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Vanishing and non-vanishing for the first L^p-cohomology of groups

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Abstract. We prove two results on the first L^p-cohomology $\overline{H}^1_{(p)}(\Gamma)$ of a finitely generated group Γ:

1) If $N \subset H \subset \Gamma$ is a chain of subgroups, with N non-amenable and normal in Γ, then $\overline{H}^1_{(p)}(\Gamma) = 0$ as soon as $\overline{H}^1_{(p)}(H) = 0$. This allows for a short proof of a result of W. Lück: if $N \lhd \Gamma$, N is infinite, finitely generated as a group, and Γ/N contains an element of infinite order, then $\overline{H}^1_{(2)}(\Gamma) = 0$.

2) If Γ acts isometrically, properly discontinuously on a proper CAT(-1) space X, with at least 3 limit points in ∂X, then for p larger than the critical exponent $e(\Gamma)$ of Γ in X, one has $\overline{H}^1_{(p)}(\Gamma) \neq 0$. As a consequence we extend a result of Y. Shalom: let G be a cocompact lattice in a rank 1 simple Lie group; if G is isomorphic to Γ, then $e(G) \leq e(\Gamma)$.

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1. Introduction

Let Γ be a countable group. Assume first that Γ admits a $K(\Gamma, 1)$-space which is a simplicial complex X finite in every dimension. Let $\tilde{X}$ be the universal cover of X. Fix $p \in [1, \infty[$. Denote by $\ell^p C^k$ the space of p-summable complex k-cochains on $\tilde{X}$, i.e. the ℓ^p-functions on the set C^k of k-simplices of $\tilde{X}$. The *L^p-cohomology of* Γ is the reduced cohomology of the complex

$$d_k \colon \ell^p C^k \to \ell^p C^{k+1},$$

where d_k is the simplicial coboundary operator; we denote it by

$$\overline{H}^k_{(p)}(\Gamma) = \operatorname{Ker} d_k / \overline{\operatorname{Im} d_{k-1}}.$$

As explained at the beginning of [Gro93], this definition only depends on Γ.

For $p=2$, the space $\overline{H}^k_{(2)}(\Gamma)$ is a module over the von Neumann algebra of Γ, and its von Neumann dimension is the *k-th L^2-Betti number of* Γ, denoted by $b^k_{(2)}(\Gamma)$; recall that $b^k_{(2)}(\Gamma)=0$ if and only if $\overline{H}^k_{(2)}(\Gamma)=0$.

For $k=1$, it is possible to define the first L^p-cohomology of Γ under the mere assumption that Γ is finitely generated. Denote by $\mathcal{F}(\Gamma)$ the space of all complex-valued functions on Γ, and by λ_Γ the left regular representation of Γ on $\mathcal{F}(\Gamma)$. Define then the space of *p-Dirichlet finite* functions on Γ:

$$D_p(\Gamma)=\{f\in\mathcal{F}(\Gamma)\mid \lambda_\Gamma(g)f-f\in\ell^p(\Gamma)\text{ for every } g\in\Gamma\}.$$

If S is a finite generating set of Γ, define a norm on $D_p(\Gamma)/\mathbb{C}$ by:

$$\|f\|^p_{D_p}=\sum_{s\in S}\|\lambda_\Gamma(s)f-f\|^p_p.$$

Denote by $i\colon \ell^p(\Gamma)\to D_p(\Gamma)$ the inclusion. The *first L^p-cohomology* of Γ is

$$\overline{H}^1_{(p)}(\Gamma)=D_p(\Gamma)/\overline{i(\ell^p(\Gamma))+\mathbb{C}}.$$

