Solved and Unsolved Problems

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I am not really doing research, just trying to cultivate myself.

Alexander Grothendieck (1928–2014)

With this opportunity, I would like to express my deepest thanks to Professor Martin Raussen, who appointed me as a member of the Editorial Board of the Newsletter of the EMS in charge of the problem corner in 2005. I would also like to express my gratitude to Professor Krzysztof Ciesielski for proposing in 2004 that I write an article in the Newsletter of the EMS, which subsequently initiated my communication with Martin Raussen, with whom I have had a wonderful and productive collaboration. I note that editors generally serve for four years and I feel deeply honoured that my membership as the problem column editor has lasted for more than 10 years. Thus, I wish to express my sincere thanks to Professors Vicente Munoz and Lucia Di Vizio, who served as Editors-in-Chief after Martin Raussen; I continued to have a wonderful collaboration with them.

The preparation of this column has been very stimulating and a source of great pleasure. From the very beginning, the "Problem Corner" has appeared in two issues per year (the March and September issues) with six proposed problems and two open problems. In every subsequent issue in which the problem corner has appeared, the solutions of the previous proposed problems have appeared together with the names of additional problem-solvers. In total, 170 problems have appeared in the problem column while I have served as its editor. Mathematicians from all over the world have participated in this effort. Going through the issues of the Newsletter of the EMS, one can see problems proposed or solved by mathematicians from Australia, Canada, China, Denmark, England, Germany, Greece, Hong-Kong, Iran, Ireland, India, Italy, Poland, Portugal, Romania, Russia, Sweden, Ukraine, USA and others.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

163. Find all positive integers m and n such that the integer

$$a_{m,n} = \underbrace{2\ldots2}_{m\text{ time}}\underbrace{5\ldots5}_{n\text{ time}}$$

is a perfect square.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

164. Prove that every power of 2015 can be written in the form $\frac{x^2+y^2}{x-y}$, with x and y positive integers.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

165. Find the smallest positive integer k such that, for any $n \geq k$, every degree n polynomial $f(x)$ over $\mathbb{Z}$ with leading coefficient 1 must be irreducible over $\mathbb{Z}$ if $|f(x)| = 1$ has not less than $\left[\frac{n}{2}\right] + 1$ distinct integral roots.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

166. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing (f not necessarily continuous). If $f(0) > 0$ and $f(100) < 100$, show that there exists $x \in \mathbb{R}$ such that $f(x) = x$.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

167. Show that, for any $a, b > 0$, we have

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

168. Let $f : I \to \mathbb{C}$ be an n-time differentiable function on the interior $\mathring{I}$ of the interval I, and $f^{(n)}$, with $n \ge 1$, be locally absolutely continuous on $\mathring{I}$. Show that, for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in \mathbb{R}\setminus\{0, 1\}$, we have the representation

$$\begin{aligned} f(x) &= (1-\lambda) f(a) + \lambda f(b) \\ &+ \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda) f^{(k)}(a)(x-a)^k + (-1)^k \lambda f^{(k)}(b)(b-x)^k\right] \\ &+ S_{n,\lambda}(x,a,b), \end{aligned} \quad (1)$$

where the remainder $S_{n,\lambda}(x, a, b)$ is given by

$$\begin{aligned} &S_{n,\lambda}(x,a,b) \\ &:= \frac{1}{n!}\left[(1-\lambda)(x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds \right. \\ &\left. + (-1)^{n+1}\lambda(b-x)^{n+1}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds\right]. \end{aligned} \quad (2)$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

II Two new open problems

169*. Find all functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation

$$[f(x) - f(y)]\,k(x+y) = [g(x) - g(y)]\,h(x+y) \quad (3)$$

for all $x, y \in \mathbb{R}$.

Remark. The above open problem appeared in the book of Sahoo and Riedel (see Section 2.7, page 80 in [2]). In a recent paper, Balogh, Ibrogimov and Mityagin [1] have given a partial solution to this open problem.

References

[1] Z. M. Balogh, O. O. Ibrogimov and B. S. Mityagin, Functional equations and the Cauchy mean value theorem, *Aequat. Math.*, (2016), DOI 10.1007/s00010-015-0395-6.
[2] P. K. Sahoo and T. Riedel, Mean Value Theorems and Functional Equations, World Scientific, Singapore, 1998.

