$$E_5 \ll q^{\mu+\nu-2\rho} + q^{\mu+\nu-\mu_0}\tau(q^{\mu_0}) + q^{\mu}\log q^{\mu_0}.$$

By (48), (60) and (46), we have $\mu_0 \ge \mu - 4\rho \ge 2\rho$ and $\nu \ge 2\rho$ so that

$$E_5 \ll \max(\log q^{\mu_0}, \tau(q^{\mu_0}))q^{\mu+\nu-2\rho}. \tag{68}$$

6.2. Estimate of $E'_4(r)$

We have

$$\begin{aligned} E'_4(r) = &\sum_{|h_0|\le H} \sum_{|h_1|\le H} b_{h_0}(q^{\mu_0-\mu_2}, H) b_{h_1}(q^{\mu_0-\mu_2}, H) \\ &\times \sum_m \sum_n \sum_{\substack{0\le u_0<q^{\mu_2-\mu_0}\\ 0\le u_1<q^{\mu_2-\mu_0}}} \mathrm{e}\left(h_0\frac{mn}{q^{\mu_2}} - h_0\frac{u_0}{q^{\mu_2-\mu_0}}\right) \mathrm{e}\left(h_1\frac{mn+mr}{q^{\mu_2}} - h_1\frac{u_1}{q^{\mu_2-\mu_0}}\right). \end{aligned}$$

We observe that for $h_0 \not\equiv 0 \bmod q^{\mu_2-\mu_0}$ we have $\sum_{0\le u_0<q^{\mu_2-\mu_0}} \mathrm{e}\left(-h_0 \frac{u_0}{q^{\mu_2-\mu_0}}\right) = 0$, and similarly for h_1. Hence we may assume $h_0 \equiv h_1 \equiv 0 \bmod q^{\mu_2-\mu_0}$. Writing

$h_0 = h'_0 q^{\mu_2-\mu_0}$ and $h_1 = h'_1 q^{\mu_2-\mu_0}$ and using the upper bound $|b_h(q^{\mu_0-\mu_2}, H)| \le 1/H$ from (15), we get

$$|E'_4(r)| \ll \frac{q^{2(\mu_2-\mu_0)}}{H^2} \sum_{|h'_0|\le H/q^{\mu_2-\mu_0}} \sum_{|h'_1|\le H/q^{\mu_2-\mu_0}} \left| \sum_m \sum_n \mathrm{e}\left(\frac{(h'_0+h'_1)mn + h'_1 mr}{q^{\mu_0}} \right) \right|.$$

The contribution to $E'_4(r)$ of the terms for which $h'_0 + h'_1 = 0$, after summation over m, is bounded by

$$\frac{q^{2(\mu_2-\mu_0)}}{H^2} N \sum_{|h'_1|\le H/q^{\mu_2-\mu_0}} \min\left(M, \left| \sin \pi \frac{h'_1 r}{q^{\mu_0}} \right|^{-1} \right).$$

Since $1 \le r < q^\rho$ and $H \le q^\mu$, we have $|h'_1 r| < q^{\mu-\mu_2+\mu_0+\rho} = q^{\mu_0-\rho}$ (by (51)) so that the values of $h'_1 r$ are all distinct modulo q^{μ_0}. Therefore we conclude that the contribution to $E'_4(r)$ of the terms for which $h'_0 + h'_1 = 0$ is bounded by

$$\frac{q^{2(\mu_2-\mu_0)}}{H^2} N(M + q^{\mu_0} \log q^{\mu_0}) \ll \frac{q^{2(\mu_2-\mu_0)}}{H^2} q^{\mu+\nu}(1 + q^{\mu_0-\mu} \log q^{\mu_0}).$$

The contribution to $E'_4(r)$ of the terms for which $h'_0 + h'_1 \ne 0$, after summation over n, is bounded by

$$\frac{q^{2(\mu_2-\mu_0)}}{H^2} \sum_{h'_0+h'_1\ne 0} \sum_m \min\left(N, \left| \sin \pi \frac{(h'_0+h'_1)m}{q^{\mu_0}} \right|^{-1} \right),$$

which, writing $h' = h'_0 + h'_1$, is less than

$$\frac{q^{\mu_2-\mu_0}}{H} \sum_{1\le|h'|\le 2H/q^{\mu_2-\mu_0}} \sum_m \min\left(N, \left| \sin \pi \frac{h'm}{q^{\mu_0}} \right|^{-1} \right).$$

The summation over m runs over at most $q^{\mu-\mu_0}$ periods modulo q^{μ_0}, which gives the upper bound

$$q^{\mu-\mu_0} \frac{q^{\mu_2-\mu_0}}{H} \sum_{1\le h'\le 2H/q^{\mu_2-\mu_0}} \sum_{0\le m'<q^{\mu_0}} \min\left(N, \left| \sin \pi \frac{h'm'}{q^{\mu_0}} \right|^{-1} \right).$$

Using (22) we find that this is bounded by

$$q^{\mu-\mu_0}(\tau(q^{\mu_0})N + q^{\mu_0} \log q^{\mu_0}) \ll q^{\nu+\mu-\mu_0}\tau(q^{\mu_0}) + q^\mu \log q^{\mu_0}.$$

We conclude that

$$|E'_4(r)| \ll \frac{q^{2(\mu_2-\mu_0)}}{H^2} q^{\mu+\nu}(1 + q^{\mu_0-\mu} \log q^{\mu_0}) + q^{\nu+\mu-\mu_0}\tau(q^{\mu_0}) + q^\mu \log q^{\mu_0}.$$

With the choice (67) and by (48) and (60), we have $\mu_0 \ge \mu - 4\rho \ge 2\rho$, so that

$$|E'_4(r)| \ll \max(\log q^{\mu_0}, \tau(q^{\mu_0}))q^{\mu+\nu-2\rho}. \tag{69}$$

By (64), (65), (68) and (69) we have

$$S_3(r,s) = S_4(r,s) + O(\max(\log q^{\mu_0}, \tau(q^{\mu_0}))q^{\mu+\nu-2\rho}). \tag{70}$$

6.3. Fourier analysis of $S_4(r,s)$

We write

$$u_0 + q^{\mu_1-\mu_0} sn \equiv u_2 \bmod q^{\mu_2-\mu_0}$$

and

$$u_1 + q^{\mu_1-\mu_0} sn + q^{\mu_1-\mu_0} sr \equiv u_3 \bmod q^{\mu_2-\mu_0}.$$

This gives

$$\begin{aligned} S_4(r,s) = &\sum_{|h_0|\le H} \sum_{|h_1|\le H} a_{h_0}(q^{\mu_0-\mu_2}, H) a_{h_1}(q^{\mu_0-\mu_2}, H) \frac{1}{q^{2(\mu_2-\mu_0)}} \\ &\times \sum_{\substack{0\le h_2<q^{\mu_2-\mu_0}\\ 0\le h_3<q^{\mu_2-\mu_0}}} \sum_{\substack{0\le u_0<q^{\mu_2-\mu_0}\\ 0\le u_1<q^{\mu_2-\mu_0}}} \mathrm{e}\left(-\frac{h_0 u_0}{q^{\mu_2-\mu_0}}\right) \mathrm{e}\left(-\frac{h_1 u_1}{q^{\mu_2-\mu_0}}\right) \overline{g}(u_1) g(u_0) \\ &\times \sum_{\substack{0\le u_2<q^{\mu_2-\mu_0}\\ 0\le u_3<q^{\mu_2-\mu_0}}} g(u_3)\overline{g}(u_2) \\ &\times \sum_{m\in I_2(M,s)} \sum_n \mathrm{e}\left(\frac{h_0 mn + h_1 mn + h_1 mr}{q^{\mu_2}}\right) \mathrm{e}\left(h_2 \frac{u_0 + q^{\mu_1-\mu_0} sn - u_2}{q^{\mu_2-\mu_0}}\right) \\ &\times \mathrm{e}\left(h_3 \frac{u_1 + q^{\mu_1-\mu_0} sn + q^{\mu_1-\mu_0} sr - u_3}{q^{\mu_2-\mu_0}}\right). \end{aligned}$$

The discrete Fourier transform of g defined by (62) is

$$\widehat{g}(h) = \frac{1}{q^{\mu_2-\mu_0}} \sum_{0\le u<q^{\mu_2-\mu_0}} g(u)\, \mathrm{e}\left(\frac{-uh}{q^{\mu_2-\mu_0}}\right), \tag{71}$$

so that

$$\begin{aligned} S_4(r,s) = q^{2(\mu_2-\mu_0)} &\sum_{|h_0|\le H} \sum_{|h_1|\le H} a_{h_0}(q^{\mu_0-\mu_2}, H) a_{h_1}(q^{\mu_0-\mu_2}, H) \\ &\times \sum_{\substack{0\le h_2<q^{\mu_2-\mu_0}\\ 0\le h_3<q^{\mu_2-\mu_0}}} \mathrm{e}\left(\frac{h_3 sr}{q^{\mu_2-\mu_1}}\right) \widehat{g}(h_0-h_2)\, \overline{\widehat{g}(h_3-h_1)}\, \overline{\widehat{g}(-h_2)}\, \widehat{g}(h_3) \\ &\times \sum_{m\in I_2(M,s)} \sum_n \mathrm{e}\left(\frac{(h_0+h_1)mn + h_1 mr + (h_2+h_3) q^{\mu_1} sn}{q^{\mu_2}}\right). \end{aligned}$$

6.4. Estimate of $S_4(r,s)$

We write

$$S_4(r,s) = S_4'(r,s) + S_4''(r,s), \tag{72}$$

where $S_4'(r,s)$ denotes the contribution of the terms for which $h_0+h_1=0$, while $S_4''(r,s)$ denotes the contribution of the terms for which $h_0+h_1\neq 0$.