Let us recall briefly why this definition is coherent with the previous one. If Γ admits a finite $K(\Gamma,1)$-space X, we can choose one such that the 1-skeleton of $\tilde{X}$ is a Cayley graph $\mathcal{G}(\Gamma,S)$ of Γ. This means that S is some finite generating subset of Γ, that $C^0=\Gamma$, and that C^1 is the set $\mathbb{E}_\Gamma$ of oriented edges:

$$\mathbb{E}_\Gamma=\{(x,sx)\mid x\in\Gamma, s\in S\}.$$

Then d_0 is the restriction to $\ell^p(\Gamma)$ of the coboundary operator

$$d_\Gamma\colon \mathcal{F}(\Gamma)\to\mathcal{F}(\mathbb{E}_\Gamma);\quad f\mapsto[(x,y)\mapsto f(y)-f(x)].$$

Since $\tilde{X}$ is contractible, by Poincaré's lemma any closed cochain is exact, i.e. any element in $\operatorname{Ker} d_1$ can be written as $d_\Gamma f$, for some $f\in D_p(\Gamma)$ defined up to an additive constant. This means that $d_\Gamma\colon D_p(\Gamma)\to\ell^p(\mathbb{E}_\Gamma)$ induces an isomorphism of Banach spaces $D_p(\Gamma)/\mathbb{C}\to \operatorname{Ker} d_1$, which maps $i(\ell^p(\Gamma))$ to $\operatorname{Im} d_0$. This shows the equivalence of both definitions of $\overline{H}^1_{(p)}(\Gamma)$.

Our first result is:

Theorem 1. *Let $N\subset H\subset\Gamma$ be a chain of groups, with H and Γ finitely generated, N infinite and normal in Γ.*

1) *If H is non-amenable and $\overline{H}^1_{(p)}(H)=0$, then $\overline{H}^1_{(p)}(\Gamma)=0$.*

2) *If $b^1_{(2)}(H)=0$, then $b^1_{(2)}(\Gamma)=0$.*

We do *not* know whether part 1) of Theorem 1 holds when H is amenable[1].

As an application of part 2) of Theorem 1, we will give a very short proof of the following result of W. Lück (Theorem 0.7 in [Lue97]):

Corollary 1. *Let Γ be a finitely generated group. Assume that Γ contains an infinite, normal subgroup N, which is finitely generated as a group, and such that Γ/N is not a torsion group. Then $b^1_{(2)}(\Gamma) = 0$.*

Using his theory of L^2-Betti numbers for equivalence relations and group actions, D. Gaboriau was able to improve the previous result by merely assuming that Γ/N is infinite (see [Gab02], Théorème 6.8). It is a challenging, and vaguely irritating question, to find a purely group cohomological proof of Gaboriau's result.

As shown by Gaboriau's result, non-vanishing of $\overline{H}^1_{(2)}$ is an obstruction for the existence of finitely generated normal subgroups. We now present a non-vanishing result. Its proof is based on an idea due to G. Elek (see [Ele97], Theorem 2).

Let X be a proper CAT(-1) space (see [BH99] for the definitions), and let Γ be an infinite, finitely generated, properly discontinuous subgroup of isometries of X. Recall that the *critical exponent* of Γ is defined as

$$e(\Gamma) = \inf \left\{ s > 0 \mid \sum_{g \in \Gamma} e^{-s|go-o|} < +\infty \right\},$$

where o is any origin in X, and where $|\cdot - \cdot|$ denotes the distance in X. In many cases, $e(\Gamma) < +\infty$; in particular, this happens when the isometry group of X is co-compact (see Proposition 1.7 in [BM96]).

Theorem 2. *Assume that $e(\Gamma)$ is finite. If the limit set of Γ in ∂X has at least 3 points, then for $p > \max\{1, e(\Gamma)\}$ the Banach space $\overline{H}^1_{(p)}(\Gamma)$ is non zero.*

When Γ is in addition co-compact, Theorem 2 was already known to Pansu and Gromov (see [Pan89] and page 258 in [Gro93]).

