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Hopf plumbing and minimal diagrams

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Abstract. We determine minimal diagrams and braid indices for fibered arborescent knots arising from plumbing positive or negative Hopf bands along trees.

1. Introduction

One of the most striking applications of the Jones polynomial concerns crossing numbers of knots. Indeed, several proofs of the first Tait conjecture were announced shortly after the discovery of the Jones polynomial (see [8], [11] and [15]). Yet there is another bound for the crossing number of knots coming from the HOMFLY polynomial (see [12] and [4]). In this paper we show that the latter estimate works especially well for fibered knots. More precisely, we exhibit a large class of fibered arborescent knots and find minimal diagrams for them. This class contains 50 knots of Rolfsen's table ([13]) and is not contained in the class of Montesinos links, for which minimal diagrams are known, already (see [9]).

Let Γ be a planar tree with signed vertices, as shown in Figure 1.

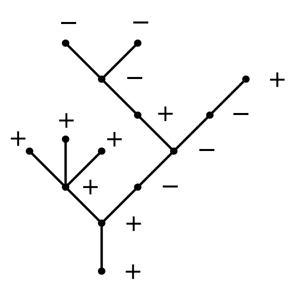


Figure 1. A planar tree with signs.

There is a well-known procedure, called Hopf plumbing, which associates a link $K(\Gamma)$ to a planar tree with signs Γ:

(1) Draw a positive or negative Hopf band (see Figure 2) at each vertex of Γ, according to its sign.

(2) Plumb the Hopf bands together along the edges of Γ, as shown in Figure 3.

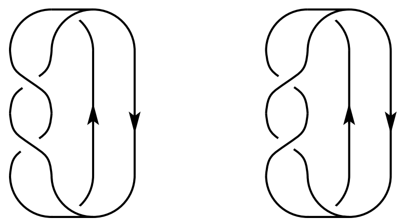


Figure 2. A positive and a negative Hopf band.

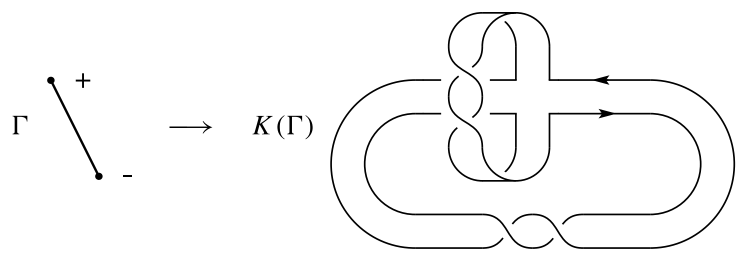


Figure 3. Plumbing along an edge.

We shall determine the minimal crossing number and draw minimal diagrams of knots associated with trees with signs. For this purpose we have to introduce two quantities of trees with signs Γ. Deleting all negative vertices of Γ we get a forest Γ_+, i.e. a union of trees (with positive signs). An analogous procedure (deleting all positive vertices of Γ) yields Γ_-. We set

$$P(\Gamma) = \sum_{T \in \Gamma_+} s(T),$$

$$N(\Gamma) = \sum_{T \in \Gamma_-} s(T),$$

where the sums run over all trees T of the forests Γ_+ or Γ_-, respectively, and the function s is defined as follows:

$$s(T) = 1 + \min\{\#S \mid S \subset E(T),\ T - S \text{ has no vertices of valency greater than two}\}.$$

Here $E(T)$ is the set of edges of the tree T. We remark that s depends on the abstract structure of a tree only, whereas P and N depend on the abstract structure and the signs of a tree. Now we are ready to state our main result about the minimal crossing number $c(K(\Gamma))$.

Theorem 1. *Let Γ be a tree with signs such that $K(\Gamma)$ is a knot with one component. Then*

$$c(K(\Gamma)) = V(\Gamma) + P(\Gamma) + N(\Gamma).$$

Here $V(\Gamma)$ is the number of vertices of Γ.