6.4.1. Contribution of $S_4'(r,s)$. Since $h_0+h_1=0$, the summations over m and n are independent. Noticing that by (49), (67), (60) and (48) we have $|h_1 r| \le HR = q^{\mu_2-\mu_0+3\rho} \le q^{\mu_2}/2$, we deduce

$$\left|\sum_{m\in I_2(M,s)} \mathrm{e}\left(\frac{h_1 m r}{q^{\mu_2}}\right)\right| \le \min\left(q^{\mu}, \left|\sin\frac{\pi h_1 r}{q^{\mu_2}}\right|^{-1}\right) \le \min\left(q^{\mu}, \frac{q^{\mu_2}}{r|h_1|}\right)$$

and

$$\left|\sum_{n} \mathrm{e}\left(\frac{(h_2+h_3)q^{\mu_1} s n}{q^{\mu_2}}\right)\right| \le \min\left(q^{\nu}, \left|\sin\frac{\pi(h_2+h_3)s}{q^{\mu_2-\mu_1}}\right|^{-1}\right).$$

Using the periodicity of $\widehat{g}$ modulo $q^{\mu_2-\mu_0}$ and writing $h=h_2+h_3$, we get

$$|S_4'(r,s)| \le S_5(r,s) \tag{73}$$

with

$$S_5(r,s) = q^{2(\mu_2-\mu_0)} \sum_{|h_1|\le H} |a_{h_1}(q^{\mu_0-\mu_2}, H)|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r|h_1|}\right) \times \sum_{0\le h<q^{\mu_2-\mu_0}} \min\left(q^{\nu}, \left|\sin\frac{\pi h s}{q^{\mu_2-\mu_1}}\right|^{-1}\right) S_6(h,h_1)$$

$$S_6(h,h_1) = \sum_{0\le h_3<q^{\mu_2-\mu_0}} |\widehat{g}(h_3-h_1-h)\widehat{g}(h_3-h_1)\widehat{g}(h_3-h)\widehat{g}(h_3)|.$$

By using the Cauchy–Schwarz inequality we have

$$S_6(h,h_1) \le \left(\sum_{h_3} |\widehat{g}(h_3-h_1-h)\widehat{g}(h_3-h)|^2\right)^{1/2} \left(\sum_{h_3} |\widehat{g}(h_3-h_1)\widehat{g}(h_3)|^2\right)^{1/2}.$$

The two quantities in the parentheses above are equal by periodicity modulo $q^{\mu_2-\mu_0}$, hence

$$S_6(h,h_1) \le S_7(h_1)$$

with

$$S_7(h_1) = \sum_{0\le h'<q^{\mu_2-\mu_0}} |\widehat{g}(h'-h_1)\widehat{g}(h')|^2. \tag{74}$$

By periodicity modulo $q^{\mu_2-\mu_1}$ and (22) we have

$$\begin{aligned}\frac{1}{S}\sum_{1\le s<S}\sum_{0\le h<q^{\mu_2-\mu_0}} &\min\left(q^{\nu}, \left|\sin\frac{\pi h s}{q^{\mu_2-\mu_1}}\right|^{-1}\right)\\ &= \frac{q^{\mu_1-\mu_0}}{S}\sum_{1\le s<S}\sum_{0\le h<q^{\mu_2-\mu_1}} \min\left(q^{\nu}, \left|\sin\frac{\pi h s}{q^{\mu_2-\mu_1}}\right|^{-1}\right)\\ &\ll q^{\mu_1-\mu_0}(q^{\nu}\tau(q^{\mu_2-\mu_1}) + q^{\mu_2-\mu_1}\log q^{\mu_2-\mu_1}).\end{aligned}$$

Hence

$$\frac{1}{S}\sum_{1\le s<S} S_5(r,s) \ll q^{\nu+\mu_1-\mu_0}(\tau(q^{\mu_2-\mu_1}) + q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1})S_8(r) \tag{75}$$

with

$$S_8(r) = q^{2(\mu_2-\mu_0)} \sum_{|h_1|\le H} |a_{h_1}(q^{\mu_0-\mu_2},H)|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r|h_1|}\right) S_7(h_1).$$

Taking (67) into account we split the summation $S_8(r)$ into three parts,

$$S_8(r) = S_8'(r) + S_8''(r) + S_8'''(r), \tag{76}$$

depending on the size of $|h_1|$: $|h_1| \le q^{2\rho}$, $q^{2\rho} < |h_1| \le q^{\mu_2-\mu_0}$ and $q^{\mu_2-\mu_0} < |h_1| \le H$. Using (15) in $S_8'(r)$ we have $|a_{h_1}(q^{\mu_0-\mu_2},H)| \le q^{\mu_0-\mu_2}$, thus

$$\begin{aligned} S_8'(r) &= q^{2(\mu_2-\mu_0)} \sum_{|h_1|\le q^{2\rho}} |a_{h_1}(q^{\mu_0-\mu_2},H)|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r|h_1|}\right) S_7(h_1) \\ &\le q^{\mu} \sum_{|h_1|\le q^{2\rho}} S_7(h_1). \end{aligned}$$

In order to prove that this short sum over h_1 is small, we will assume in this section that the following lemma holds:

Lemma 10. *If $c>0$ is the constant introduced in Definition 2 and*

$$\mu \le \left(2+\tfrac{4}{3}c\right)\rho, \tag{77}$$

then uniformly for $\lambda\in\mathbb{N}$ with $\frac{1}{3}(\mu_2-\mu_0) \le \lambda \le \frac{4}{5}(\mu_2-\mu_0)$ we have

$$\sum_{0\le h<q^{\mu_2-\mu_0}} \sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} |\widehat{g}(h+k)\widehat{g}(h)|^2 \ll q^{-\gamma_1(\lambda,\mu_1-\mu_0)}(\log q^{\mu_2-\mu_1})^2, \tag{78}$$

where

$$\gamma_1(\lambda,\mu_1-\mu_0) = (\gamma(\lambda)-\mu_1+\mu_0)/2. \tag{79}$$

Proof. Lemma 10 will be proved in Section 7. □

By (78), (51), (57), (60), (59) we have $\mu_2-\mu_0 \le 6\rho$, so that $\mu_2-\mu_0-2\rho \le \frac{2}{3}(\mu_2-\mu_0) \le \frac{4}{5}(\mu_2-\mu_0)$ and

$$\sum_{|h_1|\le q^{2\rho}} S_7(h_1) \ll q^{-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)},$$

hence

$$S_8'(r) \ll q^{\mu-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)}. \tag{80}$$

Using (15) in $S_8''(r)$ we have $|a_{h_1}(q^{\mu_0-\mu_2}, H)| \le q^{\mu_0-\mu_2}$, thus

$$\begin{aligned} S_8''(r) &= q^{2(\mu_2-\mu_0)} \sum_{q^{2\rho}<|h_1|\le q^{\mu_2-\mu_0}} |a_{h_1}(q^{\mu_0-\mu_2}, H)|^2 \min\Big(q^{\mu}, \frac{q^{\mu_2}}{r|h_1|}\Big) S_7(h_1) \\ &\le \frac{q^{\mu_2}}{r} \sum_{q^{2\rho}<|h_1|\le q^{\mu_2-\mu_0}} \frac{S_7(h_1)}{|h_1|} \le \frac{q^{\mu_2-2\rho}}{r} \sum_{|h_1|\le q^{\mu_2-\mu_0}} S_7(h_1). \end{aligned}$$

As by (25) we have

$$\sum_{0\le h<q^{\mu_2-\mu_0}} |\widehat{g}(h)|^2 = 1, \tag{81}$$

we obtain $S_8''(r) \le q^{\mu_2-2\rho}/r$, hence using (51) and (49), we get

$$\frac{1}{R}\sum_{1\le r<R} S_8''(r) \ll q^{\mu_2-2\rho}\frac{\log R}{R} = q^{\mu-\rho}\log q^{\rho}. \tag{82}$$

Using (15) in $S_8'''(r)$ we have $|a_{h_1}(q^{\mu_0-\mu_2}, H)| \le \frac{1}{\pi|h_1|}$, thus

$$\begin{aligned} S_8'''(r) &= q^{2(\mu_2-\mu_0)} \sum_{q^{\mu_2-\mu_0}<|h_1|\le H} |a_{h_1}(q^{\mu_0-\mu_2}, H)|^2 \min\Big(q^{\mu}, \frac{q^{\mu_2}}{r|h_1|}\Big) S_7(h_1) \\ &\ll q^{2(\mu_2-\mu_0)}\frac{q^{\mu_2}}{r} \sum_{q^{\mu_2-\mu_0}<|h_1|\le H} \frac{S_7(h_1)}{|h_1|^3}. \end{aligned}$$

Observing that $S_7(h_1)$ is $q^{\mu_2-\mu_0}$-periodic, we split the summation into $jq^{\mu_2-\mu_0} < |h_1| \le (j+1)q^{\mu_2-\mu_0}$ where $1 \le j < H/q^{\mu_2-\mu_0}$ and bound $|h_1|^{-3}$ by $j^{-3}q^{-3(\mu_2-\mu_0)}$:

$$S_8'''(r) \ll q^{2(\mu_2-\mu_0)}\frac{q^{\mu_2}}{r}\sum_{j\ge 1}\frac{1}{j^3 q^{3(\mu_2-\mu_0)}} \sum_{0\le h_1<q^{\mu_2-\mu_0}} S_7(h_1), \tag{83}$$

thus by (81) and (57),

$$S_8'''(r) \ll q^{-(\mu_2-\mu_0)}\frac{q^{\mu_2}}{r} = \frac{q^{\mu_0}}{r} \le \frac{q^{\mu-2\rho}}{r}.$$

It follows from (76), (80), (82), (83) that

$$\frac{1}{R}\sum_{1\le r<R} S_8(r) \ll q^{\mu-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)} + q^{\mu-\rho}\log q^{\rho},$$

and using (75) we obtain

$$\begin{aligned} \frac{1}{RS}\sum_{1\le r<R}\sum_{1\le s<S} S_5(r,s) &\ll q^{\nu+\mu_1-\mu_0}(\tau(q^{\mu_2-\mu_1}) + q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1}) \\ &\quad \times (q^{\mu-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)} + q^{\mu-\rho}\log q^{\rho}). \end{aligned}$$

Finally by (73) we get

$$\frac{1}{RS}\sum_{1\le r<R}\sum_{1\le s<S}|S_4'(r,s)| \ll q^{\mu+\nu+\mu_1-\mu_0}(q^{-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)}+q^{-\rho}\log q^{\rho}) \\ \times(\tau(q^{\mu_2-\mu_1})+q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1}). \tag{84}$$

6.4.2. Contribution of $S_4''(r,s)$. Since $h_0+h_1\neq 0$, after a summation over m, we get

$$S_4''(r,s)\ll q^{2(\mu_2-\mu_0)}\sum_{|h_0|\le H}\sum_{h_1\neq -h_0}|a_{h_0}(q^{\mu_0-\mu_2},H)a_{h_1}(q^{\mu_0-\mu_2},H)| \\ \times\sum_{0\le h_2<q^{\mu_2-\mu_0}}|\widehat{g}(h_0-h_2)\widehat{g}(-h_2)|\sum_{0\le h_3<q^{\mu_2-\mu_0}}|\widehat{g}(h_3-h_1)\widehat{g}(h_3)| \\ \times\sum_n\min\left(q^{\mu},\left|\sin\pi\frac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right).$$

Using (21) we have

$$\sum_n\min\left(q^{\mu},\left|\sin\pi\frac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right) \\ \ll\lceil q^{\nu-\mu_2}\rceil((h_0+h_1,q^{\mu_2})q^{\mu}+q^{\mu_2}\log q^{\mu_2}),$$

and observing that $|h_0+h_1|\le 2H$ we get

$$\sum_n\min\left(q^{\mu},\left|\sin\pi\frac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right)\ll\lceil q^{\nu-\mu_2}\rceil(Hq^{\mu}+q^{\mu_2}\log q^{\mu_2}).$$

By (67) we have $Hq^{\mu}\ge q^{\mu+\mu_2-\mu_0}\ge q^{\mu_2}$, so that

$$\sum_n\min\left(q^{\mu},\left|\sin\pi\frac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right)\ll\lceil q^{\nu-\mu_2}\rceil H\,q^{\mu}\log q^{\mu_2}.$$