Theorem 2 is optimal for the co-compact lattices in rank one semi-simple Lie group: for those $p > e(\Gamma)$ if and only if $\overline{H}^1_{(p)}(\Gamma) \neq 0$, thanks to a result of Pansu [Pan89]. Recall that $e(\Gamma) = 1$ for lattices Γ in SO$(2, 1)$ (and exactly for those among rank one lattices). Since L^p-cohomology of groups is an invariant of isomorphism, by combining Pansu's result with Theorem 2, we obtain the following generalisation of a result of Shalom (Theorem 1.1 in [Sha00]):

Corollary 2. *Let G be a co-compact lattice in a rank one semi-simple Lie group (other than SO$(2, 1)$). Assume that G is isomorphic to a properly discontinuous subgroup Γ of isometries of a proper CAT(-1) space X. Then $e(G) \leq e(\Gamma)$.* □

[1] When $p = 2$ and H is amenable, we appeal to the Cheeger–Gromov vanishing theorem [CG86]; to the best of our knowledge, there is no analogue of this result in L^p-cohomology for $p \neq 2$, although Gromov notices in Remark (A_2) of [Gro93], $8.A_1$, that it should be the case.

Shalom established this by different methods in the special case where X is the symmetric space associated to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$; his result also holds for non-cocompact lattices (when the Lie group is different from $\mathrm{SO}(2,1)$). In [BCG99] the authors establish Corollary 2 in the case Γ is quasi-convex, this assumption simplifies their proof but they do not really need it.

The equality case in Corollary 2, which leads to a rigidity theorem, is studied in [Bou96] and [Yue96] and in [BCG99], when Γ is in addition quasi-convex. Again methods of proofs developed in [BCG99] should apply without the quasi-convex assumption.

2. Group cohomology; proof of Theorem 1

Let V be a topological Γ-module, i.e. a real or complex topological vector space endowed with a continuous, linear representation $\pi\colon \Gamma \times V \to V\colon (g,v) \mapsto \pi(g)v$. If H is a subgroup of Γ, we denote by $V|_H$ the space V viewed as an H-module for the restricted action, and by V^H the set of H-fixed points:

$$V^H = \{v \in V \mid \pi(h)v = v \text{ for all } h \in H\}.$$

We now introduce the space of 1-cocycles and 1-coboundaries on Γ, and the 1-cohomology with coefficients in V:

- $Z^1(\Gamma, V) = \{b\colon \Gamma \to V \mid b(gh) = b(g) + \pi(g)b(h) \text{ for all } g, h \in \Gamma\}$;
- $B^1(\Gamma, V) = \{b \in Z^1(\Gamma, V) \mid \text{there exists } v \in V \text{ such that } b(g) = \pi(g)v - v \text{ for all } g \in \Gamma\}$;
- $H^1(\Gamma, V) = Z^1(\Gamma, V)/B^1(\Gamma, V)$.

Suppose that V is a Banach space. The space $Z^1(\Gamma, V)$ of 1-cocycles is a Fréchet space when endowed with the topology of pointwise convergence on Γ. The 1-reduced cohomology space with coefficients in V is

$$\overline{H^1}(\Gamma, V) = Z^1(\Gamma, V)/\overline{B^1(\Gamma, V)}.$$

Recall that V *almost has invariant vectors* if, for every finite subset F in Γ, and every $\epsilon > 0$, there exists a vector v of norm 1 in V, such that $\|\pi(g)v - v\| < \epsilon$ for every $g \in F$. The following result is due to Guichardet (Theorem 1 and Corollary 1 in [Gui72]).[2]

Proposition 1. *Let Γ be a countable group.*

[2] Strictly speaking, Guichardet proves this result for unitary Γ-modules; but his proof, only appealing to the Banach isomorphism theorem, carries over without change to Banach Γ-modules.

1) *Let V be a Banach Γ-module with $V^\Gamma = 0$. The map*

$$H^1(\Gamma, V) \to \overline{H^1}(\Gamma, V)$$

is an isomorphism if and only if V does not almost have invariant vectors.

