Dynamical dessins are dense

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Abstract. We apply a recent result of the first author to prove the following result: any continuum in the plane can be approximated arbitrarily closely in the Hausdorff topology by the Julia set of a postcritically finite polynomial with two finite postcritical points.

1. Introduction

Given compact subsets $A, B \subset \mathbb{C}$ their Hausdorff distance $d(A, B)$ is given by

$$d(A, B) := \inf \{ r : A \subset N_r(B), \ B \subset N_r(A) \}$$

where $N_r(A), N_r(B)$ denote the $r$-neighborhoods of $A$ and $B$, respectively. Given a polynomial $g \in \mathbb{C}[z]$, we denote by $g^j$ the $j$th iterate of $g$, and define its

- filled-in Julia set $K(g) := \{ z : g^j(z) \not\to \infty \}$, and
- Julia set $J(g) := \partial K(g)$.

K. Lindsey ([4], Theorem 2.2) has shown:

Theorem 1. Given any Jordan curve $\mathcal{J}$ bounding a closed topological disk $K$ and any $\epsilon > 0$, there exists a polynomial $g \in \mathbb{C}[z]$ such that

1. $d(K(g), K) < \epsilon$,
2. $d(J(g), \mathcal{J}) < \epsilon$.

The proof is constructive; the above paper illustrates the result of applying the method of proof to a Jordan domain $K$ outlining the figure of a cat, yielding a polynomial $g$ of degree 301.

In this note, a continuum is a compact connected subset of $\mathbb{C}$. It is elementary to show that any continuum can be approximated arbitrarily closely in the Hausdorff topology by a Jordan curve. Conclusion (2) of Theorem 1 then implies:

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Corollary 1. Given any continuum $K$ and any $\epsilon > 0$, there exists a polynomial $g \in \mathbb{C}[z]$ such that $d(J(g), K) < \epsilon$.

In this note, we generalize Corollary 1.

Before stating our main result, we recall some definitions. A continuum is a dendrite if it is locally connected and has empty interior. Given a complex polynomial $p \in \mathbb{C}[z]$, a complex number $c$ is a critical point of $p$ if $p'(c) = 0$; its image $p(c)$ is a critical value. We denote by $C(p) := \{c : p'(c) = 0\}$ the set of critical points of $p$. A polynomial $f$ is a Belyi polynomial if $\deg(f) > 1$ and if its set of critical values $f(C(f))$ is contained in the set $\{0, 1\}$; these have been much studied from many points of view, see, e.g., [7]. We next introduce some dynamical notions.

A polynomial $g \in \mathbb{C}[z]$ is postcritically finite if $P(g) := \{g^j(c) : c \in C(g), j > 0\}$ is finite. If $g$ is postcritically finite, the following facts are known (see, e.g., [5]): $J(g)$ is connected and locally connected, and is a dendrite if and only if no element of $C(g)$ is periodic. In [6], a Belyi polynomial $g$ is called an extra-clean dynamical Belyi polynomial if $P(g) = \{0, 1\}$, $g(0) = g(1) = 0$, and $g'(0) \neq 0, g'(1) \neq 0$; we denote the set of such polynomials by $XDBP$. Note that if $g \in XDBP$ then $J(g)$ is a dendrite. Theorem 3.6 in [3] implies that each $g \in XDBP$ is naturally a point on a zero-dimensional variety defined over $\mathbb{Q}$. It follows that if $g \in XDBP$ then the coefficients of $g$ lie in the field $\mathbb{Q}$ of algebraic numbers. Two polynomials $g_1, g_2$ are conjugate as dynamical systems if there exists $A(z) = az + b, a, b \in \mathbb{C}, a \neq 0$, such that $g_2 = A \circ g_1 \circ A^{-1}$. We denote by

$$G := \{A \circ g \circ A^{-1} : A(z) = az + b, a, b \in \mathbb{Q}, a \neq 0, g \in XDBP\} \subset \mathbb{Q}[z].$$

Since $\mathbb{Q}[z]$ is countable, so is $G$.