Moreover by the Cauchy–Schwarz inequality and (81),

$$\sum_{0\le h_2<q^{\mu_2-\mu_0}}|\widehat{g}(h_0-h_2)\widehat{g}(-h_2)|\le\Big(\sum_{h_2}|\widehat{g}(h_0-h_2)|^2\Big)^{1/2}\Big(\sum_{h_2}|\widehat{g}(-h_2)|^2\Big)^{1/2}=1,$$

and similarly

$$\sum_{0\le h_3<q^{\mu_2-\mu_0}}|\widehat{g}(h_3-h_1)\widehat{g}(h_3)|\le 1.$$

Furthermore

$$\sum_{|h|\le H}|a_h(q^{\mu_0-\mu_2},H)|\le\sum_{|h|\le q^{\mu_2-\mu_0}}\frac{1}{q^{\mu_2-\mu_0}}+\sum_{q^{\mu_2-\mu_0}<|h|\le H}\frac{1}{\pi|h|} \\ \ll\log(H/q^{\mu_2-\mu_0})\ll\log q^{\rho}.$$

Finally we obtain

$$|S_4''(r,s)| \ll (\log q)^2 \rho^2 q^{2(\mu_2-\mu_0)} \lceil q^{\nu-\mu_2} \rceil H\, q^{\mu} \log q^{\mu_2},$$

which gives, with the choice of H defined by (67),

$$|S_4''(r,s)| \ll (\log q)^3 (\mu+\nu)^3 q^{\mu+\nu+3(\mu_2-\mu_0)+2\rho}\, (q^{-\mu_2} + q^{-\nu}). \tag{85}$$

6.5. *End of the estimate of sums of type II*

By (55), (63), (70), (72), (84) and (85), for any function f with the carry property and lying in $\mathcal{F}_{\gamma,c}$ with $c \geq 10$, from (77) and (78) we obtain, uniformly for $\vartheta \in \mathbb{R}$,

$$\begin{aligned} |S_{II}(\vartheta)|^4 \ll\ & q^{4\mu+4\nu+\mu_1-\mu_0} (q^{\frac{1}{2}(\mu_1-\mu_0-\gamma(\mu_2-\mu_0-2\rho))} + q^{-\rho} \log q^{\rho}) \\ & \qquad (\tau(q^{\mu_2-\mu_1}) + q^{\mu_2-\mu_1-\nu} \log q^{\mu_2-\mu_1}) \\ & + (\log q)^3 (\mu+\nu)^3 q^{4\mu+4\nu+3(\mu_2-\mu_0)+2\rho} (q^{-\mu_2} + q^{-\nu}) \\ & + \max(\log q^{\mu_0}, \tau(q^{\mu_0})) q^{4\mu+4\nu-2\rho} \\ & + \max(\tau(q), \log q)(\mu+\nu)^{\omega(q)} q^{4\mu+4\nu-2\rho'}. \end{aligned} \tag{86}$$

By (51), (57), (60), (59), (26) and since the function γ is non-decreasing, we see that $\gamma(\mu_2-\mu_0-2\rho) \geq \gamma(\mu_2-\mu_1-2\rho) = \gamma(2\rho)$ and $\rho \geq \gamma(2\rho)$. By multiplicativity of τ we have $\tau(q^{\mu_2-\mu_1}) \leq (\mu_2-\mu_1)^{\omega(q)} \tau(q)$. By (51), (57), (46) and (48) we have $\mu_2 - \mu_1 = 4\rho \leq \mu - 2\rho \leq \nu - 2\rho$ so that $q^{\mu_2-\mu_1-\nu} \log q^{\mu_2-\mu_1} \ll 1$. This implies

$$\begin{aligned} |S_{II}(\vartheta)|^4 \ll\ & \tau(q)(\mu_2-\mu_1)^{\omega(q)} q^{4\mu+4\nu+\frac{3}{2}(\mu_1-\mu_0)-\frac{1}{2}\gamma(2\rho)} \log q^{\rho} \\ & + (\log q)^3 (\mu+\nu)^3 q^{4\mu+4\nu+3(\mu_1-\mu_0)+14\rho-\mu} \\ & + \max(\log q^{\mu_0}, \tau(q^{\mu_0})) q^{4\mu+4\nu-2\rho} \\ & + \max(\tau(q), \log q)(\mu+\nu)^{\omega(q)} q^{4\mu+4\nu-2\rho'}. \end{aligned}$$

Taking

$$\rho' = \lfloor \gamma(2\rho)/10 \rfloor, \tag{87}$$

we have (59) by (26) and by (60) $\mu_1 - \mu_0 \leq 2\rho' \leq \gamma(2\rho)/5 \leq \rho/5$ so that

$$\frac{3}{2}(\mu_1-\mu_0) - \frac{\gamma(2\rho)}{2} \leq \frac{3\gamma(2\rho)}{10} - \frac{\gamma(2\rho)}{2} \leq \frac{-\gamma(2\rho)}{5}.$$

Choosing

$$\rho = \lfloor \mu/15 \rfloor \tag{88}$$

we get, for $\mu \geq 15 \times 75 = 1125$,

$$3(\mu_1-\mu_0) + 14\rho - \mu \leq 3\rho/5 + 14\rho - 15\rho + 15 \leq -\rho/5.$$

In order to ensure (77) it is sufficient to check that

$$\mu \leq (2 + 4c/3)(\mu/15 - 1),$$

which is true for μ large enough provided $c > 39/4$. For convenience and in order to avoid that the implied constants depend on c we take

$$c \geq 10, \tag{89}$$

so that the inequality above is valid for $\mu \geq 46 \times 15 = 690$. Finally we obtain

$$|S_{II}(\vartheta)|^4 \ll \max(\tau(q)\log q, \log^3 q)(\mu+\nu)^{1+\max(\omega(q),2)} q^{4\mu+4\nu-\gamma(2\lfloor \mu/15\rfloor)/5}, \tag{90}$$

which completes the proof of (47) and Proposition 2.

It remains to prove Lemma 10, which is the goal of Section 7.

7. Distribution of the discrete Fourier transform

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function satisfying $\lim_{\lambda\to\infty}\gamma(\lambda) = \infty$, and let $f : \mathbb{N} \to \mathbb{U}$ be a function with the carry property and lying in $\mathcal{F}_{\gamma,c}$ for some $c \geq 10$ (see Definitions 1 and 2). The discrete Fourier transform of the q^λ-periodic function $u \mapsto f_{\mu_1,\mu_2}(\mathrm{r}_\lambda(u)q^{\mu_0})$ is a q^λ-periodic function $G_{\mu_0,\lambda}$ defined for $t \in \mathbb{R}$ by

$$G_{\mu_0,\lambda}(t) = \frac{1}{q^\lambda} \sum_{0\leq u<q^\lambda} f_{\mu_1,\mu_2}(uq^{\mu_0})\,\mathrm{e}\left(-\frac{ut}{q^\lambda}\right). \tag{91}$$

By (25) we have, for any $t \in \mathbb{R}$ and $\lambda \in \mathbb{N}$,

$$\sum_{0\leq h<q^\lambda} |G_{\mu_0,\lambda}(h+t)|^2 = 1, \tag{92}$$

and by (62) and (71), for $h \in \mathbb{Z}$ we have

$$\widehat{g}(h) = G_{\mu_0,\mu_2-\mu_0}(h). \tag{93}$$

To prove Lemma 10 we will need the following:

Lemma 11. *If μ and ρ satisfy (77) then we have, uniformly for $\lambda \in \mathbb{N}$ with*

$$\tfrac{1}{3}(\mu_2-\mu_0) \leq \lambda \leq \tfrac{4}{5}(\mu_2-\mu_0) \tag{94}$$

and uniformly for $t \in \mathbb{R}$,

$$\sum_{0\leq k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0}(k+t)|^2 \ll q^{\frac{1}{2}(\mu_1-\mu_0-\gamma(\lambda))}(\log q^{\mu_2-\mu_1})^2.$$

Proof. For $0 \leq \lambda \leq \mu_2 - \mu_0$ and $t \in \mathbb{R}$ we can write

$$G_{\mu_0,\mu_2-\mu_0}(t) = \frac{1}{q^{\mu_2-\mu_0}} \sum_{0\leq u<q^\lambda} \sum_{0\leq v<q^{\mu_2-\mu_0-\lambda}} f_{\mu_1,\mu_2}((u+vq^\lambda)q^{\mu_0})\, \mathrm{e}\left(-\frac{(u+vq^\lambda)t}{q^{\mu_2-\mu_0}}\right).$$

Hence for $\mu_1 - \mu_0 \leq \lambda \leq \mu_2 - \mu_0$, observing that $0 \leq u + vq^\lambda < q^{\mu_2-\mu_0}$ and $(u+vq^\lambda)q^{\mu_0} \equiv uq^{\mu_0} \bmod q^{\mu_1}$, using (56) we get, for $0 \leq u < q^\lambda$ and $0 \leq v < q^{\mu_2-\mu_0-\lambda}$,

$$\begin{aligned} f_{\mu_1,\mu_2}((u+vq^\lambda)q^{\mu_0}) &= f_{\mu_2}((u+vq^\lambda)q^{\mu_0})\overline{f_{\mu_1}((u+vq^\lambda)q^{\mu_0})} \\ &= f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_1}(uq^{\mu_0})}, \end{aligned}$$

and this yields

$$\begin{aligned} G_{\mu_0,\mu_2-\mu_0}(t) = \frac{1}{q^{\mu_2-\mu_0-\lambda}} &\sum_{0\leq v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\, \mathrm{e}\left(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}\right) \\ &\times \frac{1}{q^\lambda} \sum_{0\leq u<q^\lambda} f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f(vq^{\mu_0+\lambda})}\,\overline{f_{\mu_1}(uq^{\mu_0})}\, \mathrm{e}\left(-\frac{ut}{q^{\mu_2-\mu_0}}\right). \end{aligned}$$

Given $\rho_3 \in \mathbb{N}$ such that

$$1 \leq \rho_3 \leq \mu_2 - \mu_0 - \lambda, \tag{95}$$

by Definition 1 the number of $v \in \{0, \ldots, q^{\mu_2-\mu_0-\lambda} - 1\}$ such that there exists u in $\{0, \ldots, q^\lambda - 1\}$ for which

$$f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f(vq^{\mu_0+\lambda})} \neq f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(vq^{\mu_0+\lambda})}$$

is $O(q^{\mu_2-\mu_0-\lambda-\rho_3})$. Hence the set $\widetilde{\mathcal{W}}_\lambda$ of (u, v) with this property satisfies

$$\operatorname{card} \widetilde{\mathcal{W}}_\lambda \ll q^{\mu_2-\mu_0-\rho_3}. \tag{96}$$