Remark. The class of knots associated with trees with signs contains the class of slalom divide knots (see N. A'Campo [1] for a definition of slalom divide knots). Indeed, according to M. Hirasawa ([5], see also M. Ishikawa [6]), we know that slalom divide knots correspond to knots of certain slalom trees with positive signs only. Thus Theorem 1 proves a conjecture of M. Ishikawa ([7]) about the minimal crossing number of slalom divide knots.

It is well-known that every knot can be represented as the closure of a braid. The minimal number of strands among all the braids representing K is called the braid index $b(K)$ of the knot K.

Theorem 2. *Let Γ be a tree with signs such that $K(\Gamma)$ is a knot with one component. Then*

$$b(K(\Gamma)) = P(\Gamma) + N(\Gamma) + 1.$$

In the following section we prove a recursive formula for the function s for trees, which we shall use in the proof of Theorem 1. In the third section we present a lower bound for the minimal crossing number of knots and prove the inequality

$$c(K(\Gamma)) \geqslant V(\Gamma) + P(\Gamma) + N(\Gamma).$$

The proofs of Theorems 1 and 2 will be accomplished in the fourth section.

2. The function s for trees

Let T be a tree, $E(T)$ its set of edges. We call $S \subset E(T)$ a *grinding subset* for T, if $T - S$ has no vertex of valency greater than two. Thus the function $s(T)$ equals one plus the minimal cardinality among all grinding subsets for T.

Let k be an outer edge of T. Figure 4 shows a section of T near k.

Two vertices of T are incident with k; they are labelled v_1 and v_2. The trees of the forest $T - \{v_1, v_2\}$ are labelled $T_1, T_2, \dots, T_m$. They are attached to the vertex v_2 by the edges $k_1, k_2, \dots, k_m$.

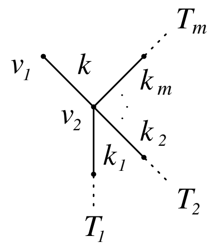


Figure 4. A section of T.

Proposition.

$$s(T) = \max\left\{s(T - v_1),\ \textstyle\sum_{i=1}^{m} s(T_i)\right\}.$$

Proof. First we observe that

$$s(T - v_1) \leqslant s(T) \leqslant 1 + s(T - v_1).$$

Further, a careful inspection of the definition of s shows that

$$\sum_{i=1}^{m} s(T_i) \leqslant s(T) \leqslant 1 + \sum_{i=1}^{m} s(T_i).$$

Indeed, if S is a grinding subset for T, then $S - \{k, k_1, k_2, \dots k_m\}$ is a grinding subset for the forest $T - \{v_1, v_2\}$. This implies the first inequality, since $\#S \cap \{k, k_1, k_2, \dots k_m\} \geqslant m - 1$. Conversely, if S is a grinding subset for the forest $T - \{v_1, v_2\}$, then $S \cup \{k_1, k_2, \dots k_m\}$ is a grinding subset for T. This implies the second inequality.

In order to prove the proposition, we have to exclude the case

$$s(T) = 1 + s(T - v_1) = 1 + \sum_{i=1}^{m} s(T_i).$$

Suppose $s(T) = 1 + s(T - v_1)$. Let S be a minimal grinding subset for $T - v_1$, i.e. a grinding subset for $T - v_1$ of minimal cardinality. Since $s(T) = 1 + s(T - v_1)$, the vertex v_2 must have valency two in $T - v_1 - S$; say $k_1, k_2 \in E(T - v_1 - S)$. We claim that $S \cap E(T_i)$ is a minimal grinding subset for T_i, $1 \leqslant i \leqslant m$. If $i \neq 1, 2$, this is obvious, since then $k_i \notin E(T - v_1 - S)$. The case $i = 1$ (and, analogously, the case $i = 2$) needs a special consideration. Suppose T_1 had a grinding subset $S_1 \subset E(T_1)$ of smaller cardinality than $S \cap E(T_1)$. Then the subset