2) *Let $p \in [1, \infty[$. Assume that Γ is infinite. The map*

$$H^1(\Gamma, \ell^p(\Gamma)) \to \overline{H^1}(\Gamma, \ell^p(\Gamma))$$

is an isomorphism if and only if Γ is non-amenable. □

We will prove:

Proposition 2. *Let $p \in [1, \infty[$. Let $N \subset H \subset \Gamma$ be a chain of groups, where Γ finitely generated and N is infinite and normal in Γ. If $H^1(H, \ell^p(H)) = 0$, then $H^1(\Gamma, \ell^p(\Gamma)) = 0$.*

The following link between $\overline{H}^1_{(p)}(\Gamma)$ and $H^1(\Gamma, \ell^p(\Gamma))$ has been noticed by several people – see e.g. Lemma 3 in [BV97] (for $p = 2$ and Γ non-amenable), or in [Pul03] (in general). We give the easy argument for completeness.

Lemma 1. *For finitely generated Γ, there are isomorphisms*

$$D_p(\Gamma)/(i(\ell^p(\Gamma)) + \mathbb{C}) \simeq H^1(\Gamma, \ell^p(\Gamma)) \quad \textit{and} \quad \overline{H}^1_{(p)}(\Gamma) \simeq \overline{H^1}(\Gamma, \ell^p(\Gamma)).$$

Proof. The map $D_p(\Gamma) \to Z^1(\Gamma, \ell^p(\Gamma))\colon f \mapsto [g \mapsto \lambda_\Gamma(g)f - f]$ is continuous, with kernel the space $\mathbb{C}$ of constant functions, and the image of $i(\ell^p(\Gamma))$ is exactly $B^1(\Gamma, \ell^p(\Gamma))$. Moreover this map is onto because of the classical fact that $H^1(\Gamma, \mathcal{F}(\Gamma)) = 0$. □

Before proving Proposition 2 (for which we will actually give two proofs), we explain how to deduce Theorem 1 from it.

Proof of Theorem 1 *from Proposition* 2. 1) In view of Lemma 1, the assumption of Theorem 1 reads $\overline{H^1}(H, \ell^p(H)) = 0$. Since H is non-amenable, by Proposition 1 we have $H^1(H, \ell^p(H)) = 0$. By Proposition 2 we deduce $H^1(\Gamma, \ell^p(\Gamma)) = 0$. By Lemma 1 again, we get the conclusion.

2) If H is non-amenable, the result is a particular case of the first part. If H is amenable, then so is N, and the result follows from the Cheeger–Gromov vanishing theorem [CG86]: if a group Γ contains an infinite, amenable, normal subgroup, then *all* L^2-Betti numbers of Γ are zero. □

Important remark. Cheeger and Gromov [CG86] defined L^2-Betti numbers of a group Γ without any assumption on Γ, in particular not assuming Γ to be finitely generated. Using their definition, D. Gaboriau has shown us (private communication) a proof that $b^1_{(2)}(\Gamma) = 0$ always implies $\overline{H}^1(\Gamma, \ell^2(\Gamma)) = 0$. As a consequence, part 2) of Theorem 1 holds *even if H is not finitely generated*.

Our first proof of Proposition 2 will require the following lemma, which is classical for $p = 2$.

Lemma 2. *Let $p \in [1, \infty[$. Let H be a countable group. Let X be a countable set on which H acts freely. The following statements are equivalent:*

i) *H is amenable.*

ii) *The permutation representation λ_X of H on $\ell^p(X)$, almost has invariant vectors.*

Proof. We recall (see [Eym72]) that a group Γ is amenable if and only if it satisfies Reiter's condition (P_p), i.e. for every finite subset $F \subset \Gamma$ and $\epsilon > 0$, there exists $f \in \ell^p(\Gamma)$ such that $f \geq 0$, $\|f\|_p = 1$, and $\|\lambda_\Gamma(g)f - f\|_p < \epsilon$ for $g \in F$. In particular $\ell^p(\Gamma)$ almost has invariant vectors.

So if H is amenable, then $\ell^p(X)$ almost has invariant vectors since it contains $\ell^p(H)$ as a sub-module. This proves (i) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (i), we assume that $\ell^p(X)$ almost has invariant vectors and prove in 3 steps that H satisfies Reiter's property (P_1), so is amenable. So fix a finite subset $F \subset H$, and $\epsilon > 0$; find $f \in \ell^p(X)$, $\|f\|_p = 1$, such that $\|\lambda_X(h)f - f\|_p < \frac{\epsilon}{2p}$ for $h \in F$.