(Prasanna K. Sahoo, University of Louisville, Louisville, USA)

170*. Consider the operators

$$P_n^\alpha(f,x) = \sum_{k=0}^{\infty} v_{n,k}^\alpha(x) f(k/n),$$

where

$$v_{n,k}^\alpha(x) = \binom{n+k-1}{k}\frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}, \quad \text{for } x \in [0,\infty),$$

and $x^{[k,-\alpha]} = x(x+\alpha)\cdots(x+(k-1)\alpha)$. In the case $\alpha = 1/n$, we can write this in an alternative form as

$$\begin{aligned} v_{n,k}^{1/n}(x) &= \binom{n+k-1}{k}\frac{(nx)_k.(2n)!}{2(n!)(nx+n)_{n+k}} \\ &= \frac{(n)_k}{k!}.\frac{(nx)_k.(2n)!}{2(n!)(nx+n)_n(nx+2n)_k}. \end{aligned}$$

If we denote the m-th order moment by

$$T_{n,m}(x) = \sum_{k=0}^{\infty} v_{n,k}^{1/n}(x)\left(\frac{k}{n}\right)^m \quad (4)$$

then, by simple computation, we have $T_{n,0}(x) = 1, T_{n,1}(x) = \frac{nx}{n-1}$. Examine whether a recurrence relation can be obtained for $T_{n,m}(x)$ between the moments.

(Vijay Gupta, Netaji Subhas Institute of Technology, New Delhi, India)

III Solutions

152. Let G be an arbitrary group written multiplicatively. Let $\sigma : G \to G$ be an anti-homomorphism (i.e., $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$) satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. Let $\mathbb{C}$ be the field of complex numbers.

(i) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(xy) + f(\sigma(y)x) = 2 f(x) \quad (5)$$
for all $x, y \in G$.

(ii) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(xy) - f(x\sigma(y)) = 2 f(y) \quad (6)$$
for all $x, y \in G$.

(iii) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(x\sigma(y)) = f(x) f(y) \quad (7)$$
for all $x, y \in G$.

(*Prasanna K. Sahoo, University of Louisville, Louisville, USA*)

Solution of problem 152 (i), (ii) and (iii). Let $Hom(G, \mathbb{C})$ be the set of all homomorphisms from group G to the additive group $(\mathbb{C}, +)$ of $\mathbb{C}$ and $Hom(G, \mathbb{C}^\star)$ be the set of all homomorphisms from group G to the multiplicative group of non-zero complex numbers $\mathbb{C}^\star$. A function $f : G \to \mathbb{C}$ is said to be σ-even if and only if $f(\sigma(x)) = f(x)$ for all $x \in G$. Similarly, a function $f : G \to \mathbb{C}$ is said to be σ-odd if and only if $f(\sigma(x)) = -f(x)$ for all $x \in G$. A function $f : G \to \mathbb{C}$ is called a central function if and only if $f(xy) = f(yx)$ for all $x, y \in G$.

First, we determine all central functions $f : G \to \mathbb{C}$ satisfying functional equations (5) and (6) respectively. Then, we find all functions $f : G \to \mathbb{C}$ that satisfy functional equation (7).

Solution of problem (i). The central solution of functional equation (5) is of the form

$$f(x) = \phi(x) + \alpha, \quad \forall\, x \in G, \tag{6}$$

where ϕ is a σ-odd function in $Hom(G, \mathbb{C})$ and $\alpha \in \mathbb{C}$ is an arbitrary constant. The converse is also true.