$$\widetilde{S} = S - (S \cap E(T_1)) \cup S_1 \cup \{k_1\} \subset E(T - v_1)$$

were a minimal grinding subset for $T - v_1$. Furthermore, the vertex v_2 would have valency one in $T - v_1 - \widetilde{S}$, which is a contradiction to the assumption $s(T) = 1 + s(T - v_1)$. Finally, $S \cup \{k\}$ is a minimal grinding subset for T with

$$\#S \cup \{k\} = m - 1 + \sum_{i=1}^{m} \#S \cap E(T_i) = m - 1 + \sum_{i=1}^{m} (s(T_i) - 1).$$

Hence

$$s(T) = \sum_{i=1}^{m} s(T_i). \qquad \square$$

3. The HOMFLY polynomial and crossing numbers

The HOMFLY polynomial is a Laurent polynomial in two variables a and z, which is normalized to one for the trivial knot and satisfies the following relation (see [2]):

$$\frac{1}{a} P_{\times}(a, z) - a P_{\times}(a, z) = z P_{)(}(a, z).$$

We denote the minimal degree in the variable a of $P_K(a, z)$ by $e(K)$, the maximal degree by $E(K)$. The following theorem is mainly due to K. Murasugi ([12]); an explicit formulation can be found in H. Gruber (see Lemma 3.2. and Corollary 6.1. in [4]).

Theorem 3. (i) $c(K) \geqslant \frac{1}{2}(E(K) - e(K)) + 2g(K)$.

(ii) *If* $c(K) = \frac{1}{2}(E(K) - e(K)) + 2g(K)$, *then*

$$b(K) = \frac{1}{2}(E(K) - e(K)) + 1.$$

Here $g(K)$ *is the minimal genus among all orientable surfaces spanning the knot* K.

Proof. Let D be a minimal diagram of the knot K. There is a natural Seifert surface $S(D)$ associated with D. Its Euler characteristic $\chi(S(D))$ equals $s(D) - c(D)$, where $s(D)$ and $c(D)$ denote the number of Seifert circles and the number of crossings of the diagram D, respectively. We conclude

$$c(K) = c(D) = s(D) - \chi(S(D)) \geqslant b(K) + 2g(K) - 1.$$

The last inequality follows from a theorem of S. Yamada ([17]) which asserts that the number of Seifert circles in any diagram of a knot K cannot be smaller than the braid index of K. An alternative simple proof of this fact is due to P. Vogel ([16]). Now both statements of Theorem 3 follow immediately by Morton's inequality (see [10]; a thorough discussion on braid index criteria can be found in A. Stoimenow [14]):

$$b(K) \geqslant \frac{1}{2}(E(K) - e(K)) + 1. \qquad \square$$

We observe that the plumbing construction of Hopf bands along a tree provides a natural fiber surface of minimal genus (see [3]). Therefore

$$2g(K(\Gamma)) = V(\Gamma),$$

in case $K(\Gamma)$ is a knot. Comparing the first statement of Theorem 3 and Theorem 1, we see that it remains to settle the equation

$$E(K(\Gamma)) - e(K(\Gamma)) = 2P(\Gamma) + 2N(\Gamma)$$

to establish the desired lower bound for the crossing number of Theorem 1. It turns out that the range in the variable a of $P_K(a, z)$ is not affected by the specialization $z = 1$. Therefore we restrict our computations to the polynomial in one variable

$$O_K(a) = P_K(a, 1).$$

Further we redefine $e(K)$ and $E(K)$ as the minimal and maximal degree of $O_K(a)$, respectively. In order to compute $e(K)$ and $E(K)$, we have to keep book on the signs of the extremal coefficients very carefully. Thus let us write $\sigma_e(K)$ and $\sigma_E(K)$ for the signs of the minimal and maximal coefficients of $O_K(a)$, respectively.