Our main result is:

Theorem 2. Given any continuum $K \subset \mathbb{C}$ and any $\epsilon > 0$, there exists a polynomial $g \in G$ with $d(J(g), K) < \epsilon$.

A key ingredient in our proof is an approximation result of the first author wherein continua are approximated by sets of the form $f^{-1}([0, 1])$, where $f$ is a Belyi polynomial and $[0, 1] \subset \mathbb{C}$ is the unit interval.

In this paragraph, we introduce some terminology and perspective related to Belyi polynomials; see [7]. We denote by $BP$ the set of Belyi polynomials. If $f \in BP$, its dessin is $D(f) := f^{-1}([0, 1])$. By ibid. Lemma 3.4, $D(f)$ is a tree with vertices $V(f) := f^{-1}((0, 1))$; an edge $e$ of $D(f)$ is the closure of a component of $f^{-1}((0, 1))$. Thinking of $[0, 1]$ as a tree with a single edge and with two vertices $v_0 = 0, v_1 = 1$, the map $f : D(f) \to [0, 1]$ sends a closed edge $e$ of $D(f)$ homeomorphically to the edge $[0, 1]$. Thus the valence of a vertex $\bar{v}$ of $D(f)$, defined as the number of edges incident to $\bar{v}$, coincides with the local degree $\deg(f, \bar{v})$ of $f$ at $\bar{v}$, defined as the multiplicity of the zero of the polynomial $z \mapsto f(z) - f(\bar{v})$. A leaf of $D(f)$ is a vertex $\bar{v}$ of valence 1. Hence a vertex $\bar{v}$ of $D(f)$ is a critical point of $f$ if and only if it is not a leaf.
The approximation result we use is the following theorem.

**Theorem 3.** Given any continuum $K \subset \mathbb{C}$ and any $\epsilon > 0$, there exists $f \in \mathcal{B}P$ for which (i) $d(D(f), K) < \epsilon$, (ii) for each $\tilde{v} \in V(f)$, $\deg(f, \tilde{v}) \leq 4$, and (iii) the coefficients of $f$ belong to $\overline{\mathbb{Q}}$.

**Proof.** Conclusion (i) is Theorem 1.1 in [2]; (ii) follows from its proof; see op. cit. §3, paragraph 3. We now prove (iii). Let $f \in \mathcal{B}P$ satisfy (i) with $d(D(f), K) < \epsilon/2$ and also (ii). Belyi’s theorem and the Grothendieck correspondence [7] imply that there exists $h_0(z) = a_0z + b_0$, $a_0, b_0 \in \mathbb{C}, a_0 \neq 0$, for which $f \circ h_0 \in \overline{\mathbb{Q}}[z]$. Using the density of $\overline{\mathbb{Q}}$ in $\mathbb{C}$, choose $a_1, b_1 \in \overline{\mathbb{Q}}$ with $a_1 \approx a_0, b_1 \approx b_0$ so that

$$\max\{|(h_1 \circ h_0^{-1})(z) - z| : z \in D(f)\} < \epsilon/2,$$

and put $f_1 := f \circ h_0 \circ h_1^{-1} \in \overline{\mathbb{Q}}[z]$. Then $f_1$ satisfies conditions (ii) and (iii), and (i) holds since $D(f_1) = (h_1 \circ h_0^{-1})(D(f))$ and

$$d(D(f_1), K) \leq d(D(f_1), D(f)) + d(D(f), K) < \epsilon. \quad \square$$

The proof of our main result, Theorem 2, has two steps. Suppose $K \subset \mathbb{C}$ is a continuum and $\epsilon > 0$ is given.

1. We apply Theorem 3 to obtain a polynomial $f \in \mathcal{B}P \cap \overline{\mathbb{Q}}[z]$ satisfying both $d(D(f), K) < \epsilon/2$ and the valence condition (ii).

2. We define a sequence of polynomials $g_n \in G$ such that $d(J(g_n), D(f)) \to 0$ as $n \to \infty$. The convergence will be proven in Lemma 1; it is here we use the valence condition on $f$. Then, choosing $n$ such that $d(J(g_n), D(f)) < \epsilon/2$ will establish that $d(J(g_n), K) < \epsilon$, completing the proof.