1) Replacing f with $|f|$, we may assume that $f \geq 0$.

2) Set $g = f^p$, so that $g \in \ell^1(X)$, $\|g\|_1 = 1$, $g \geq 0$. For $h \in F$, we have:

$$\begin{aligned}
\|\lambda_X(h)g - g\|_1 &= \sum_{x\in X} |f(h^{-1}x)^p - f(x)^p| \\
&\leq p \sum_{x\in X} |f(h^{-1}x) - f(x)|(f(h^{-1}x)^{p-1} + f(x)^{p-1}) \\
&\leq p\Big(\sum_{x\in X} |f(h^{-1}x) - f(x)|^p\Big)^{\frac{1}{p}} \Big(\sum_{x\in X}(f(h^{-1}x)^{p-1} + f(x)^{p-1})^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \\
&\leq p\|\lambda_X(h)f - f\|_p \Big(2^{\frac{1}{p-1}} \sum_{x\in X}(f(h^{-1}x)^p + f(x)^p)\Big)^{\frac{p-1}{p}} \\
&= 2p\|\lambda_X(h)f - f\|_p < \epsilon
\end{aligned}$$

where we have used consecutively[3] the inequalities

[3] The expert will recognize here the argument to pass from property (P_p) to property (P_1), as in [Eym72].

- $|a^p - b^p| \leq p|a-b|(a^{p-1} + b^{p-1})$ for $a, b > 0$,
- Hölder's inequality,
- $(a+b)^{\frac{p}{p-1}} \leq 2^{\frac{1}{p}}(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}})$ for $a, b > 0$,

and the fact that $\|f\|_p = 1$.

3) Let $(x_n)_{n\geq 1}$ be a set of representatives for the orbits of H in X. Define a function g_n on H by $g_n(h) = g(hx_n)$, and set $G = \sum_{n=1}^{\infty} g_n$. Then $G \geq 0$ and $\|G\|_1 = \sum_{h\in H}\sum_{n=1}^{\infty} g(hx_n) = \sum_{x\in X} g(x) = 1$. Moreover, for $h \in F$:

$$\|\lambda_H(h)G - G\|_1 = \sum_{\gamma\in H}\Big|\sum_{n=1}^{\infty}(g(h^{-1}\gamma x_n) - g(\gamma x_n))\Big| \leq \|\lambda_X(h)g - g\| < \epsilon$$

by the previous step. This establishes property (P_1) for H. □

First proof of Proposition 2 (homological algebra)

Claim. $H^1(H, \ell^p(\Gamma)|_H) = 0$. Choosing representatives for the right cosets of H in Γ, we identify $\ell^p(\Gamma)|_H$ in an H-equivariant way with the ℓ^p-direct sum $\oplus\ell^p(H)$ of $[\Gamma : H]$ copies of $\ell^p(H)$. Since cohomology commutes with finite direct sums, the claim is clear if $[\Gamma : H] < \infty$. So assume that $[\Gamma, H] = \infty$. If $b \in Z^1(H, \ell^p(\Gamma)|_H)$, write $b = (b_k)_{k\geq 1}$ where $b_k \in Z^1(H, \ell^p(H))$ for every $k \geq 1$. By assumption, for each k, there is a function $f_k \in \ell^p(H)$ such that $b_k(h) = \lambda_H(h)f_k - f_k$ for every $h \in H$. Set

$$B_N(h) = (\lambda_H f_1 - f_1, \ldots, \lambda_N(h)f_N - f_N, 0, 0, \ldots)$$

so that $B_N \in B^1(H, \ell^p(\Gamma)|_H)$ and B_N converges to b pointwise on H, for $N \to \infty$. This already shows that $\overline{H}^1(H, \ell^p(\Gamma)|_H) = 0$. Notice now that, by Proposition 1 (2), the assumption $H^1(H, \ell^p(H)) = 0$ implies that H is non-amenable. By Lemma 2 applied to $X = \Gamma$, this means that $\ell^p(\Gamma)|_H$ does not almost have invariant vectors. By Proposition 1 (1), we get $H^1(H, \ell^p(\Gamma)|_H) = 0$, proving the claim.