Lemma. *Let Γ be a tree with signs (here $K(\Gamma)$ may have several components).*

(i) $e(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) - 2N(\Gamma)$.

(ii) $E(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) + 2P(\Gamma)$.

(iii) $\sigma_e(K(\Gamma)) = (-1)^{V(\Gamma_-)+N(\Gamma)}$.

(iv) *$O_{K(\Gamma)}(a)$ is either even or odd and alternating, i.e. there are natural numbers $c_k \neq 0$, $e \leqslant k \leqslant E$ such that*

$$O_{K(\Gamma)}(a) = (-1)^{\sigma_e} \sum_{l=0}^{\frac{1}{2}(E-e)} (-1)^l c_{e+2l}\, a^{e+2l}.$$

In particular, $\sigma_E(K(\Gamma)) = (-1)^{V(\Gamma_-)+P(\Gamma)}$.

Proof. First we observe that the lemma is true if Γ has one vertex only. Indeed,

$$O_{K(\bullet\, +)} = 2a - a^3,$$

$$O_{K(\bullet\,-)} = a^{-3} - 2a^{-1}.$$

Now let us assume that all the statements of the lemma are true for all trees Γ with n ($\geqslant 1$) vertices, at most. We have to verify the statements for a tree Γ with $n+1$ ($\geqslant 2$) vertices. Let k be an outer edge of Γ, as shown in Figure 4. As before, two vertices (v_1 and v_2) are incident with k. In addition, each vertex carries a sign. Now let us remember the defining relation for $O_K(a)$:

$$\frac{1}{a} O_{\times}(a) - a O_{\times}(a) = O_{)(}(a). \tag{1}$$

We shall apply this relation to a crossing of the Hopf band at the vertex v_1. For this purpose, we have to understand the effect of the following two operations:

(1) Smoothing a crossing at v_1 (in an oriented manner) causes a small collapse; the Hopf band at v_1 disappears. At this point, it may be helpful to look at Figure 3.

$$K(\Gamma) \longrightarrow K(\Gamma - \{v_1\}).$$

(2) A crossing change at v_1 causes a big collapse; the Hopf bands at v_1 and v_2 disappear, and we end up with a connected sum of all the arborescent links corresponding to the trees of the forest $\Gamma - \{v_1, v_2\}$. We notice that these trees $T_1, T_2, \ldots, T_m$ are trees with signs.

$$K(\Gamma) \longrightarrow K(T_1)\# \ldots \# K(T_m),\ T_i \in \Gamma - \{v_1, v_2\}.$$

After these preparations, we are in a position to carry out the inductive step. However, we have to consider four cases corresponding to the signs of the vertices v_1 and v_2, separately.

Case 1. v_1 and v_2 carry negative signs. Relation (1) for $K(\Gamma)$ at v_1 reads

$$\frac{1}{a} O_{K(T_1)}(a) \ldots O_{K(T_m)}(a) - a O_{K(\Gamma)}(a) = O_{K(\Gamma - \{v_1\})}(a),$$

since the HOMFLY polynomial is multiplicative under the connected sum operation. Thus

$$O_{K(\Gamma)}(a) = \frac{1}{a^2} O_{K(T_1)}(a) \ldots O_{K(T_m)}(a) - \frac{1}{a} O_{K(\Gamma - \{v_1\})}(a). \tag{2}$$

We remark that

$$P(\Gamma) = P(\Gamma - \{v_1\}) = \sum_{i=1}^{m} P(T_i).$$

Therefore, the maximal degrees and the signs of the maximal coefficients of $\frac{1}{a^2} O_{K(T_1)}(a) \ldots O_{K(T_m)}(a)$ and $-\frac{1}{a} O_{K(\Gamma - \{v_1\})}(a)$ agree, and we conclude that $O_{K(\Gamma)}(a)$ is either even or odd and alternating. Furthermore,

$$E(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) + 2P(\Gamma),$$

$$\sigma_E(K(\Gamma)) = (-1)^{V(\Gamma_-)+P(\Gamma)}.$$

As to the minimal degree, we need the statement of the proposition in the second section:

$$N(\Gamma) = \max\left\{N(\Gamma - \{v_1\}),\ \textstyle\sum_{i=1}^m N(T_i)\right\}.$$