In the next two paragraphs, we construct the polynomials $g_n$.

Let $q(z) := 4z(1 - z)$. Note that $q \in \mathcal{B}P$, that $q([0, 1]) = q^{-1}([0, 1]) = [0, 1]$, and that $q(0) = q(1) = 0$, with $C(q) = \{1/2\}$. For each $n \in \mathbb{N}$, $n \geq 1$, we have $q^n \circ f \in \mathcal{B}P \cap \overline{\mathbb{Q}}[z]$ and $D(q^n \circ f) = D(f)$ as subsets of $\mathbb{C}$. Their tree structures differ: each edge of $D(f)$ is a union of $2^n$ edges of $D(q^n \circ f)$. It is easy to see that the set of leaves of $D(q^n \circ f)$ coincides with the set of leaves of $D(f)$, and that if $\tilde{v}$ is such a leaf then $(q^n \circ f)(\tilde{v}) = 0$. Lemma 2 will say that we can make edges of $q^n \circ f$ as small as we like by choosing $n$ sufficiently large. Since $D(q^n \circ f) = D(f)$ as sets, the valence of the tree $D(q^n \circ f)$ remains bounded above by 4.

We now turn $q^n \circ f$ into a dynamical system; cf. [6]. Suppose $v_0, v_1 \in V(f)$ are leaves of $D(f)$, that is, vertices of valence 1. By replacing $f$ with $q \circ f$, we may assume that $f(v_0) = f(v_1) = 0$. The assumption $f \in \overline{\mathbb{Q}}[z]$ implies $v_0, v_1 \in \overline{\mathbb{Q}}$. Let $A(z) = (v_1 - v_0)z + v_0$, so that $A(0) = v_0, A(1) = v_1$. Fix $n \in \mathbb{N}$. Let $g_n := A \circ q^n \circ f$.

The paragraph below discusses the properties of the polynomials $g_n$.

By construction, $g_n \in \overline{\mathbb{Q}}[z]$ and $g_n$ has two critical values, namely $v_0$ and $v_1$. We have $D(f) = D(q^n \circ f) = g^{-1}_n([v_0, v_1])$ as sets. As trees, now an edge $e$ of $D(q^n \circ f)$ is the closure of a component of $g^{-1}_n((v_0, v_1))$, where $(v_0, v_1)$ is the interval $[v_0, v_1]$ minus its endpoints. Abusing notation slightly, we denote by $V(g_n) := g^{-1}_n\{v_0, v_1\}$ the set of vertices of $D(q^n \circ f)$. Each critical point of $g_n$ maps under $g_n$
either to $v_0$ or to $v_1$; by construction, $v_0 = g_n(v_0) = g_n(v_1)$, and $g'_n(v_0) \neq 0$, $g'_n(v_1) \neq 0$. It follows that $P(g_n) = \{v_0, v_1\} \subset \mathbb{Q}$, so that $g_n$ is postcritically finite, and that every critical point lands on the fixed point $v_0$ under iteration of $g_n$. It is a general fact that all fixed points of a postcritically finite map $g_n$ are either critical points or they lie in the Julia set. We conclude $v_0 \in J(g_n)$. Since $g_n(v_1) = v_0$, we have $v_1 \in J(g_n)$ too. Hence $V(g_n) = g_n^{-1}(\{v_0, v_1\}) \subset J(g_n)$ by invariance of $J(g_n)$; moreover, $J(g_n)$ is a dendrite. The valence condition on $f$ implies that the local degree of $g_n$ at any point is at most 4. Since $A^{-1} \circ g_n \circ A \in XDBP$ and $A \in \mathbb{Q}[z]$, we conclude $g_n \in G$.

![Figure 1](image_url)

The proof of Theorem 2 then rests upon establishing the closeness that Figure 1 suggests:

**Lemma 1.** The Hausdorff distance $d(J(g_n), D(f)) \to 0$ as $n \to \infty$.

### 2. Proof of Lemma 1

Suppose $f, q, n, g_n$ are as in step 2 of the outline given in the Introduction.