Thus for any $t \in \mathbb{R}$ we can write

$$G_{\mu_0,\mu_2-\mu_0}(t) = G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t) + G_{\mu_0,\mu_2-\mu_0,\lambda,2}(t)$$

with

$$\begin{aligned} G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t) &= \frac{1}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\leq v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\, \mathrm{e}\left(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}\right) \\ \times \frac{1}{q^\lambda} \sum_{0\leq u<q^\lambda} & f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(vq^{\mu_0+\lambda})}\,\overline{f_{\mu_1}(uq^{\mu_0})}\, \mathrm{e}\left(-\frac{ut}{q^{\mu_2-\mu_0}}\right), \end{aligned}$$

$$G_{\mu_0,\mu_2-\mu_0,\lambda,2}(t) = \frac{1}{q^{\mu_2-\mu_0}} \sum_{(u,v)\in\widetilde{\mathcal{W}}_\lambda} f(vq^{\mu_0+\lambda})\overline{f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\Big(-\frac{(u+vq^\lambda)t}{q^{\mu_2-\mu_0}}\Big)$$
$$\times\big(f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f(vq^{\mu_0+\lambda})} - f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(vq^{\mu_0+\lambda})}\big).$$

Let us introduce in $G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)$ the residue w of v modulo q^{ρ_3} in order to make the variables u and v independent:

$$G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)$$
$$= \sum_{0\le w<q^{\rho_3}} \frac{1}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\Big(\frac{-vt}{q^{\mu_2-\mu_0-\lambda}}\Big)\frac{1}{q^{\rho_3}}\sum_{0\le \ell<q^{\rho_3}}\mathrm{e}\Big(\ell\frac{v-w}{q^{\rho_3}}\Big)$$
$$\times\frac{1}{q^\lambda}\sum_{0\le u<q^\lambda} f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+wq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(wq^{\mu_0+\lambda})}\,\overline{f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\Big(\frac{-ut}{q^{\mu_2-\mu_0}}\Big).$$

This gives

$$G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t) = \sum_{0\le \ell<q^{\rho_3}} \frac{\widetilde{c}_\ell(t)}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\Big(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}+\frac{v\ell}{q^{\rho_3}}\Big)$$

with

$$\widetilde{c}_\ell(t) = \frac{1}{q^{\rho_3}}\sum_{0\le w<q^{\rho_3}} c_\lambda(w,t)\,\mathrm{e}\Big(-\frac{w\ell}{q^{\rho_3}}\Big)$$

and

$$c_\lambda(w,t)$$
$$= \frac{1}{q^\lambda}\sum_{0\le u<q^\lambda} f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+wq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(wq^{\mu_0+\lambda})}\,\overline{f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\Big(\frac{-ut}{q^{\mu_2-\mu_0}}\Big).$$

By the Cauchy–Schwarz inequality we get

$$|G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)|^2 \le \Big(\sum_{0\le \ell<q^{\rho_3}}|\widetilde{c}_\ell(t)|^2\Big)$$
$$\times\sum_{0\le \ell<q^{\rho_3}}\Big|\frac{1}{q^{\mu_2-\mu_0-\lambda}}\sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\Big(\frac{-vt}{q^{\mu_2-\mu_0-\lambda}}+\frac{v\ell}{q^{\rho_3}}\Big)\Big|^2.$$

But

$$\sum_{0\le \ell<q^{\rho_3}}|\widetilde{c}_\ell(t)|^2 = \frac{1}{q^{2\rho_3}}\sum_{0\le w<q^{\rho_3}}\sum_{0\le w'<q^{\rho_3}} c_\lambda(w,t)\overline{c_\lambda(w',t)}\sum_{0\le \ell<q^{\rho_3}}\mathrm{e}\Big(-\frac{(w-w')\ell}{q^{\rho_3}}\Big)$$
$$= \frac{1}{q^{\rho_3}}\sum_{0\le w<q^{\rho_3}}|c_\lambda(w,t)|^2.$$

Since $f_{\mu_1}(u_0q^{\mu_0}+u_1q^{\mu_1}) = f_{\mu_1}(u_0q^{\mu_0})$ for $0 \le u_0 < q^{\mu_1-\mu_0}$ and $0 \le u_1 < q^{\lambda-\mu_1+\mu_0}$, we can write

$$\begin{aligned} c_\lambda(w,t) = \overline{f_{\mu_0+\lambda+\rho_3}(wq^{\mu_0+\lambda})}\, \mathrm{e}\left(\frac{wq^\lambda t}{q^{\mu_2-\mu_0}}\right) \frac{1}{q^{\mu_1-\mu_0}} \sum_{0\le u_0<q^{\mu_1-\mu_0}} \overline{f_{\mu_1}(u_0q^{\mu_0})} \\ \times \frac{1}{q^{\lambda-\mu_1+\mu_0}} \sum_{0\le u_1<q^{\lambda-\mu_1+\mu_0}} f_{\mu_0+\lambda+\rho_3}(u_0q^{\mu_0}+u_1q^{\mu_1}+wq^{\mu_0+\lambda}) \\ \times \mathrm{e}\left(-\frac{(u_0+u_1q^{\mu_1-\mu_0}+wq^\lambda)t}{q^{\mu_2-\mu_0}}\right). \end{aligned}$$

The sum over u_1 may be written as a sum over u' such that $0 \le u' < q^{\lambda+\rho_3}$ and $u' = u_0 + u_1q^{\mu_1-\mu_0} + wq^\lambda$ for some u_1 with $0 \le u_1 < q^{\lambda-\mu_1+\mu_0}$. Hence this last line is

$$\begin{aligned} \sum_{0\le \ell'<q^{\lambda+\rho_3}} \left(\frac{1}{q^{\lambda-\mu_1+\mu_0}} \sum_{0\le u_1<q^{\lambda-\mu_1+\mu_0}} \mathrm{e}\left(\ell' \frac{u_0+u_1q^{\mu_1-\mu_0}+wq^\lambda}{q^{\lambda+\rho_3}}\right)\right) \\ \times \frac{1}{q^{\lambda+\rho_3}} \sum_{0\le u'<q^{\lambda+\rho_3}} f(u'q^{\mu_0})\, \mathrm{e}\left(-\frac{u't}{q^{\mu_2-\mu_0}} - \frac{\ell' u'}{q^{\lambda+\rho_3}}\right). \end{aligned}$$

In order to use (7) with $\kappa = \mu_0$ and λ replaced by $\lambda+\rho_3$ we need to check that $\mu_0 \le c(\lambda+\rho_3)$. By (60) we have $\mu_0 \le \mu_1$, so that by (94) a sufficient condition would be that $\mu_1 \le \frac{c}{3}(\mu_2-\mu_1)$. By (57) and (51) this is equivalent to (77). We are now ready to use (7) with $\kappa = \mu_0$ and λ replaced by $\lambda+\rho_3$. We obtain, uniformly for $t \in \mathbb{R}$ and $0 \le w < q^{\rho_3}$,

$$|c_\lambda(w,t)| \ll \frac{q^{-\gamma(\lambda+\rho_3)}}{q^{\lambda-\mu_1+\mu_0}} \sum_{0\le \ell'<q^{\lambda+\rho_3}} \min\left(q^{\lambda-\mu_1+\mu_0}, \left|\sin\pi\left(\frac{\ell' q^{\mu_1-\mu_0}}{q^{\lambda+\rho_3}}\right)\right|^{-1}\right).$$

The sum over ℓ' runs over $q^{\mu_1-\mu_0}$ periods modulo $q^{\lambda+\rho_3-\mu_1+\mu_0}$, thus

$$|c_\lambda(w,t)| \ll \frac{q^{-\gamma(\lambda+\rho_3)}}{q^{\lambda-\mu_1+\mu_0}} q^{\lambda+\rho_3} \log q^{\lambda+\rho_3-\mu_1+\mu_0} = q^{\rho_3+\mu_1-\mu_0-\gamma(\lambda+\rho_3)} \log q^{\lambda+\rho_3-\mu_1+\mu_0}.$$

Since the function γ is non-decreasing, and by (95), this gives, uniformly for $t \in \mathbb{R}$ and $0 \le w < q^{\rho_3}$,

$$|c_\lambda(w,t)| \ll q^{\rho_3+\mu_1-\mu_0-\gamma(\lambda)} \log q^{\mu_2-\mu_1}.$$

We deduce that

$$\sum_{0\le \ell<q^{\rho_3}} |\widetilde{c}_\ell(t)|^2 \ll q^{2\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)} (\log q^{\mu_2-\mu_1})^2$$

and

$$\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0,\lambda,1}(k+t)|^2 \ll q^{2\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)}(\log q^{\mu_2-\mu_1})^2$$
$$\times \sum_{0\le \ell<q^{\rho_3}} \frac{1}{q^{2(\mu_2-\mu_0-\lambda)}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v'<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\overline{f(v'q^{\mu_0+\lambda})}$$
$$\times \mathrm{e}\left(-\frac{(v-v')t}{q^{\mu_2-\mu_0-\lambda}}+\frac{(v-v')\ell}{q^{\rho_3}}\right) \sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} \mathrm{e}\left(-\frac{(v-v')k}{q^{\mu_2-\mu_0-\lambda}}\right)$$
$$\ll q^{2\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)}(\log q^{\mu_2-\mu_1})^2 \sum_{0\le \ell<q^{\rho_3}} \frac{1}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} 1$$
$$\ll q^{3\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)}(\log q^{\mu_2-\mu_1})^2.$$

Denoting by $\mathcal{W}_\lambda$ the set of integers $w=u+q^\lambda v$ such that $(u,v)\in\widetilde{\mathcal{W}}_\lambda$, we observe that

$$G_{\mu_0,\mu_2-\mu_0,\lambda,2}(t)=\frac{1}{q^{\mu_2-\mu_0}}\sum_{w<q^{\mu_2-\mu_0}} c'_\lambda(w)\,\mathrm{e}\left(-\frac{wt}{q^{\mu_2-\mu_0}}\right),$$

where $|c'_\lambda(w)|\le 2$ for $0\le w<q^{\mu_2-\mu_0}$ and $c'_\lambda(w)=0$ for $w\notin\mathcal{W}_\lambda$. Therefore for any $t\in\mathbb{R}$ we have

$$\sum_{0\le k\le q^{\mu_2-\mu_0}} |G_{\mu_0,\mu_2-\mu_0,\lambda,2}(k+t)|^2$$
$$=\frac{1}{q^{2(\mu_2-\mu_0)}}\sum_{w<q^{\mu_2-\mu_0}}\sum_{w'<q^{\mu_2-\mu_0}} c'_\lambda(w)\overline{c'_\lambda(w')}\,\mathrm{e}\left(-\frac{(w-w')t}{q^{\mu_2-\mu_0}}\right)$$
$$\times\sum_{0\le k\le q^{\mu_2-\mu_0}} \mathrm{e}\left(-\frac{(w-w')k}{q^{\mu_2-\mu_0}}\right)$$
$$=\frac{1}{q^{\mu_2-\mu_0}}\sum_{w<q^{\mu_2-\mu_0}}|c'_\lambda(w)|^2=\frac{1}{q^{\mu_2-\mu_0}}\sum_{w\in\mathcal{W}_\lambda}|c'_\lambda(w)|^2\le\frac{1}{q^{\mu_2-\mu_0}}\sum_{w\in\mathcal{W}_\lambda}2^2\ll q^{-\rho_3},$$

where we take

$$\rho_3=\max\left(1,\left\lfloor\tfrac{1}{2}(\gamma(\lambda)-\mu_1+\mu_0)\right\rfloor\right).$$

By (95) this is admissible at least if $\lambda+\gamma(\lambda)/2\le\mu_2-\mu_0$. By (26) to ensure this inequality it is sufficient that $\lambda\le\frac{4}{5}(\mu_2-\mu_0)$. Then we get

$$\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0}(k+t)|^2$$
$$\le 2\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}}|G_{\mu_0,\mu_2-\mu_0,\lambda,1}(k+t)|^2+2\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}}|G_{\mu_0,\mu_2-\mu_0,\lambda,2}(k+t)|^2$$
$$\ll q^{\frac{1}{2}(\mu_1-\mu_0-\gamma(\lambda))}(\log q^{\mu_2-\mu_1})^2,$$

which completes the proof of Lemma 11. □

It follows from Lemma 11 and (81) that (78) holds with γ_1 defined by (79), which completes the proof of Lemma 10.