In any case, we conclude

$$e(K(\Gamma)) = V(\Gamma_+) - V(\Gamma_-) - 2N(\Gamma).$$

Case 2. v_1 carries a negative sign, v_2 carries a positive sign. As in the first case, equation (2) holds for $O_{K(\Gamma)}(a)$. The crucial quantities are

$$P(\Gamma) = P(\Gamma - \{v_1\}),$$

$$\left|P(\Gamma) - \sum_{i=1}^m P(T_i)\right| \leqslant 1,$$

$$N(\Gamma) = 1 + N(\Gamma - \{v_1,\}) = 1 + \sum_{i=1}^m N(T_i).$$

Thus $\frac{1}{a^2}O_{K(T_1)}(a)\dots O_{K(T_m)}(a)$ contributes to the required minimal degree of $O_{K(\Gamma)}(a)$ with the correct sign, whereas $-\frac{1}{a}O_{K(\Gamma-\{v_1\})}(a)$ contributes to the required maximal degree of $O_{K(\Gamma)}(a)$ with the correct sign. In particular, $O_{K(\Gamma)}(a)$ is either even or odd and alternating.

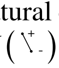
The remaining two cases (i.e. v_1 carries a positive sign) can be treated analogously; no new phenomena occur. Alternatively, we may consider the mirror image of $K(\Gamma)$ and replace the variable a of $O_{K(\Gamma)}(a)$ by $-a^{-1}$. □

As an immediate consequence of the lemma, the width of $O_{K(\Gamma)}(a)$ equals $2P(\Gamma) + 2N(\Gamma)$. However, the width of $P_{K(\Gamma)}(a, z)$ in the variable a could still be greater than $2P(\Gamma) + 2N(\Gamma)$. In any case, due to the first statement of Theorem 3, we get the required lower bound for the number of crossings:

$$c(K(\Gamma)) \geqslant V(\Gamma) + P(\Gamma) + N(\Gamma),$$

in case $K(\Gamma)$ is a knot.

4. Construction of minimal diagrams

In this section, we construct minimal diagrams for knots associated with trees with signs and accomplish the proofs of Theorems 1 and 2. First we remark that a knot $K(\Gamma)$ has a natural diagram with $2V(\Gamma)$ crossings. This fact is illustrated in Figure 5 for the knot $K\big($$\big)$.

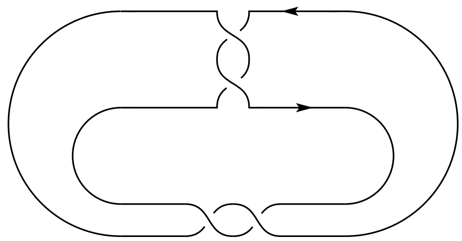


Figure 5. A natural diagram of $K(\Gamma)$.

If the signs of the vertices of Γ are distributed in an alternating pattern, then $P(\Gamma) + N(\Gamma) = V(\Gamma)$, and Theorem 1 is true. This is no surprise, since in this case, the natural diagram of $K(\Gamma)$ is alternating. Now suppose Γ contains an edge k with two positive vertices. Then we can change the natural diagram of $K(\Gamma)$ in order to eliminate one crossing, as shown in Figure 6. The two squares contain the parts of the diagram of $K(\Gamma)$ corresponding to $\Gamma - k$; we may possibly have to flype a square through the corresponding twisted band before we can apply such a reduction move.