**Lemma 2.** The maximum diameter of an edge $e$ of $D(q^n \circ f)$ tends to zero as $n \to \infty$.

**Proof.** An easy exercise shows the conclusion holds when $f = q$. Now suppose $f \in BP$. Since the inverse branches of $f$ are uniformly continuous on $(0, 1)$, the general conclusion holds. $\square$

Let $D := D(f)$. We recall from step 2 the following: $D = g_n^{-1}(\{v_0, v_1\})$; the set $g_n^{-1}(\{v_0, v_1\})$ is the set of vertices of the tree $D$; the edges of $D$ are the closures of the components of $g_n^{-1}(v_0, v_1)$, where $(v_0, v_1)$ is the Euclidean segment $[v_0, v_1]$ minus its endpoints.

We are going to cover $D$ by a certain pair of Jordan domains $W_i$ with the property that $W_i \cap \{v_0, v_1\} = v_i, i = 0, 1$. See Figure 2.
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Figure 2. Caricature of $W_1$. The domain $W_0$ is similar. The disk shown is $B := B \left( \frac{-1 + \sqrt{5}}{2}, 10M \right)$. The domain $\tilde{W}_1$ is the portion of the disk to the right of the longer vertical segment. The figure is not to scale; one should imagine that $v_0, v_1$ appear much closer together compared to the diameter of $B$, and that $D$ is contained in the smaller disk $\frac{1}{M}B$ with the same center and $\frac{1}{M}$ the radius.

Their precise definition is a bit technical; we will give it later. Let $W$ denote either of the domains $W_0, W_1$, and let $\tilde{W}$ be a connected component of $g_n^{-1}(W)$; it will also be a Jordan domain. We will show $\text{diam} \tilde{W} \to 0$ uniformly in $n$ (Lemma 3). Lemma 1 will then follow easily.

In order to control the diameters of the domains $\tilde{W}$, we will thicken the domains $W_0, W_1$ to Jordan domains $\hat{W}_0, \hat{W}_1$ so that $\hat{W}_i \supset W_i \cap \{v_0, v_1\}$ and in addition $\hat{W}_i \cap \{v_0, v_1\} = W_i \cap \{v_0, v_1\} = v_i, i = 0, 1$. Now suppose $W, \tilde{W}$ are as in the previous paragraph. Let $\hat{W}$ be the thickening of $W$. There is a unique component $\hat{\tilde{W}}$ of $g_n^{-1}(\hat{W})$ that contains $\tilde{W}$; it is a thickening of $\tilde{W}$. The “Koebe space” $\hat{\tilde{W}} \setminus \tilde{W}$ will allow us to control distortion and relate the diameter of $\tilde{W}$ to the diameter of the edge it meets.

Suppose $W, \tilde{W}, \hat{W}, \hat{\tilde{W}}$ are as in the previous two paragraphs. Choose a point $v := W \cap \{v_0, v_1\}$; it is a branch value of $g_n$. Since $g_n$ is a polynomial, we obtain a map of pairs $g_n : (\hat{W}, \tilde{W}) \to (\hat{\tilde{W}}, \hat{W})$ in which each restriction is proper and each domain is a Jordan domain. Since $\tilde{W}$ contains exactly one branch value of $g_n$, the preimage $\tilde{W} \cap g_n^{-1}(v)$ consists of a single point, which we will denote by $\tilde{v}$, which is a vertex of $D$. Since $v \in W$, we have $\tilde{v} \in \tilde{W}$. Let $k := \text{deg}(g_n, \tilde{v})$. Since the ramification of $g_n : \tilde{W} \to \hat{W}$, if there is any, occurs at the unique point $\tilde{v}$, we have $\text{deg}(g_n : \tilde{W} \to \hat{W}) = k$ as well. The control on the local degrees of the polynomial $f$ in Theorem 3 shows that $k \leq 4$. Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. Up to precomposition with a rotation about the origin, there exists a unique Riemann map $\phi : (\mathbb{D}, 0) \to (\hat{\tilde{W}}, \tilde{v})$. Since $g_n : \tilde{W} \to \hat{W}$ is ramified only possibly at $\tilde{v}$, we obtain a Riemann map $\hat{\phi} : (\mathbb{D}, 0) \to (\hat{\tilde{W}}, \tilde{v})$ such that the following diagram
commutes:

\[
\begin{align*}
\left( \tilde{W}, \tilde{v} \right) & \xrightarrow{2^n} \left( \hat{W}, v \right) \\
\phi & \quad \phi \\
(D, 0) & \xrightarrow{z \to z^k} \left( D, 0 \right)
\end{align*}
\]