8. Proof of Theorems 1 and 2

By Proposition 1 we have

$$S_I(\vartheta) \ll (\log q)^{5/2}(\mu+\nu)^2\, q^{\mu+\nu-\frac{1}{2}\gamma(\frac{\mu+\nu}{3})},$$

and Proposition 2 (with the observation that (46) implies $\frac{\mu+\nu}{60} \leq \frac{\mu}{15}$) yields

$$\begin{aligned}|S_{II}(\vartheta)| &\ll \max(\tau(q)\log q, \log^3 q)^{1/4}(\mu+\nu)^{\frac{1}{4}(1+\max(\omega(q),2))} q^{\mu+\nu-\gamma(2\lfloor(\mu+\nu)/60\rfloor)/20}\\ &\ll \max(\tau(q)\log q, \log^{10} q)^{1/4}(\mu+\nu)^{\max(2,\frac{1+\omega(q)}{4})} q^{\mu+\nu-\gamma(2\lfloor(\mu+\nu)/60\rfloor)/20}.\end{aligned}$$

Since $(\mu+\nu)/3 \geq 2\lfloor(\mu+\nu)/60\rfloor$, we also have

$$S_I(\vartheta) \ll \max(\tau(q)\log q, \log^{10} q)^{1/4}(\mu+\nu)^{\max(2,\frac{1+\omega(q)}{4})} q^{\mu+\nu-\gamma(2\lfloor(\mu+\nu)/60\rfloor)/20},$$

so that by applying [24, Lemma 1], or its analogue in the case of μ obtained using (13.40) instead of (13.39) from [15], we obtain

$$\Big|\sum_{x/q<n\leq x}\Lambda(n) f(n)\,\mathrm{e}(\vartheta n)\Big| \ll c_1(q)(\log x)^{c_2(q)} x q^{-\gamma(2\lfloor(\log x)/60\log q\rfloor)/20},$$
$$\Big|\sum_{x/q<n\leq x}\mu(n) f(n)\,\mathrm{e}(\vartheta n)\Big| \ll c_1(q)(\log x)^{c_2(q)} x q^{-\gamma(2\lfloor(\log x)/60\log q\rfloor)/20},$$

with $c_1(q)$ and $c_2(q)$ defined in Theorem 1.

Now we replace x by x/q^k and sum over k. Let $K \in \mathbb{N}$ be such that $q^K \leq x^{1/4} < q^{K+1}$. Since γ is non-decreasing, we have

$$\begin{aligned}\sum_{k\leq K}\frac{x}{q^k} q^{-\gamma(2\lfloor\log(xq^{-k})/60\log q\rfloor)/20} &\leq q^{-\gamma(2\lfloor\log x^{3/4}/60\log q\rfloor)/20}\sum_{k\leq K}\frac{x}{q^k}\\ &\ll x q^{-\gamma(2\lfloor\log x/80\log q\rfloor)/20},\end{aligned}$$

while

$$\sum_{k>K}\frac{x}{q^k} q^{-\gamma(2\lfloor\log(xq^{-k})/60\log q\rfloor)/20} \leq \sum_{k\geq 0}\frac{x^{3/4}}{q^k} \ll x^{3/4} \ll x q^{-\gamma(2\lfloor\log x/80\log q\rfloor)/20}.$$

Then Theorems 1 and 2 follow.

9. Proof of Corollaries 1, 2 and 3

In order to prove Corollaries 1 and 2 we use a classical partial summation. It follows (for example) from [24, Lemma 11] that if $f : \mathbb{N} \to \mathbb{C}$ is such that $|f(n)| \leq 1$ for any $n \in \mathbb{N}$ then

$$\left|\sum_{p\leq x} f(p)\right| \leq \frac{2}{\log x} \max_{t\leq x}\left|\sum_{n\leq t} \Lambda(n) f(n)\right| + O(\sqrt{x}). \tag{97}$$

To prove Corollary 1 we observe first that if $\alpha \in \mathbb{Q}$, then the sequence $(\alpha b(p))_{p\in\mathcal{P}(a,m)}$ takes a finite number of values modulo 1, and therefore is not equidistributed modulo 1. If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, then for any $h \in \mathbb{Z}$ such that $h \neq 0$, we have $h\alpha \in \mathbb{R}\setminus\mathbb{Q}$, so that the function $n \mapsto \mathrm{e}(h\alpha b(n))$ has the carry property and is in $\mathcal{F}_{\gamma,c}$ for some $c \geq 10$ (see Definitions 1 and 2). By Theorem 1, for any $0 \leq j < m$ we have

$$\sum_{n\leq x} \Lambda(n)\, \mathrm{e}\left(h\alpha b(n) + \frac{jn}{m}\right) = o(x),$$

hence

$$\sum_{\substack{n\leq x\\ n\equiv a \bmod m}} \Lambda(n)\, \mathrm{e}(h\alpha b(n)) = \frac{1}{m} \sum_{0\leq j<m} \mathrm{e}\left(-\frac{ja}{m}\right) \sum_{n\leq x} \Lambda(n)\, \mathrm{e}\left(h\alpha b(n) + \frac{jn}{m}\right) = o(x).$$

By (97) and the Prime Number Theorem in arithmetic progressions (see for example [2, Theorem 9.12]), we can write, for $\gcd(a, m) = 1$,

$$\sum_{\substack{p\leq x\\ p\equiv a \bmod m}} \mathrm{e}(h\alpha b(p)) = o\left(\frac{x}{\log x}\right) + O(\sqrt{x}) = o(\pi(x; a, m)),$$

which proves that the sequence $(\alpha b(p))_{p\in\mathcal{P}(a,m)}$ is equidistributed modulo 1 according to Weyl's criterion (see for example [26, Chapter 1, p. 1]).

In order to prove Corollary 2, we write

$$\begin{aligned}\sum_{\substack{p\leq x\\ p\equiv a \bmod m\\ b(p)\equiv a' \bmod m'}} 1 &= \sum_{p\leq x} \frac{1}{mm'} \sum_{\substack{0\leq j<m\\ 0\leq j'<m'}} \mathrm{e}\left(\frac{j}{m}(p-a) + \frac{j'}{m'}(b(p)-a')\right)\\ &= \frac{\pi(x; a, m)}{m'} + \frac{1}{mm'} \sum_{\substack{0\leq j<m\\ 1\leq j'<m'}} \mathrm{e}\left(-\frac{aj}{m} - \frac{a'j'}{m'}\right) \sum_{p\leq x} \mathrm{e}\left(\frac{jp}{m} + \frac{j'b(p)}{m'}\right).\end{aligned}$$

Since for $1 \leq j' < m'$ the functions $n \mapsto \mathrm{e}\big(\frac{j'}{m'}b(n)\big)$ have the carry property and are in $\mathcal{F}_{\gamma,c}$ for some $c \geq 10$, by Theorem 1 and using (97), for any $0 \leq j < m$ and $1 \leq j' < m'$ we obtain

$$\sum_{p\leq x} \mathrm{e}\left(\frac{j'}{m'}b(p) + \frac{j}{m}p\right) = o(\pi(x)).$$

As the integers m and m' are fixed, it follows by using again the Prime Number Theorem in arithmetic progressions that for $\gcd(a, m) = 1$,

$$\sum_{\substack{p \le x \\ p \equiv a \bmod m \\ b(p) \equiv a' \bmod m'}} 1 = (1 + o(1)) \frac{\pi(x; a, m)}{m'},$$

which proves Corollary 2.

In order to prove Corollary 3 we observe first that if $\vartheta \in \mathbb{Q}$, then the sequence $(\vartheta p)_{p \in \mathcal{B}(a,m,a',m')}$ takes a finite number of values modulo 1, and therefore is not equidistributed modulo 1. If $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$, we write

$$\begin{aligned}\sum_{\substack{p \le x \\ p \equiv a \bmod m \\ b(p) \equiv a' \bmod m'}} \mathrm{e}(h\vartheta p) &= \sum_{p \le x} \frac{1}{mm'} \sum_{\substack{0 \le j < m \\ 0 \le j' < m'}} \mathrm{e}\left(h\vartheta p + \frac{j}{m}(p-a) + \frac{j'}{m'}(b(p) - a') \right) \\ &= \frac{1}{mm'} \sum_{0 \le j < m} \mathrm{e}\left(-\frac{aj}{m} \right) \sum_{p \le x} \mathrm{e}\left(\left(h\vartheta + \frac{j}{m} \right) p \right) \\ &\quad + \frac{1}{mm'} \sum_{\substack{0 \le j < m \\ 1 \le j' < m'}} \mathrm{e}\left(-\frac{aj}{m} - \frac{a'j'}{m'} \right) \sum_{p \le x} \mathrm{e}\left(\left(h\vartheta + \frac{j}{m} \right) p + \frac{j'}{m'} b(p) \right).\end{aligned}$$

It follows from [8] that for any $h \in \mathbb{Z} \setminus \{0\}$, $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$ and $0 \le j < m$ we have

$$\sum_{p \le x} \mathrm{e}\left(\left(h\vartheta + \frac{j}{m} \right) p \right) = o(\pi(x)).$$

Since for $1 \le j' < m'$ the functions $n \mapsto \mathrm{e}\big(\frac{j'}{m'} b(n)\big)$ have the carry property and are in $\mathcal{F}_{\gamma,c}$ for some $c \ge 10$, by Theorem 1 and using (97), for any $h \in \mathbb{Z} \setminus \{0\}$, $\vartheta \in \mathbb{R}$, $0 \le j < m$ and $1 \le j' < m'$ we obtain

$$\sum_{p \le x} \mathrm{e}\left(\frac{j'}{m'} b(p) + \left(h\vartheta + \frac{j}{m} \right) p \right) = o(\pi(x)).$$

It follows that

$$\sum_{\substack{p \le x \\ p \equiv a \bmod m \\ b(p) \equiv a' \bmod m'}} \mathrm{e}(h\vartheta p) = o(\pi(x)),$$

which, as the integers m and m' are fixed, proves that the sequence $(\vartheta p)_{p \in \mathcal{B}(a,m,a',m')}$ is equidistributed modulo 1 according again to Weyl's criterion.