It is easy to verify that f given by (6) satisfies (5). It is left to show that (6) is the only solution of (5). Let a, b and c be three arbitrary elements in G. Letting $x = ab$ and $y = c$ in (5), we have

$$f(abc) + f(\sigma(c)ab) = 2f(ab). \tag{7}$$

Next, letting $x = \sigma(c)a$ and $y = b$ in (5), we obtain

$$f(\sigma(c)ab) + f(\sigma(cb)a) = 2f(\sigma(c)a). \tag{8}$$

Use of (7) in (8) yields

$$2f(ab) - f(abc) + f(\sigma(cb)a) = 2f(\sigma(c)a). \tag{9}$$

Using (5), we see that $f(\sigma(cb)a) = 2f(a) - f(acb)$ and $f(\sigma(c)a) = 2f(a) - f(ac)$. In view of these, equation (9) gives rise to

$$f(abc) + f(acb) = 2f(ab) + 2f(ac) - 2f(a).$$

Letting $a = e$ (the identity in group G), we have

$$f(bc) + f(cb) = 2f(b) + 2f(c) - 2f(e).$$

Defining $\phi : G \to \mathbb{C}$ by $\phi(x) := f(x) - \alpha$, where $\alpha := f(e)$, the last equation reduces to

$$\phi(bc) + \phi(cb) = 2\phi(b) + 2\phi(c).$$

Since f is central, ϕ is also central and hence we have $\phi \in Hom(G, \mathbb{C})$. From the definition of ϕ, we obtain $f = \phi + \alpha$. Using this form of f in equation (5), we have $\phi(\sigma(y)) + \phi(y) = 0$ for all $y \in G$. Hence, ϕ is σ-odd.

Solution of problem (ii). If $f : G \to \mathbb{C}$ is any central function that satisfies functional equation (6) for all $x, y \in G$ then f is a σ-odd function in $Hom(G, \mathbb{C})$. The converse is also true.

It is easy to check that any σ-odd homomorphism f from G to $\mathbb{C}$ satisfies functional equation (6). Next, we show that it is the only solution of (6). Let a, b and c be any three arbitrary elements in G. With $x = ab$ and $y = c$ in (6), we get

$$f(abc) - f(ab\sigma(c)) = 2f(c). \tag{10}$$

Next, substitute $x = a$ and $y = b\sigma(c)$ in (6) to obtain

$$f(ab\sigma(c)) - f(ac\sigma(b)) = 2f(b\sigma(c)). \tag{11}$$

Adding (10) and (11), we see that

$$f(abc) - f(ac\sigma(b)) = 2f(c) + 2f(b\sigma(c)). \tag{12}$$

Using (6), we get $f(ac\sigma(b)) = f(acb) - 2f(b)$ and $f(b\sigma(c)) = f(bc) - 2f(c)$. Hence, (12) can be rewritten as

$$2f(bc) + f(acb) - f(abc) = 2f(c) + 2f(b).$$

Letting $a = e$ in the last equation, we obtain $f(bc) + f(cb) = 2f(b) + 2f(c)$ and, since f is central, we have $f \in Hom(G, \mathbb{C})$. Since $f \in Hom(G, \mathbb{C})$, from equation (6) we have $f(x) + f(y) - f(x) - f(\sigma(y)) = 2f(y)$, which proves that f is a σ-odd function in $Hom(G, \mathbb{C})$.

Remark 1. (a) Note that in (i) and (ii) the group G can be replaced by a unital semigroup S.

(b) We have provided the solution of (i) and (ii) assuming f to be a central function. Without this assumption on f, we do not know the solutions of (i) and (ii).

Solution of problem (iii). Every function $f : G \to \mathbb{C}$ that satisfies functional equation (7) is either a zero function or a σ-even function in $Hom(G, \mathbb{C}^\star)$. The converse of this is also true.

It is easy to verify that a zero function or every non-zero σ-even function in $Hom(G, \mathbb{C}^\star)$ satisfies functional equation (7). Next, we show that these are the only solutions of (7).

If f is a constant function then, from (7), we get $f = 0$ or $f = 1$. If $f = 1$ then $f \in Hom(G, \mathbb{C}^\star)$. Further, this f is σ-even. Next, assume that f is a non-constant function. For arbitrary elements $a, b, c \in G$, letting $x = a$ and $y = b\sigma(c)$ in (7), we have $f(ac\sigma(b)) = f(a)f(b\sigma(c))$. Using (5), the last equality can be rewritten as $f(ac)f(b) = f(a)f(b)f(c)$. Thus, $f(b)[f(ac) - f(a)f(c)] = 0$. Since f is non-constant, this implies that $f(ac) = f(a)f(c)$. Hence, $f \in Hom(G, \mathbb{C}^\star)$. Since $f \in Hom(G, \mathbb{C}^\star)$ and non-constant, we have from (7) that f is σ-even.