We will apply the Koebe distortion principle to the map \( \tilde{\phi} \) and conclude that the diameter of \( \tilde{W} \) is bounded from above in terms of the diameters of the edges of \( D \); by Lemma 2, these tend to zero as \( n \to \infty \).

We now construct the domains \( W_0, W_1 \). First, denote \( M := \text{diam}(D) \) and \( B(a, r) := \{ z \in \mathbb{C} : |z - a| < r \} \). Next see Figure 2.

We now give the definitions of the sets \( W_i \) and \( \hat{W}_i \). Let

\[
\begin{align*}
v_0' & := \frac{7v_0 + v_1}{8}, \quad v_0'' = \frac{3v_0 + v_1}{4} \\
v_1' & := \frac{v_0 + 7v_1}{8}, \quad v_1'' = \frac{v_0 + 3v_1}{4} \\
\hat{W}_{-1} & := B \left( \frac{v_1 + v_0}{2}, 10M \right) \cap \{|z - v_1'| < |z - v_1|\}, \ i = 0, 1 \\
W_{-1} & := B \left( \frac{v_0 + v_1}{2}, 9M \right) \cap \{|z - v_1'| < |z - v_1|\}, \ i = 0, 1.
\end{align*}
\]

By construction,

- \( \hat{W}_i \cap \{ v_0, v_1 \} = W \cap \{ v_0, v_1 \} = v_i, \ i = 0, 1 \);
- \( D \subset W_0 \cup W_1 \);
- \( \hat{W}_i \setminus W_i \) is an annulus, \( i = 0, 1 \).

**Lemma 3.** The maximum diameter of a component \( \hat{W} \) tends to zero as \( n \to \infty \).

**Proof.** Suppose \( g_n : (\tilde{W}, \tilde{W}) \to (\hat{W}, W) \) is a map of pairs as in the preceding paragraphs; we adopt the notation used there. Up to precomposition with rotations about the origin, the map \( \phi \) is one of only two possible Riemann maps. Hence there exist \( 0 < r < s < 1 \) such that if \( U := \phi^{-1}(W) \), then

\[
B(0, r) \subset U \subset B(0, s) \subset D.
\]

Denote

\[
\bar{U} := \{ z \in \mathbb{D} \mid z^k \in U \}.
\]

From the second part of Theorem 3 we have \( 1 \leq k \leq 4 \). Hence

\[
r \leq \bar{r} := r^{1/k}, \quad \bar{s} := s^{1/k} \leq s^{1/4},
\]

and

\[
B(0, r) \subset B(0, \bar{r}) \subset \bar{U} \subset B(0, \bar{s}) \subset B(0, s^{1/4}) \subset D;
\]

(2.1)
note that \( r \) and \( s^{1/4} \) do not depend on the choice of component \( \tilde{W} \). By definition, the following diagram commutes:

\[
\begin{array}{ccc}
(W, v) & \xrightarrow{g_n} & (\tilde{W}, \tilde{v}) \\
\phi & & \phi \\
(U, 0) & \xrightarrow{\tilde{v}} & (\tilde{U}, 0)
\end{array}
\]

The rescaled map \( \psi := |\tilde{\phi}'(0)|^{-1}(\tilde{\phi} - \tilde{\phi}(0)) \) is an element of the class of so-called \textit{Schlicht functions}: injective holomorphic maps \( \psi : \mathbb{D} \to \mathbb{C} \) with the normalization \( \psi(0) = 0, \psi'(0) = 1 \). By Theorem 5.3 in [1], for all \( z \in \mathbb{D} \) and all Schlicht functions \( \psi \),

\[
|z|(1 + |z|)^{-2} \leq |\psi(z)| \leq |z|(1 - |z|)^{-2}.
\]