Recall from group cohomology (see e.g. 8.1 in [Gui80]) that, for any Γ-module V, there is an exact sequence

$$0 \to H^1(\Gamma/N, V^N) \stackrel{i_*}{\to} H^1(\Gamma, V) \xrightarrow{\mathrm{Rest}_\Gamma^N} H^1(N, V|_N)^{\Gamma/N}$$

where $i\colon V^N \to V$ denotes the inclusion. In particular, if $V^N = 0$, then the restriction map

$$\mathrm{Rest}_\Gamma^N\colon H^1(\Gamma, V) \to H^1(N, V|_N)$$

is injective. We apply this with $V = \ell^p(\Gamma)$ (noticing that $V^N = 0$ as N is infinite).

Consider then the composition of restriction maps

$$H^1(\Gamma, \ell^p(\Gamma)) \xrightarrow{\mathrm{Rest}_\Gamma^H} H^1(H, \ell^p(\Gamma)|_H) \xrightarrow{\mathrm{Rest}_H^N} H^1(N, \ell^p(\Gamma)|_N);$$

this composition is Rest_Γ^N, which is injective as we just saw. On the other hand, by the claim this composition is also the zero map. So $H^1(\Gamma, \ell^p(\Gamma)) = 0$, as was to be established. □

Second proof of Proposition 2 (geometry). This proof works under the extra assumption that H is finitely generated. Fix finite generating sets T for H, S for Γ, with $T \subset S$, and consider the Cayley graph $\mathcal{G}(\Gamma, S)$ and its coboundary operator $d_\Gamma : \mathcal{F}(\Gamma) \to \mathcal{F}(\mathbb{E}_\Gamma)$. Then $D_p(\Gamma) = \{f \in \mathcal{F}(\Gamma) : d_\Gamma f \in \ell^p(\mathbb{E}_\Gamma)\}$. Similarly, let d_H be the coboundary operator associated with the Cayley graph $\mathcal{G}(H, T)$.

Fix $f \in D_p(\Gamma)$; the goal is to show that $f \in \ell^p(\Gamma) + \mathbb{C}$. Let $(g_i)_{i \in I}$ be a set of representatives for the right cosets of H in Γ, so that $\Gamma = \coprod_{i \in I} Hg_i$. For $i \in I$, set $f_i(x) = f(xg_i)$ $(x \in H)$. Then

$$\begin{aligned} \|d_H(f_i)\|_p^p &= \sum_{x \in H} \sum_{s \in T} |f(sxg_i) - f(xg_i)|^p \\ &\leq \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p \\ &= \|d_\Gamma f\|^p < \infty, \end{aligned}$$

i.e. $f_i \in D_p(H)$. Using our assumption and Lemma 1, we may write

$$f_i = h_i + u_i$$

where $h_i \in \ell^p(H)$ and $u_i \in \mathbb{C}$. Define functions h and u on Γ by $h(xg_i) = h_i(x)$ and $u(xg_i) = u_i$ $(x \in H)$.

First claim. $h \in \ell^p(\Gamma)$. Indeed, since H is non-amenable (by Proposition 1), there exists a constant $C > 0$ (depending only on p, H, T) such that for every $i \in I$:

$$\|h_i\|_p \leq C \|d_H(h_i)\|_p.$$

Then summing over i we obtain

$$\begin{aligned} \|h\|_p^p &= \sum_{i \in I} \|h_i\|_p^p \\ &\leq C^p \sum_{i \in I} \|d_H(f_i)\|_p^p = C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |h_i(sx) - h_i(x)|^p \\ &= C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |f_i(sx) - f_i(x)|^p = C^p \sum_{x \in \Gamma} \sum_{s \in T} |f(sx) - f(x)|^p \end{aligned}$$

$$\leq C^p \sum_{x\in\Gamma}\sum_{s\in S} |f(sx) - f(x)|^p$$
$$= C^p \|d_\Gamma(f)\|_p^p < \infty.$$