Remark 2. Note that in (iii) the group G can also be replaced by a unital semigroup. □

Notes.

1. John N. Daras, (pupil, Lyceum of Filothei, Athens, Greece) also solved problems 131 and 142.
2. G. C. Greubel (Newport News, Virginia, USA) also solved problems 149, 153* and 154*.

155. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, with $a, b \in \mathring{I}$ (interior of I), $a < b$ and $\nu \in [0, 1]$. Show that

$$\begin{aligned}(0 \le)&\nu(1-\nu)(b-a)\left[f'_+((1-\nu)a+\nu b) - f'_-((1-\nu)a+\nu b)\right] \\ &\le (1-\nu)f(a) + \nu f(b) - f((1-\nu)a+\nu b) \\ &\le \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right],\end{aligned} \tag{8}$$

where $f'_\pm$ are the lateral derivatives of the convex function f.
In particular, for any $a, b > 0$ and $\nu \in [0, 1]$, show that the following reverses of Young's inequality are valid:

$$(0 \le)(1-\nu)a + \nu b - a^{1-\nu}b^\nu \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{9}$$

and

$$(1 \le)\ \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^\nu} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{10}$$

where K is Kantorovich's constant *defined by*

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{11}$$

(*Sever S. Dragomir, Victoria University, Melbourne City, Australia*)

Solution by the proposer. The case $\nu = 0$ or $\nu = 1$ reduces to equality in (8).

Since f is convex on I, it follows that the function is differentiable on $\mathring{I}$ except at a countable number of points, the lateral derivatives $f'_\pm$ exist at each point of $\mathring{I}$, they are increasing on $\mathring{I}$ and $f'_- \le f'_+$ on $\mathring{I}$.

For any $x, y \in \mathring{I}$, we have, for the Lebesgue integral,

$$f(x) = f(y) + \int_y^x f'(s)\,ds$$
$$= f(y) + (x-y)\int_0^1 f'((1-t)y + tx)dt. \qquad (12)$$

Assume that $v \in (0,1)$. By (12), we have

$$f((1-v)a + vb)$$
$$= f(a) + v(b-a)\int_0^1 f'\Big((1-t)a + t((1-v)a + vb)\Big)dt \qquad (13)$$

and

$$f((1-v)a + vb)$$
$$= f(b) - (1-v)(b-a)\int_0^1 f'((1-t)b + t((1-v)a + vb))\,dt. \qquad (14)$$

If we multiply (13) by $1-v$, (14) by v and add the obtained equalities then we get

$$f((1-v)a + vb) = (1-v)f(a) + vf(b)$$
$$+ (1-v)v(b-a)\int_0^1 f'\Big((1-t)a + t((1-v)a + vb)\Big)dt$$
$$- (1-v)v(b-a)\int_0^1 f'\Big((1-t)b + t((1-v)a + vb)\Big)dt,$$

which is equivalent to

$$(1-v)f(a) + vf(b) - f((1-v)a + vb)$$
$$= (1-v)v(b-a)$$
$$\times \int_0^1 \Big[f'\Big((1-t)b + t((1-v)a + vb)\Big)$$
$$- f'\Big((1-t)a + t((1-v)a + vb)\Big)\Big]dt. \qquad (15)$$

This is an equality of interest in itself.

Since $a < b$ and $v \in (0,1)$, we have $(1-v)a + vb \in (a,b)$ and

$$(1-t)a + t((1-v)a + vb) \in [a, (1-v)a + vb]$$

while

$$(1-t)b + t((1-v)a + vb) \in [(1-v)a + vb, b]$$

for any $t \in [0,1]$.

By the monotonicity of the derivative, we have

$$f'_+((1-v)a + vb) \le f'\Big((1-t)b + t((1-v)a + vb)\Big) \le f'_-(b) \qquad (16)$$

and

$$f'_+(a) \le f'\Big((1-t)a + t((1-v)a + vb)\Big) \le f'_-((1-v)a + vb) \qquad (17)$$

for almost every $t \in [0,1]$.