10. Application to Rudin–Shapiro sequences

10.1. Rudin–Shapiro sequences of order δ

For any $n \in \mathbb{N}$ we denote by

$$n = \sum_{k \geq 0} \varepsilon_k(n)$$

its representation in base 2, where $\varepsilon_k(n)$ denotes the k-th least significant digit of n in base 2. Let $\delta \in \mathbb{N}$ and $\beta_\delta(n)$ the number of occurrences of patterns $1w1$ (where $w \in \{0,1\}^\delta$) in the representation of n in base 2:

$$\beta_\delta(n) = \sum_{k \geq \delta+1} \varepsilon_{k-\delta-1}(n)\varepsilon_k(n).$$

For $\alpha \in \mathbb{R}$ we consider in this section $f(n) = \mathrm{e}(\beta_\delta(n)\alpha)$. By (5) for any $\lambda \geq \delta + 2$ we have

$$f_\lambda = \mathrm{e}\Big(\alpha \sum_{\delta+1 \leq i < \lambda} \varepsilon_{i-\delta-1}\varepsilon_i\Big).$$

Therefore considering $f_{\kappa+\rho}$ in (6), the inequality may occur only by carry propagation when the digits of $\ell q^\kappa + k_1$ of indices $\kappa, \ldots, \kappa + \rho - 1$ are equal to 1, i.e. for integers ℓ with $\gg 2^\rho$ least significant digits equal to 1. It follows that f has the carry property. For $\delta + 2 \leq \mu_1 < \mu_2$, by (56) we have

$$f_{\mu_1,\mu_2} = f_{\mu_2}\overline{f_{\mu_1}} = \mathrm{e}\Big(\alpha \sum_{\mu_1 \leq i < \mu_2} \varepsilon_{i-\delta-1}\varepsilon_i\Big).$$

It follows that $f_{\mu_1,\mu_2}(n)$ depends only on the digits of n of indices $\mu_1 - \delta - 1, \mu_1, \ldots, \mu_2 - 1$. Therefore in (60) we can choose any $\mu_0 \leq \mu_1 - \delta - 1$ and (61) will be satisfied for any $\rho' \leq \rho$, which makes the choice (87) admissible. The aim of Proposition 3 is to show that for any $\alpha \in \mathbb{R}$, the function $n \mapsto \mathrm{e}(\alpha\beta_\delta(n))$ belongs to some $\mathcal{F}_{\gamma,c}$ (observe that $\beta_\delta(2^\kappa n) = \beta_\delta(n)$ for any $\kappa \in \mathbb{N}$).

Proposition 3. *For any $\delta, \lambda \in \mathbb{N}$ and $\alpha, \vartheta \in \mathbb{R}$ we have*

$$\Big|2^{-\lambda} \sum_{0 \leq n < 2^\lambda} \mathrm{e}(\alpha\beta_\delta(n) + \vartheta n)\Big| \leq 2^{(\delta+1)/2}\left(\frac{1 + |\cos \pi\alpha|}{2}\right)^{\lambda/2}.$$

Remark 3. This is Theorem 3.1 of [1], but we will present here a direct proof.

Proof of Proposition 3. For $0 \leq i < 2^{\delta+1}$ we write

$$\Gamma^{[i]}(n) = \sum_{k=0}^{\delta} \varepsilon_k(n)\varepsilon_{\delta-k}(i), \quad S_\lambda^{[i]}(\alpha,\vartheta) = \sum_{0 \leq n < 2^\lambda} \mathrm{e}(\alpha\Gamma^{[i]}(n) + \alpha\beta_\delta(n) + \vartheta n),$$

and

$$S_\lambda(\alpha, \vartheta) = \begin{pmatrix} S_\lambda^{[0]}(\alpha, \vartheta) \\ \vdots \\ S_\lambda^{[2^{\delta+1}-1]}(\alpha, \vartheta) \end{pmatrix}.$$

If $0 \le i < 2^\delta$, we have $\varepsilon_\delta(i) = 0$, so that for any $n \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$,

$$\begin{aligned} \Gamma^{[i]}(2n+\varepsilon) &= \sum_{k=0}^{\delta} \varepsilon_k(2n+\varepsilon)\varepsilon_{\delta-k}(i) = \sum_{k=1}^{\delta} \varepsilon_k(2n+\varepsilon)\varepsilon_{\delta-k}(i) \\ &= \sum_{k=1}^{\delta} \varepsilon_{k-1}(n)\varepsilon_{\delta-k+1}(2i) = \sum_{k=0}^{\delta-1} \varepsilon_k(n)\varepsilon_{\delta-k}(2i) \\ &= \sum_{k=0}^{\delta} \varepsilon_k(n)\varepsilon_{\delta-k}(2i) = \Gamma^{[2i]}(n) \end{aligned}$$

and $\varepsilon_\delta(2^\delta + i) = 1$, so that

$$\begin{aligned} \Gamma^{[2^\delta+i]}(2n+\varepsilon) &= \sum_{k=0}^{\delta} \varepsilon_k(2n+\varepsilon)\varepsilon_{\delta-k}(2^\delta+i) \\ &= \varepsilon_0(2n+\varepsilon)\cdot 1 + \sum_{k=1}^{\delta} \varepsilon_k(2n+\varepsilon)\varepsilon_{\delta-k}(2^\delta+i) \\ &= \varepsilon + \sum_{k=1}^{\delta} \varepsilon_{k-1}(n)\varepsilon_{\delta-k}(i) = \varepsilon + \sum_{k=1}^{\delta} \varepsilon_{k-1}(n)\varepsilon_{\delta-k+1}(2i) \\ &= \varepsilon + \Gamma^{[2i]}(n). \end{aligned}$$

Furthermore

$$\begin{aligned} \Gamma^{[2i+1]}(n) &= \sum_{k=0}^{\delta} \varepsilon_k(n)\varepsilon_{\delta-k}(2i+1) = \varepsilon_\delta(n)\varepsilon_0(2i+1) + \sum_{k=0}^{\delta-1} \varepsilon_k(n)\varepsilon_{\delta-k}(2i+1) \\ &= \varepsilon_\delta(n) + \sum_{k=0}^{\delta-1} \varepsilon_k(n)\varepsilon_{\delta-k}(2i) = \varepsilon_\delta(n) + \Gamma^{[2i]}(n). \end{aligned}$$

It follows from the definition of β_δ that

$$\begin{aligned} \beta_\delta(2n) &= \sum_{k\ge\delta+1} \varepsilon_{k-\delta-1}(2n)\varepsilon_k(2n) \\ &= \varepsilon_0(2n)\varepsilon_{\delta+1}(2n) + \sum_{k\ge\delta+2} \varepsilon_{k-\delta-2}(n)\varepsilon_{k-1}(n) = \beta_\delta(n) \end{aligned}$$

and

$$\begin{aligned}\beta_\delta(2n+1) &= \sum_{k\geq\delta+1} \varepsilon_{k-\delta-1}(2n+1)\varepsilon_k(2n+1)\\ &= \varepsilon_0(2n+1)\varepsilon_{\delta+1}(2n+1) + \sum_{k\geq\delta+2} \varepsilon_{k-\delta-2}(n)\varepsilon_{k-1}(n) = \varepsilon_\delta(n) + \beta_\delta(n),\end{aligned}$$

so that for any $i \in \{0, \dots, 2^\delta - 1\}$ and $\lambda \in \mathbb{N}$ we have

$$\begin{aligned}S_{\lambda+1}^{[i]}(\alpha,\vartheta) &= \sum_{0\leq n<2^{\lambda+1}} \mathrm{e}(\alpha\Gamma^{[i]}(n) + \alpha\beta_\delta(n) + \vartheta n)\\ &= \sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha\Gamma^{[i]}(2n) + \alpha\beta_\delta(2n) + 2n\vartheta)\\ &\quad + \sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha\Gamma^{[i]}(2n+1) + \alpha\beta_\delta(2n+1) + (2n+1)\vartheta)\\ &= \sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha\Gamma^{[2i]}(n) + \alpha\beta_\delta(n) + 2n\vartheta)\\ &\quad + \mathrm{e}(\vartheta)\sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha\Gamma^{[2i]}(n) + \alpha\varepsilon_\delta(n) + \alpha\beta_\delta(n) + 2n\vartheta)\\ &= S_\lambda^{[2i]}(\alpha, 2\vartheta) + \mathrm{e}(\vartheta)S_\lambda^{[2i+1]}(\alpha, 2\vartheta)\end{aligned}$$

and

$$\begin{aligned}S_{\lambda+1}^{[2^\delta+i]}(\alpha,\vartheta) &= \sum_{0\leq n<2^{\lambda+1}} \mathrm{e}(\alpha\Gamma^{[2^\delta+i]}(n) + \alpha\beta_\delta(n) + \vartheta n)\\ &= \sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha\Gamma^{[2^\delta+i]}(2n) + \alpha\beta_\delta(2n) + 2n\vartheta)\\ &\quad + \sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha\Gamma^{[2^\delta+i]}(2n+1) + \alpha\beta_\delta(2n+1) + (2n+1)\vartheta)\\ &= \sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha\Gamma^{[2i]}(n) + \alpha\beta_\delta(n) + 2n\vartheta)\\ &\quad + \mathrm{e}(\vartheta)\sum_{0\leq n<2^\lambda} \mathrm{e}(\alpha + \alpha\Gamma^{[2i]}(n) + \alpha\varepsilon_\delta(n) + \alpha\beta_\delta(n) + 2n\vartheta)\\ &= S_\lambda^{[2i]}(\alpha, 2\vartheta) + \mathrm{e}(\alpha+\vartheta)S_\lambda^{[2i+1]}(\alpha, 2\vartheta).\end{aligned}$$

This yields

$$S_{\lambda+1}(\alpha,\vartheta) = M(\alpha,\vartheta)S_\lambda(\alpha, 2\vartheta), \tag{98}$$

where $M(\alpha, \vartheta)$ denotes the $2^{\delta+1} \times 2^{\delta+1}$ matrix

$$M(\alpha, \vartheta) = \begin{pmatrix} 1 & e(\vartheta) & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & e(\vartheta) & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & e(\vartheta) & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & e(\vartheta) \\ 1 & e(\alpha+\vartheta) & 0 & & & \cdots & & & 0 \\ 0 & 0 & 1 & e(\alpha+\vartheta) & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & e(\alpha+\vartheta) & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & e(\alpha+\vartheta) \end{pmatrix}.$$

Writing

$$A(\alpha, \vartheta) = \begin{pmatrix} 2 & e(\vartheta) + e(\alpha+\vartheta) \\ e(-\vartheta) + e(-\alpha-\vartheta) & 2 \end{pmatrix}$$

we observe that ${}^t\overline{M(\alpha, \vartheta)}M(\alpha, \vartheta)$ is the block matrix

$${}^t\overline{M(\alpha, \vartheta)}M(\alpha, \vartheta) = \begin{pmatrix} A(\alpha, \vartheta) & & 0 \\ & \ddots & \\ 0 & & A(\alpha, \vartheta) \end{pmatrix}.$$