Second claim. u is constant. Indeed, since $f = h + u$, and $d_\Gamma(f), d_\Gamma(h) \in \ell^p(\mathbb{E}_\Gamma)$, we have $d_\Gamma(u) \in \ell^p(\mathbb{E}_\Gamma)$. In particular this implies, for fixed indices $i, j \in I$:

$$\begin{aligned}\infty > \sum_{x\in N} |u((g_j g_i^{-1})xg_i) - u(xg_i)|^p &= \sum_{x\in N} |u((g_j g_i^{-1})xg_i) - u_i|^p \\ &= \sum_{x\in N} |u(x(g_j g_i^{-1})g_i) - u_i|^p\end{aligned}$$

since N is normal in Γ. The latter sum is equal to

$$\sum_{x\in N} |u_j - u_i|^p < \infty.$$

Since N is infinite, this forces $u_i = u_j$, i.e. u is constant.

The first and the second claim together prove Proposition 2. □

3. Some results of W. Lück

The following result was obtained by Lück in [Lue94], Theorem 2.1. We recall his short, elegant argument.

Lemma 3. *Let N be a finitely generated group, and let α be an automorphism of N. Let $H = N \rtimes_\alpha \mathbb{Z}$ be the corresponding semi-direct product. Then $b^1_{(2)}(H) = 0$.*

Proof. The proof depends on two classical properties of the L^2-Betti numbers for a finitely generated group Γ:

- $b^1_2(\Gamma) \leq d(\Gamma)$, where $d(\Gamma)$ denotes the minimal number of generators of Γ;
- if Λ is a subgroup of finite index d in Γ, then $b^k_{(2)}(\Lambda) = d \cdot b^k_{(2)}(\Gamma)$.

Let then $p\colon H \to \mathbb{Z}$ denote the quotient map; for $n \geq 1$, set $H_n = p^{-1}(n\mathbb{Z})$, a subgroup of index n in H. Then:

$$n \cdot b^1_{(2)}(H) = b^1_{(2)}(H_n) \leq d(H_n) \leq d(N) + 1.$$

Since this holds for every $n \geq 1$, the lemma follows. □

Proof of Corollary 1. Since Γ/N is not a torsion group, we find a subgroup H of Γ, containing N, such that H/N is infinite cyclic. Since N is finitely generated, we have $b^1_{(2)}(H) = 0$, by Lemma 3. The result follows then immediately from Theorem 1. □

Example 1. We point out that Lemma 3 has no analogue in L^p-cohomology, with $p \neq 2$. To see it, let M be a 3-dimensional, compact, hyperbolic manifold which fibers over the circle. Denote by Σ_g the fiber of that fibration: this is a closed Riemann surface of genus $g \geq 2$. Then the fundamental group $\Gamma = \pi_1(M)$ admits a semi-direct product decomposition $\Gamma = \pi_1(\Sigma_g) \rtimes \mathbb{Z}$, so that $\overline{H}^1_{(2)}(\Gamma) = 0$ by Lemma 2. However

$$\inf\{p \geq 1 : \overline{H}^1_{(p)}(\Gamma) \neq 0\} = 2,$$

as was proven by Pansu [Pan89].

4. Proof of Theorem 2

Denote by ∂X the (Gromov) boundary of X. Let $\Lambda = \overline{\Gamma o} \cap \partial X$ be the limit set of Γ in ∂X (the closure of Γo is taken in the compact set $X \cup \partial X$).