By integrating the inequalities (16) and (17), we get

$$f'_+((1-v)a + vb) \le \int_0^1 f'((1-t)b + t((1-v)a + vb)\Big)dt \le f'_-(b)$$

and

$$f'_+(a) \le \int_0^1 f'\Big((1-t)a + t((1-v)a + vb)\Big)dt \le f'_-((1-v)a + vb),$$

which implies that

$$f'_+((1-v)a + vb) - f'_-((1-v)a + vb)$$
$$\le \int_0^1 f'\Big((1-t)b + t((1-v)a + vb)\Big)dt$$
$$- \int_0^1 f'\Big((1-t)a + t((1-v)a + vb)\Big)dt$$
$$\le f'_-(b) - f'_+(a).$$

Making use of equality (15), we obtain the desired result (8).

If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$ then, for any $a, b \in \mathring{I}$ and $v \in [0,1]$, we have

$$(0 \le)(1-v)f(a) + vf(b) - f((1-v)a + vb) \qquad (18)$$
$$\le v(1-v)(b-a)[f'(b) - f'(a)].$$

If we write inequality (18) for the convex function $f : \mathbb{R} \to (0,\infty)$, $f(x) = \exp(x)$, then we have

$$(0 \le)(1-v)\exp(x) + v\exp(y) - \exp((1-v)x + vy) \qquad (19)$$
$$\le v(1-v)(x-y)[\exp(x) - \exp(y)]$$

for any $x, y \in \mathbb{R}$ and $v \in [0,1]$.

Let $a, b > 0$. If we take $x = \ln a$, $y = \ln b$ in (19) then we get the desired inequality (9).

Now, if we write inequality (18) for the convex function $f : (0,\infty) \to \mathbb{R}$, $f(x) = -\ln x$, then we get

$$(0 \le)\ln((1-v)a + vb) - (1-v)\ln a - v\ln b \le v(1-v)\frac{(b-a)^2}{ab},$$

namely

$$\ln\left[\frac{(1-v)a + vb}{a^{1-v}b^v}\right] \le v(1-v)\frac{(b-a)^2}{ab}.$$

This is equivalent to the desired result (10). □

Also solved by Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Socratis Varelogiannis (National Technical University of Athens, Greece)

156. Evaluate

$$\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})}{\sqrt{e}}\right]^n.$$

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

Solution by the proposer. Recall that

$$\lim_{x\to 0}\frac{\ln(1+x) - x}{x^2} = -\frac{1}{2}.$$

Hence, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every real number x with $|x| < \delta$, we have

$$-\frac{1}{2} - \varepsilon < \frac{\ln(1+x) - x}{x^2} < -\frac{1}{2} + \varepsilon.$$

Choose an integer n_0 such that, for $n \ge n_0$,

$$\frac{1}{n} < \delta.$$

Therefore, we have

$$\frac{k}{n^2} \le \frac{n}{n^2} = \frac{1}{n} < \delta,$$

implying, for $n \ge n_0$, that

$$-\frac{1}{2} - \varepsilon < \frac{\ln(1+\frac{k}{n^2}) - \frac{k}{n^2}}{\frac{k^2}{n^4}} < -\frac{1}{2} + \varepsilon, k = 0, 1, \ldots, n.$$

Then,

$$-\frac{1}{2}-\varepsilon<\frac{\sum_{k=1}^{n}[\ln(1+\frac{k}{n^2})-\frac{k}{n^2}]}{\sum_{k=1}^{n}\frac{k^2}{n^4}}<-\frac{1}{2}+\varepsilon.$$

Hence,

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[\ln(1+\frac{k}{n^2})-\frac{k}{n^2}]}{\sum_{k=1}^{n}\frac{k^2}{n^4}}=-\frac{1}{2}. \tag{20}$$

On the other hand, using the well-known formula

$$\sum_{k=1}^{n}\frac{k^2}{n^4}=\frac{n(n+1)(2n+1)}{6n^4},$$

we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3}=\frac{1}{3}$$

and from (20) we obtain

$$\lim_{n\to\infty} n\cdot\sum_{k=1}^{n}\left[\ln(1+\frac{k}{n^2})-\frac{k}{n^2}\right]=-\frac{1}{6},$$

that is,

$$\lim_{n\to\infty} n\cdot\left[\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})-\frac{n+1}{2n}\right]=-\frac{1}{6}.$$