Denoting by $\rho(A(\alpha, \vartheta))$ the spectral radius of $A(\alpha, \vartheta)$, it follows that

$$\|M(\alpha, \vartheta)\|_2 = \sqrt{\rho(A(\alpha, \vartheta))} = \sqrt{2 + |\,e(\vartheta) + e(\alpha+\vartheta)|} = \sqrt{2(1 + |\cos \pi\alpha|)}.$$

By (98) this gives

$$\|S_{\lambda+1}(\alpha, \vartheta)\|_2 \leq \|M(\alpha, \vartheta)\|_2 \|S_\lambda(\alpha, 2\vartheta)\|_2 \leq \sqrt{2(1 + |\cos \pi\alpha|)}\, \|S_\lambda(\alpha, 2\vartheta)\|_2$$

and by induction we get

$$\Big| \sum_{0 \leq n < 2^\lambda} e(\alpha\beta_\delta(n) + \vartheta n) \Big| = |S_\lambda^{[0]}(\alpha, \vartheta)| \leq \|S_\lambda(\alpha, \vartheta)\|_2$$
$$\leq (2 + 2|\cos \pi\alpha|)^{\lambda/2} \|S_0(\alpha, 2^\lambda\vartheta)\|_2 = 2^{(\delta+1)/2}(2 + 2|\cos \pi\alpha|)^{\lambda/2},$$

which ends the proof of Proposition 3. □

Observing that $\beta_\delta(u2^\kappa) = \beta_\delta(u)$, Proposition 3 shows that $f(n) = e(\beta_\delta(n)\alpha)$ belongs to $\mathcal{F}_{\gamma,c}$ for any $c > 0$ and

$$\gamma(\lambda) = -\frac{\lambda}{2\log 2} \log\left(\frac{1 + |\cos \pi\alpha|}{2}\right) - \frac{\delta+1}{2}. \tag{99}$$

Applying Theorems 1 and 2 we obtain

Theorem 3. *For any $\delta \in \mathbb{N}$, $\alpha, \vartheta \in \mathbb{R}$ and $x \geq 2$ we have*

$$\Big|\sum_{n\leq x} \Lambda(n)\,\mathrm{e}(\beta_\delta(n)\alpha + \vartheta n)\Big| \ll x(\log x)^{11/4} 2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20} \tag{100}$$

$$\Big|\sum_{n\leq x} \mu(n)\,\mathrm{e}(\beta_\delta(n)\alpha + \vartheta n)\Big| \ll x(\log x)^{11/4} 2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20}, \tag{101}$$

where γ is defined by (99).

10.2. Rudin–Shapiro sequences of degree d

Let $d \in \mathbb{N}$ with $d \geq 2$ and $b_d(n)$ denote the number of occurrences of blocks of d consecutive 1's in the representation of n in base 2:

$$b_d(n) = \sum_{k\geq d-1} \varepsilon_{k-d+1}(n)\cdots\varepsilon_k(n).$$

For $\alpha \in \mathbb{R}$ we consider in this section $f(n) = \mathrm{e}(b_d(n)\alpha)$. By (5) for any $\lambda \geq d$ we have

$$f_\lambda = \mathrm{e}\Big(\alpha \sum_{d-1\leq i<\lambda} \varepsilon_{i-d+1}\cdots\varepsilon_{i-1}\varepsilon_i\Big).$$

Therefore considering $f_{\kappa+\rho}$ in (6), the inequality may occur only by carry propagation when the digits of $\ell 2^\kappa + k_1$ of indices $\kappa, \ldots, \kappa+\rho-1$ are equal to 1, i.e. for integers ℓ with $\gg 2^\rho$ least significant digits equal to 1. It follows that f has the carry property. For $d \leq \mu_1 < \mu_2$, by (56) we have

$$f_{\mu_1,\mu_2} = f_{\mu_2}\overline{f_{\mu_1}} = \mathrm{e}\Big(\alpha \sum_{\mu_1\leq i<\mu_2} \varepsilon_{i-d+1}\cdots\varepsilon_{i-1}\varepsilon_i\Big).$$

It follows that $f_{\mu_1,\mu_2}(n)$ depends only on the digits of n of indices $\mu_1-d+1, \ldots, \mu_2-1$. Given any ρ' satisfying $(d-1)/2 \leq \rho' \leq \rho$ (which implies (59)), we can choose $\mu_0 = \mu_1 - d + 1$ so that (60) and (61) are satisfied. This makes the choice (87) admissible.

The aim of Proposition 4 is to show that for any $\alpha \in \mathbb{R}$, the function $n \mapsto \mathrm{e}(\alpha b_d(n))$ belongs to some $\mathcal{F}_{\gamma,c}$ (observe that $b_d(2^\kappa n) = b_d(n)$ for any $\kappa \in \mathbb{N}$).

Proposition 4. *For any $d \geq 2$, $\alpha, \vartheta \in \mathbb{R}$ and $\lambda \in \mathbb{N}$ we have*

$$\Big|2^{-\lambda} \sum_{0\leq n<2^\lambda} \mathrm{e}(b_d(n)\alpha + n\vartheta)\Big| \leq \left(1 - 2^{3-d}\left(\sin\frac{\pi\|\alpha\|}{4}\right)^2\right)^{\lfloor\lambda/d\rfloor}.$$

Proof. For any $k \in \mathbb{N}$ we define $\chi_k : \mathbb{N} \to \{0, 1\}$ by $\chi_k(n) = 1$ if the k least significant digits of n are 1's, and $\chi_k(n) = 0$ otherwise. This means that $\chi_0 = 1$ and for $k \geq 1$, $\chi_k(n) = \varepsilon_{k-1}(n)\cdots\varepsilon_0(n)$. In particular $\chi_k(n) = 0$ for $k \geq 1$ and $n < 2^{k-1}$.

For $n \in \mathbb{N}$ we define

$$\chi_d^{[1]}(n) = 0,$$
$$\chi_d^{[i]}(n) = \chi_{d-i+1}(n) + \cdots + \chi_{d-1}(n) \quad \text{for } i \in \{2, \ldots, d\}.$$

For $i \in \{1,\dots,d\}$, we define

$$S_\lambda^{[i]}(\alpha,\vartheta) = \sum_{0\le n<2^\lambda} \mathrm{e}(\chi_d^{[i]}(n)\alpha + b_d(n)\alpha + n\vartheta)$$

and notice that we are interested in $|S_\lambda^{[1]}(\alpha,\vartheta)|$. For $n \in \mathbb{N}$ we have

$$b_d(2n) = b_d(n), \qquad b_d(2n+1) = b_d(n) + \chi_{d-1}(n),$$

and for $k \ge 1$,

$$\chi_k(2n) = 0, \qquad \chi_k(2n+1) = \chi_{k-1}(n),$$

so that for $i \in \{1,\dots,d\}$,

$$\chi_d^{[i]}(2n) = 0 = \chi_d^{[1]}(n), \qquad \chi_d^{[1]}(2n+1) = 0 = \chi_d^{[2]}(n) - \chi_{d-1}(n),$$

and for $i \in \{2,\dots,d-1\}$,

$$\begin{aligned}\chi_d^{[i]}(2n+1) &= \chi_{d-i}(n) + \cdots + \chi_{d-2}(n) = \chi_d^{[i+1]}(n) - \chi_{d-1}(n),\\ \chi_d^{[d]}(2n+1) &= \chi_0(n) + \cdots + \chi_{d-2}(n) = 1 + \chi_d^{[d]}(n) - \chi_{d-1}(n).\end{aligned}$$

It follows that for $i = 1,\dots,d-1$,

$$\begin{aligned}S_{\lambda+1}^{[i]}(\alpha,\vartheta) &= S_\lambda^{[1]}(\alpha,2\vartheta) + \mathrm{e}(\vartheta)S_\lambda^{[i+1]}(\alpha,2\vartheta),\\ S_{\lambda+1}^{[d]}(\alpha,\vartheta) &= S_\lambda^{[1]}(\alpha,2\vartheta) + \mathrm{e}(\alpha+\vartheta)S_\lambda^{[d]}(\alpha,2\vartheta).\end{aligned}$$

Let us introduce the $d \times d$ matrix $M(\alpha,\vartheta)$ and the vector $S_\lambda(\alpha,\vartheta)$ defined by

$$M(\alpha,\vartheta) = \begin{pmatrix} 1 & \mathrm{e}(\vartheta) & 0 & \cdots & \cdots & 0\\ 1 & 0 & \mathrm{e}(\vartheta) & \cdots & \cdots & 0\\ \vdots & & & \ddots & & \vdots\\ 1 & 0 & & 0 & \ddots & 0\\ 1 & 0 & & & 0 & \mathrm{e}(\vartheta)\\ 1 & 0 & & & 0 & \mathrm{e}(\alpha+\vartheta)\end{pmatrix}, \qquad S_\lambda(\alpha,\vartheta) = \begin{pmatrix} S_\lambda^{[1]}(\alpha,\vartheta)\\ \vdots\\ S_\lambda^{[d]}(\alpha,\vartheta)\end{pmatrix}.$$

We have

$$S_{\lambda+1}(\alpha,\vartheta) = M(\alpha,\vartheta)S_\lambda(\alpha,2\vartheta),$$

and writing

$$M^{[\lambda]}(\alpha,\vartheta) = M(\alpha,\vartheta)M(\alpha,2\vartheta)\cdots M(\alpha,2^{\lambda-1}\vartheta),$$

we get

$$S_\lambda(\alpha,\vartheta) = M^{[\lambda]}(\alpha,\vartheta)S_0(\alpha,2^\lambda\vartheta).$$

Therefore

$$|S_\lambda^{[1]}(\alpha,\vartheta)| \le \|S_\lambda(\alpha,\vartheta)\|_\infty \le \|M^{[\lambda]}(\alpha,\vartheta)\|_\infty,$$

so that Proposition 4 follows from the lemma below. □

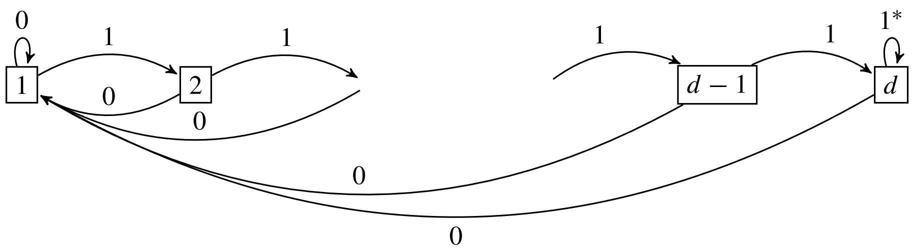
Lemma 12. *We have*

$$\|M^{[d]}(\alpha,\vartheta)\|_\infty \le 2^d - 8\big(\sin\tfrac{\pi\|\alpha\|}{4}\big)^2.$$

Proof. Let us first observe that $(M^{[d]}(0,0))_{i,j} = ((M(0,0))^d)_{i,j} = \max(2, 2^{d-j})$: indeed, it easy to show by induction on k that for $1 \le k \le d-1$,

$$(M(0,0))^k = \begin{pmatrix} 2^{k-1} & 2^{k-2} & \dots & 2^0 & 1 & 0 & \dots & \dots & 0 \\ 2^{k-1} & 2^{k-2} & & 2^0 & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \ddots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots \\ 2^{k-1} & 2^{k-2} & \dots & 2^0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

We remark that for any $(i,j) \in \{1,\dots,d\}^2$ the coefficient $(M^{[d]}(\alpha,\vartheta))_{i,j}$ is a sum of complex numbers of modulus 1. The number of them is precisely $(M^{[d]}(0,0))_{i,j} = \max(2, 2^{d-j})$ while the argument of each of them is the coding of a path of length d going from vertex $\boxed{i}$ to vertex $\boxed{j}$ in the following graph:



The coding (given by the rules of matrix product) is the following: for any path of length d from $\boxed{i}$ to $\boxed{j}$ and for any $t \in \{1,\dots,d\}$:

- crossing an arc labeled by 0 at step t adds 0 to the argument;
- crossing an arc labeled by 1 at step t adds $2^{t-1}\vartheta$ to the argument;
- crossing an arc labeled by 1* at step t adds $2^{t-1}\vartheta + \alpha$ to the argument.