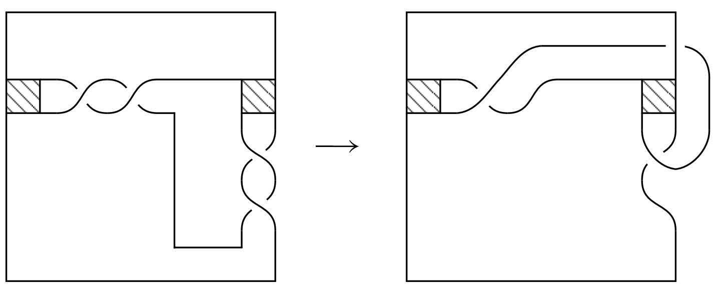


Figure 6. Eliminating a crossing at a positive edge.

However, the Hopf bands of the two involved vertices are damaged by this process; they both 'lose one crossing'. Hence we cannot apply this procedure to more than two edges containing the same vertex. If T is a subtree of Γ_+, then we can choose $V(T) - P(T)$ edges of T such that their union does not contain any vertex of valency greater than 2 (by definition of the function P). Thus we can eliminate $V(T) - P(T)$ of the crossings associated with T, even if T is a single point (since then $V(T) - P(T) = 1 - 1 = 0$). The same is true for subtrees of Γ_-, of course, replacing $P(T)$ by $N(T)$. Altogether, we can eliminate $V(\Gamma) - P(\Gamma) - N(\Gamma)$ crossings of the natural diagram of $K(\Gamma)$, ending up with

$$2V(\Gamma) - (V(\Gamma) - P(\Gamma) - N(\Gamma)) = V(\Gamma) + P(\Gamma) + N(\Gamma)$$

crossings. As a consequence, Theorem 1 is true. Finally, Theorem 2 is a consequence of the second statement of Theorem 3.

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5. Table

The subsequent table lists all knots associated with trees with signs up to ten crossings. We need only consider eight types of trees, on account of Theorem 1. They are drawn in the first column of Table 1. The second column lists all knots with 10 or fewer crossings that arise from the trees of the first column, depending on different choices of signs. We use Rolfsen's notation ([13]).

We may ask ourselves for which knots the inequality

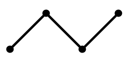
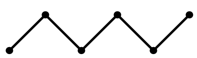
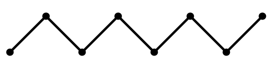
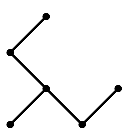
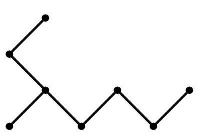
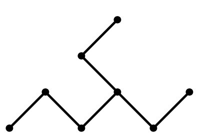
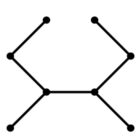
$$c(K) \geqslant \frac{1}{2}(E(K) - e(K)) + 2g(K)$$

is actually an equality. A result of K. Murasugi says that equality holds for alternating fibered knots (see [12], Theorem A and Corollary 2). It turns out that equality holds for 86 knots, up to ten crossings, namely the 50 knots of Table 1, and 36 more knots:

8_{16}, 8_{17}, 8_{18}, 9_{29}, 9_{32}, 9_{33}, 9_{34}, 9_{40}, 9_{47}, 10_5, 10_9, 10_{17}, 10_{45}, 10_{69}, 10_{75}, 10_{81}, 10_{82}, 10_{85}, 10_{88}, 10_{89}, 10_{91}, 10_{94}, 10_{96}, 10_{99}, 10_{100}, 10_{104}, 10_{105}, 10_{106}, 10_{107}, 10_{109},10_{110}, 10_{112}, 10_{115}, 10_{116}, 10_{118}, 10_{123}.

We remark that all these knots are fibered. However, there exist non-fibered knots whose minimal crossing number equals $\frac{1}{2}(E(K) - e(K)) + 2g(K)$, e.g. the knots $11a263$ and $14n6302$ (here we use the Dowker–Thistlethwaite numbering). These examples were found by A. Stoimenow.