Hence,

$$\lim_{n\to\infty}\left[\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})^n-\frac{n+1}{2}\right]=-\frac{1}{6}.$$

It follows that

$$\lim_{n\to\infty}\left[2\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})^n-n\right]=1-\frac{1}{3}=\frac{2}{3}$$

and we obtain

$$\lim_{n\to\infty}\left[\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})^n-\ln(\sqrt{e})\right]=\frac{1}{3}.$$

The last relation is equivalent to

$$\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})}{\sqrt{e}}\right]^n=\sqrt[3]{e}.$$

□

Also solved by Ulrich Abel (University of Applied Sciences, Friedberg, Germany), Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Mihaly Bencze (Brasov, Romania), Albero Bersani (Sapienza Universita di Roma, Italy), John N. Daras, (pupil, Lyceum of Filothei, Athens, Greece), Jorge Mozo Fernandez, (Universidad de Valladolid, Spain), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Edward Omey (KU Leuven, Brussels, Belgium), Angel Plaza (University of Las Palmas de Gran Canaria, Spain) Socratis Varelogiannis (National Technical University of Athens, Greece).

157. Let X be a compact space and $f : X \to X$ be continuous and expansive, that is,

$$d(f(x), f(y)) \geq d(x, y) \quad \forall\, x, y \in X\,.$$

What can be said about the function f?

(*W. S. Cheung, University of Hong Kong, Pokfulam, Hong Kong*)

Solution by the proposer.

(i) Observe that f is clearly $1-1$.

(ii) $f^{-1} : f(X) \to X$ is continuous.
In fact, $\forall\,\varepsilon > 0$, let $\delta := \varepsilon$. Then,

$$d(f(x), f(y)) < \delta \Rightarrow d(x, y) \leq d(f(x), f(y)) < \delta = \varepsilon\,.$$

(iii) f is onto.
In fact, since X is compact and f is continuous, $f(X)$ is compact. If there exists $x \in X \setminus f(X)$, we must have

$$d(x, f(X)) = \text{some positive number } d > 0\,.$$

For any $m, n \in \mathbb{N}$,

$$d \leq d(x, f^m(x)) \leq d(f(x), f^{m+1}(x)) \leq \cdots \leq d(f^n(x), f^{n+m}(x))$$

and therefore $\{f^n(x)\}$ is a sequence in X without accumulation point, which violates the assumption that X is compact.

Combining (i), (ii) and (iii), f is a homeomorphism. □

Also solved by Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Mihaly Bencze (Brasov, Romania), Jorge Mozo Fernandez (Universidad de Valladolid, Spain), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Socratis Varelogiannis (National Technical University of Athens, Greece).

158. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$xf'(x) + kf(-x) = x^2 \quad \forall\, x \in \mathbb{R},$$

where $k > 0$ is an integer.

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*)

Solution by the proposer. We prove that such functions are of the following form:

$$f(x)=\begin{cases}\frac{x^2}{k+2} & \text{if } k \text{ is even,}\\ \frac{Cx^k}{2}+\frac{x^2}{k+2}, \quad C\in\mathbb{R} & \text{if } k \text{ is odd.}\end{cases}$$

We replace x by $-x$ and we have

$$\begin{cases}xf'(x)+kf(-x)=x^2,\\ -xf'(-x)+kf(x)=x^2\end{cases}$$

and this implies, by subtraction, that

$$x(f'(x)+f'(-x))+k(f(-x)-f(x))=0.$$

Let $g:\mathbb{R}\to\mathbb{R},\ g(x)=f(x)-f(-x)$. The previous equation implies that

$$xg'(x)-kg(x)=0,\ \forall\, x\in\mathbb{R}.$$

This implies that $g(x)=Cx^k$, for all $x\in\mathbb{R}$. It follows that

$$f(x)-f(-x)=Cx^k,\ \forall\, x\in\mathbb{R}.$$

Replacing $f(-x)=f(x)-Cx^k$ in the initial differential equation, we get that

$$xf'(x)+kf(x)=Ckx^k+x^2.$$

We multiply this equation by x^{k-1} and we get that

$$\left(x^kf(x)\right)'=Ckx^{2k-1}+x^{k+1},\ \forall\, x\in\mathbb{R},$$

which implies that

$$x^kf(x)=\frac{Cx^{2k}}{2}+\frac{x^{k+2}}{k+2}+C_1.$$

We let $x = 0$ in the previous equality and we get that $C_1 = 0$. This implies that

$$f(x) = \frac{Cx^k}{2} + \frac{x^2}{k+2}, \quad C \in \mathbb{R}. \tag{21}$$

Now we check that if k is an odd integer, functions of the form (21) verify the differential equation and if k is an even integer then functions in (21) verify the differential equation if $C = 0$. The problem is solved. □