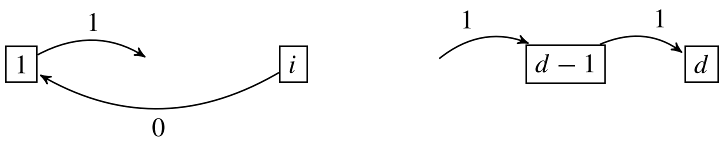
For any $i \in \{1,\dots,d\}$ there are exactly two paths of length d going from vertex $\boxed{i}$ to vertex $\boxed{d}$:

1. the path

that corresponds to the coding

$$\vartheta + 2\vartheta + \cdots + 2^{d-i-1}\vartheta + (2^{d-i}\vartheta + \alpha) + \cdots + (2^{d-1}\vartheta + \alpha) = (2^d - 1)\vartheta + i\alpha;$$

2. the path



that corresponds to the coding

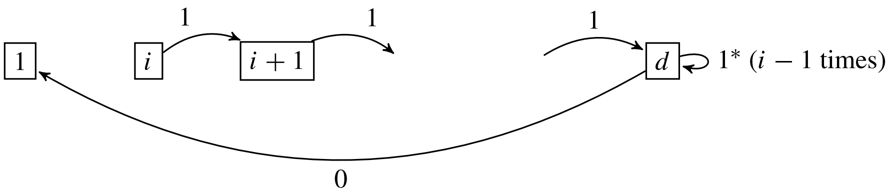
$$\vartheta + 2\vartheta + \cdots + 2^{d-1}\vartheta = (2^d - 2)\vartheta.$$

It follows that for any $i \in \{1, \dots, d\}$ we have

$$(M^{[d]}(\alpha, \vartheta))_{i,d} = \mathrm{e}((2^d - 2)\vartheta) + \mathrm{e}((2^d - 1)\vartheta + i\alpha). \tag{102}$$

For any $i \in \{1, \dots, d\}$ there are exactly 2^{d-1} paths of length d going from vertex $\boxed{i}$ to vertex $\boxed{1}$ among which we have the following two:

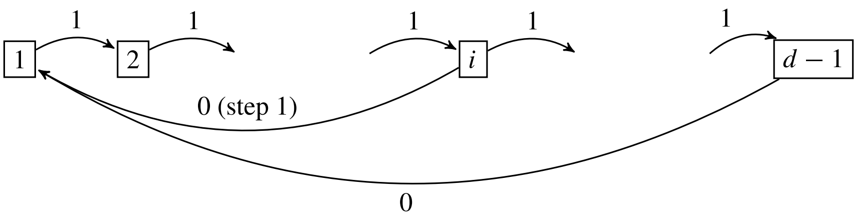
1. the path



that corresponds to the coding

$$\vartheta+2\vartheta+\cdots+2^{d-i-1}\vartheta+(2^{d-i}\vartheta+\alpha)+\cdots+(2^{d-2}\vartheta+\alpha)+0 = (2^{d-1}-1)\vartheta+(i-1)\alpha;$$

2. the path



that corresponds to the coding

$$0 + 2\vartheta + \cdots + 2^{d-2}\vartheta + 0 = (2^{d-1} - 2)\vartheta.$$

It follows that for any $i \in \{1, \ldots, d\}$ we have

$$(M^{[d]}(\alpha, \vartheta))_{i,1} = \mathrm{e}((2^{d-1} - 2)\vartheta) + \mathrm{e}((2^{d-1} - 1)\vartheta + (i-1)\alpha) \\ + (2^{d-1} - 2) \text{ terms of modulus } 1. \tag{103}$$

It follows from (102) and (103) that for any $i \in \{1, \ldots, d\}$ we have

$$\sum_{j=1}^{d} |(M^{[d]}(\alpha, \vartheta))_{i,j}| \leq \sum_{j=2}^{d-1} 2^{d-j} + (2^{d-1} - 2) \\ + |\mathrm{e}((2^{d-1} - 2)\vartheta) + \mathrm{e}((2^{d-1} - 1)\vartheta + (i-1)\alpha)| \\ + |\mathrm{e}((2^{d} - 2)\vartheta) + \mathrm{e}((2^{d} - 1)\vartheta + i\alpha)|,$$

which implies

$$\sum_{j=1}^{d} |(M^{[d]}(\alpha, \vartheta))_{i,j}| \leq 2^d - 4 + 2|\cos \pi(\vartheta + (i-1)\alpha)| + 2|\cos \pi(\vartheta + i\alpha)|.$$

For any $x \in \mathbb{R}$ we have

$$|\cos \pi x| + |\cos \pi(x+\alpha)| = |\cos \pi x + \cos \pi(x + \alpha')| \leq |e^{i\pi x} + e^{i\pi(x+\alpha')}| = 2\left|\cos \tfrac{\pi\alpha'}{2}\right|$$

with $\alpha' = \alpha$ if both cosines have the same sign, and $\alpha' = \alpha + 1$ otherwise. Hence

$$|\cos \pi x| + |\cos \pi(x+\alpha)| \leq 2 \max\left(\left|\cos \tfrac{\pi\alpha}{2}\right|, \left|\sin \tfrac{\pi\alpha}{2}\right|\right),$$

and observing that $\alpha = n \pm \|\alpha\|$ with $n \in \mathbb{Z}$ and $0 \leq \|\alpha\| \leq 1/2$, we get

$$\max\left(\left|\cos \tfrac{\pi\alpha}{2}\right|, \left|\sin \tfrac{\pi\alpha}{2}\right|\right) = \max\left(\cos \tfrac{\pi\|\alpha\|}{2}, \sin \tfrac{\pi\|\alpha\|}{2}\right) = \cos \tfrac{\pi\|\alpha\|}{2} = 1 - 2\left(\sin \tfrac{\pi\|\alpha\|}{4}\right)^2.$$

Taking $x = \vartheta + (i-1)\alpha$ we obtain

$$\sum_{j=1}^{d} |(M^{[d]}(\alpha, \vartheta))_{i,j}| \leq 2^d - 8\left(\sin \tfrac{\pi\|\alpha\|}{4}\right)^2,$$

which completes the proof of Lemma 12. □

We will now prove Proposition 4. We have

$$\left|2^{-\lambda} \sum_{0 \leq n < 2^\lambda} \mathrm{e}(b_d(n)\alpha + n\vartheta)\right| = 2^{-\lambda} |S_\lambda^{[1]}(\alpha, \vartheta)| \leq 2^{-\lambda} \|M^{[\lambda]}(\alpha, \vartheta)\|_\infty$$

and

$$M^{[\lambda]}(\alpha, \vartheta) = M(\alpha, \vartheta) M(\alpha, 2\vartheta) \cdots M(\alpha, 2^{\lambda-1}\vartheta) \\ = M^{[d]}(\alpha, \vartheta) M^{[d]}(\alpha, 2^d\vartheta) \cdots M^{[d]}(\alpha, 2^{d\lfloor \lambda/d \rfloor}\vartheta) \\ \times M(\alpha, 2^{d\lfloor \lambda/d \rfloor}\vartheta) \cdots M(\alpha, 2^{\lambda-1}\vartheta).$$

Using the submultiplicativity of the matrix norm, the bound $\|M(\alpha,\vartheta')\|_\infty = 2$ and Lemma 12, we get

$$\|M^{[\lambda]}(\alpha,\vartheta)\|_\infty \le 2^{\lambda-d\lfloor\lambda/d\rfloor}\big(2^d - 8\big(\sin\tfrac{\pi\|\alpha\|}{4}\big)^2\big)^{\lfloor\lambda/d\rfloor},$$

so that

$$\Big|2^{-\lambda}\sum_{0\le n<2^\lambda} \mathrm{e}(b_d(n)\alpha + n\vartheta)\Big| \le \big(1-2^{3-d}\big(\sin\tfrac{\pi\|\alpha\|}{4}\big)^2\big)^{\lfloor\lambda/d\rfloor},$$

which completes the proof of Proposition 4.

Taking into account that $b_d(n2^k) = b_d(n)$ for any $k \in \mathbb{N}$, $d \ge 2$ and $0 \le \|\alpha\| \le 1/2$, it follows from Proposition 4 that for $f(n) = \mathrm{e}(b_d(n)\alpha)$ and for $k \in \mathbb{N}$ and any $\vartheta \in \mathbb{R}$,

$$\begin{aligned}\Big|2^{-\lambda}\sum_{0\le u<2^\lambda} f(u2^k)\,\mathrm{e}(u\vartheta)\Big| &\le \big(1-2^{3-d}\big(\sin\tfrac{\pi\|\alpha\|}{4}\big)^2\big)^{\lfloor\lambda/d\rfloor}\\ &\le \big(1-2^{3-d}\big(\sin\tfrac{\pi}{8}\big)^2\big)^{-1}\big(1-2^{3-d}\big(\sin\tfrac{\pi\|\alpha\|}{4}\big)^2\big)^{\lambda/d},\\ &\le \sqrt{2}\big(1-2^{3-d}\big(\sin\tfrac{\pi\|\alpha\|}{4}\big)^2\big)^{\lambda/d}.\end{aligned}$$

It follows that f belongs to $\mathcal{F}_{\gamma,c}$ (Definition 2) for any $c > 0$ and

$$\gamma(\lambda) = \frac{-\lambda}{d\ \log 2}\log\left(1-2^{3-d}\left(\sin\frac{\pi\|\alpha\|}{4}\right)^2\right) - \frac{1}{2}. \tag{104}$$

Applying Theorems 1 and 2 we obtain

Theorem 4. *For any $d \in \mathbb{N}$ with $d \ge 2$, $\alpha, \vartheta \in \mathbb{R}$ and $x \ge 2$ we have*

$$\Big|\sum_{n\le x}\Lambda(n)\,\mathrm{e}(b_d(n)\alpha + \vartheta n)\Big| \ll x(\log x)^{11/4}2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20}, \tag{105}$$

$$\Big|\sum_{n\le x}\mu(n)\,\mathrm{e}(b_d(n)\alpha + \vartheta n)\Big| \ll x(\log x)^{11/4}2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20}, \tag{106}$$

where γ is defined by (104).

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