Since X is a CAT(-1) space, its boundary carries a natural metric d (called a *visual metric*) which can be defined as follows (see [Bou95], Théorème 2.5.1); for every ξ and η in ∂X:

$$d(\xi, \eta) = e^{-(\xi|\eta)},$$

where $(\cdot\,|\,\cdot)$ denotes the Gromov product on ∂X based on o, namely

$$(\xi|\eta) = \lim_{(x,y)\to(\xi,\eta)} \frac{1}{2}(|o - x| + |o - y| - |x - y|).$$

Observe that there exists a constant B such that for every $g \in \Gamma$ there is a point ξ in ∂X with $d(go, [o, \xi)) \leq B$. Indeed this property does not depend on the choice of the origin o. So we choose o on a bi-infinite geodesic (η_1, η_2). Then go belongs to $(g\eta_1, g\eta_2)$. Now since X is Gromov-hyperbolic, one of the two points $g\eta_1$ or $g\eta_2$ satisfies the claim.

Let u be a Lipschitz function of $(\partial X, d)$ which is non-constant on Λ ; such functions do exist since Λ is not reduced to a point. Following G. Elek [Ele97], let f be the function on Γ defined by $f(g) = u(\xi_g)$, where ξ_g is a point in ∂X such that $d(g^{-1}o, [o, \xi_g)) \leq B$.

Claim. $f \in D_p(\Gamma)$ for $p > \max\{1, e(\Gamma)\}$. Indeed we have

$$\begin{aligned}\|f\|_{D_p}^p &= \sum_{s\in S}\sum_{g\in\Gamma} |f(sg) - f(g)|^p = \sum_{s\in S}\sum_{g\in\Gamma} |u(\xi_{sg}) - u(\xi_g)|^p \\ &\le C \sum_{s\in S}\sum_{g\in\Gamma} [d(\xi_{sg}, \xi_g)]^p = C \sum_{g\in\Gamma}\sum_{s\in S} e^{-p(\xi_{sg}|\xi_g)} \\ &\le D \sum_{g\in\Gamma} e^{-p|g^{-1}o-o|} < +\infty,\end{aligned}$$

where C, D are constants depending only on u, B and S. The details for the first inequality in the last line are the following. Observe that $|(sg)^{-1}o - g^{-1}o| = |s^{-1}o - o|$ is bounded above by an absolute constant. This implies that if x_g and x_{sg} respectively denote the points on $[o, \xi_g)$ and $[o, \xi_{sg})$ whose distance from o is equal to $|g^{-1}o - o|$, then $|x_g - x_{sg}|$ is bounded above by an absolute constant. Now with the triangle inequality

$$|x - y| \le |x - x_{sg}| + |x_{sg} - x_g| + |x_g - y|,$$

and from the definition of the Gromov product, it follows that

$$(\xi_{sg}|\xi_g) \ge \frac{1}{2}(|o - x_{sg}| + |o - x_g| - |x_{sg} - x_g|),$$

so that $(\xi_{sg}|\xi_g)$ is bounded below by $|g^{-1}o - o|$ plus an absolute additive constant. This proves the claim.

Since Λ has at least 3 points, the group Γ is non-amenable (namely it is well-known that Λ is a minimal set, and that an amenable group stabilises one or two points in ∂X). So by Proposition 1 and by Lemma 1, we must prove that f does not belong to $i(\ell^p(\Gamma)) + \mathbb{C}$. Assume it does, then $f(g)$ tends to a constant number when the length of g in Γ tends to $+\infty$. This contradicts the fact that u is non-constant on Λ. □

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Note added in proof. The following example, suggested by F. Paulin, shows that Corollary 2 fails for lattices in SO(2, 1). Start with the free group $\mathbb{F}_2$ on two generators. Embed it as a lattice G in SO(2, 1), so that $e(G) = 1$. On the other hand, let X_λ be the regular tree of degree 4, with edge length $\lambda > 0$. This is a proper CAT(−1) space. Let $\mathbb{F}_2$ act as a properly discontinuous group Γ of isometries of X_λ, by viewing X_λ as the Cayley tree of $\mathbb{F}_2$. Then $e(\Gamma) = \frac{\log 3}{\lambda}$, which is less than 1 for λ large enough.

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