Also solved by Ulrich Abel (University of Applied Sciences, Friedberg, Germany), Mihaly Bencze (Brasov, Romania), Jorge Mozo Fernandez (Universidad de Valladolid, Spain), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece), Sotirios E. Louridas (Athens, Greece), Socratis Varelogiannis (National Technical University of Athens, Greece).

159. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$ (interior of I). If there exist the constants d, D such that

$$d \le f''(t) \le D \text{ for any } t \in \mathring{I}, \tag{22}$$

show that

$$\frac{1}{2}\nu(1-\nu)\,d\,(b-a)^2 \le (1-\nu)\,f(a) + \nu f(b) - f((1-\nu)\,a + \nu b) \le \frac{1}{2}\nu(1-\nu)\,D\,(b-a)^2 \tag{23}$$

for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$.
In particular, for any $a, b > 0$ and $\nu \in [0,1]$, show that the following refinements and reverses of Young's inequality are valid:

$$\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\} \le (1-\nu)\,a + \nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\} \tag{24}$$

and

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \le \frac{(1-\nu)\,a + \nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right]. \tag{25}$$

(*Sever S. Dragomir, Victoria University, Melbourne City, Australia*)

Solution by the proposer. We consider the auxiliary function $f_D : I \subset \mathbb{R} \to \mathbb{R}$ defined by

$$f_D(x) = \frac{1}{2}Dx^2 - f(x).$$

The function f_D is differentiable on $\mathring{I}$ and $f_D''(x) = D - f''(x) \ge 0$, showing that f_D is a convex function on $\mathring{I}$.

By the convexity of f_D, we have, for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$, that

$$\begin{aligned}
0 &\le (1-\nu)f_D(a) + \nu f_D(b) - f_D((1-\nu)a + \nu b)\\
&= (1-\nu)\Big(\frac{1}{2}Da^2 - f(a)\Big) + \nu\Big(\frac{1}{2}Db^2 - f(b)\Big)\\
&\qquad - \Big(\frac{1}{2}D((1-\nu)a + \nu b)^2 - f_D((1-\nu)a + \nu b)\Big)\\
&= \frac{1}{2}D[(1-\nu)a^2 + \nu b^2 - ((1-\nu)a + \nu b)^2]\\
&\qquad - (1-\nu)f(a) - \nu f(b) + f_D((1-\nu)a + \nu b)\\
&= \frac{1}{2}\nu(1-\nu)D(b-a)^2 - (1-\nu)f(a) - \nu f(b) + f_D((1-\nu)a + \nu b),
\end{aligned}$$

which implies the second inequality in (23).

The first inequality follows in a similar way by considering the auxiliary function $f_d : I \subset \mathbb{R} \to \mathbb{R}$ defined by $f_d(x) = f(x) - \frac{1}{2}dx^2$, which is twice differentiable and convex on $\mathring{I}$.

If we write inequality (23) for the convex function $f : \mathbb{R} \to (0,\infty)$,

$$f(x) = \exp(x),$$

then we have

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)(x-y)^2 \min\{\exp x, \exp y\} \\
&\quad\le (1-\nu)\exp(x) + \nu\exp(y) - \exp((1-\nu)x + \nu y)\\
&\quad\le \frac{1}{2}\nu(1-\nu)(x-y)^2 \max\{\exp x, \exp y\}
\end{aligned} \tag{26}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $a, b > 0$. If we take $x = \ln a$, $y = \ln b$ in (26) then we get the desired inequality (24).