Table 1. Trees and knots up to 10 crossings.

tree	knots
	3_1, 4_1
	5_1, 6_2, 6_3, 7_6, 7_7, 8_{12}
	7_1, 8_2, 8_7, 8_9, 9_{11}, 9_{17}, 9_{20}, 9_{26}, 9_{27}, 9_{31}, 10_{29}, 10_{41}, 10_{42}, 10_{43}, 10_{44}
	9_1, 10_2
	8_5, 8_{10}, 8_{19}, 9_{22}, 9_{24}, 9_{28}, 9_{30}, 9_{36}, 9_{43}, 10_{59}, 10_{60}, 10_{70}, 10_{71}, 10_{73}, 10_{78}, 10_{138}
	10_{46}, 10_{47}, 10_{48}, 10_{124}
	10_{62}, 10_{64}, 10_{139}
	10_{79}, 10_{152}

References

[1] N. A'Campo, Planar trees, slalom curves and hyperbolic knots. *Inst. Hautes Études Sci. Publ. Math.* **88** (1998), 171–180. Zbl 0960.57008 MR 1733330

[2] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet, A. Ocneanu, A new polynomial invariant of knots and links. *Bull. Amer. Math. Soc.* **12** (1985), 239–246. Zbl 0572.57002 MR 0776477

[3] D. Gabai, Genera of the arborescent links. *Mem. Amer. Math. Soc.* **59** (1986), no. 339, i–viii and 1–98. Zbl 0585.57003 MR 0823442

[4] H. Gruber, Estimates for the minimal crossing number. `arXiv: math.GT/0303273`.

[5] M. Hirasawa, Visualization of A'Campo's fibered links and unknotting operations. *Topology Appl.* **121** (2002), 287–304. Zbl 1016.57006 MR 1903697

[6] M. Ishikawa, Plumbing constructions of connected divides and the Milnor fibers of plane curve singularities. *Indag. Math.* **13** (2002), 499–514. Zbl 1036.57011 MR 2015834

[7] M. Ishikawa, Links of plane curves, fiberedness, quasipositivity and Jones polynomials. Doctoral thesis, 2001, `www.math.titech.ac.jp/~ishikawa/index_e.html`.

[8] L. H. Kauffman, State models and the Jones polynomial. *Topology* **26** (1987), 395–407. Zbl 0622.57004 MR 0899057

[9] W. B. R. Lickorish, M. B. Thistlethwaite, Some links with nontrivial polynomials and their crossing-numbers. *Comment. Math. Helv.* **63** (1988), 527–539. Zbl 0686.57002 MR 0966948

[10] H. R. Morton, Seifert circles and knot polynomials. *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 107–109. Zbl 0588.57008 MR 0809504

[11] K. Murasugi, Jones polynomials and classical conjectures in knot theory. *Topology* **26** (1987), 187–194. Zbl 0628.57004 MR 0895570

[12] K. Murasugi, On the braid index of alternating links. *Trans. Amer. Math. Soc.* **326** (1991), 237–260. Zbl 0751.57008 MR 1000333

[13] D. Rolfsen, *Knots and Links.* Mathematics Lecture Series 7, Publish or Perish, Berkeley, Ca, 1976. Zbl 0339.55004 MR 0515288

[14] A. Stoimenow, On the crossing number of positive knots and braids and braid index criteria of Jones and Morton-Williams-Franks. *Trans. Amer. Math. Soc.* **354** (2002), 3927–3954. Zbl 1002.57013 MR 1926860

[15] M. B. Thistlethwaite, A spanning tree expansion of the Jones polynomial. *Topology* **26** (1987), 297–309. Zbl 0622.57003 MR 0899051

[16] P. Vogel, Representation of links by braids: a new algorithm. *Comment. Math. Helv.* **65** (1990), 104–113. Zbl 0703.57004 MR 1036132

[17] S. Yamada, The minimal number of Seifert circles equals the braid index of a link. *Invent. Math.* **89** (1987), 347–356. Zbl 0634.57004 MR 0894383

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