Hence upon setting \( \rho := r(1 + r)^{-2} \), \( \sigma := s^{1/4}(1 - s^{1/4})^{-2} \), \( \delta := |\tilde{\phi}'(0)| \)

we have by (2.1) that

\[
B(\tilde{v}, \rho \delta) \subset \tilde{W} \subset B(\tilde{v}, \sigma \delta).
\]

Let \( e \) be any one of the \( k \) components of \( g_n^{-1}((v_0, v_1)) \) whose closure meets \( \tilde{v} \); the closure of \( e \) is an edge of \( D \) containing \( \tilde{v} \). Since \( (v_0, v_1) \not\subset W \), we have \( e \not\subset \tilde{W} \), so

\[
\rho \delta < \text{diam}(e)
\]

which implies

\[
\sigma \delta < \text{diam}(e) \frac{\sigma}{\rho}
\]

and so

\[
\text{diam}(\tilde{W}) \leq 2\sigma \delta < 2\text{diam}(e) \frac{\sigma}{\rho} \to 0
\]

as \( n \to \infty \), by Lemma 2. The constants \( \rho, \sigma \) are independent of \( n \) and of the choice of \( \tilde{v} \), so the proof of Lemma 3 is complete.

\[\square\]

Proof of Lemma 1. Let \( W_0, W_1 \) be the domains as defined above, and let \( \tilde{W}_\tilde{v} \), \( \tilde{v} \in V := g_n^{-1}((v_0, v_1)) \) denote the components of preimages \( g_n^{-1}(W_i), i \in \{0, 1\} \). Denote \( J := J(g_n) \). Pick \( \epsilon < \frac{1}{4} \inf\{|a - b| : a \in D, b \in \mathbb{C} \setminus W_0 \cup W_1\} \). Apply Lemma 3 to obtain \( n \) so that \( \text{diam}(\tilde{W}_\tilde{v}) < \epsilon \) for all \( \tilde{v} \in V(g_n) \). Each \( \tilde{W}_\tilde{v} \) is a Jordan domains, so it has the same diameter as its closure.

On the one hand, by our choice of \( \epsilon \),

\[
g_n^{-1}(\tilde{W}_0 \cup \tilde{W}_1) = \bigcup_{\tilde{v} \in V} \tilde{W}_\tilde{v} \subseteq_{\text{Lemma 3}} N_{\epsilon}(D) \subset \tilde{W}_0 \cup \tilde{W}_1
\]

and so \( \tilde{W}_0 \cup \tilde{W}_1 \) is backward-invariant under \( g_n \). It is a general fact that \( J \) may be equivalently defined as the smallest closed subset of \( \mathbb{C} \) satisfying \( \#J > 1 \) and \( g_n^{-1}(J) \subset J \); see [5].
Thus \( J \subset W_0 \cup W_1 \). By invariance of \( J \) we have then

\[
J \subset g_n^{-1}(W_0 \cup W_1) = \bigcup_{\tilde{\epsilon} \in V} \tilde{\epsilon} W_{\tilde{\epsilon}} \subset N_\epsilon(D).
\]

On the other hand, recalling the last sentence of Step 2, we have \( V \subset J \), and \([v_0, v_1] \subset W_0 \cup W_1\) implies \( D = g_n^{-1}([v_0, v_1]) \subset g_n^{-1}(W_0 \cup W_1) = \bigcup_{\tilde{\epsilon} \in V} \tilde{\epsilon} W_{\tilde{\epsilon}}\), so by our choice of \( \epsilon \) and \( n \), we have

\[
N_\epsilon(J) \supset N_\epsilon(V) \supset \bigcup_{\tilde{\epsilon} \in V} \tilde{\epsilon} W_{\tilde{\epsilon}} \supset D.
\]

This completes the proof of Lemma 1 and establishes Theorem 2. \( \square \)

References


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