Now, if we write inequality (23) for the convex function $f : (0,\infty) \to \mathbb{R}$, $f(x) = -\ln x$, then we get, for any $a, b > 0$ and $\nu \in [0,1]$, that

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}} \le \ln((1-\nu)a + \nu b) - (1-\nu)\ln a - \nu\ln b \le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}. \tag{27}$$

Now, since

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2$$

and

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2,$$

we have that (27) is equivalent to the desired result (25). □

Also solved by Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea).

160. Let p be the partition function (counting the ways to write n as a sum of positive integers), extended so that $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Prove that, for $n \geqslant 0$,

$$1 \leqslant \frac{2p(n+2) - p(n+3)}{p(n)} \leqslant \frac{3}{2}.$$

(*Mircea Merca, University of Craiova, Romania*)

Solution by the proposer. To prove this double inequality, we consider the generating function of $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \qquad |q| < 1,$$

and Euler's identity

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \qquad |q|, |z| < 1,$$

where

$$(a;q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{for } n > 0 \end{cases}$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

The left side of the double inequality is equivalent to

$$p(n) - 2p(n-1) + p(n-3) \leqslant 0, \qquad n \neq 0.$$

To prove this inequality, we need to show that the coefficient of q^n in the series

$$\sum_{n=0}^{\infty} (p(n) - 2p(n-1) + p(n-3))q^n = \frac{1-2q+q^3}{(q;q)_\infty}, \quad |q| < 1,$$

is non-positive for $n > 0$. We have

$$\begin{aligned} \frac{1-2q+q^3}{(q;q)_\infty} &= \frac{(1-q)(1-q-q^2)}{(1-q)(q^2;q)_\infty} \\ &= \frac{1}{(q^3;q)_\infty} - \frac{q}{(q^2;q)_\infty} \\ &= \sum_{n\geq 0} \frac{q^{3n}}{(q;q)_n} - \sum_{n\geq 0} \frac{q^{2n+1}}{(q;q)_n} \\ &= 1 - q + \sum_{n\geq 2} \frac{q^{2n+1}}{(q;q)_n}\left(q^{n-1} - 1\right) \\ &= 1 - q - \sum_{n\geq 2} \frac{q^{2n+1}}{(q;q)_{n-2}(1-q^n)} \end{aligned}$$

and we see, for $n > 0$, that the coefficient of q^n is non-positive. We have invoked the fact that

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n, \quad |q| < 1.$$

The right side of the double inequality is equivalent to

$$p(n) - 2p(n-1) + \frac{3}{2}p(n-3) \geqslant 0, \quad n \neq 1.$$

Moreover, considering the trivial inequality

$$2p(n-1) - p(n) \geqslant 0, \quad n > 0,$$

we can write

$$\begin{aligned} &p(n) - 2p(n-1) + \frac{3}{2}p(n-3) \\ &\qquad \geqslant p(n) - 2p(n-1) + 2p(n-3) - p(n-4), \quad n \neq 3. \end{aligned}$$

We show that, except for the coefficient of q, all the coefficients in the series

$$\begin{aligned} &\sum_{n=0}^{\infty} (p(n) - 2p(n-1) + 2p(n-3) - p(n-4))q^n \\ &\qquad = \frac{1-2q^2+2q^3-q^4}{(q;q)_\infty}, \quad |q| < 1, \end{aligned}$$

are non-negative. We have

$$\begin{aligned} \frac{1-2q^2+2q^3-q^4}{(q;q)_\infty} &= \frac{(1-q)^2(1-q^2)}{(q;q)_\infty} \\ &= \frac{1-q}{(q^3;q)_\infty} \\ &= (1-q)\sum_{n=0}^{\infty} \frac{q^{3n}}{(q;q)_n} \\ &= 1 - q + \sum_{n=1}^{\infty} \frac{q^{3n}}{(q^2;q)_{n-1}}. \end{aligned}$$

Clearly, the coefficient of q^0 is 1, the coefficient of q^1 is -1 and, for $k > 1$, all the coefficients of q^k are non-negative. In other words, the inequality

$$p(n) - 2p(n-1) + 2p(n-3) - p(n-4) \geqslant 0$$

is valid for $n \neq 1$. This concludes the proof. □

Also solved by Mihaly Bencze (Brasov, Romania)

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *mathematical analysis*.